A Failure of Representative Democracy

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Workshop on Judgment Aggregation and Voting Theory

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Motivating Example

A population has 3 policy alternatives: $\{a, b, c\} = X$

Each voter has strict preferences over X

How should the population make policy decisions?

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1. Direct Democracy
2. Representative Democracy

1. Direct Democracy

A choice problem arrives, $A \subseteq X$

Population votes over **alternatives**

Aggregate votes to select a policy outcome: $a \in A$

2. Representative Democracy

Set of candidates

For us, candidate $\equiv$ ordinal ranking of alternatives

- a binding, contingent plan of action
- example: candidate $\pi = abc$

Population votes over **candidates**

Aggregate votes to select a candidate $\pi \in \Pi$

A choice problem arrives, $A \subseteq X$

Choice is made according to candidate's ordering, $c_\pi(A) \in A$

Comparing the two:

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Direct Democracy

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Representative Democracy

- population's policy choices will be rational
- **more practical**: fewer transaction costs
- **very common** in practice

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What is the **relationship between direct and representative** democracy?

Does representative democracy implement the policy choices that would be made under direct democracy?

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What is the **relationship between direct and representative** democracy?

Does representative democracy implement the policy choices that would be made under direct democracy?

- Say direct democracy choices are consistent with an ordering *abc* ...
- Does representative democracy select candidate *abc*?

To answer this question, need models of direct and representative democracy

- which **policies are chosen** under direct democracy?
- which **candidates are chosen** under representative democracy?

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- which **policies are chosen** under direct democracy?
- which **candidates are chosen** under representative democracy?

We'll build **models that are as similar as possible** to one another - case **most likely to lead to consistency** between the two methods

We use **tournament theory** to model each

What is tournament theory?

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A **tournament**, $T(X)$, is a complete, asymmetric binary relation on X

A solution of T is a set of best elements, $S(T) \subseteq X$. $S : T \rightarrow 2^x \backslash \phi$

Examples of solutions: the Condorcet winner, the top cycle, or the uncovered set

There is a large, well-known literature on tournament theory (see Laslier 1997)

Model of **direct democracy**: the **majority rule tournament** over the alternatives

- policies are compared pairwise
- policy *a* beats policy *b* if it earns a majority of the votes

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Our populations will have majority tournaments **consistent with an ordering**

- most straightforward case - easy to determine policy choices under direct democracy
- provides **a clear normative** recommendation

To model **representative democracy**, need a theory of **how population votes over candidates**

- assume a voter votes for the candidate with whom she is **most likely to agree** about the policy choice from a randomly-selected set of alternatives
- this rule generates a **tournament over the candidates (or orderings)**
- we define and analyze this new type of tournament

Outline

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- Introduce **models of direct and representative** democracy
- Analyze **structure and solutions** of these tournaments
- **Compare outcomes** across direct and representative democracy

Notation

- space of **alternatives**, $X = \{a, b, ..., n\}$
- **preferences** strict orders on X, identified with permutations π
  - these are also our candidates
- space of **preferences (**and candidates**)**, $\Pi = \{\pi_1, ..., \pi_{n!}\}$
- **choices** made according to ordering, $c_\pi(A) =$ first element of A in the order π
- a **population** λ is a distribution over Π
  - thinking of large populations, won't worry about ties

Notation

- a **tournament** is a complete, asymmetric binary relation
- for tournaments on X, we use $\Gamma(X)$
  - write $a \Gamma b$ if a beats b in $\Gamma(X)$
- for tournaments on Π, we use the traditional $T(\Pi)$
  - write $\pi T \pi'$ if π beats π' in $T(\Pi)$

Figure: Our Approach

$$\lambda$$

Figure: Map a Population into Two Tournaments

$$
\begin{tikzcd}
 & \lambda \arrow[dl] \arrow[dr] & \\
\Gamma(X) & & T(\Pi)
\end{tikzcd}
$$

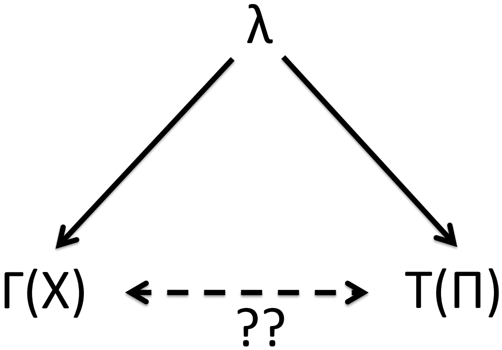
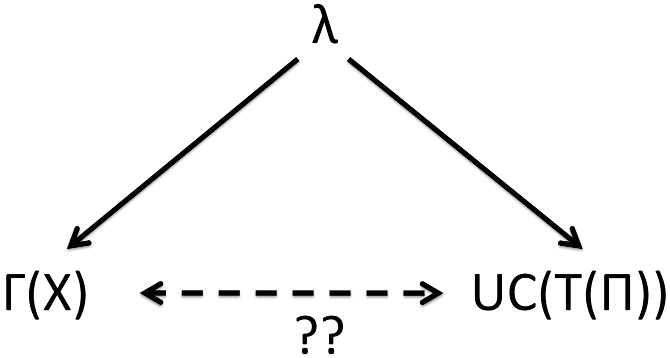


Figure: What is the Relationship between the Two Tournaments?

Figure: How are the Solutions of the Two Tournaments Related?



Model of direct democracy

Use the tournament generated by majority rule

Population votes over elements in X

We have $a \Gamma b$ if $\sum (\lambda(\pi) | a \succ b) > \frac{1}{2}$

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Focus on populations where **majority preferences are consistent with an ordering**

- $a\Gamma b$, $b\Gamma c$, and $a\Gamma c$
- clear normative prediction: representative democracy should elect candidate abc

Model of representative democracy

Voters have preferences over alternatives

Key question: how do **preferences over alternatives map into preferences over candidates**?

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We use **likelihood of agreement** with candidate

- suppose a voter faces a choice between candidate π and candidate π'
- assume she votes for candidate π if she is **more likely to agree** with the choices made by π

What is **more likely to agree**?

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- a voter with ordering π **agrees** with a candidate π' about the choice from A if and only if $c_\pi(A) = c_{\pi'}(A)$
- assume there is a **probability distribution over choice problems** of alternatives
- a voter chooses the candidate with whom she is most likely to agree given this probability distribution

This more likely to agree idea is captured by **choice-based metrics** (Baldiga and Green)

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Under choice-based metrics,

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Let ν be a probability distribution over $\mathcal{X}$,

- then $f(\pi, \pi'; \nu) = \nu\{A \in \mathcal{X} | c_\pi(A) \neq c_{\pi'}(A)\}$

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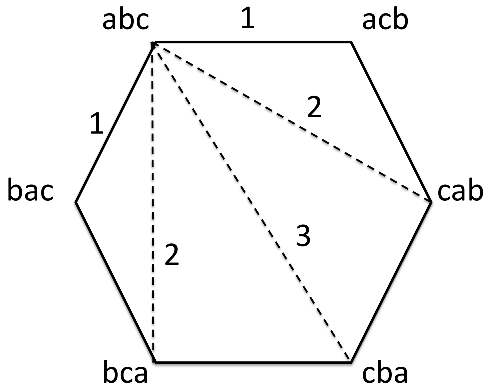
- this choice-based metric is **Kemeny's distance** (aka the bubble sort distance, the Kendall distance)
  - we'll write $f(\pi, \pi'; \mu^K)$
- distance = probability of disagreement about choice from randomly chosen pair of alternatives
- most likely to agree = closest to under the Kemeny distance

Figure: Graphical Representation of the Kemeny Distance



Direct democracy depends only on population's **pairwise preferences** over alternatives - use a model of representative democracy that also depends only on pairwise preferences over alternatives

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- models of direct and representative democracy are as similar as possible
- a **best case** analysis
- prove negative results for this best case
- results won't depend heavily on the Kemeny distance assumption

Tournaments over orderings: models of political competition

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Voters with true preference π'' will vote for π' **if they are more likely to agree** with π' than π.

- π' attracts voters with preference π'' iff $f(\pi'', \pi'; \nu) < f(\pi'', \pi; \nu)$
- π' **attracts a majority against** π if there exists a subset of preferences Π_1 with $\lambda(\Pi_1) > \frac{1}{2}$ such that $f(\pi'', \pi'; \nu) < f(\pi'', \pi; \nu)$ for all $\pi'' \in \Pi_1$

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This form of political action defines a **tournament** $T^{\lambda,\nu}$on Π :

- $\pi' T^{\lambda,\nu} \pi$ if π'attracts a majority against π
- a new type of tournament: a tournament over orderings

Tournament Solutions

T typically has many cycles - there may be no Condorcet winner

We need a **tournament solution** for $T, S(T) \subseteq X$. $S : T \rightarrow 2^x \backslash \phi$

The top cycle of T is very large and therefore not a difficult test to pass.

We will focus on the **uncovered set**.

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- A subset of the top cycle
- Condorcet consistent
- Contains only Pareto undominated elements
- Characterizes the outcomes under several majoritarian voting procedures (Miller, Shepsle and Weingast)
- Contains most other popular tournament solutions (Laslier)

The Uncovered Set of the Tournament

Define the covering relation of T

For a given $T^{\lambda,\nu}$, we say π **covers** π' iff:

(a) $\pi T^{\lambda,\nu} \pi'$, *and*

(b) $\forall \pi'' \in \Pi$, $\pi' T^{\lambda,\nu} \pi'' \Rightarrow \pi T^{\lambda,\nu} \pi''$

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The **uncovered set of** $T^{\lambda,\nu}$ is the set of maximal elements of the covering relation

$\pi \in UC(T^{\lambda,\nu})$ iff $\nexists \pi' \in \Pi$ such that π' covers π

Overview of Results

Result 1: The mapping $\lambda \rightarrow T^{\lambda,\mu^K}$ is not onto. That is, we cannot in general generate an arbitrary tournament, T^{λ,μ^K}.

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On the structure of tournaments over orderings

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There is additional structure that limits the space of tournaments that can be generated

On the structure of tournaments over orderings

There is a **link between** a population's **tournament over alternatives** and its **tournament over orderings**

- Recall that Kemeny distance $\equiv$ # of pairwise disagreements
- Majority preference on $a \succ b$ determines T^{λ,μ^K} for pairs of orderings that disagree on only (a, b)

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Example: $a \succ b$ implies: $abc\ T\ bac$ and $cab\ T\ cba$

All the people with $a \succ b$ are closer to abc (cab) than bac (cba)

Therefore, $abc\ T\ bac \Leftrightarrow cab\ T\ cba$

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Constrains the space of tournaments we can generate

For $n = 3$, only 24 of 32,768 possible $T^{\lambda,\mu^{\kappa}}$ are achievable

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Result 2: Consistency for n=3

Focus on case of $n = 3$

Proposition: Suppose majority rule over alternatives is consistent with an ordering $\pi = abc$. Then, $UC(T^{\lambda,\mu^K}) = \pi = abc$.

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Reach the **same outcomes under direct or representative democracy**

We'll prove this result through a series of diagrams

Step 1: Map majority pairwise preferences into $T^{\lambda,\mu^{\kappa}}$

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Inequality implies a set of T relations:

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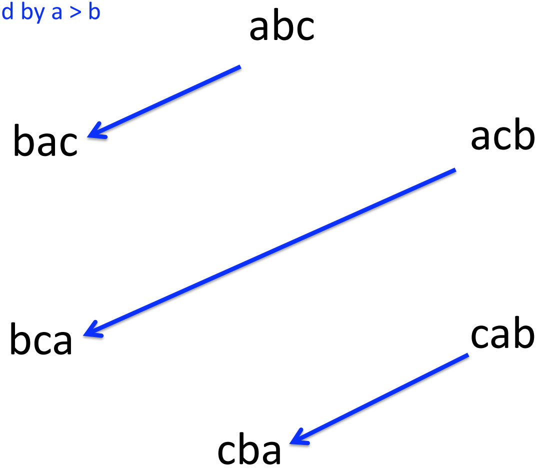
Figure: Result 2

Transitive Majority Rule

- T determined by a > b



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Transitive Majority Rule

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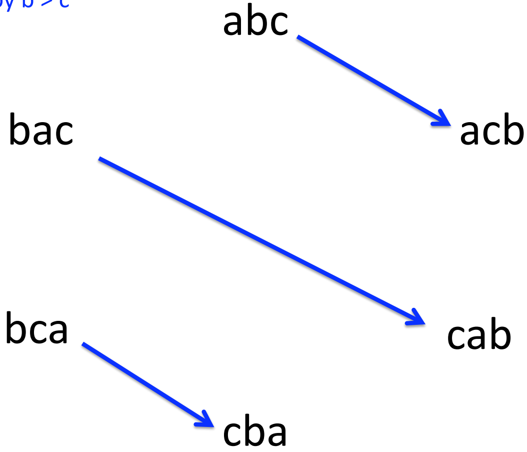

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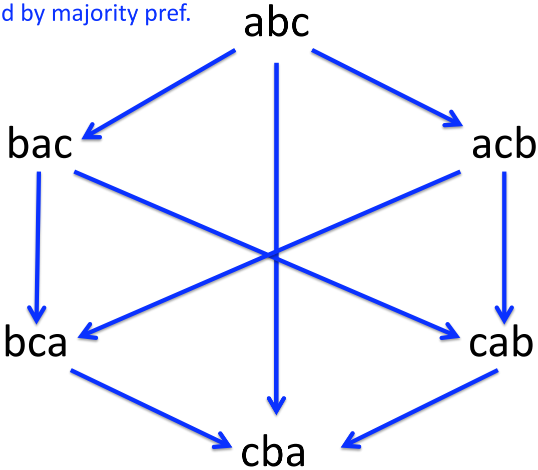
Transitive Majority Rule

- T determined by majority pref.



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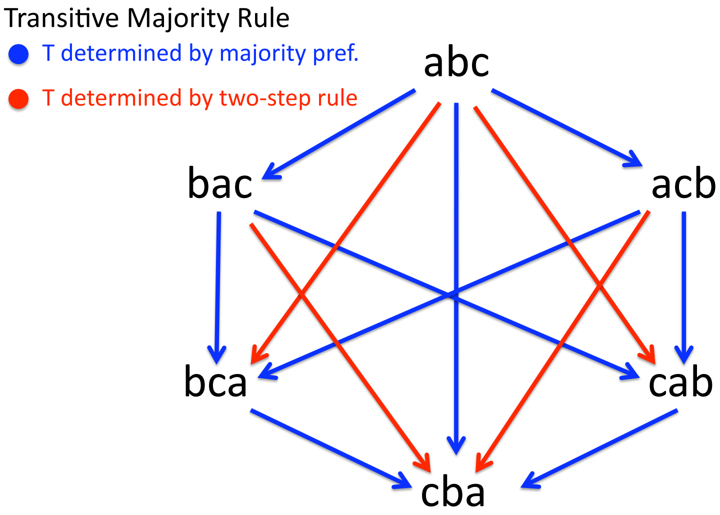

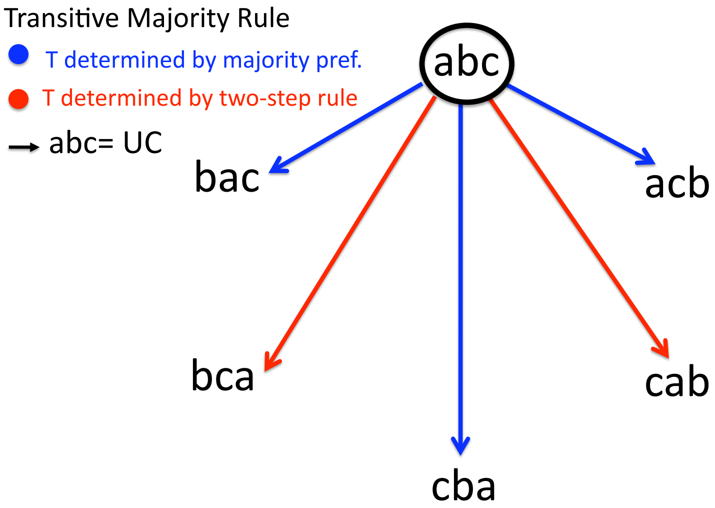

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Choices under direct democracy $\neq$ Choices under representative democracy

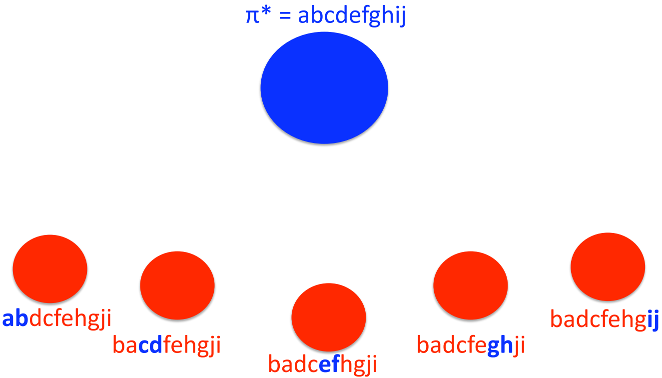
Counterexample: Let $n = 10$

Focus on 5 pairs of mutually exclusive, adjacent elements: $(ab), (cd), (ef), (gh)$, and (ij)

π	$\lambda(\pi)$
$\pi^* = abcdefghij$	$\frac{1}{2} - \varepsilon$
$\mathbf{ab}dcfehgji$	$\frac{\frac{1}{2}+\varepsilon}{5}$
$ba\mathbf{cd}fehgji$	$\frac{\frac{1}{2}+\varepsilon}{5}$
$badc\mathbf{ef}hgji$	$\frac{\frac{1}{2}+\varepsilon}{5}$
$badcfe\mathbf{gh}ji$	$\frac{\frac{1}{2}+\varepsilon}{5}$
$badcfehg\mathbf{ij}$	$\frac{\frac{1}{2}+\varepsilon}{5}$

Majority preferences consistent with $\pi^* = abcdefghij$ on all pairs of alternatives

Figure: Result 3: The Population



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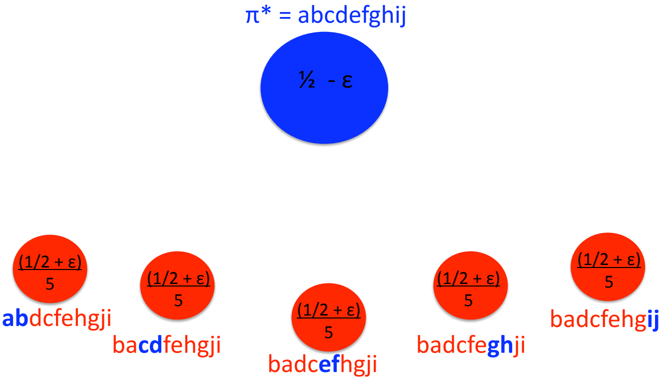

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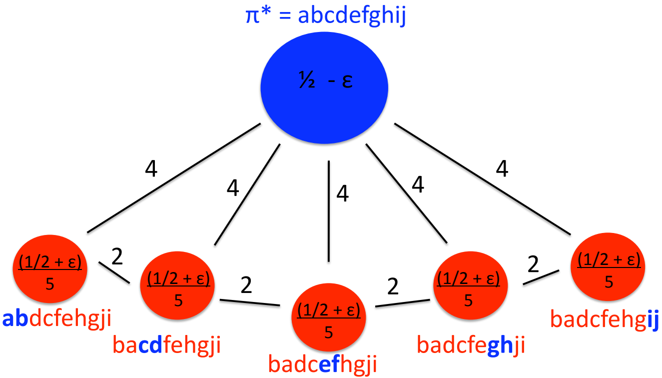

Check that majority rule is consistent with π^* :

- For each of the 5 pairs $\{ab, cd, ef, gh, ij\}$, we have that $(\frac{1}{2} - \varepsilon) + (\frac{\frac{1}{2}+\varepsilon}{5}) > \frac{1}{2}$ agrees with π^*
- For all other pairs, unanimous agreement with π^*

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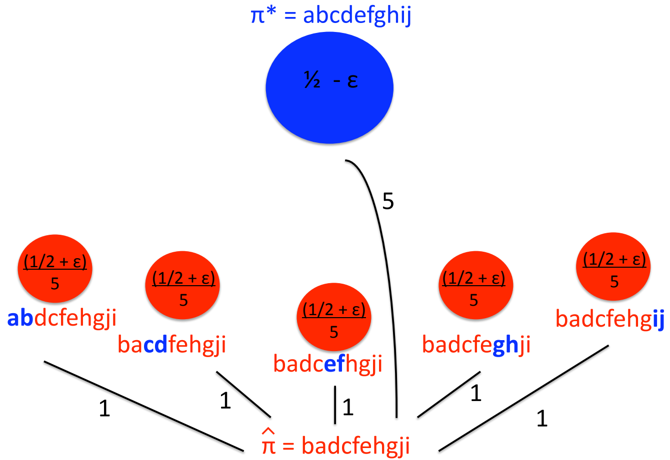
- For all other pairs, unanimous agreement with π^*

Now we'll show that $\pi^* \notin UC(\lambda, \mu^K)$

Let $\hat{\pi} = badcfehgji$

- disagrees with majority preferences on the 5 pairs $\{ab, cd, ef, gh, ij\}$
- agrees with the majority preference on all other pairs

Figure: Result 3: The Population



Claim: $\hat{\pi}$ covers π^*. To prove, we NTS:

(a) $\hat{\pi} T^{\lambda,\mu^K} \pi^*$, *and*

(b) $\forall \pi' \in \Pi$, $\pi^* T^{\lambda,\mu^K} \pi'' \Rightarrow \hat{\pi} T^{\lambda,\mu^K} \pi''$

First show that $\hat{\pi} T \pi^*$

- For all 5 orderings in population other than π^*, we have $f(\pi, \hat{\pi}) = 1$ and $f(\pi, \pi^*) = 4$
- $\frac{1}{2} + \varepsilon$ of the population that is closer to $\hat{\pi}$ than $\pi^* \to \hat{\pi} T \pi^*$.

Now NTS there cannot exist a π' such that $\pi^* T \pi'$ but $\pi' T \hat{\pi}$. Suppose there did exist such a π'

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  - $f(\pi^*, \pi) = 4 \rightarrow f(\pi', \pi) \geq 4$ for at least one of the orderings π in population other than π^*
- $\pi' T \hat{\pi} \rightarrow$ for at least one of the orderings π in population other than π^*, $f(\pi', \pi) \leq f(\hat{\pi}, \pi)$
  - $f(\hat{\pi}, \pi) = 1 \rightarrow f(\pi', \pi) \leq 1$ for at least one of the orderings π in the population other than π^*

Now NTS there cannot exist a π' such that $\pi^* T \pi'$ but $\pi' T \hat{\pi}$. Suppose there did exist such a π'

- $\pi^* T \pi' \rightarrow$ for at least one of the orderings π in population other than π^*, $f(\pi^*, \pi) \leq f(\pi', \pi)$
  - $f(\pi^*, \pi) = 4 \rightarrow f(\pi', \pi) \geq 4$ for at least one of the orderings π in population other than π^*
- $\pi' T \hat{\pi} \rightarrow$ for at least one of the orderings π in population other than π^*, $f(\pi', \pi) \leq f(\hat{\pi}, \pi)$
  - $f(\hat{\pi}, \pi) = 1 \rightarrow f(\pi', \pi) \leq 1$ for at least one of the orderings π in the population other than π^*
- $f(\pi_i, \pi_j) \leq 2$ for any π_i, π_j in population not equal to π^*

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Representative democracy fails to implement choices consistent with π^*

Intuition for result

- Consider the population
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  - Five smaller minority groups, more similar to each other than to the mainstream
- When voting issue-by-issue, **minorities never vote all together** on a contested issue, so mainstream voters are able to implement their preferred choices
- But, under representative democracy, there exist candidates which all five minority groups prefer to π^*, so they can **elect a compromise candidate** $\hat{\pi}$, which disagrees with mainstream preference on all five of the contested issues

Connections to other results

Intuition reveals similarities to papers on issue bundling

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- Besley and Coate (2008) show that citizens' initiatives and electoral competition of representatives may lead to different outcomes in a two-dimensional policy space
- Ahn and Oliveros (2010) study distortions imposed by simultaneous decision of multiple issues

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- **not** driven by **cyclical** collective preferences
  - no tension between collective preferences and producing an ordering
- **not** driven by **tension between majoritarian** methods **and positional** methods
  - all scoring rules applied to example λ would select the ordering π^*

Why use n=10?

We need a candidate that covers π^*. It must beat π^* **and** everything that π^* beats. Following proposition tells us something about which candidates can cover π^*.

Proposition 1: Let majority preferences be consistent with π^*.

1. Take an ordering π. Obtain π' by performing one transposition of alternatives that appeared in the natural order in π. Then, we have $\pi T^{\lambda,\mu^K} \pi'$ for all such π'.
2. Take any such π'. Obtain π'' by performing one transposition of alternatives that appeared in the natural order in π and π'. Then, we have $\pi T^{\lambda,\mu^K} \pi''$.

Corollary: $\pi^* T^{\lambda,\mu^K} \pi$ for all π such that $f(\pi^*, \pi) \leq 2$

Figure: An Illustration of the Proposition

$$\pi^* = abcdefgh$$

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$\pi^* = abcdefgh$

*ba*cdefgh

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π* = abcdefgh

$\downarrow$

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π^* = abcdefgh

$\downarrow$

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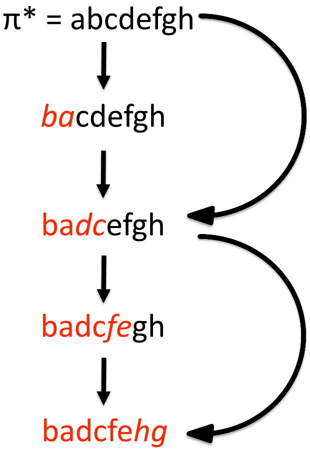
ba*dc*efgh

Figure: An Illustration of the Proposition

$$\begin{tikzcd}
\pi^* = abcdefgh \arrow[d] \arrow[dd, bend left] \\
bacdefgh \arrow[d] \\
badcefgh
\end{tikzcd}$$

Figure: An Illustration of the Proposition



Why use n=10?

Diagram tells us if $f(\pi^*, \hat{\pi}) \leq 4$, $\hat{\pi}$ can't cover π^*. There will always be an ordering π in between such that $\pi^* T \pi$ and $\pi T \hat{\pi}$.

So why $n = 10$?

- Proposition 1 told us we must have $f(\pi^*, \hat{\pi}) > 4$
- So, $\hat{\pi}$ must disagree with π^* on at least 5 pairs
- It is much easier to work with pairs that don't overlap - preferences over each issue are independent from one another
- To have 5 non-overlapping pairs, we need 10 alternatives

There may be a counterexample for $n < 10$ which uses overlapping pairs.

Conclusions

Our contributions:

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Open questions and future work:

- Better understanding of structure and solutions for $n > 3$?
- What is minimal number of alternatives needed for main result?
- Connections to other voting models, applying our multi-issue framework
- Party/coalition formation in multi-dimensional settings