Dealing with the inconsistencies of judgment aggregation and social choice: A general proposal based on Theophrastus principle

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Inconsistencies:

when using the majority rule

Doctrinal paradox:

inconsistency with the doctrine $t \leftrightarrow p \wedge q$

The doctrinal paradox

$t \leftrightarrow p \land q$

	p	q	t
45%	Y	N	N
30%	N	Y	N
25%	Y	Y	Y
Y - N	70 - 30	55 - 45	25 - 75

Inconsistencies:

when using the majority rule

Doctrinal paradox:

inconsistency with the doctrine $t \leftrightarrow p \wedge q$

Preferential voting:

inconsistency with transitivity (Condorcet)

Approval-preferential voting:

inconsistency between approval and prefs

Approval-preferential voting

Approving-disapproving + ranking

40%	a | b > c
30%	b > c | a
25%	c | a > b
5%	a > c | b
Majority	c a b, b > c

60 55 70 70

?

Brams + Sanver 2009: c

Inconsistencies:

when using the majority rule

Doctrinal paradox: DP

inconsistency with the doctrine $t \leftrightarrow p \land q$

Preferential voting: PV

inconsistency with transitivity

Approval-preferential voting: APV

inconsistency between approval and prefs

How to arrive at consistent decisions?

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Theophrastus principle

Modal logic, degrees of belief

"Peiorem semper conclusio sequitur partem"
the conclusion follows the weakest premise

$$p \wedge q \rightarrow t$$

Theophrastus principle

Modal logic, degrees of belief

"Peiorem semper conclusio sequitur partem"
the conclusion follows the weakest premise

$$\mathbf{p} \wedge q \rightarrow t$$
$$\mathbf{r} \wedge s \rightarrow t$$

scale from 0 to 1

arises when aggregating many individual views

The doctrinal paradox

$t \leftrightarrow p \wedge q$

	p	q	t
45%	Y	N	N
30%	N	Y	N
25%	Y	Y	Y
Y - N	70 - 30	55 - 45	25 - 75

$\bar{t} \wedge q \rightarrow \bar{p}$	$\bar{t} \wedge p \rightarrow \bar{q}$	$p \wedge q \rightarrow t$
70 - 55	55 - 70	55 - 75

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Basic propositions (issues)

Examples

DP t: the accused is guilty; p, q

PV p_{xy}: x is preferable to y ($x,y \in A$)

APV p_{xy} ($x,y \in A$); g_x: x is good ($x \in A$)

Π: set of basic propositions + their negations ("literals")

$\bar{p}$: opposite of p $\bar{\bar{p}} = p$

Constraints (feasibility)

Examples

DP $t \leftrightarrow (p \wedge q)$

PV $p_{xy} \leftrightarrow \overline{p}_{yx}$, $(p_{xy} \wedge p_{yz}) \rightarrow p_{xz}$

APV $p_{xy} \leftrightarrow \overline{p}_{yx}$, $(g_x \wedge \overline{g}_y) \rightarrow p_{xy}$

In general: Several compound propositions
(basic propositions combined by $\neg$ $\wedge$ $\vee$ $\rightarrow$ $\leftrightarrow$)
that are required/assumed to hold

Constraints (feasibility)

Examples

DP $t \leftrightarrow (p \wedge q)$

PV $p_{xy} \leftrightarrow \overline{p}_{yx}$, $(p_{xy} \wedge p_{yz}) \rightarrow p_{xz}$

APV $p_{xy} \leftrightarrow \overline{p}_{yx}$, $(g_x \wedge \overline{g}_y) \rightarrow p_{xy}$

In general: *A* compound proposition

(basic propositions combined by $\neg\ \wedge\ \vee\ \rightarrow\ \leftrightarrow$)

that is required /assumed to hold

"Doctrine"

Valuation (profile)

$$v : \Pi \to [0,1]$$
$$p \mapsto v_p$$

ignorance

$$v_p + v_{\bar{p}} \;\; \begin{smallmatrix} < \\ = \\ > \end{smallmatrix} \;\; 1$$

contradiction

$$v = \sum_k \alpha_k v^k \quad \left(\sum_k \alpha_k = 1\right)$$

Decision associated to v

p accepted & $\bar{p}$ rejected iff $v_p > v_{\bar{p}}$

p & $\bar{p}$ undecided iff $v_p = v_{\bar{p}}$

Valuation (profile)

$$v : \Pi \to [0,1]$$
$$p \mapsto v_p$$

$$v_p + v_{\bar{p}} \; \begin{matrix} \text{ignorance} \\ < \\ = \\ > \\ \text{contradiction} \end{matrix} \; 1$$

$$v = \sum_k \alpha_k v^k \quad \left(\sum_k \alpha_k = 1\right)$$

Decision associated to v (if $v_p + v_{\bar{p}} = 1$)

p accepted & $\bar{p}$ rejected iff $v_p > 1/2$

p & $\bar{p}$ undecided iff $v_p = 1/2$

Valuation (profile)

$$v : \Pi \to [0,1]$$
$$p \mapsto v_p$$

$$v_p + v_{\bar{p}} \begin{matrix} \text{ignorance} \\ < \\ = \\ > \\ \text{contradiction} \end{matrix} 1$$

$$v = \sum_k \alpha_k v^k \quad \left(\sum_k \alpha_k = 1\right)$$

Decision associated to v (margin η)

p accepted & $\bar{p}$ rejected iff $v_p - v_{\bar{p}} > \eta$

p & $\bar{p}$ undecided iff $|v_p - v_{\bar{p}}| \leq \eta$

The problem

We are given a valuation v,
possibly inconsistent with the doctrine.
Want to make a consistent decision.
Which one is most suitable to v ?

Main idea

Revise v using Theophrastus principle,
along the implications of the doctrine

To get all the implications :
Rewrite the doctrine in
conjunctive normal form
(a conjunction of disjunctions of literals)

$$t \leftrightarrow (p \wedge q)$$

$$\equiv$$

$$(t \rightarrow (p \wedge q)) \wedge ((p \wedge q) \rightarrow t) \qquad \alpha \rightarrow \beta$$

$$\equiv \qquad\qquad \equiv$$

$$(\bar{t} \vee (p \wedge q)) \wedge (\bar{p} \vee \bar{q} \vee t) \qquad \bar{\alpha} \vee \beta$$

$$\equiv$$

$$(\bar{t} \vee p) \wedge (\bar{t} \vee q) \wedge (\bar{p} \vee \bar{q} \vee t)$$

clause clause clause

In general : $\bigwedge_{\mathcal{C}\in\mathcal{D}} \bigvee_{p\in\mathcal{C}} p$ $\qquad (p \vee \bar{p})$

"tertium non datur"

true $\downarrow$

for any p and $\mathcal{C}$ with $p \in \mathcal{C} \in \mathcal{D}$:

$$p \leftarrow \bigwedge_{\substack{\alpha\in\mathcal{C} \\ \alpha\neq p}} \bar{\alpha} \qquad p \leftarrow p$$

Theophrastus principle $\downarrow$

$$v'_p \geq \min_{\substack{\alpha\in\mathcal{C} \\ \alpha\neq p}} v_{\bar{\alpha}} \qquad v'_p \geq v_p$$

$$v'_p = \max_{\substack{\mathcal{C}\in\mathcal{D} \\ \mathcal{C}\ni p}} \min_{\substack{\alpha\in\mathcal{C} \\ \alpha\neq p}} v_{\bar{\alpha}}$$

* The iteration $v \rightarrow v' \rightarrow v'' \ldots$ eventually reaches an invariant state **v^* ("upper revised valuation")**

* **Characterization.** v^* is the lowest valuation w that lies above v and satisfies $w' = w$ **(consistency)**

* **Consistency of the associated decisions.** For any η in the interval $0 \leq \eta \leq 1$, the decision of margin η associated with v^* is definitely consistent with the doctrine:

  $\forall \mathcal{C}\in\mathcal{D}, \forall p\in\mathcal{C}$:

  all $\alpha\in\mathcal{C}\setminus\{p\}$ rejected $\Rightarrow$ p accepted

added
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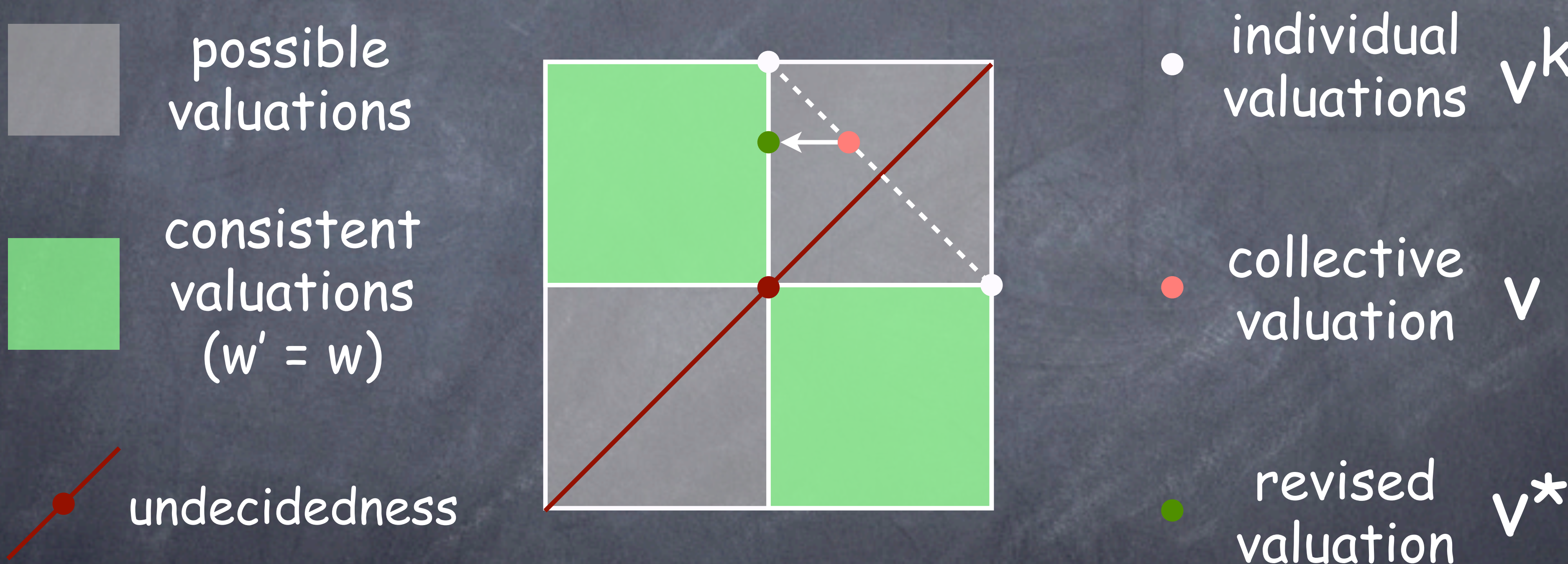
* **Respect for consistent majority decisions.** Assume that every p satisfies either $v_p > 1/2 > v_{\bar{p}}$ (p accepted) or $v_{\bar{p}} > 1/2 > v_p$ (p rejected). Assume also that this decision is consistent. In that case, v^* arrives at the same decision.

* **Respect for unanimity.** If v is an aggregate of consistent truth assignments and $v_p = 1$, then p is accepted by the basic decision associated with v^*

* **Monotonicity.** If v_p grows while v_α is kept constant for $\alpha \neq p$, then the acceptability of p, namely $v^*_p - v^*_{\bar{p}}$, either increases or stays constant

We did a sort of non-convex projection



Which conjunctive normal form?

Not unique

They can lead to different v* !

Example: Adding $(p \vee q \vee r)$ besides $(q \vee r)$

"Implicate": any clause implied by the doctrine

Include only "prime" implicates
Include all of them

} "Blake canonical form"

Unique, its computation is finite (though may take long)

(Blake 1937, Quine 1955–59)

C prime $\equiv \overline{C}$ "critical (forbidden) fragment" (Nehring+Puppe)

$\equiv \overline{C}$ "minimal inconsistent set" (Dietrich+List)

DP The doctrinal paradox

$t \leftrightarrow p \wedge q$

	p	q	t
45%	Y	N	N
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25%	Y	Y	Y
v	70 - 30	55 - 45	25 - 75
v*	70 - 55	55 - 70	55 - 75

v* "conclusion"-based criterion = v* "premise"-based criterion !

PV Preferential voting

$$v^*(p_{xy}) = \text{Max}\ \min\left(v(p_{x_0x_1}), v(p_{x_1x_2}), \ldots , v(p_{x_{n-1}x_n})\right)$$

Max : all (non-cyclic) paths $x_0, x_1, \ldots x_n$
of length $n \geq 1$ from $x_0 = x$ to $x_n = y$

The method of "paths" (Schulze 1997, 2011)

Other good properties:

- Condorcet-Smith
- Clone consistency
- Can be extended to a "continuous rating method" (CMS 2011)

PV Approval-preferential voting

$$v^*(g_x) = \mathrm{Max}\ \min\Big(v(p_{x_0 x_1}), v(p_{x_1 x_2}), \ldots, v(p_{x_{n-1} x_n}), v(g_{x_n})\Big)$$

$$v^*(\bar{g}_x) = \mathrm{Max}\ \min\Big(v(\bar{g}_{x_0}), v(p_{x_0 x_1}), v(p_{x_1 x_2}), \ldots, v(p_{x_{n-1} x_n})\Big)$$

Max : all (non-cyclic) paths $x_0, x_1, \ldots x_n$
of length $n \geq 0$ from $x_0 = x$ to $x_n = y$

Other good properties:

- Monotonicity

PV Approval-preferential voting

40% a | b > c, 30% b > c | a, 25% c | a > b, 5% a > c | b

v

	g_x	$\bar{g}_x$
a	45	55
b	30	70
c	60	40

p_{xy}

a	70	45
30	b	70
55	30	c

accepts

g_c $\bar{g}_b$ p_{bc}

against

$\bar{g}_b \wedge p_{bc} \rightarrow \bar{g}_c$

v*

	g_x	$\bar{g}_x$
a		
b		
c		70

p_{xy}

a		
	b	
		c

PV Approval-preferential voting

40% a | b > c, 30% b > c | a, 25% c | a > b, 5% a > c | b

v

	g_x	$\bar{g}_x$
a	45	55
b	30	70
c	60	40

p_{xy}

a	70	45
30	b	70
55	30	c

accepts

$g_c \quad \bar{g}_b \quad p_{bc}$

against

$\bar{g}_b \wedge p_{bc} \rightarrow \bar{g}_c$

v^*

	g_x	$\bar{g}_x$
a	60	55
b	60	70
c	60	70

p_{xy}

a	70	60
55	b	70
55	60	c

Our Choice: a

Concluding remarks

- Can be applied to any set of constraints
- It reveals the logic behind
  a variety of known methods

  plurality, minimax, maximin, approval ← binary logic
  paths (Schulze) median rate ← graded logic
  single link (aggregation of equivalence relations)
- Produces new interesting methods
- Incomplete valuations are welcome

References

Rosa Camps, Xavier Mora, Laia Saumell, 2010.
A general method for deciding about logically constrained issues
http://arxiv.org/abs/1007.2534

Rosa Camps, Xavier Mora, Laia Saumell, 2011.
Choosing and ranking · Let's be logical about it
http://arxiv.org/abs/1109.4335