A measure of distance between judgment sets

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Workshop on Judgment Aggregation and Voting Theory

The doctrinal paradox.

	p	q	$r \leftrightarrow (p \wedge q)$	r
Judge 1	T	T	T	T
Judge 2	F	T	T	F
Judge 3	T	F	T	F
Majority judgment	**T**	**T**	**T**	**F**

Table: The doctrinal paradox.

Majoritarianism.

- Majorities exist in this example.
- Determining collective judgments by taking majorities produces inconsistent judgments.
- Just like Condorcet voting paradox.
- What should the collective judgments be at this profile?

Distance-based approach.

- One approach to determining the collective judgments is to minimise the distance between the collective judgments and the individual judgments.
- N is the set of individuals and d is a metric.
- Map each profile of individual judgment sets $(A_1, \ldots, A_n)$ to a consistent and complete collective judgment set A that minimises the total distance from the individual judgment sets, i.e. selects an A such that $\sum_{i \in N} d(A, A_i)$ is minimised.
- This is called the distance-based approach to judgment aggregation.

Independence.

- Distance-based rules will violate independence.
- Independence says that the collective judgment on p should depend only on individual judgments on p.
- Literature suggests it is hard to combine independence and unanimity/monotonicity with a desire for logically consistent collective judgments.

Choose a metric.

- A central question is the choice of d.
- Hamming's metric is the most common.
- This is the number of propositions over which two judgment sets disagree.
- The distance between $\{p, q, p \wedge q\}$ and $\{p, \neg q, \neg(p \wedge q)\}$ is 2.

Double counting.

- Idea of logical interconnectedness is central to judgment aggregation.
- But with $\{p, q, p \wedge q\}$ and $\{p, \neg q, \neg(p \wedge q)\}$ the disagreement over $p \wedge q$ is logically entailed by the disagreement over q.
- Seems odd that distance is 2, intuitively it should be 1.
- Hamming's metric is double counting because it ignores interconnectedness.

An alternative metric.

- The metric is based around the idea of "betweenness".
- We follow Kemeny and Snell (1962) who use a similar betweenness concept to characterise Kemeny's measure of distance between preference rankings.
- Judgment set C is between judgment sets A and B if A, B and C are distinct and, on each proposition, C agrees with A or with B (or both).

A graph.

- Imagine a graph where each feasible judgment set is a vertex.
- We join two judgment sets with an edge if there is no other judgment set between them.
- The distance between two judgment sets is the length of the shortest path from one to the other.

An example.

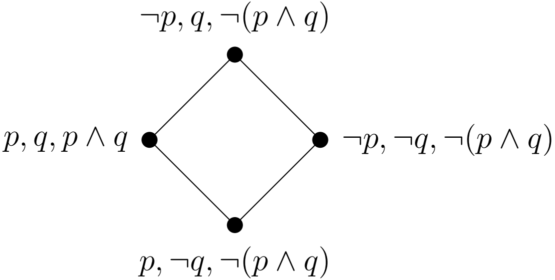

Figure: Graph with judgment sets as vertices.

Another way to think about the metric.

- To switch from $\{p, q, p \wedge q\}$ to $\{p, \neg q, \neg(p \wedge q)\}$ requires just one change in belief (q to $\neg q$, for instance).
- The distance between two judgment sets is the smallest number of logically coherent changes needed to convert one into the other.

The doctrinal paradox.

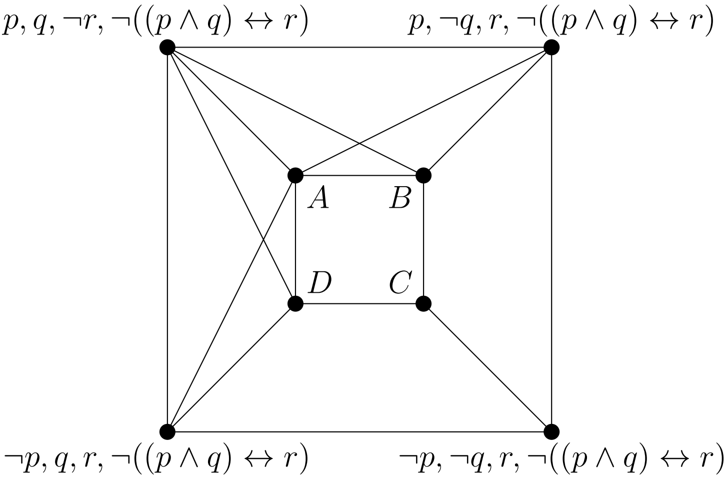

Figure: Graph for the doctrinal paradox agenda.

The doctrinal paradox (continued).

- $A = \{p, q, r, (p \wedge q) \leftrightarrow r\}$,
- $B = \{p, \neg q, \neg r, (p \wedge q) \leftrightarrow r\}$,
- $C = \{\neg p, \neg q, \neg r, (p \wedge q) \leftrightarrow r\}$ and
- $D = \{\neg p, q, \neg r, (p \wedge q) \leftrightarrow r\}$.

Hamming.

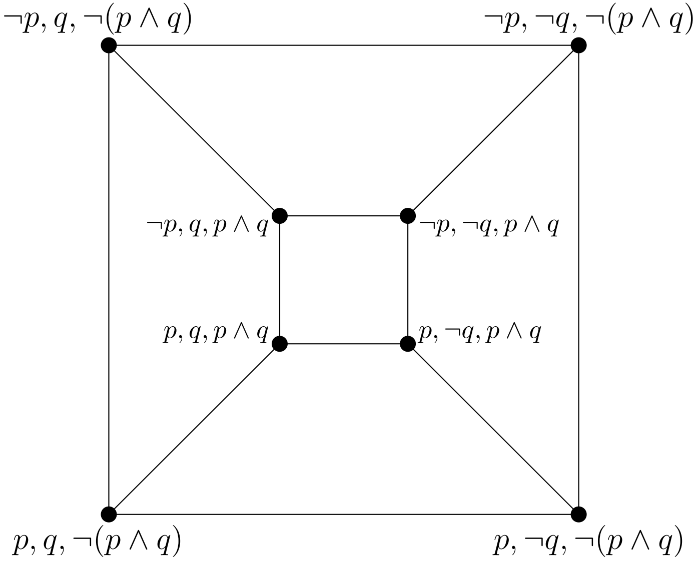

Figure: Hamming graph.

Our metric.

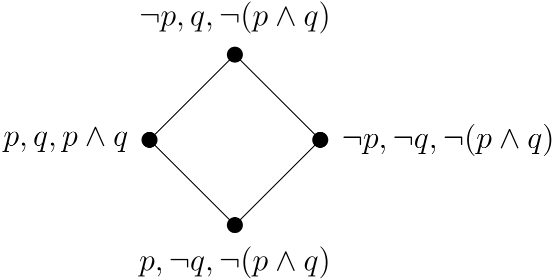

Figure: Geometric vs. logical.

Axiomatisation.

- For any given agenda, let $\mathscr{F}$ be the set of all complete and consistent judgment sets.
- A measure of distance between judgment sets A and B will be denoted by $d(A,B)$ where $d : \mathscr{F} \times \mathscr{F} \to \mathbb{R}$.

Axiom 1. $d(A,B) = 0$ if and only if $A = B$.

Axiom 2. $d(A,B) = d(B,A)$.

Axiom 3. $d(A,B) \leq d(A,C) + d(C,B)$.

- For all $A, B, C \in \mathscr{F}$, we say that C is between A and B if and only if $A \neq C \neq B$ and $(A \cap B) \subset C$.
- We use the word "between" here not in the geometric sense but rather to mean that C represents a compromise between A and B (logical betweenness).

Axiom 4.

Axiom 4. If there is a judgment set between A and B then there exists $C \in \mathscr{F} - \{A, B\}$ such that $d(A, B) = d(A, C) + d(C, B)$.

Interpretation of Axiom 4.

- The axiom says that when there is a judgment set between A and B, then there should also be a judgment set between them in the geometric sense.
- We do not go so far as to make the additional requirement that the judgment sets that are between A and B in the former sense are the same as the judgment sets that are between A and B in the latter sense.
- Hamming's metric, h, satisfies Axiom 4.
- C is between A and B iff $h(A, B) = h(A, C) + h(C, B)$.

Axiom 5.

Axiom 5. If there is no judgment set between A and B, with $A \neq B$, then $d(A, B) = 1$.

Consequence of Axiom 5.

- Hamming's metric violates Axiom 5.
- No judgment set exists between $\{p, q, p \wedge q\}$ and $\{p, \neg q, \neg(p \wedge q)\}$ yet the Hamming distance between them is 2.

Justification.

- Take two distinct judgment sets A and B that have no judgment set between them.
- Therefore, if one accepts the propositions in $A \cap B$ then one must either accept every proposition in $A - B$ or reject every proposition in $A - B$.
- In other words, if the propositions in $A \cap B$ are true, then the propositions in $A - B$ are logically equivalent.
- Since A and B both accept the propositions in $A \cap B$, the disagreement between A and B over the propositions in $A - B$ should simply count as one disagreement. Hence, $d(A, B) = 1$.

Example.

- The agenda is $\{p, \neg p, q, \neg q, r, \neg r, ((p \wedge q) \leftrightarrow r), \neg((p \wedge q) \leftrightarrow r)\}$.
- Let $A = \{p, q, \neg r, \neg((p \wedge q) \leftrightarrow r)\}$ and $B = \{\neg p, q, \neg r, ((p \wedge q) \leftrightarrow r)\}$.
- $A \cap B = \{q, \neg r\}$ and there is no judgment set in $\mathscr{F} - \{A, B\}$ that is a superset of $A \cap B$.
- Note that $A - B = \{p, \neg((p \wedge q) \leftrightarrow r)\}$.
- Consider the following truth table.

q	$\neg r$	p	$\neg((p \wedge q) \leftrightarrow r)$
T	T	T	T
T	T	F	F

Formal description of metric.

- Let $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ be a graph with the same number of vertices as there are judgment sets in $\mathscr{F}$, and a set of edges $\mathscr{E}$ defined as follows.
- Let v be a bijection from $\mathscr{F}$ to the set of vertices $\mathscr{V}$. For all $A, B \in \mathscr{F}$, the set of edges $\mathscr{E}$ contains $\{v(A), v(B)\}$ if and only if there is no judgment set between A and B.
- Define a function g with domain $\mathscr{F} \times \mathscr{F}$ as follows. For all $A, B \in \mathscr{F}$, $g(A, B)$ is equal to the length of a shortest path from $v(A)$ to $v(B)$ in graph $\mathscr{G}$.

Theorem.

Theorem

The function g is unique in satisfying all of the axioms.

Application: judgment aggregation.

	p	q	$p \wedge q$
Two individuals	T	T	T
Two individuals	T	F	F
Three individuals	F	T	F
Majority judgment	**T**	**T**	**F**
Our metric	**T**	**T**	**T**
Hamming's metric	**F**	**T**	**F**
Premiss-based rule	**T**	**T**	**T**

Table: Different outcome to Hamming's.

Application: judgment aggregation.

	p	q	$p \leftrightarrow q$
Two individuals	T	T	T
Two individuals	T	F	F
Three individuals	F	T	F
Our metric	**F**	**T**	**F**
Premiss-based	**T**	**T**	**T**

Table: A different outcome than the premiss-based approach.

The doctrinal paradox.

	p	q	$r \leftrightarrow (p \wedge q)$	r
Judge 1	T	T	T	T
Judge 2	F	T	T	F
Judge 3	T	F	T	F
Our metric	**T**	**T**	**T**	**T**

Table: The doctrinal paradox.

A measure of distance between preferences.

- Kemeny distance is the application of Hamming's metric to preferences.
- Hamming distance between judgment sets A and B is $|A - B|$.
- Kemeny distance between preference rankings P and P' is $|P - P'| + |P' - P|$. P is the asymmetric part of a weak ordering.
- Addition of $|P' - P|$ required because judgment sets and binary relations treat negation differently.

A measure of distance between preferences.

- Let T be a two-place predicate and X a finite set of alternatives.
- Construct an agenda containing, for all $x, y \in X$, the propositions xTy and $\neg xTy$.
- Given a binary relation P over X, let us define its judgment set counterpart $J(P)$ as follows. For all $x, y \in X$, if xPy then $xTy \in J(P)$, otherwise $\neg xTy \in J(P)$. The set of feasible judgment sets contains all and only those judgment sets that correspond to preference rankings over X.
- We can now see that the Kemeny distance from P to P' is equal to the Hamming distance from $J(P)$ to $J(P')$.

A measure of distance between preferences.

- An important feature of our metric is that when there is no judgment set between A and B the distance from A to B is 1.
- This means that our metric, when applied to preference rankings, must be different to Kemeny's.
- To see this, suppose that P and P' are preference rankings over $\{x, y, z\}$, with P ranking x first, and y and z tied second, while P' places all three items in the same equivalence class.
- The Kemeny distance is 2.
- However, there is no preference ranking between P and P'.

A measure of distance between preferences.

- Consider the relations $P = \{(x,y),(y,z),(x,z)\}$ and $P' = \{(y,x),(z,y),(z,x)\}$.
- One way of thinking about this distance is that the Kemeny metric k counts 6 steps:

$$\begin{array}{c} x \\ y \\ z \end{array} \rightarrow \begin{array}{c} x \\ yz \end{array} \rightarrow \begin{array}{c} x \\ z \\ y \end{array} \rightarrow \begin{array}{c} xz \\ y \end{array} \rightarrow \begin{array}{c} z \\ x \\ y \end{array} \rightarrow \begin{array}{c} z \\ xy \end{array} \rightarrow \begin{array}{c} z \\ y \\ x \end{array}$$

- Our metric g counts 4 steps:

$$\begin{array}{c} x \\ y \\ z \end{array} \rightarrow \begin{array}{c} x \\ yz \end{array} \rightarrow xyz \rightarrow \begin{array}{c} z \\ xy \end{array} \rightarrow \begin{array}{c} z \\ y \\ x \end{array}$$

- The distances determined by k and g do not always differ.
- Consider $P'' = \{(y,x),(y,z),(x,z)\}$.
- We find that $k(P,P'') = g(P,P'') = 2$.
- When we model preferences by linear orders then the difference between k and g disappears.

Conclusions.

- Proposed an alternative to Hamming's metric.
- Explored the implications of the metric for judgment aggregation.
- Can use the metric to measure distance between preferences, when the latter are converted into judgment sets.
- Do better metrics exist, and what are their implications for judgment aggregation?
- We have not provided any justification for the distance-based procedure itself. Can one be found?