Binary Aggregation with Integrity Constraints

Ulle Endriss
Institute for Logic, Language and Computation
University of Amsterdam

[joint work with Umberto Grandi]

Preference Aggregation

Individual 1: $\triangle \succ \bigcirc \succ \square$

Individual 2: $\square \succ \triangle \succ \bigcirc$

Individual 3: $\bigcirc \succ \square \succ \triangle$

?

Judgment Aggregation

	p	$p \to q$	q
Judge 1:	True	True	True
Judge 2:	True	False	False
Judge 3:	False	True	False
		?	

Multiple Referenda

	fund museum?	*fund school?*	*fund metro?*
Voter 1:	Yes	Yes	No
Voter 2:	Yes	No	Yes
Voter 3:	No	Yes	Yes

?

[Constraint: we have money for *at most two projects*]

General Perspective

The last example is arguably the clearest. We can rephrase many aggregation problems as problems of *binary aggregation:*

- *Do you rank option $\triangle$ above option $\bigcirc$?* Yes/No
- *Do you believe formula "$p \rightarrow q$" is true?* Yes/No
- *Do you want the new school to get funded?* Yes/No

Each problem domain comes with its own *integrity constraints:*

- *Rankings should be transitive and not have any cycles.*
- *The accepted set of formulas should be logically consistent.*
- *We should fund at most two projects.*

The *paradoxes* we have seen show that the *majority rule* does not *lift* our integrity constraints from the *individual* to the *collective* level.

Talk Outline

- Framework: *binary aggregation with integrity constraints*
- Focus on *language* used to express IC ($\rightsquigarrow$ feasible outcomes)
- Idea: characterise aggregators via langauge of IC's it can lift
- Applications of that idea

The Model

Basic terminology and notation:

- Finite set of *issues* $\mathcal{I} = \{1, \ldots, m\}$, defining a boolean combinatorial *domain* $\mathcal{D} = D_1 \times \cdots \times D_m$, with $D_i = \{0, 1\}$.
- Each of a finite set of *individuals* $\mathcal{N} = \{1, \ldots, n\}$ votes by supplying a *ballot* $B_i \in \mathcal{D}$. $\rightsquigarrow$ *profile* $\boldsymbol{B} = (B_1, \ldots, B_n) \in \mathcal{D}^{\mathcal{N}}$
- A *binary aggregator* is a function $F : \mathcal{D}^{\mathcal{N}} \to \mathcal{D}$.

We can define *axioms* in the usual manner, possibly restricting their scope to some (feasible) subdomain $X \subseteq \mathcal{D}$. Example:

- F is *unanimous* on $X \subseteq \mathcal{D}$, if for any $(B_1, \ldots, B_n) \in X^{\mathcal{N}}$ and any $x \in \{0, 1\}$, if $b_{i,j} = x$ for all $i \in \mathcal{N}$, then $F(B_1, \ldots, B_n)_j = x$.

Integrity Constraints

Rather than defining the subdomain $X \subseteq \mathcal{D}$ extensionally, we want to give an intentional characterisation, by means of integrity constraints.

- Introduce a *propositional variable* p_i for each issue $i \in \mathcal{I}$ and consider the *propositional language* $\mathcal{L}_{PS}$ over $PS = \{p_1, \ldots, p_m\}$ (closed under $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$).
- Any given *integrity constraint* (formula) $\mathrm{IC} \in \mathcal{L}_{PS}$ defines a domain of aggregation $X = \mathrm{Mod}(\mathrm{IC}) := \{B \in \mathcal{D} \mid B \models \mathrm{IC}\}$.
- Ballots are models (truth assignments) for formulas in $\mathcal{L}_{PS}$. Call ballot $B \in \mathcal{D}$ *rational* wrt. $\mathrm{IC} \in \mathcal{L}_{PS}$ if $B \models \mathrm{IC}$.

Recall the three-project example:

$\mathrm{IC} = \neg(p_1 \wedge p_2 \wedge p_3)$ = "we cannot afford all three projects"

Voter 1:	$B_1 = (1,1,0)$	$\rightsquigarrow$	$B_1 \models \mathrm{IC}$	(rational)
Majority:	$M = (1,1,1)$	$\rightsquigarrow$	$M \not\models \mathrm{IC}$	(irrational)

What's a paradox?

As a first application, we can give a generic definition of "paradox":

> A *paradox* is a triple $(F, \boldsymbol{B}, \mathrm{IC})$ consisting of an aggregator F, a profile $\boldsymbol{B}$, and an integrity constraint IC such that $B_i \models \mathrm{IC}$ for all $i \in \mathcal{N}$ but $F(\boldsymbol{B}) \not\models \mathrm{IC}$.

Examples:

- Preference aggregation:
  - $p_{ab} \leftrightarrow \neg p_{ba}$ for all $a \neq b$ and $\neg p_{aa}$ for all a
  - $p_{ab} \wedge p_{bc} \rightarrow p_{ac}$ for all a, b, c
- Judgment aggregation:
  - $p_{\varphi} \vee p_{\bar{\varphi}}$ for all complementary $\varphi, \bar{\varphi}$
  - $\neg \bigwedge_{\varphi \in S} p_{\varphi}$ for all minimally inconsistent sets $S \subseteq \textsc{Agenda}$

Collective Rationality wrt. a Language

Collective rationality wrt. an integrity constraint:

- An aggregator F is *collectively rational* wrt. $\mathrm{IC} \in \mathcal{L}_{PS}$ if $B_i \models \mathrm{IC}$ for all $i \in \mathcal{N}$ implies $F(B_1, \ldots, B_n) \models \mathrm{IC}$ (F can "lift" IC).
- Thus: F is CR wrt. $\mathrm{IC} \Leftrightarrow \nexists \boldsymbol{B}$ s.t. $(F, \boldsymbol{B}, \mathrm{IC})$ is a paradox

Now consider a *language* $\mathcal{L} \subseteq \mathcal{L}_{PS}$ of integrity constraints, e.g.,

- the language of *cubes* (conjunctions of literals),
- the language of *clauses* of length $\leqslant 2$, etc.

Collective rationality wrt. a language:

- An aggregator F is *collectively rational* wrt. $\mathcal{L} \subseteq \mathcal{L}_{PS}$ if F is collectively rational wrt. every $\mathrm{IC} \in \mathcal{L}$.

Template for Results

Two ways of defining classes of aggregators:

- The class of aggregators that *lift* all integrity constraints in $\mathcal{L}$:

$$\mathcal{CR}[\mathcal{L}] \quad := \quad \{F : \mathcal{D}^{\mathcal{N}} \to \mathcal{D} \mid F \text{ is collectively rational wrt. } \mathcal{L}\}$$

- The class of aggregators defined by a given list of *axioms* AX:

$$\mathcal{F}_{\mathcal{L}}[\mathrm{AX}] \quad := \quad \{F : \mathcal{D}^{\mathcal{N}} \to \mathcal{D} \mid F \text{ satisfies AX on all } \mathcal{L}\text{-domains}\}$$

What we want:

$$\boxed{\mathcal{CR}[\mathcal{L}] \; = \; \mathcal{F}_{\mathcal{L}}[\mathrm{AX}]}$$

Example for a Characterisation Result

Cubes (= conjunctions of literals) are lifted by an aggregator *iff* that aggregator satisfies *unanimity:*

$$\mathcal{CR}[cubes] \quad = \quad \mathcal{F}_{cubes}[\text{Unanimity}]$$

More Results

Characterisation results:

- $\mathcal{CR}[p \leftrightarrow q] \;=\; \mathcal{F}_{\leftrightarrow}[\text{Issue-Neutrality}]$
- $\mathcal{CR}[p \text{ XOR } q] \;=\; \mathcal{F}_{\text{XOR}}[\text{Domain-Neutrality}]$

Negative results:

- there exists no language $\mathcal{L}$ such that $\mathcal{CR}[\mathcal{L}] \;=\; \mathcal{F}[\text{Anonymity}]$
- there exists no language $\mathcal{L}$ such that $\mathcal{CR}[\mathcal{L}] \;=\; \mathcal{F}[\text{Independence}]$

Characterisation within a noncharacterisable class:

- $\mathcal{CR}[k\textit{-pclauses}] \cap \underset{\substack{\uparrow \\ \text{quoata rules}}}{\mathrm{QR}} \;=\; \mathrm{QR}[\sum q_i < n + k] \cup \mathrm{QR}[\prod q_i = 0]$

Application: Preference Aggregation

Call a preference aggregator *imposed* if there exist x and y such that x is collectively preferred to y in every profile. A theorem:

> *Any anonymous, independent and monotonic preference aggregator for $\geqslant 3$ alternatives and $\geqslant 2$ individuals is imposed.*

Proof:

- Adapt Dietrich-List result on quota rules in JA to show that any A-I-M aggregator must be a *quota rule*.
- IC's for preference aggregation entail two *3-clauses:*

$$p_{ba} \vee p_{cb} \vee p_{ac} \qquad p_{ab} \vee p_{bc} \vee p_{ca}$$

- Apply our *lifting theorem* to derive a constraint on the quotas:

$$\sum q_i < n + 3 \quad \text{or} \quad \prod q_i = 0 \ [\Leftrightarrow \text{imposed}]$$

- Rewriting of LHS (and $p_{xy} + p_{yx} = n + 1$) yields contradiction. ✓

Application: Good Binary Aggregators

Is there an aggregator that will lift *every* integrity constraint? *Yes!*

F will lift *every* $\mathrm{IC} \in \mathcal{L}_{PS}$ *iff* F is a *generalised dictatorship*, i.e., *iff* there exists a function $g : \mathcal{D}^{\mathcal{N}} \to \mathcal{N}$ such that always $F(B_1, \ldots, B_n) = B_{g(B_1,\ldots,B_n)}$.

The class of generalised dictatorships includes:

- proper *dictatorships* $F_i : (B_1, \ldots, B_n) \mapsto B_i$ for each $i \in \mathcal{N}$
- *distance-based generalised dictatorships* mapping $(B_1, \ldots, B_n)$ to that B_i that minimises the sum of the Hamming distances to the others (+ tie-breaking). An attractive procedure!

Last Slide

Binary aggregation with integrity constraints:

- *language* to express *rationality assumptions* in binary aggregation
- concept of *collective rationality* with respect to a language
- characterisation results, relating *axioms* and *languages*
- *applications:* preference + judgment aggregation, good procedures

U. Grandi and U. Endriss. Lifting Rationality Assumptions in Binary Aggregation. Proc. 24th AAAI Conference on Artificial Intelligence, 2010.

U. Grandi and U. Endriss. Binary Aggregation with Integrity Constraints. Proc. 22nd International Joint Conference on Artificial Intelligence (IJCAI), 2011.