

# Which Voting Rule Is More Manipulable? Results from Simulation Studies

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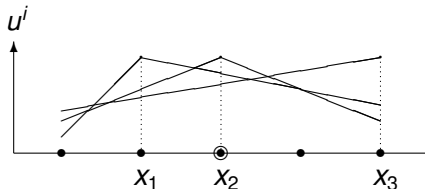
- A **voting rule**  $F$  on a set  $X$  of social states selects a (unique)  $x \in X$  for every profile  $(\succ^1, \succ^1, \dots, \succ^n)$  of preferences over  $X$ .
- $F$  is **strategy-proof** if  $F(\succ^1, \dots, \succ^i, \dots, \succ^n) \succeq^i F(\succ^1, \dots, \succ^{i'}, \dots, \succ^n)$

## Theorem (Gibbard-Satterthwaite)

*If  $\#X \geq 3$ , then the only strategy-proof voting rule on an unrestricted preference domain is the dictatorship of one individual.*

## Theorem (Black 1942/58, Moulin 1980)

Suppose that social states can be ordered from left to right such that all preferences are **single-peaked**, then there exist non-degenerate strategy-proof voting rules. E.g. choosing the median of the individual peaks defines an anonymous, neutral and strategy-proof voting rule.



- Nehring/Puppe (2007/2010) demonstrate the existence of non-dictatorial and strategy-proof voting rules for classes of generalized single-peaked preferences.
- Unfortunately, for many economically relevant domains all strategy-proof voting rules are dictatorial, even under generalized single-peakedness.

- What happens in such domains?
- If all non-dictatorial voting rules are manipulable, which of those are **less** manipulable than others? (Throughout, we will assume that  $F$  respects **unanimity**: if all individuals happen to agree that  $x$  is best, then  $F(\dots) = x$ .)
- What is a good notion of “**less manipulable**”?

- 1 Voting on the Simplex
  - Peaks-Only Preference Aggregation
  - Voting Rules
- 2 Setting of the Simulation Studies
  - Basic Settings and Parameters
  - Variables Measuring Manipulability
- 3 Preliminary Results

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## Definition ((K,L)-Simplex)

A **(K,L)-Simplex** is the set

$$X = \{x \in \mathbb{R}^K : \sum_{j=1}^K x_j = L, x_j \geq 0\}$$

Economic Applications:

- budget allocation problem where  $x_j$  is the money amount spent on (public) good  $j$  and  $L$  the total budget,
- aggregation of probability distributions ( $L = 1$ ), where  $x_j$  is the probability of  $j$

If  $K \geq 3$ , all strategy-proof voting rules on  $X$  are dictatorial, even under generalized single-peaked preferences (See Nehring and Puppe 2010).

# Peaks-Only Preference Aggregation

- Let  $d(x, y) = \frac{1}{2} \sum_{j=1}^K |x_j - y_j|$  denote the **distance** between  $x$  and  $y$ , e.g. in the public goods context the number of dollars that have to be shifted to get from allocation  $x$  to  $y$ .
- We assume that an agent  $i$  has (generalized) single-peaked preferences on  $X$  (with respect to that distance) and submits a **proposal**  $w^i \in X$  (the peak).
- Denote by  $p(w)$  the number of agents who proposed  $w$ .

## Definition (Voting Rule on the Simplex)

A **voting rule** on the simplex is a mapping

$$F(w^1, w^2, \dots, w^n) = x \in X$$

which assigns to each profile of peaks  $(w^1, w^2, \dots, w^n)$  an element  $x$  in the simplex.

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- A midpoint minimizes the sum of distances to every peak.

## Definition (Midpoint)

An allocation  $m(p) \in X$  is a **midpoint** if

$$m(p) = \operatorname{argmin}_{x \in X} \sum_{w \in X} p_w d(x, w)$$

- Let  $M(p)$  denote the set of midpoints. Evidently,  $M(p)$  need not be a singleton but it is always non-empty.

Natural single valued selection for Midpoint Rule: the “shadow voter selection”. Give *every point* in the simplex in addition a mass of  $\epsilon$  (shadow voter):  $\tilde{p}(w) := \epsilon + p(w)$

## Theorem

For  $\lim_{\epsilon \rightarrow 0}$  we have  $\tilde{M}(p) \subseteq M(p)$  and  $|\tilde{M}(p)| = 1$ , i.e. a unique midpoint.

# Limited Manipulability of the Set of Midpoints

The possibility to influence the set of Midpoints is restricted.

## Theorem

*Under the midpoint rule, an agent cannot move the midpoint (metrically) closer to his/her own peak by misrepresentation.*

- This rule selects the mean of peaks:

## Definition (Mean Rule)

$$F^{Mean}(w^1, w^2, \dots, w^n) := (\bar{w}_1, \bar{w}_2, \dots, \bar{w}_K)$$

where

$$\bar{w}_j = \frac{1}{n} \sum_{i=1}^n w_j^i$$

- The “Normalized Median Rule” takes the coordinate wise median and normalizes it to the simplex:

## Definition (NMedian Rule)

$$F^{NMedian}(w^1, w^2, \dots, w^n) := \left( \frac{w_1^{med}}{c} L, \frac{w_2^{med}}{c} L, \dots, \frac{w_K^{med}}{c} L \right)$$

where  $w_j^{med}$  is the median of coordinate  $j$  and  $c = \sum_{j=1}^K w_j^{med}$



- The “Sequential Median Rule” takes the coordinate wise median and adjusts it to the simplex **in a fixed order**:

## Example

Let  $K = 3$ ,  $L = 21$  and  $w_1^{med} = 7$ ,  $w_2^{med} = 10$ ,  $w_3^{med} = 6$ . Note that  $\sum_{j=1}^3 w_j^{med} > L$ . The  $w_j^{med}$  are adjusted in ascending order (1, 2, 3).

$$F^{SeqMedian}(7, 10, 6) = (7, 10, 4)$$

## Example

Let  $K = 3$ ,  $L = 21$  and  $w_1^{med} = 3$ ,  $w_2^{med} = 4$ ,  $w_3^{med} = 8$ . Note that  $\sum_{j=1}^3 w_j^{med} < L$ . The  $w_j^{med}$  are adjusted in ascending order (1, 2, 3).

$$F^{SeqMedian}(3, 4, 8) = (3, 4, 14)$$

- Evidently, the SeqMedian Rule depends on the order of coordinates.

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- written in Java, using parallel computing
- $L = 1$ , variation of  $K$
- discretization of the simplex by a grid, e.g.  $K = 3, L = 1$  by a  $(3, 99)$ -grid (voting over 5050 alternatives).

- 1 The (true) peaks of the agents are drawn by a pseudorandom number generator
- 2 Agents are acting in a random order and determine their best responses (i.e. their utility maximizing announced peak) given the announced peaks of the other agents
- 3 Say that a voting situation *converges*, if no agent changes his or her announced peak after a finite number of manipulations
- 4 We stopped a simulation run after 200 manipulations, interpreting this as lack of convergence

- $K$  and grid size (normalized to  $L = 1$ )
- peak distribution: uniform, Dirichlet, bi-modal
- preferences of the agents: Cobb-Douglas, CES, “metric” single-peaked
- number of agents

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- number of agents manipulating
- number of pivotal agents
- number of manipulations
- maximum utility gain of an agent given the true peaks

- distance between outcome under true peaks and outcome under announced peaks (after convergence)
- average deviation (of an agent) between true and announced peaks
- sum of utility gains/losses
- comparison of sum of utilities to utilitarian maximum and expected utility under random dictatorship
  - normalize an agent's utility such that  $u^i(\vec{0}) := 0$  and  $u^i(w^i) := 1$  and compute  $s(x) := \sum_i^n u^i(x)$
  - scale  $s(x)$  such that 1 is the utilitarian maximum and 0 the expectation under random dictatorship
  - **for later:** "utility loss" = loss in this scale

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- $K = 3$ , grid size 99 (i.e. 5050 possible peaks)
- sample size 1000
- Cobb-Douglas preferences
- Dirichlet-(1, 2, 3) distribution of the true peaks (to avoid artefacts from (very special) symmetry of distribution)
- number of agents: 3 to 45

Table: number of agents = 5

	MANIPULATIONS	MAXUTILGAIN	PERCAGMANI
MEAN	12,11	7,41%	99,98%
MIDPOINT	16,19	1,94%	59,60%
NMEDIAN	2,4	0,38%	34,96%
SEQMEDIAN	2,46	0,74%	38,54%

Table: number of agents = 45

	MANIPULATIONS	MAXUTILGAIN	PERCAGMANI
MEAN	53,09	1,87%	100,00%
MIDPOINT	58,09	0,78%	39,05%
NMEDIAN	1,61	0,08%	3,53%
SEQMEDIAN	5,54	0,39%	12,18%

Table: number of agents = 5

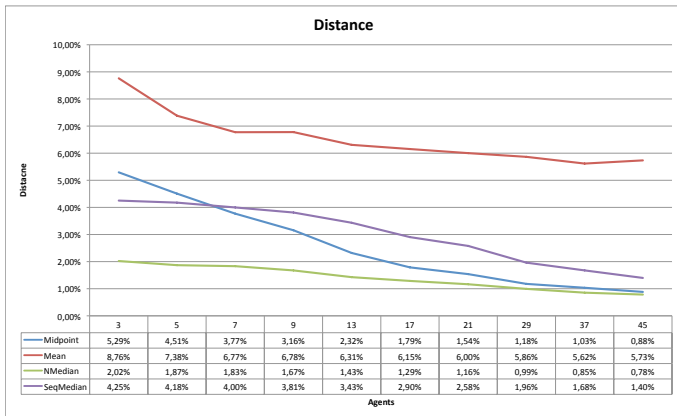
	AVDEV	DISTANCE	UTLOSS
MEAN	39,28%	7,38%	1,60%
MIDPOINT	4,86%	4,51%	1,05%
NMEDIAN	1,93%	1,87%	1,05%
SEQMEDIAN	1,74%	4,18%	1,35%

Table: number of agents = 45

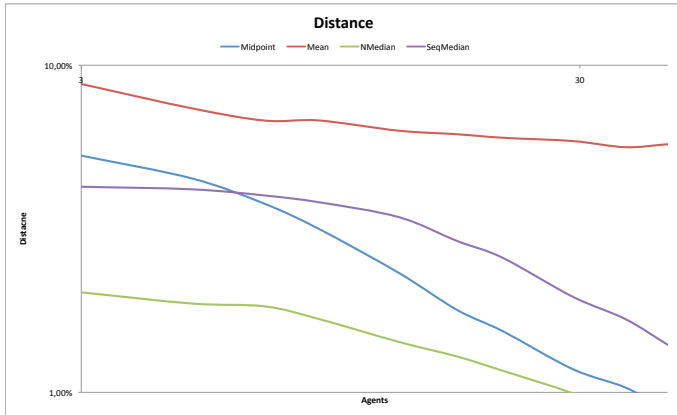
	AVDEV	DISTANCE	UTLOSS
MEAN	43,09%	5,73%	1,02%
MIDPOINT	2,96%	0,88%	0,20%
NMEDIAN	0,07%	0,78%	0,40%
SEQMEDIAN	0,44%	1,40%	0,56%



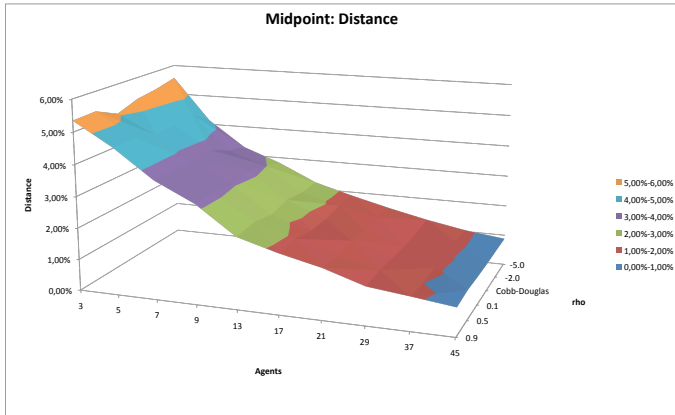
# Effect of Manipulation



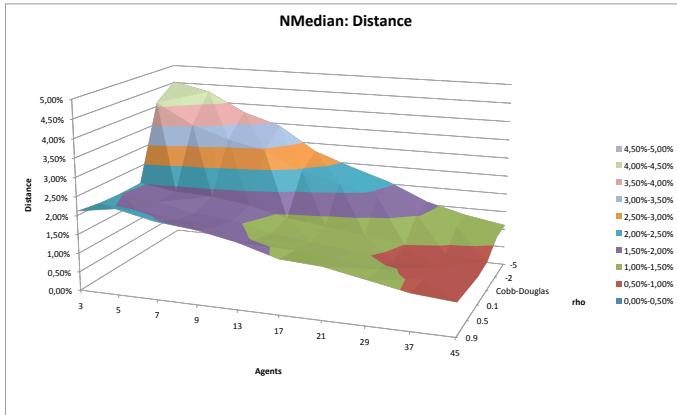
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# Effect of Manipulation



- Midpoint seems to be invariant against variations of the CES- $\rho$  while NMedian does not
- Similarities between rules which use the median (number of agents manipulating, manipulations)
- Mean is highly manipulable in all situations
- Good welfare properties of the Midpoint and the NMedian

- What are the forces driving the extent of manipulation under different rules?
- What are characteristics of manipulable profiles for different rules?
- Is there a chance to predict the manipulability of a profile (under a certain rule)?