Some Aggregative Theory of Dichotomous Evaluations (VERY PROVISIONAL VERSION)

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Dichotomous preferences

- Dichotomous preference orderings are those orderings which have at most two indifference classes.
- They have specific, essentially favourable aggregative properties that have long been noted in voting theory. They bear a natural connection with an attractive rule, *approval voting*, whereby individuals cast votes for as many candidates as they wish, giving no more than one vote to each of them, and those candidates with the greatest vote total are elected.
- With dichotomous preferences, the set of Condorcet winners is never empty (Inada, 1969). Brams and Fishburn (1977) further show that approval voting on this domain exactly selects the set of Condorcet winners, which the plurality voting rule does not always do.
- With dichotomous preferences, approval voting is strategically well-behaved. Brams and Fishburn (1977, 1981) show it to be less open to voters' manipulations than any other non-ranking voting system, including plurality voting.

- For more on the voting theory of dichotomous preferences, see Brams and Fishburn (2002) and recent work by Vorsatz (2007, 2008). All of this adopts the formalism of *choice functions*, i.e., mappings from profiles of preferences to subsets of the alternative set.
- We approach the aggregation of dichotomous preferences in the formalism of social welfare functions, here *collective preference functions*. This side of the topic has been neglected, perhaps because of the emphasis on manipulability, perhaps because of the (incorrect) view that the results in one formalism automatically trtanslate into the other.
- We distinguish between the *simply dichotomous* case, in which only individual preferences are dichotomous, and the *twice dichotomous* case, in which collective preferences also are.

- The TWICE DICHOTOMOUS case is plagued with an aggregative impossibility. Theorem 1 is a version of Arrow's (1963): *dictatorship follows from Independence of Irrelevant Alternatives and Weak Pareto*.
- This is a novel result (see Le Breton and Weymark's 2002 survey of special Arrovian domains) and it is surprising inasmuch as no cyclical profile of individual preferences is available for the proof.

- The SIMPLY DICHOTOMOUS case delivers possibilities. Proposition 1 *characterizes approval voting in terms of Independence of Irrelevant Alternatives, Anonymity and the Strong Pareto Principle*.
- This generalizes Arrow's possibility theorem for two alternatives and May's related characterization of majority voting for this special case.
- Some characterizations of approval voting (eg, Vorsatz, 2007) in terms of choice functions are akin to Proposition 1, and this suggests that the choice-theoretic approach to dichotomous preferences is closer to the simply dichotomous case than to the twice dichotomous case.
- The unfinished part of the paper aims at drawing connections with the judgment aggregation theory.

The framework

- Usual notation for preference orderings: R, P, I. An *indifference class for* R is one of the equivalence classes of I.
- A particular case of R, a *dichotomous preference ordering* has one or two indifference classes. (Brams and Fishburn define it as having exactly two classes, this will prove to be immaterial to the results.)
- The obvious translation rules in terms of classes are:

$$\begin{aligned} xPy &\Leftrightarrow x \in H, y \in L \\ xIy &\Leftrightarrow \text{either } x, y \in H \text{ or } x, y \in L \text{ or } x, y \in C. \end{aligned}$$

  where H (L) is the higher (lower) class where there are two classes and C is the single class otherwise.
- $\mathcal{O}$ is the set of all preference orderings on X, and $\mathcal{D} \subset \mathcal{O}$ the set of dichotomous preference orderings

- The set of social alternatives X has $\mid X \mid \geq 3$ and the population $n \geq 2$ individuals. A *collective preference function* is a mapping

$$F : (R_1, ..., R_n) \longmapsto R.$$

- Two cases to be considered: $F : \mathcal{D}^n \rightarrow \mathcal{D}$ (*twice dichotomous*) and $F : \mathcal{D}^n \rightarrow \mathcal{O}$ (*simply dichotomous*).

A list of properties that F may satisfy.

Condition 1 *Independence of irrelevant alternatives (IIA): For all* $(R_1, ..., R_n)$, $(R'_1, ..., R'_n)$ *and all* $x, y \in X$, *if* $xR_iy \Leftrightarrow xR'_iy$ *and* $yR_ix \Leftrightarrow yR'_ix$ *for all* $i = 1, ..., n$, *then* $xRy \Leftrightarrow xR'y$.

Condition 2 *Weak Pareto (WP): For all* $(R_1, ..., R_n)$ *and all* $x, y \in X$, *if* xP_iy *for all* $i = 1, ..., n$, *then* xPy.

Condition 3 *Dictatorship (D): There is* $j = 1, ..., n$ *such that for all* $(R_1, ..., R_n)$ *and all* $x, y \in X$, *if* xP_jy, *then* xPy.

Condition 4 *Anonymity (A):For all* $(R_1, ..., R_n)$ *and all permutations* σ *of* $\{1, ..., n\}$, $F(R_1, ..., R_n) = F(R_{\sigma(1)}, ..., R_{\sigma(n)})$.

Condition 5 *Neutrality (N): For all* $(R_1, ..., R_n)$, $(R'_1, ..., R'_n)$ *and all* $x, y, z, w \in X$, *if* $xR_iy \Leftrightarrow zR'_iw$ *and* $yR_ix \Leftrightarrow wR'_iz$ *for all* $i = 1, ..., n$, *then* $xRy \Leftrightarrow zR'w$.

Condition 6 *Pareto Indifference (PI): For all* $(R_1, ..., R_n)$ *and all* $x, y \in X$*, if* xI_iy *for all* $i = 1, ..., n$*, then* xIy*.*

Condition 7 *Strict Pareto (SP): For all* $(R_1, ..., R_n)$ *and all* $x, y \in X$*, if* xR_iy *for all* $i = 1, ..., n$ *and* xP_iy *for some* i*, then* xPy*.*

Condition 8 *Pareto Preference (PP): For all* $(R_1, ..., R_n)$ *and all* $x, y \in X$*, if* xR_iy *for all* $i = 1, ..., n$*, then* xRy*.*

Condition 9 *Positive Responsiveness 1 (PR1): For all* $(R_1, ..., R_n)$, $(R'_1, ..., R'_n)$ *and all* $x, y \in X$, *if* $xP_iy \Rightarrow xP'_iy$ *and* $xI_iy \Rightarrow xR'_iy$ *for all* $i = 1, ..., n$, *and* yP_ix *and* xR'_iy, *or* xI_iy *and* xP_iy, *for some* i, *then* $xPy \Rightarrow xP'y$.

Condition 10 *Positive Responsiveness 2 (PR2): For all* $(R_1, ..., R_n)$, $(R'_1, ..., R'_n)$ *and all* $x, y \in X$, *if* $xP_iy \Rightarrow xP'_iy$ *and* $xI_iy \Rightarrow xR'_iy$ *for all* $i = 1, ..., n$, *and* yP_jx *and* xR'_jy, *or* xI_jy *and* xP_jy, *for some* i, *then* $xRy \Rightarrow xP'y$.

Condition 11 *Positive Responsiveness 3 (PR3): For all* $(R_1, ..., R_n)$, $(R'_1, ..., R'_n)$ *and all* $x, y \in X$, *if* $xP_iy \Rightarrow xP'_iy$ *and* $xI_iy \Rightarrow xR'_iy$ *for all* $i = 1, ..., n$, *and* yP_jx *and* xR'_jy *for some* i, *then* $xRy \Rightarrow xP'y$.

Obviously, *N* ⇒*IIA*, *SP*⇒*WP*,*(PR2)*⇒*(PR1)*, *(PR2)*⇒*(PR3)*. The *Strong Pareto Principle* (*SPP*) is *SP&PI*.

The twice dichotomous case

Theorem 1 *If a collective preference function* $F : \mathcal{D}^n \longrightarrow \mathcal{D}$ *satisfies IIA and WP, it satisfies D.*

The proof goes by three lemmas.

Lemma 1 *If a collective preference function* $F : \mathcal{D}^n \longrightarrow \mathcal{D}$ *satisfies IIA and WP, it satisfies PI and N.*

Lemma 2 *If a collective preference function* $F : \mathcal{D}^n \longrightarrow \mathcal{D}$ *satisfies N and WP, it satisfies PP and PR1.*

Lemma 3 *If a collective preference function* $F : \mathcal{D}^n \longrightarrow \mathcal{D}$ *satisfies IIA, WP and PP, there exist* j *and* $(x, y) \in X^2$ *such that* $\{j\}$ *is semi-decisive on* (x, y)*.*

Individual j of Lemma 3 is semi-decisive on any pair by *N* and is a dictator by *PR1*.
Basically the classic proof strategy except for the use of *PI* and *PP* (obtained from the $\mathcal{D}$ range).

- The complete indifference ordering $CI \in \mathcal{D}$ does not occur in the proof unless $\mid X \mid = 3$. So Theorem 1 extends to the preference domain $\mathcal{D}^- = \mathcal{D} \setminus \{CI\}$ if $\mid X \mid \geq 4$; cf. Brams and Fishburn.
- The theorem does not hold if the range is $\mathcal{O}$.

For any profile $(R_1, ..., R_n)$ in $\mathcal{O}$ or $\mathcal{D}$, define $N(xP_iy) = \{i : xP_iy\}$ and $n(xP_iy) = |\{i : xP_iy\}|$ (similarly for R_i, I_i), and $N(x \in H_i) = \{i : x \in H_i\}$ and $n(x \in H_i) = |\{i : x \in H_i\}|$ (similarly for L_i, C_i).

Definition 1 *F is approval voting* if, for all $(R_1, ..., R_n) \in \mathcal{D}^n$, $xPy \Leftrightarrow n(x \in H_i) > n(y \in H_i)$.*

With $\mid X \mid \geq 3$, the range is $\mathcal{O}$, and since *IIA* and *WP* hold, the simply dichotomous case is seen to differ from the twice dichotomous case.

We may redefine approval voting so that it becomes twice dichotomous. For any profile $(R_1, ..., R_n)$ in $\mathcal{D}^n$, we put

$$Max(R_1, ..., R_n) = \{x \in X : N(x \in H_i) \geq N(y \in H_i), \forall y \in X\}.$$

Definition 2 *F is approval voting** if, for every* $(R_1, ..., R_n) \in \mathcal{D}^n$*,*

- *if* $Max(R_1, ..., R_n) \neq X$*, then* $H = Max(R_1, ..., R_n)$ *and* $L = X \diagdown Max(R_1, ..., R_n)$*,*
- *if* $Max(R_1, ..., R_n) = X$*, then* $R = CI$*.*

This has range $\mathcal{D}$ and satisfies *WP*, but not *IIA*:
Take xP_1yI_1z, zP_2yI_2x and $xI_1'zP_1'y$, $zP_2'yI_2'x$. Hence $x, z \in H, y \in L$, so xPy, and $z \in H', x, y \in L'$, so $xI'y$.

The simply dichotomous case

Proposition 1 *A collective preference function* $F : \mathcal{D}^n \to \mathcal{O}$ *is approval voting* if and only if it satisfies IIA, SPP and A.*

The sufficiency part relies on two lemmas.

Lemma 4 *If* $F : \mathcal{D}^n \to \mathcal{O}$ *satisfies IIA, WP and PI, it satisfies N.*

Lemma 5 *If* $F : \mathcal{D}^n \to \mathcal{O}$ *satisfies N and SP, it satisfies PR2.*

The end of proof makes essential use of *PR2* (as against other *PR* conditions).

Vorsatz (2007, Theorem 1) characterizes approval voting for the simply dichotomous case as a social choice function satisfying Anonymity, Neutrality, Strategyproofness and Strict Monotonicity.

The conditions are heuristically related: Vorsatz's Neutrality corresponds with *IIA* and *PI*, and his Strict Monotonicity with *PR3*, here a consequence of *PR3*, or *N* and *SP*.

We go farther into the Arrovian foundations of approval voting, Proposition 1 being the positive counterpart to Theorem 1.

Compare also with Arrow's (1963, ch. V) "possibility theorem" for two alternatives and May's (1952) derivative characterization of majority voting for this case: it satisfies *N*, *PR3* and *A*.
Majority voting (or *plurality voting* if $|X| \geq 3$) is defined here by the collective preference function: for all $(R_1, ..., R_n)$, xRy if $n(xR_iy) \geq n(yR_ix)$, or equivalently $n(xP_iy) \geq n(yP_ix)$.
With $\mathcal{D}^n$, this is equivalent to approval voting*, so Proposition 1 generalizes the early findings. The relevant restriction is not on the number of alternatives, but of equivalence classes available to the individuals.