LATTICE THEORY of CONSENSUS (AGGREGATION)
An overview

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First a little precision

In their kind invitation letter, Klaus and Clemens wrote

"Like others in the judgment aggregation community, we are aware of the existence of a **sizeable amount of work** of you and other – **mainly French** – authors on generalized aggregation models".

Indeed, there is a sizeable amount of work and I will only present some main directions and some main results.

Now here a list of the main contributors:

Bandelt H.J.	Germany
Barbut, M.	France
Barthélemy, J.P.	France
Crown, G.D.,	USA
Day W.H.E.	Canada
Janowitz, M.F.	USA
Mulder H.M.	Germany
Powers, R.C.	USA
Leclerc, B.	France
Monjardet, B.	France
McMorris F.R.	USA
Neumann, D.A.	USA
Norton Jr. V.T	USA
Powers, R.C.	USA
Roberts F.S.	USA

LATTICE THEORY of CONSENSUS (AGGREGATION) : An overview

OUTLINE

ABSTRACT AGGREGATION THEORIES: WHY? HOW

The LATTICE APPROACH

LATTICES: SOME RECALLS

The CONSTRUCTIVE METHOD

The federation consensus rules

The AXIOMATIC METHOD

Arrowian results

The OPTIMISATION METHOD

Lattice metric rules and the median procedure

The "good" lattice structures for medians:

Distributive lattices

Median semilattice

ABSTRACT CONSENSUS THEORIES: WHY?

"since Arrow's 1951 theorem, there has been a flurry of activity designed to prove analogues of this theorem in other contexts, and to establish contexts in which the rather dismaying consequences of this theorem are not necessarily valid. The resulting theories have developed somewhat independently in a number of disciplines, and one often sees the same theorem proved differently in different contexts. What is needed is a general mathematical model in which these matters may be disposed of in a common setting. That is to say, we forget about the exact nature of the objects and, **using some abstract structure on various sets of objects under consideration**, **concern ourselves instead with ways in which the structure can be used to summarize a given family of objects**".

excerpt of the introduction of Barthélemy and Janowitz's 1991 paper

ABSTRACT CONSENSUS THEORIES: HOW

The different approaches result of the different abstract structures on "the sets of objects under consideration":

Which abstract structure exist (or can be put) on the set of objects to aggregate ?

ANSWER

- Logical structure
- Combinatorial structure (graph, hypergraph…)
- Algebraic structure (vector space…)
- Order and especially lattice structure
- Metric space structure

etc

ABSTRACT CONSENSUS THEORIES: HOW

- LOGICAL APPROACH

Guilbaud 1952 (*..le problème logique de l'agrégation*), Murakami 1958 (*Logic and Social Choice*)

Judgment aggregation theory since the 2000s

- COMBINATORIAL APPROACH

Wilson 1975 (*covers and frames*) Bandelt & Barthélemy 1984 (*median graphs*)

Nehring & Puppe (since) 2002 (*properties spaces = separating copair hypergraphs ≈ separating split systems), median spaces = separating Helly copair hypergraphs*)

- ALGEBRAIC APPROACH

Rubinstein and Fishburn 1986 (vector spaces)

- LATTICE APPROACH

since the 90s, see the continuation

LINKS between STRUCTURES

EXAMPLES

median graphs ↔ median spaces ↔ median semilattices

n-dimensional vector spaces on GF(2) ↔ Boolean algebras $\underline{2}^n$

↔

distributive and complemented lattices

Property spaces ↔ Subsets of Bolean algebras

(classical) propositional calculus ↔ Boolean calculus

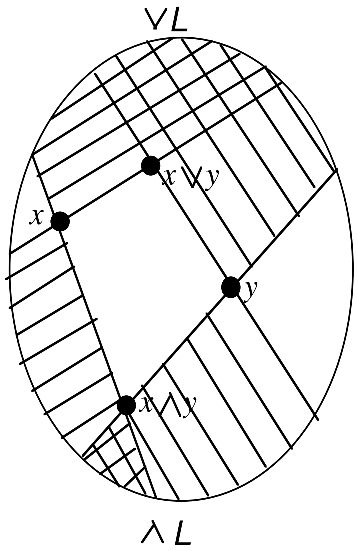
These links (should) imply

LINKS between APPROACHES

but…

LATTICES: SOME RECALLS

A *LATTICE* $(L, \leq)$ is a partially ordered set (poset) such that the *greatest lower bound* and the *least upper bound* of x and y, exist for all $x, y \in L$



NOTATION $x \wedge y$ = glb(x,y) is the *meet* of x and y,

$x \vee y$ = lub(x,y) is the *join* of x and y

L (finite) lattice imply:

for every $X \subseteq L$, $\mathrm{glb}(X) = \wedge X$ and $\mathrm{lub}(X) = \vee X$ exist

In particular

L has a least element $\wedge L$

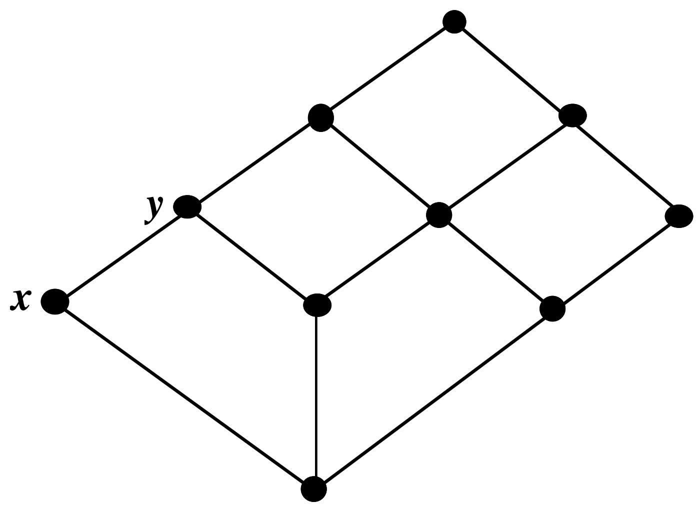
and a greatest element $\vee L$

A lattice (as any poset) can be vizualized by a (*Hasse*) *DIAGRAM*

representing the *covering relation* $\prec$ of $\leq$:

x is covered by y if there is no element *between* x and y

(formally: $x \prec y$ if $x < y$ and $x \leq z < y$ implies $x = z$)



a diagram of a lattice

More generally

A *JOIN-SEMILATTICE* $(L, \leq)$ is a poset such that
the greatest lower bound
of x and y, exist for all $x, y \in L$.

A *MEET-SEMILATTICE* $(L, \leq)$ is a poset such that
the least upper bound
of x and y, exist for all $x, y \in L$.

Then, a lattice is a meet- and a join-semilattice

SOME EXAMPLES of (SEMI)LATTICES of "OBJECTS"

The set of all binary relations (on a set) wrt to the *inclusion* order

$$R \subseteq R'$$

The set of all order relations (on a set) wrt to the *inclusion* order

$$O \subseteq O'$$

The set of all partitions (of a set) wrt to the *refinement* order

$P \leq Q$ (if any class of P is contained in a class of Q)

The set of all choice functions (on a set) wrt to the *pointwise* order

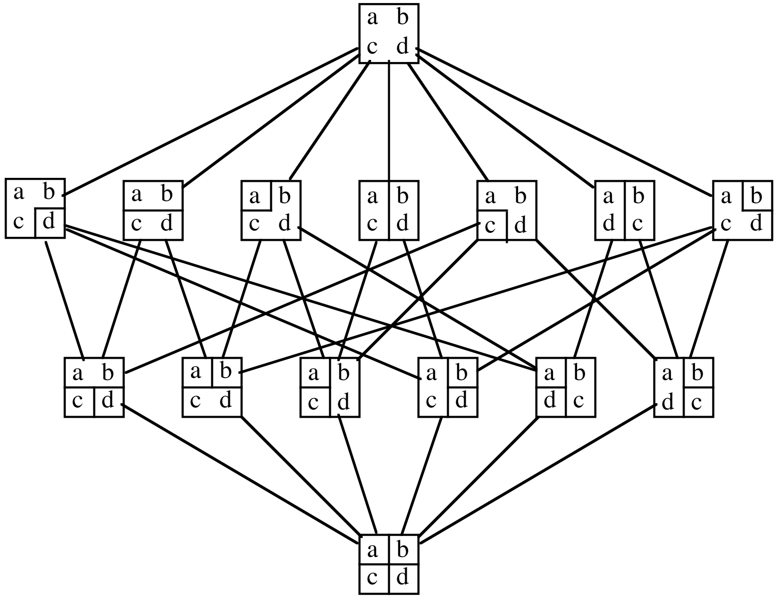
$c \leq c'$ (if ($c(A) \subseteq c'(A)$ for every subset A)

The set of all choice functions satisfying the *heredity* property (= Chernoff axiom = α condition = etc) wrt to the *pointwise* order

The set of all (e)valuations of n propositional variables

$(x_1,\ldots,x_n) \leq (x'_1,\ldots,x'_n)$ (if $x_i \leq x'_i$ for every i)

Etc, etc



The lattice of partitions of $\{a,b,c,d\}$

You find that a partially ordered set is a lattice

What must you do ?

To search if this lattice belongs to one of the many well studied particular classes of lattices such that
Boolean,
distributive,
locally distributive,
modular,
semimodular,
geometric,
bounded,
pseudo-complemented,
etc

How to find the properties of a lattice ?

- direct check

Example: since binary relations are sets, the lattice of binary relations is distributive:

$$R_1 \cup (R_2 \cap R_3) = (R_1 \cup R_2) \cap (R_1 \cup R_3)$$

- to determine the (join or meet) *irreducible elements* of the lattice and the *arrow relations* between them

many properties of a lattice depend only of properties of these arrow relations (see Darmstadt' school, Wille & co)

WHAT ARE the JOIN (MEET)-IRREDUCIBLES ?

An element j of a lattice L is *JOIN-IRREDUCIBLE* if
it is not join of other elements of L
(formally: $j = \vee X$ implies $j \in X$)

or, **equivalently**

j covers a unique element of L
(formally: $\exists!\, x \in L$ such that $x \prec j$)

FACT

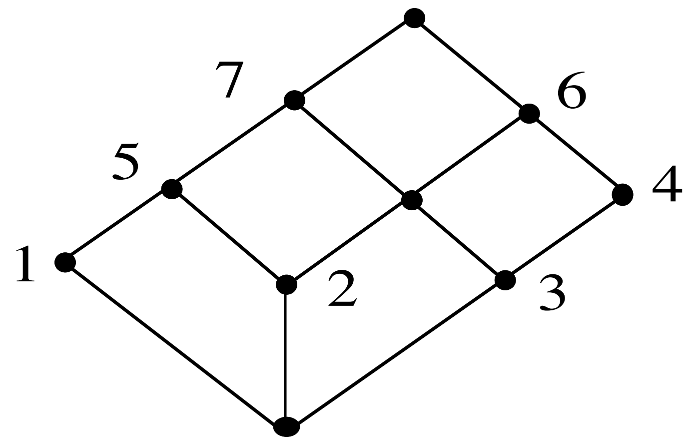
Any element of a lattice L is join of join-irreducibles of L

$$x = \vee J_x = \vee \{j \in J_L : \ j \leq x\}$$

The join-irreducible elements of a lattice are the "bricks" whose the elements of the lattice are formed

Dually,

An element m of L is MEET-IRREDUCIBLE if it is not meet of other elements of L and equivalently if it is covered by a unique element
Any element of a lattice is meet of meet-irreducible elements



L

$J_L = \{1,2,3,4\}$ $M_L = \{1,4,5,6,7\}$

N.B.

1,2,3, the elements covering the least element of L are called its *atoms*

(6,7 are its *coatoms*)

SOME EXAMPLES for (SEMI)LATTICES of "OBJECTS"

The lattice of all binary relations (on a set S)

$$J = \{(x, y), x,y \in S\}$$

The (semi)lattice of all order relations (on a set S)

$$J = \{(x<y), x,y \in S\}$$

The lattice of all partitions (of a set S)

$$J = \{(A/x/y/.../z), A \subset S, |A| > 1\}$$

The set of all (e)valuations of n propositional variables

$$J = \{(0,\ldots 0,1,0\ldots 0)\}$$

The lattice of all choice functions (on a set S)

$$J = \{c_{U,x} \text{ with } c_{U,x}(A) = x \text{ if } A = U \text{ and } c_{U,x}(A) = \varnothing \text{ if not, for any } U \subseteq S \text{ and any } x \in U\}$$

The lattice of all choice functions (on a set) satisfying the *heredity* property

$J = \ldots$

Etc (but not always easy to determine)

The LATTICE APPROACH: GENERAL FRAMEWORK

The "objects" are elements of a lattice L
There is a set $N = \{1,\ldots n\}$ of "*voters*". Each voter choses an element of L
So, one has n-tuples (*profiles*) $\pi = (x_1,.. x_i...x_n)$ of elements of L

One search to define "good" procedures aggregating any n-tuple π into one -or several- *consensus object(s)*

A *(lattice) consensus function* is a map

$$F: \quad L^n \rightarrow L$$
$$\pi = (x_1,\ldots,x_n) \rightarrow F(\pi) = x$$

or

$$F: \quad L^n \rightarrow 2^L$$
$$\pi = (x_1,\ldots,x_n) \rightarrow F(\pi) \subseteq L$$

HOW to DEFINE CONSENSUS FUNCTIONS?

constructive method

the consensus map uses the structure defined on the objects (*mean*, *median*..for numbers)

axiomatic method

"to sit in an armchair and think of desirable properties that a consensus rule should possess, and then attempt to find the rules satisfying these properties" (McMorris 1985)

optimisation method

the consensus objects must optimize a given criteria measuring their remoteness to the profile

What is often the more interesting is to find the relations between these methods

for instance: can we characterize axiomatically an optimisation method and or define it algebraically ?

The CONSTRUCTIVE METHOD
the FEDERATION CONSENSUS FUNCTIONS

Federation (*simple game*) on $N = \{1,\ldots n\}$ (the set of "*voters*"):
a family $\mathcal{F}$ of subsets of N (the "*winning coalitions*")

such that $[V \in \mathcal{F}, W \supseteq V] \Rightarrow [W \in \mathcal{F}]$.

$\mathcal{F} \rightarrow$ $F_{\mathcal{F}}$ *federation consensus function associated to* $\mathcal{F}$:

$$\boxed{\pi = (x_1,\ldots,x_n) \in L^n \rightarrow \quad F_{\mathcal{F}}(\pi) = \bigvee_{W \in \mathcal{F}} (\bigwedge_{i \in W} x_i)}$$

$F_{\mathcal{F}}$ is given by a *lattice polynomial*

(very useful) FACT

a federation consensus function $F_{\mathcal{F}}$ is obtained as a join of join-irreducibles:

j join-irreducible of L

$\pi = (x_1,\ldots,x_n) \in L^n$

$N_\pi(j) = \{i \in N : j \leq x_i\} \subseteq N$
(the set of voters which "vote" for j)

$n_\pi(j) = |N_\pi(j)|$
(the number of voters which "vote" for j)

$$\boxed{\pi \to F_{\mathcal{F}}(\pi) = \vee_{W \in \mathcal{F}} (\wedge_{i \in W} x_i) = \vee\{j \in J : N_\pi(j) \in \mathcal{F}\}}$$

$(MAJ(\pi) = \cup\{(x,y) : |\{i \in N : xR_iy\}| \geq n/2\} = \cup_{|W| \geq n/2}\{\cap\{R_i, i \in W\}\})$

EXAMPLES

Majority rule $\mathcal{F} = \{W \subseteq N : |W| \geq n/2\}$

$$x_{\mathrm{MAJ}}(\pi) = \vee\{j \in J : n_\pi(j) \geq n/2\}$$

Strict Majority rule $\mathcal{F} = \{W \subseteq N : |W| > n/2\}$

$$x_{\mathrm{SMAJ}}(\pi) = \vee\{j \in J : n_\pi(j) > n/2\}$$

"*Meet-projection* ("oligarchic rule") $\mathcal{F}$ is the *filter* $\{W \subseteq N : M \subseteq W\}$

$$F_{\mathcal{F}}(\pi) = \wedge_{i \in M} x_i = \vee\{j \in J : N_\pi(j) \supseteq M\}$$

Projection ("dictatorial rule") $\mathcal{F}$ is the *ultrafilter* $\{W \subseteq N : i \in W\}$)

$$F_{\mathcal{F}}(\pi) = x_i$$

The AXIOMATIC METHOD
SOME PROPERTIES ("axioms")

$$F : L^n \rightarrow L$$

$\pi = (x_1,\ldots,x_n) \in L^n$, j join-irreducible of L $N_\pi(j) = \{i \in N : j \leq x_i\}$

F is *decisive* if for every $j \in J$ and for all $\pi, \pi' \in L^n$,

$$[N_j(\pi) = N_j(\pi')] \Rightarrow [j \leq F(\pi) \Leftrightarrow j \leq F(\pi')]$$

F is *neutral monotonic* if for all $j, j' \in J$ and for all $\pi, \pi' \in L^n$,

$$[N_j(\pi) \subseteq N_{j'}(\pi')] \Rightarrow [j \leq F(\pi) \Rightarrow j' \leq F(\pi')]$$

F is *meet compatible (Paretian)* if for every $\pi \in L^n$,

$$\wedge\{x_j, i \in N\} = \vee\{j \in J : N_\pi(j) = N\} \leq F(\pi)$$

The AXIOMATIC METHOD: a FIRST RESULT

$F : L^n \rightarrow L$ *a consensus function on a lattice* L

(1) *If* L *is distributive, then F is a federation consensus function if and only F is neutral monotonic and Paretian.*

(2) *If* L *is not distributive, then F is a meet projection (oligarchic) consensus function if and only if F is neutral monotonic and Paretian.*

(3) *If* L *is δ-strong, then F is a meet-projection (oligarchic) consensus function if and only F it is decisive and Paretian.*

BM 1990

WHAT IS the DEPENDENCE RELATION δ ?

$j, j' \in J = J_L =$ {join-irreducible elements of L}

$j\delta j' \Leftrightarrow j \neq j'$ and there exists $x \in L$ such that $j, j' \nleq x$ and $j < j' \vee x$

(J, δ) is a directed graph

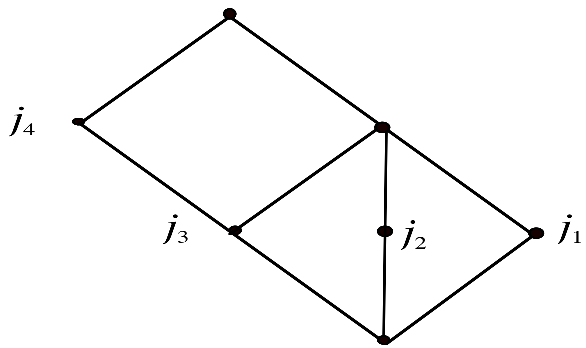
L is said *δ-strong* if (J, δ) is strongly connected

N.B.

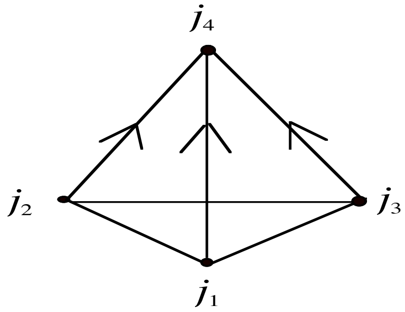
$j < j' \Rightarrow j\delta j'$ $(j < j' \vee 0_L)$

$\delta = <_J$ if and only if L is a distributive lattice

DEPENDENCE RELATION EXAMPLES



$$j_3 < j_4 \quad j_3 < j_1 \vee j_2 \quad j_1 < j_2 \vee j_3 < j_2 \vee j_4 j_2 \quad j_2 < j_1 \vee j_3 < j_1 \vee j_4$$



δ

DEPENDENCE RELATION EXAMPLES

- In the semilattice O of orders

$$(x<z)\delta(x<y) \text{ and } (x<z)\delta(y<z)$$

since $(x<z) < (x<y)\vee(y<z)$ $(= x<y<z)$ (transitivity!)

So, O is *δ-strong*

-The same for the lattice of partitions

SOME COROLLARIES

BROWN's RESULT (for orders) F: $\mathcal{O}^n \rightarrow \mathcal{O}$

F is a decisive and Pareto $\mathcal{O}$-consensus function

$\Leftrightarrow$

F is a $\cap$-projection ("oligarchic")

MIRKIN-LECLERC's RESULT (for partitions) F: $\mathcal{P}^n \rightarrow \mathcal{P}$

F is an independent and Pareto $\mathcal{P}$-consensus function

$\Leftrightarrow$

F is a meet-projection ($F(\pi) = \wedge_{i\in M} P_i$)

NEUMANN and-NORTON's RESULT (for partitions) F: $\mathcal{P}^n \rightarrow \mathcal{P}$

Characterization of join-projections $F(\pi) = \vee_{i\in M} P_i$

AIZERMAN AND ALESKEROV's RESULT (for choice functions satisfying H) F: $\mathcal{C}^n \rightarrow \mathcal{C}$

F is a neutral monotonic and Paretian $\mathcal{H}$-consensus function

$\Leftrightarrow$

F is a federation consensus function

Etc, other classes of choice functions, valued (fuzzy) preorders, ultrametrics, hierarchies….

The AXIOMATIC METHOD: a RECENT RESULT

$F : L^n \to L$ *consensus function on an atomistic and δ-strong lattice*

F is decisive and Paretian

$\Leftrightarrow$

F is neutral monotonic and Paretian

$\Leftrightarrow$

F is a $\wedge$*-morphism* $(F(\pi \wedge \pi') = F(\pi) \wedge F(\pi'))$ *and Paretian*

$\Leftrightarrow$

F is a residual map and $j \leq F((j,\dots j))$ *for any* $j \in J$

$\Leftrightarrow$

F is a meet-projection (oligarchic)

BLeclerc BM *Order* 2012

COROLLARIES

δ-strong atomistic lattices:

- the lattice of partitions

  Dimitrov D., Marchant T., Mishra N.: Separability and Aggregation of Equivalence Relations, *Economic Theory*, 2011

  Chambers C.P., Miller A.D. : Rules for Aggregating Information. *Social Choice and Welfare*. 2011

- the lattice of orders

- the lattice of preorders

GENERALIZATIONS to ORDERS

The notion of (join or meet) irreducible extends to any ordered set

The notion of dependency relation may be extended in several ways

So, several results may be extended in some ordered sets

See (up to 2003), Day W.H.E., McMorris F.R. *Axiomatic Consensus Theory in Group Choice and Biomathematics*. SIAM.

OPTIMISATION METHOD (METRIC CRITERION)

L (meet) semilattice ; $\pi = (x_1,\ldots,x_n) \in L^n$

d *distance* on L

$R(x, \pi)$ *remoteness* (depending on d) between π and $x \in L$

$$\pi \rightarrow \{x \in L\colon R(x, \pi) \text{ MIN}\}$$

Examples of remoteness:

$$R(x, \pi) = \Sigma_{i = 1,\ldots n} d(x_i, x)$$

$$R(x, \pi) = \Sigma_{i = 1,\ldots n} d^2(x_i, x)$$

$$R(x, \pi) = \text{Max}_{i = 1,\ldots n} d(x_i, x)$$

The *d*-MEDIAN PROCEDURE

L (meet) semilattice ; $\pi = (x_1,\ldots,x_n) \in L^n$

d distance on L $R(x, \pi) = \Sigma_{i=1,\ldots n} d(x_i, x)$

The *d-median procedure* is:

$$\pi \rightarrow \mathrm{Med}_d(\pi) = \{x \in L : \Sigma_{i=1,\ldots,n}\, \boldsymbol{d}(x_i, x) \text{ MIN}\}$$
$$= \{\boldsymbol{d}\text{-}medians \text{ of } \pi\}$$

(a profile has generally several medians)

the Δ-MEDIAN PROCEDURE

$$x \in L \rightarrow J_x = \{j \in J_L:\ j \leq x\}$$

$x, x' \in L$, **Δ** is the *symmetric difference distance* between J_x and $J_{x'}$:

$$\Delta(x, x') = |J_x \Delta J_{x'}| = |\{j \in J\text{: } [j \in J_x \text{ and } j \notin J_{x'}] \text{ or } [j \notin J_x \text{ and } j \in J_{x'}]\}|$$

L (meet) semilattice ; $\pi = (x_1,\ldots,x_n) \in L^n$

$$\mathrm{Med}_\Delta(\pi) = \{x \in L\text{: } \Sigma_{i=1,\ldots n}\ \Delta(x_i, x) \text{ MIN}\}$$
$$= \{(\Delta\text{-})\textit{medians} \text{ of } \pi\}$$

the Δ-MEDIAN PROCEDURE

$$x \in L \rightarrow J_x = \{j \in J_L:\ j \leq x\}$$

$x, x' \in L$, Δ is the *symmetric difference distance* between J_x and $J_{x'}$

$$\Delta(x, x') = |J_x \Delta J_{x'}| = |\{j \in J: [j \in J_x \text{ and } j \notin J_{x'}] \text{ or } [j \notin J_x \text{ and } j \in J_{x'}]\}|$$

L (meet) semilattice ; $\pi = (x_1,\ldots,x_n) \in L^n$

$$\mathrm{Med}(\pi) = \{x \in L: \Sigma_{i=1,\ldots n}\ \Delta(x_i, x) \text{ MIN}\} = \{(\Delta\text{-})\textit{medians} \text{ of } \pi\}$$

$$j \in J_L \quad n_\pi(j) = |\{i \in N : j \leq x_i\})|$$

$$\boxed{\mathrm{Med}_\Delta(\pi) = \{x \in L: A(\pi,x) = \Sigma_{j \leq x} n_\pi(j) \text{ MAX}\}}$$

The computation of medians is a problem of combinatorial optimization, and -generally- a difficult problem

"GOOD" LATTICE STRUCTURES for (Δ-)MEDIANS

I Distributive lattices

A lattice L is *distributive* if each one of the meet and join operations is distributive over the other, for instance, if for all $x,y,z \in L$,

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

equivalently if for all $x,y,z \in L$,

$$(x \wedge y) \vee (x \wedge z) \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \wedge (y \vee z)$$

This element is called the *algebraic median* of x,y,z

A lattice L is distributive if each one of the meet and join operations is distributive over the other, for instance, if for all $x,y,z \in L$,

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

equivalently, if for all $x,y,z \in L$,

$$(x \wedge y) \vee (x \wedge z) \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \wedge (y \vee z)$$

This element is called the algebraic median of x,y,z

More generally, the *algebraic median* of $\pi = (x_1,\ldots,x_n)$ is obtained by the majority rule ($n_\pi(j) = |\{i \in N : j \leq x_i\}|$)

$$x_{\mathrm{SMAJ}}(\pi) = \vee_{S \subseteq N,\ |S| > n/2}(\wedge_{i \in S} x_i) = \wedge_{S \subseteq N,\ |S| > n/2}(\vee_{i \in S} x_i)$$
$$= \vee_{j \in J}\{n_\pi(j) > n/2\} \quad (n_\pi(j) = |\{i \in N : j \leq x_i\}|$$
$$= \vee_{j \in J}\{\textit{strict majority}\ \text{join-irreducible}\}$$

One also considers

$$x_{\mathrm{MAJ}}(\pi) = \vee_{j \in J}\{n_\pi(j) \geq n/2\}$$
$$= \vee_{j \in J}\{\textit{majority}\ \text{join-irreducible}\}$$

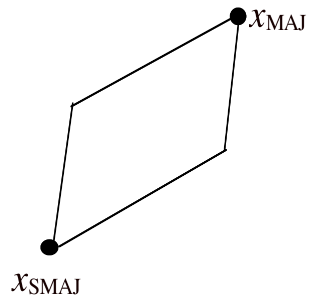
then

$$x_{\mathrm{SMAJ}}(\pi) \leq x_{\mathrm{MAJ}}(\pi)$$

MEDIANS in DISTRIBUTIVE LATTICES

If L is a distributive lattice,
the set of medians of any profile is an interval of L
containing its algebraic median as least element

$$\pi \in L^n \;\Rightarrow\; \mathrm{Med}(\pi) = [x_{\mathrm{SMAJ}}(\pi), x_{\mathrm{MAJ}}(\pi)]$$

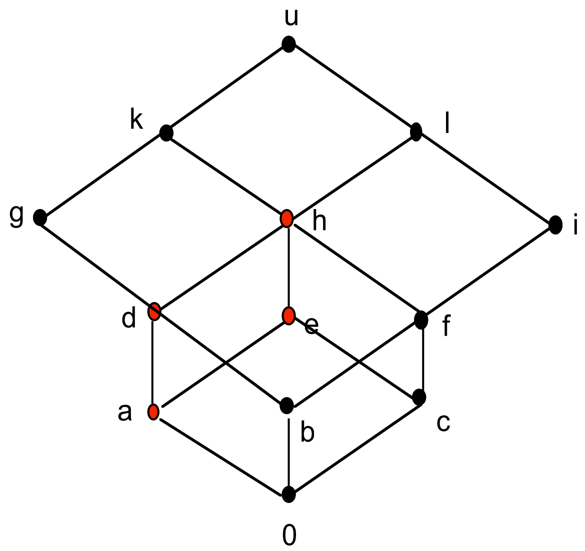


where

$$x_{\mathrm{SMAJ}}(\pi) = \vee\{\, j \in J : n_\pi(j) > n/2\} \;\leq\; x_{\mathrm{MAJ}}(\pi) = \vee\{\, j \in J : n_\pi(j) \geq n/2\}$$

$$(n_\pi(j) = |\{i \in N : j \leq x_i\}|)$$

EXAMPLE



$$\pi = (a, a, e, f, g, l)$$

$$x_{\mathrm{SMAJ}}(\pi) = a \qquad x_{\mathrm{MAJ}}(\pi) = h$$

$$\mathrm{Med}(\pi) = [a,h] = \{a,d,b,h\}$$

NB 1 if n is odd

$$\mathrm{Med}(\pi) = x_{SMAJ}(\pi)$$

the metric median = the algebraic median of π

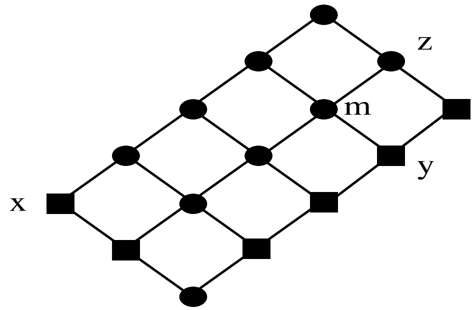
For instance,

$$\mathrm{Med}(x,y,z) = (x\wedge y)\vee(x\wedge z)\vee(y\wedge z)\ (=(x\vee y)\wedge(x\vee z)\wedge(y\vee z))$$

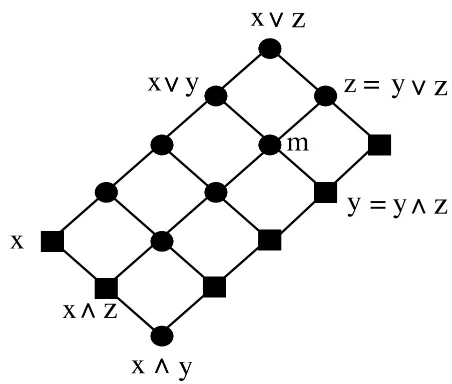
minimizes $\Delta(x,t) + \Delta(y,t) + \Delta(z,t)$

NB 2 In a distributive lattice L,

$\Delta(x,x')$ = minimum path length between x and x' in the covering graph of L



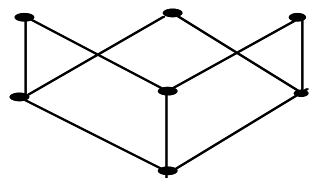
$$\Delta(x,m) + \Delta(y,m) + \Delta(z,m) = 6 < \Delta(x,t) + \Delta(y,t) + \Delta(z,t) \text{ for any } t \neq m$$



$$\mathrm{Med}(x,y,z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$$

"GOOD" LATTICE STRUCTURES for MEDIANS
II Median semilattices

A meet semilattice L is *lower distributive* if, for any $x \in L$,
the order ideal $\{x' \in L\colon x' \leq x\}$ is a distributive lattice

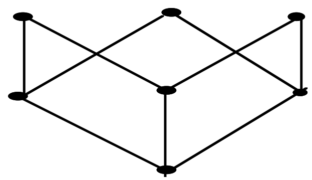


a lower distributive meet semilattice

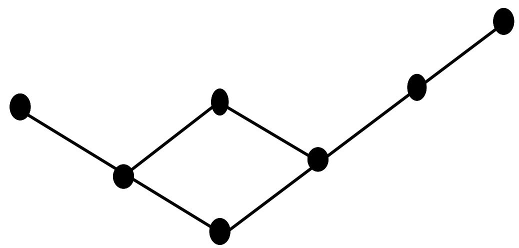
A meet semilattice L is *lower distributive* if, for any $x \in L$,
the order ideal $\{x' \in L : x' \leq x\}$ is a distributive lattice

A *median semilattice* is a lower distributive meet semilattice L in which, for all $x_1, x_2, x_3 \in L$,

$$x_1 \vee x_2,\ x_1 \vee x_3 \text{ and } x_2 \vee x_3 \text{ all exist} \quad \Rightarrow \quad x_1 \vee x_2 \vee x_3 \text{ exists}$$



Counter-example



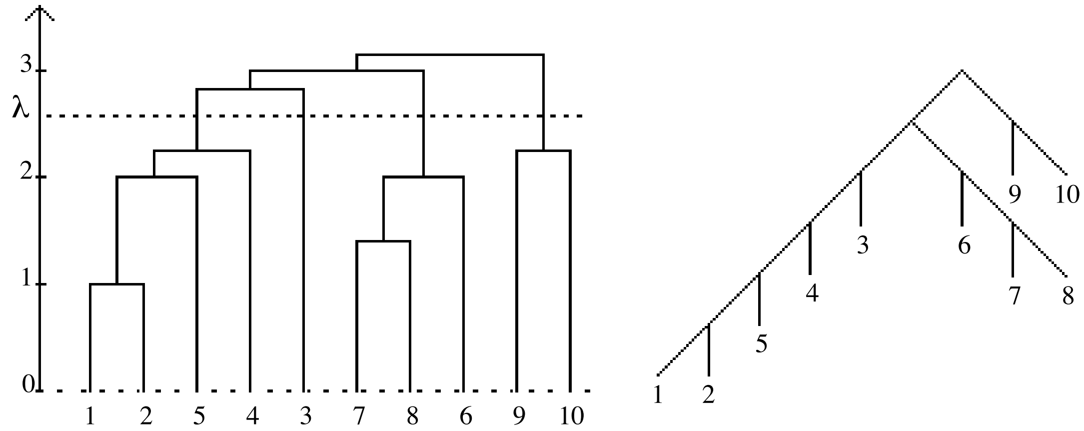
example

MEDIAN SEMILATTICES EXAMPLES

Distributive lattices
(oriented) Trees

Poset of all the cliques of an (unoriented) graph :
Poset of all the chains of a poset
Poset of all the antichains of a poset

Poset of all the *hierarchies* on a set



MEDIAN SEMILATTICES

L median semilattice $\Rightarrow$

for any $\pi = (x_1, x_2, \ldots x_n) \in L^n$, its algebraic median

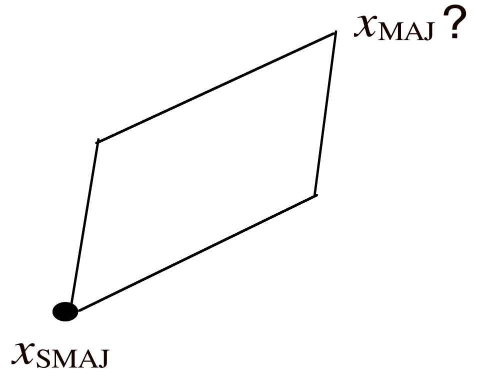
$x_{\mathrm{SMAJ}}(\pi) = \vee\{\, j \in J\colon n_\pi(j) > n/2\}$ $(= \vee_{S \subseteq N,\ |S| > n/2}(\wedge_{i \in S} x_i))$ exists

and in particular

for all $x_1, x_2, x_3 \in L$, their algebraic median

$(x_1 \wedge x_2)(x_2 \wedge x_3) \vee (x_3 \wedge x_1)$ exists

BUT the element $x_{MAJ}(\pi)$ does not necessarily exist



L median semilattice $\Rightarrow$

for any $\pi = (x_1, x_2, \ldots x_n) \in L^n$, its algebraic median

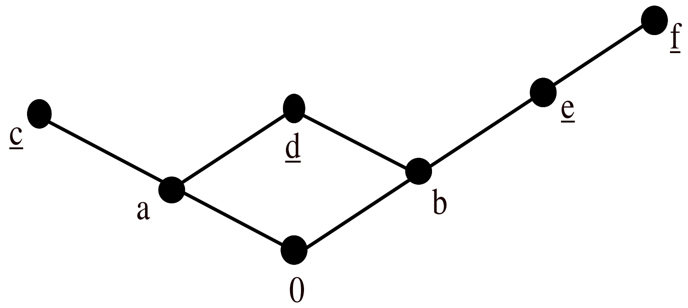
$$x_{SMAJ}(\pi) = \vee\{\, j \in J: n_\pi(j) > n/2\}\ \text{exists}$$

and in particular

for all $x_1, x_2, x_3 \in L$, their algebraic median

$$(x_1 \wedge x_2)(x_2 \wedge x_3) \vee (x_3 \wedge x_1)\ \text{exists}$$

but the element $x_{MAJ}(\pi)$ does not necessarily exist



$$\pi = (c,d,e,f) \quad x_{SMAJ}(\pi) = \vee_{j \in J}\{n_\pi(j) \geq 3\} = b$$

$$x_{MAJ}(\pi) = \vee_{j \in J}\{n_\pi(j) \geq 2\} = a \vee b \vee e \text{ does not exist}$$

MEDIANS in MEDIAN SEMILATTICES

$\pi = (x_1,\ldots,x_n) \in L^n$ $\quad$ j join-irreducible $\quad$ $n_\pi(j) = |\{i \in N : j \leq x_i\}|$

$C(\Pi) = \{j \in J: n_\pi(j) > n/2\}$ (strict majority join-irreducible)

$\subseteq$

$B(\Pi) = \{j \in J : n_\pi(j) \geq n/2\}$ (majority join-irreducible)

L median semilattice and $\pi \in L^n$

if n is odd,

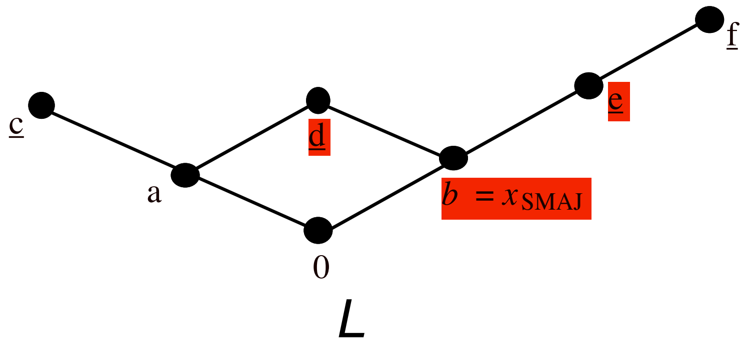
$$\mathrm{Med}(\pi) = x_{\mathrm{SMAJ}}(\pi) = \vee \{ j \in J: n_\pi(j) > n/2\}$$

if n is even,

$$\mathrm{Med}(\pi) = \{\vee K : C(\Pi) \subseteq K \subseteq B(\Pi) \subseteq J \text{ and } \vee K \text{ exists}\}$$

Bandelt and Barthélemy,1984

COMPUTATION of MEDIANS in a MEDIAN SEMILATTICE I

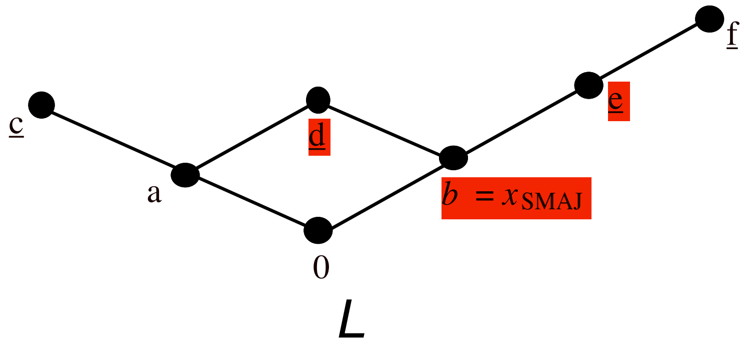


$$\pi = (c,d,e,f) \in L^4 \quad \longrightarrow \quad \mathrm{Med}(\pi) = \{b,d,e\}$$

$C(\Pi) = \{j \in J\colon n_\pi(j) \geq 3\} = b$
$B(\Pi) = \{j \in J\colon n_\pi(j) \geq n/2\} = \{a,b,e\}$

$$K = C(\Pi) = \{b\} \rightarrow b,$$
$$K = \{a,b\} \rightarrow a \vee b = d,$$
$$K = \{b,e\} \rightarrow b \vee e = e$$

COMPUTATION of MEDIANS in a MEDIAN SEMILATTICE II



$$\pi = (c,d,e,f) \in L^4 \quad \longrightarrow \quad \mathrm{Med}(\pi) = \{b,d,e\}$$

In a median semilattice L, $\Delta(x,x')$ = minimum path length between x and x' in the covering graph of L

$$\Delta(c,b) + \Delta(d,b) + \Delta(e,b) + \Delta(f,b) = 7$$

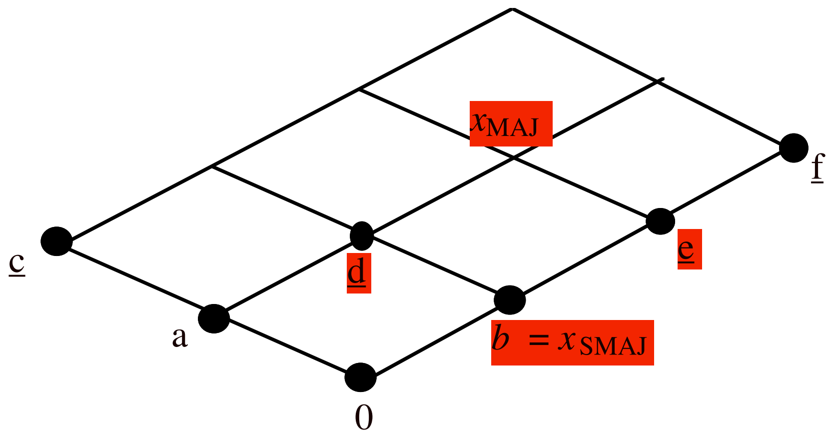
COMPUTATION of MEDIANS in a MEDIAN SEMILATTICE III
Sholander's embedding

Any median semilattice L can be embedded in a distributive lattice D so that L is an (order) ideal of D

Sholander 1954

Then

$$\mathbf{MED}(\pi) = [x_{\mathrm{SMAJ}}(\pi), x_{\mathrm{MAJ}}(\pi)]_D \cap L$$



MEDIAN SEMILATTICES

AXIOMATICS of the MEDIAN PROCEDURE

L median semilattice $\quad L^* = \cup\{L^n, n \in \mathbb{N}\}$

m: $L^* \to (2^L \setminus \{\emptyset\})$ an aggregation (multi)procedure

m is the median procedure $\Leftrightarrow$ m satisfies the following

three properties:

"Condorcet": $\pi \in L^n$ with even n,

$j \in J_L$, $2n(j) = n$, $t \in L$ and $t \vee j$ exists

$\Rightarrow \quad [t \vee j^- \in m(\pi) \Leftrightarrow t \vee j \in m(\pi)]$

Consistency: $\pi, \pi' \in L^*$ and $m(\pi) \cap m(\pi') \neq \emptyset$

$\Rightarrow \quad m(\pi\pi') = m(\pi) \cap m(\pi')$

Faithfulness: $\pi = (t) \in L \Rightarrow m(\pi) = \{t\}$

Barthelemy, Janowitz (1991) McMorris, Mulder, Powers (2000)

the MEDIAN PROCEDURE for other DISTANCES or/and other SEMILATTICES

Many results on the properties of the median procedure for **Δ** or other distances in other classes of semilattices and on the determination of the medians.

Example (Leclerc, *Discrete Applied Mathematics*, 2003):

The median procedure in the semilattice of orders is Paretian for the distance **Δ** but not Paretian for many other distances *d*:

There exist profiles $\pi = (O_1, .. O_i ... O_n)$ of orders and (median) orders M minimizing $\Sigma_{i=1,...n} d(O_i, M)$ such that

$\cap\{Oi, i \in N\}$ □ M

PARETO PROPERTY and MEDIANS

Question: does $\wedge\{x_i\} \leq m$ hold for any median m of any profile π of the (semi)lattice L?

Type of (semi)lattice	weight metrics d	metric Δ	metric ∂
distributive lattice	Yes (1)	Yes (♦)	Yes (♦)
modular lattice	?	?	Yes (2)
LLD lattice	No (∅)	No (∅)	No (3)
lower semimodular lattice	No (↑)	No (↑)	No (↑)
upper semimodular lattice	No (¬)	No (¬)	Yes (2)
geometric lattice	No (∅)	No (4,5)	Yes (↑)

partition lattice	No (4)	Yes (6)	Yes (↑)
median semilattice	Yes (7)	Yes (♦)	Yes (♦)
distributive semilattice	No (∅)	No (4)	No (♦)
LLD semilattice	No (↑)	No (↑)	No (↑)
order semilattice	No (8)	Yes (8)	Yes (♦)

(1) Monjardet (1980); (2) Leclerc (1990); (3) Li (1996); (4) Leclerc (1994); (5) Barthélemy and Leclerc (1995); (6) Régnier (1965); (7) Leclerc (1993); (8) Leclerc (1999); (♦) from the entry at left; (∅) from the entry at right; (↑) from the entry above; (¬) from the entry below.

The metric ∂ is the "lattice metric" (minimum path length in the unvalued undirected covering graph of L); Δ is the symmetric difference metric on the representations by join-irreducibles; these join-irreducible may be weighted to give the "weight metrics".

(more or less) RELATED WORKS

P. Gärdenfors, 2006, A Representation Theorem For Voting With Logical Consequences , *Economics and Philosophy*, 22, 181 – 190

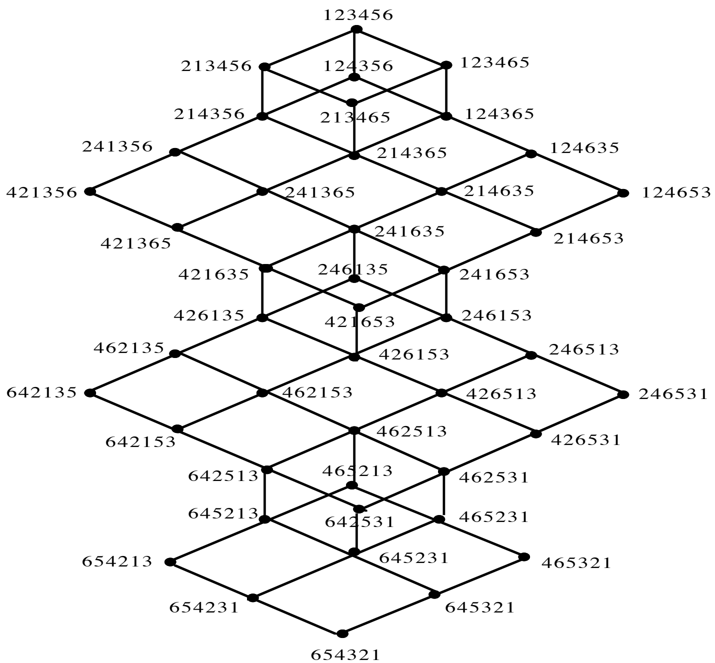
Alternatives = elements of an (atomless) Boolean algebra

T.R. Daniels, E. Pacuit (2009) A General Approach to Aggregation Problems *Journal of Logic and Computation* 19(3), 517-536.

Alternatives = Judgment sets
= deductively closed (wrt a consequence relation) subsets of a language
= elements of a (rather special) lattice

El. Dokow, R. Holzman (2010) Aggregation of binary evaluations, *Journal of Economic Theory*, 145 (2), 495-511.
Alternatives = (feasible) binary evaluations = elements of 2^n
(Boolean aggregators : Guilbaud, Eckert & bm)

BONUS



A "Condorcet" (*consistent*) domain of maximum size 64 for linear orders on $n = 6$

Most of the domains of linear orders where majority rule works well (no "voting paradox") are distributive lattices (sublattices of the *permutoedre lattice*)

(see Acyclic domains of linear orders: a survey, in *The Mathematics of Preference, Choice and Order, Essays in honor of Peter C. Fishburn*, S. Brams, W. V. Gehrlein & F. S. Roberts (Editors), Springer, 2009, 139-160)

First example: single-peaked domains are distributive lattices

$$
\begin{tikzcd}
 & & & & BACDE \arrow[r] & ABCDE \\
 & & & CBADE \arrow[r] & BCADE \arrow[u] & \\
 & DCBAE \arrow[r] & CDBAE \arrow[r] & CBDAE \arrow[r] \arrow[u] & BCDAE \arrow[u] & \\
 & DCBEA \arrow[r] \arrow[u] & CDBEA \arrow[r] \arrow[u] & CBDEA \arrow[r] \arrow[u] & BCDEA \arrow[u] & \\
 & DCEBA \arrow[r] \arrow[u] & CDEBA \arrow[u] & & & \\
EDCBA \arrow[r] & DECBA \arrow[u] & & & &
\end{tikzcd}
$$

Guilbaud 1952

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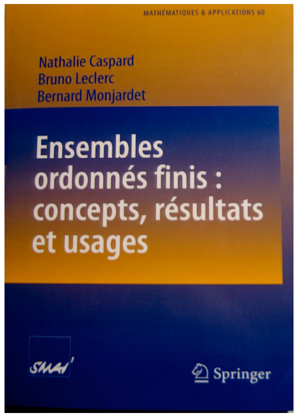

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