Impartial ranking of peers

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evaluation by peers:

- *index* of influence of journals, by the pattern of mutual citations
- reputation *scores* in peer-to-peer systems
- *ranking* of universities by surveys of alumni
- *ranking* of web pages by search engines

axiomatic literature:

- on scoring methods:

Palacios-Huerta, L. and O. Volij, *The measurement of intellectual influence*, Econometrica 2004

Slutzki G. and O. Volij, *Scoring of web pages and tournaments-axiomatizations*, Social Choice and Welfare, 2006

Ohseto, S., *Exclusion of self evaluations in peer ratings: monotonicity versus unanimity*, forthcoming Social Choice and Welfare, 2011

axiomatic literature:

- on ranking systems:

Altman A. and M. Tennenholtz, *On the axiomatic foundations of ranking systems*, 19th Int. Joint Confce on Artificial Intelligence, 2005

Demange G., *Collective attention and ranking methods*, working paper PSE, 2010

Impartiality a.k.a. Strategyproofness:

assume: I am selfishly concerned about my own ranking, not about that of my peers

require: my message does not affect my own ranking *related literature:*

- *(closely)* impartial methods to select one or more winners

Alon, N., Fischer, F., Procaccia, A., and M. Tennenholtz, *Sum of us: strategyproof selection from the selectors,* 2009

Holzman R. and H. Moulin, *Impartial award of a prize*, working paper 2010

- *(loosely)* preventing strategic cloning

Cheng A. and E. Friedman, *Sybilproof reputation mechanisms*, SIGCOMM 05, August 2005

model: peer ranking

assign the n agents in N to n ranks

notation:

$[n] = \{1, 2, \cdots, n\}$

a ranking of N is a bijection $\sigma : N \to [n]$

$\Sigma(N, [n])$ is the set of all rankings

i's message $m_i \in M_i$

ranking method:

choose a message space M_i for each agent i

for each profile of messages $m = (m_i) \in M_N$, the method selects $\theta(m) = \sigma \in \mathbf{\Sigma}(N, [n])$

- **Impartiality**: $\theta(m|^i m_i)[i] = \theta(m|^i m'_i)[i]$, for all m, i, m_i, m'_i

for $n = 3$ the only impartial ranking methods are constant, or fix the rank of an agent, who decides about the other two

$\rightarrow$ for $n \geq 4$, is there an impartial method satisfying

- **Full Ranks :**

for all $i \in N$, $a \in [n]$, there is some $m \in M_N$ s.t. $\theta(m)[i] = a$

- **No Dummy** :

for all $i \in N$, there is some m, m_i, m'_i s.t. $\theta(m|^i m_i) \neq \theta(m|^i m'_i)$

?

YES

for $n = 4$, we can use binary messages for all i: $M^i = \{0, 1\}$

the canonical method $\widetilde{\theta}^4$:

$(0,0,0,0) \longrightarrow 1234$; $(1,0,0,0) \longrightarrow 1432$; $(0,0,0,1) \longrightarrow 1324$; $(1,0,0,1) \longrightarrow 1423$

$(0,0,1,0) \longrightarrow 2134$; $(0,1,1,0) \longrightarrow 2143$; $(0,0,1,1) \longrightarrow 2314$; $(0,1,1,1) \longrightarrow 2341$

$(1,1,0,0) \longrightarrow 3412$; $(1,1,1,0) \longrightarrow 3142$; $(1,1,0,1) \longrightarrow 3421$; $(1,1,1,1) \longrightarrow 3241$

$(0,1,0,0) \longrightarrow 4213$; $(0,1,0,1) \longrightarrow 4321$; $(1,0,1,0) \longrightarrow 4132$; $(1,0,1,1) \longrightarrow 4213$

for $n = 4p$ we can partition the agents arbitrarily in four groups:

$$N = \cup_{k=1}^{4} N_k, \text{ where } |N_k| = p \text{ for } k = 1, \cdots, 4$$

and use a two step procedure:

step 1: for each k, the agents in N_k *jointly select* $m_k = 0$ or 1; then $\widetilde{\theta}^4(m)$ determines the ranking of the four groups; for instance

$N_3 N_1 N_4 N_2$ means: the first p ranks to N_3; the next p to N_1, etc..

step 2: for each k, the agents in $\cup_{j \neq k} N_j$ *jointly* assign the block of ranks found in step 1 to agents in N_k

- this procedure can be adapted to deal with any $n \geq 4$

- each agent can be assigned any rank, but not every *ranking* in $\Sigma(N,[n])$ is feasible: for instance $\widetilde{\theta}^4$ reaches 15 rankings out of 24

- fairly symmetric treatment of the agents, but

- the messages have no clear meaning

a more realistic model:

agent i reports a ranking $r_i \in \mathbf{\Sigma}(N, [n])$

the *aggregation method* f: $f(r) = \sigma$, maps $\mathbf{\Sigma}(N, [n])^N$ into $\mathbf{\Sigma}(N, [n])$

Impartiality: same definition

the aggregation terminology is justified if f satisfies:

- **Full Rankings:** for all $\sigma \in \Sigma(N, [n])$ there is some $r \in \Sigma(N, [n])^N$: $\sigma = f(r)$

- **Monotonicity**: if j goes up in r_i, ceteris paribus, the rank of j in $f(r)$ does not increase

Open question: can we construct an impartial, monotonic aggregation method satisfying Full Rankings?

We have two weaker positive statements:

- *for* $n \geq 4$, *there are impartial, monotonic aggregation method satisfying Full Ranks*

- *for* $n \geq 6$, *there are impartial aggregation methods satisfying Unanimity*

  Unanimity: if $r_i = \sigma$ for all i, then $f(r) = \sigma$

Note that Unanimity implies Full Rankings

$\longrightarrow$ our example for $n = 4$ of an impartial, monotonic aggregator with full ranks, translates a reported ranking into a binary input of the method $\widetilde{\theta}^4$; thus it aggregates very little information from the reports

$\longrightarrow$ our construction, for any $n \geq 6$, of an impartial and unanimous aggregator delivers a much more palatable method *fix once for all three "leaders", labeled agents* $1, 2, 3$

step 1: the reports of leaders select impartially three ranks k_1, k_2, k_3, for themselves; the key property is that *all assignments of* $\{1, 2, 3\}$ *to* $[n]$ *are feasible*

step 2: the three leaders select a fourth agent i_4 and his rank k_4, where the rank k_4 has not been assigned in step 1; next $\{1, 2, 3, i_4\}$ assign a new agent i_5 to a rank k_5 not yet assigned; and so on

Impartiality is clearly preserved is step 2 *explaining step 1*

$\rightarrow$ a *separating family* in A is a subset $\mathcal{S}$ of subsets of A s.t.

$$\text{for all } a, b \in A, a \neq b, \text{ there exists } S \in \mathcal{S} : a \in S, b \notin \mathcal{S}$$

Lemma: for $|A| \geq 6$, we can find three **pairwise disjoint** separating families in A: $\mathcal{S}_k, k = 1, 2, 3$, such that all subsets in $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$ are of identical size.

(for $|A| \leq 5$, we can find at most two such disjoint families) example for $|A| = 6$

	$\mathcal{S}_1$	$\mathcal{S}_2$	$\mathcal{S}_3$
	abc	abd	abe
	bcd	bce	bcf
$A = \{a, b, c, d, e, f\}$	cde	cdf	acd
	def	ade	bde
	aef	bef	cef
	abf	acf	adf

example for $|A| = 7$

$A = \{a, b, c, d, e, f, g\}$

$\mathcal{S}_1$	$\mathcal{S}_2$	$\mathcal{S}_3$
ab	ac	ad
bc	bd	be
cd	ce	cf
de	df	dg
ef	eg	ae
fg	af	bf
ag	bg	cg

step 1 continued

fix once and for all three separating families $\mathcal{S}_k$ of $[n]$, $k = 1, 2, 3$, *as in the Lemma*

transform mechanically (no influence from other reports) leader k's report r_k into an element S_k of $\mathcal{S}_k$

given $(S_1, S_2, S_3) \in \mathcal{S}_1 \times \mathcal{S}_2 \times \mathcal{S}_3$

assign 1 to a rank in $S_3 \cap S_2^c$

assign 2 to a rank in $S_1 \cap S_3^c$

assign 3 to a rank in $S_2 \cap S_1^c$

each set is non empty by the identical size assumption break ties (if any)

in $S_3 \cap S_2^c$ by an onto vote of leaders 2 and 3

in $S_1 \cap S_3^c$ by an onto vote of leaders 1 and 3

in $S_1 \cap S_3^c$ by an onto vote of leaders 1 and 3

Impartiality is clear

checking Unanimity

converting the report of a leader in step 1

leader k chooses S_k containing the rank $r_k(k+1)$ but not $r_k(k-1)$

(using cyclical addition)

this guarantees that all assignments of $\{1,2,3\}$ to $[n]$ are feasible

in step 2 we make sure that an agent reported at the same rank by $1, 2, 3, \cdots, i_l$, if any, is selected at that rank

Unanimity follows

there are many variants in both steps

in step 1 we can select four leaders; the choice by leader k of a subset of ranks S_k forces the final rank of leader $(k+1)$ in S_k, and that of every other leader outside S_k

the construction of step 1 violates Monotonicity our impartial aggregators do not treat agents equally, but that is inevitable in the self evaluation context

randomizing the selection of leaders restores such horizontal equity, while preserving Impartiality and Unanimity

it is then possible to test numerically the lack of Monotonicity, and other desirable properties of aggregators