Majority Rule in the Absence of a Majority

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  - Principle that **the “most widely shared” view should prevail**

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  1. **The Analytical Question:**
     What is "the most widely shared" view?
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  1. **The Analytical Question:**
     What is "the most widely shared" view?
     - on complex issues, there may be none (total indeterminacy), or only a set of views can be identified as more or less predominant (partial indeterminacy)
  2. **The Normative Question:**
     Why should the most widely shared view prevail?
     - may invoke principles of democracy, self-governance, political stability etc.

- Here we shall focus on analytical question:
  What is Majority Rule without a Majority?

- stay agnostic about normative question
- in practice, many institutions seem to adopt majoritarian procedures
  - prima facie case for majoritarian committments,
    - but not clear how deep it is.

Framework I

- standard JA framework:
  individuals (voters) and the group hold judgments on a set of interdependent issues ("views")
  - K set of *issues*
  - $X \subseteq \{\pm 1\}^K$ set of *feasible* **views**
  - $x \in X$ particular *views ("sets of judgments") on* $x \in X$.
- shall describe anonymous **profiles** of views by measures $\mu \in \Delta(X)$
  - allow profiles to be real-valued
- (K, X, μ) "**JA problem**"

Framework II

- Systematic criteria to select among views in JA problems described by **aggregation rules**
  - Aggregation rule $F : (X, \mu) \mapsto F(X, \mu) \subseteq X$.
  - will consider different domains
    - X frequently fixed
  - leave domain unspecified for now to emphasize **single-profile issue**: *what views are majoritarian in the JA problem* (X, μ)?

Tally Vectors

- Central role: *tally vector* $\widetilde{\mu} \in [0,1]^K$ given by

$$\widetilde{\mu}_k := \sum_{x \in X} x_k \mu(x)$$

  - E.g.: If 57% affirm proposition k at μ, $\widetilde{\mu}_k = 0.14$

- Set of feasible tally vectors:

$$\{\widetilde{\mu} : \mu \in \Delta(X)\} = conv(X).$$

  - Aside: a lot of the technical difficulties arise from need to consider general 0-1-polytopes, rather than $[0,1]^K$

Majority Rule in the Absence of a Majority I

Axiom

(Binary Majoritarianism)

If $\mu(x) > \frac{1}{2}$, *then* $F(X, \mu) = \{x\}$.

- "Majority Rule in the presence of a majority"
- If reject BM, simply reject Majoritarianism.

- Evident Problem: premise rarely satisfied if $K > 1$.

Condorcet Consistency I

- $\mathcal{M}(x, \mu) := \{k \in K : x_k \widetilde{\mu}_k \geq 0\}$
  - those issues in which x aligned with majority
- Condorcet Consistency: if majority judgment on each issue is consistent, this is the majority view.
  - $Maj(\mu) := \{x \in \{\pm 1\}^K : \mathcal{M}(x, \mu) = K\}$

Axiom (Condorcet Consistency)

If $Maj(\mu) \cap X \neq \varnothing$, then $F(X, \mu) \subseteq Maj(\mu)$.

- Obvious Limitation: easily $Maj(\mu) \cap X = \varnothing$.
  - Condorcet Paradox

Condorcet Admissibility I

- Condorcet Admissible Set (NPP 2011):

$$x \in Cond(X, \mu) \text{ iff, for no } y \in X,\ \mathcal{M}(v, \mu) \supsetneq \mathcal{M}(x, \mu).$$

Axiom

Minimal Majoritarianism $F(X, \mu) \subseteq Cond(X, \mu)$.

- Claim in NPP 2011: this captures normative implications of Majoritarianism *per se.*

- Problem: except for particular spaces (mainly median spaces), $Cond(X, \mu)$ may be set-valued, and often large

Condorcet Admissibility II

- But selection from $Cond(X, \mu)$ not matter of indifference
  - there may be further considerations that favor some Condorcet admissible views over another
  - these may not flow from Majoritarianism per se, but appeal to Majoritarian among others.

  - here: refine *Cond* based on considerations of "parity" among issues.

Supermajoritarian Efficiency I

- Premise: Majoritarianism plus Parity
- **Parity**: "each issue counts equally"
  - sometimes, Parity may be justified by symmetries of judgment space X
    - e.g. preference aggregation, equivalence relations
  - but Parity has broader applicability
  - Parity not always plausible, e.g. truth-functional aggregation

Supermajoritarian Efficiency II

Example: (Preference Aggregation over 3 Alternatives)

- $A = \{a, b, c\}$
- $X = X_A^{pr}$; (3-Permutahedron)
  - $K = \{ab, bc, ca\}$
- $\mu(a \succ b) = 0.7$;
  $\mu(b \succ c) = 0.6$;
  $\mu(b \succ a) = 0.55$
- $Cond(X, \mu) = \{abc, bca, cab\}$.

- Each Condorcet admissible ordering overrides one majority preference
- *Arguably, the ordering abc is the most widely supported (hence "most majoritarian") since it overrides the weakest majority*

Supermajoritarian Efficiency III

- Argument via "Supermajoritarian Dominance"
  - compare *bca* to *cab*
    - *bca* has advantage over *cab* on *bc* (at 0.6);
      cab has advantage over *bac* on *bc* (at 0.55);
    - since 0.6>0.55, *bca* **supermajority-dominates** *cab*

Supermajoritarian Efficiency IV

- General idea: x supermajority dominates y at μ if it sacrifices smaller majorities for larger majorities.
  - assumes that each proposition $k \in K$ counts equally.

- For any threshhold $q \in [0, 1]$,

$$\gamma_{\mu,x}(q) := \#\{k \in K : x_k \widetilde{\mu}_k \geq q\}.$$

- x **supermajority-dominates** y at μ ($\boxed{\text{“}x \rhd_\mu y\text{”}}$)
  if, for all $q \in [0, 1]$, $\gamma_{\mu,x}(q) \geq \gamma_{\mu,y}(q)$, and,
  for some $q \in [0, 1]$, $\gamma_{\mu,x}(q) > \gamma_{\mu,y}(q)$.
  - for economists: note analogy to first-order stochastic dominance.

Supermajoritarian Efficiency V

- x is **supermajoritarian efficient** at μ ($\boxed{\text{“}x \in SME\,(X,\mu)\text{”}}$) if, for no $y \in X$, $y \rhd_{\mu} x$.
  - SM efficiency normatively transparent since *single-profile* criterion
    - “WYSWYG”
- In example: $SME\,(X,\mu) = \{abc\}$.

Supermajoritarian Efficiency VI

- In 3-permutahedron, this is general situation

- x is **SM equivalent** to y at μ ($\boxed{\text{“}x \approx_\mu y\text{”}}$)

  if, for all $q \in [0,1]$, $\gamma_{\mu,x}(q) = \gamma_{\mu,y}(q)$.

- μ is **supermajoritarian determinate** if $x \approx_\mu y$ for any $x, y \in SME(X, \mu)$.
  - for these profiles, SM efficiency is maximally selective.

- X is **supermajoritarian determinate** if μ is supermajoritarian determinate for all $\mu \in \Delta(X)$.

Observation. If $\#A = 3$, X_A^{pr} is SM determinate.

Supermajoritarian Efficiency VII

- Does not generalize to $\#A > 3$.

SME in 4-Permutahedron:

1. If μ has top cycle B of size $<= 3$, then μ is SM determinate
2. If μ has top cycle B of size 4, then μ *may be* SM determinate.

- In case (2), wlog

$$Maj(\mu) = \{ab, bc, cd, da, ac, bd\} \tag{1}$$

- $Cond(\mu) = \{abcd, bcda, cdab, dabc\}$
  - $\#M(abcd) = 5$,
  - $\#M(bcda) = 4$,
  - $\#M(cdab) = 3$,
  - $\#M(dabc) = 4$.

Supermajoritarian Efficiency VIII

- When $abcd \rhd_\mu bcda$?
  - Iff $\widetilde{\mu}_{da} \leq \max(\widetilde{\mu}_{ab}, \widetilde{\mu}_{ac})$
- When $bcda \rhd_\mu abcd$?
  - Never, since second-lowest tally of $abcd$ $>$0.5 while second-lowest tally of $bcda$ $<$0.5
- If $\widetilde{\mu}_{da} > \max(\widetilde{\mu}_{ab}, \widetilde{\mu}_{ac})$, then **tradeoff** between overriding one larger or two smaller majorities
  - this tradeoff not governed by SME

- For any μ satisfying (1), $abcd \in SME(\mu)$.
- There exists μ satisfying (1) such that $Y = SME(\mu)$ if and only if $abcd \in Y$.
  - In particular, there exists μ such that $SME(\mu)$ is issue-wise indeterminate – but much rarer than for $Cond(\mu)$

Which spaces are SM determinate? I

- $d(x,y) := \{k \in K : x_k \neq y_k\}$
  - "Hamming distance"
- Basic observation: If $x, y \in SME(X,\mu)$ and not $x \approx_\mu y$, then $d(x,y) \geq 3$.

1. Any X with $\#K \leq 3$
   - parallel argument to 3-Permutahedron
2. Median Spaces
   - these are *majoritarian determinate:* $x \approx_\mu y$ for any $x, y \in Cond\,(X,\mu)$.
3. General answer: "proximal" spaces

Which spaces are SM determinate? II

- $\{x, y\}$ is an *edge* of the polyhedron $conv(X)$ if there exists $\mathbf{c} \in \mathbb{R}^K$ such that, for all $z \in X \setminus \{x, y\}$

$$x \bullet \mathbf{c} = y \bullet \mathbf{c} > \mathbf{z} \bullet \mathbf{c}.$$

  $\{x, y\}$ is an **internal edge** if $\mathbf{c}$ can be chosen from $conv(X)$.

- X is **proximal** if, for any internal edge $\{x, y\}$, $d(x, y) \leq 2$.

Theorem

a) If X is proximal, it is SM determinate.
b) If X is SM determinate and $int(conv(X)) \neq \varnothing$, then it is proximal.

- The assumption that $int(conv(X)) \neq \varnothing$ cannot simply be dropped, since any X with $\#X = 2$ is SM determinate.

Which spaces are SM determinate? III

Example

Comittee spaces $X_{I,J} := \{x \in \{\pm 1\}^K : I \leq \#\{k : x_k = 1\} \leq J\}$.

- here K : set of candidates

- more general: Resource Allocation spaces
  - e.g. allocation of public good (NPP, LNP)

Additive Support Rules I

- Majoritarianism plus Parity = SME?
  - doubtful, since SME exploits only ordinal tally information; ignores cardinal differences in strength of majorities

- To select among SME views, need to make tradeoffs between number and strength of majorities overruled
  - systematic tradeoff criterion described by "additive support rules"

Rules

1. Fixed space X
   - An **aggregation rule** for X is a correspondence $F : \Delta(X) \rightrightarrows X$
2. Variable spaces $X \in \mathfrak{X}$:
   - An **aggregation rule** for domain of spaces $\mathfrak{X}$ is a correspondence $F : \bigsqcup_{X \in \mathfrak{X}} \Delta(X) \rightrightarrows \bigsqcup_{X \in \mathfrak{X}} X$ such that $\mu \in \Delta(X)$ implies $F(\mu) \subseteq X$.
     - write here $F(X, \mu)$ to highlight underlying space.

Gain Functions I

- key ingredient: **gain function** ϕ
  - first cut: $\phi : [-1, +1] \to \mathbb{R}$, *increasing*; induces **additive support rule** F_ϕ via

$$F_\phi(\mu) := \arg\max_{x \in X} \sum_{k \in K} \phi\left(x_k \widetilde{\mu}_k\right).$$

  - $x_k \widetilde{\mu}_k$ "majority advantage" for x on issue k
  - $\phi\left(x_k \widetilde{\mu}_k\right)$ is the support for x on issue k; ϕ measures how much majorities of different sizes *count*.
  - $\sum_{k \in K} \phi\left(x_k \widetilde{\mu}_k\right)$ is total support for x

Gain Functions II

Remark

Since ϕ increasing, $F_\phi \subseteq SME$.

Example

(Median Rule: $\phi = id$);

$$F_{med}(\mu) := F_{id}(\mu) = \arg\max_{x \in X} \sum_{k \in K} x_k \widetilde{\mu}_k$$

- maximizes total number of votes for x over all issues.
  - in preference aggregation: Kemeny rule.
  - widely studied as general-purpose aggregation rule (Barthelemy, Monjardet, Janowitz, ...)

Gain Functions III

Example

(Slater Rule) $\phi(r) = sgn(r)$;

$$F_{slat}(\mu) := F_{sgn}(\mu) = \arg\max_{x \in X} \#\{k : x_k \widetilde{\mu}_k > 0\}.$$

- maximizes number of propositions in which there is majority support
- in general, $F_{slat} \not\subseteq SME$, since ϕ is not increasing

Gain Functions - Oddness I

- Gain functions *odd* wlog.

- $\phi : [-1, +1] \to \mathbb{R}$ is **odd** if $\phi(-r) = -\phi(r)$ for all $r \in [-1, +1]$

Observation. *For any ϕ, let $\widetilde{\phi}$ be given by*

$$\widetilde{\phi}(r) = \phi(r) - \phi(-r).$$

Then $\widetilde{\phi}$ is odd and increasing, and $F_\phi = F_{\widetilde{\phi}}$.

- Upshot: positive and negative parts of ϕ do not have independently meaningful choice content.
  - $\widetilde{\phi}(r)$ describes gain from *realizing rather than overrriding* majority of size r.
  - Hence: will assume gain-functions to be odd throughout.

Gain Functions - Oddness II

Observation. *If* ϕ odd, then

$$F_{\phi}(\mu) = \arg\max_{x \in X} \sum_{k \in K} x_k \phi\left(\widetilde{\mu}_k\right) = \arg\max_{x \in X} x \bullet \phi\left(\widetilde{\mu}\right).$$

- As it were,

$$F_{\phi}(\mu) = F_{med}\left(\phi\left(\widetilde{\mu}\right)\right). \qquad (2)$$

  - F_{ϕ}-maximization linear programming problem with integer constraints.
- But technically (2) false, since feasible gain vectors $\phi\left(\widetilde{\mu}\right)$ for F_{med} given by $conv(X)$, and for F_{ϕ} given by $\phi\left[conv(X)\right]$
  - source of significant technical problems.

Gain Functions - Oddness III

- Can understand ϕ in terms of relative gains for realizing supermajorities of various sizes.

- ϕ **inverse-S shaped**: large supermajorities count disproportionately
  "**consensus favoring**";
  ϕ **S-shaped**: size of supermaj. less important, cardinality of majority propositions more important.
  "**cardinality favoring**"

- contrast well-illustrated with homogeneous rules

A One-Parameter Family

- "homogeneous rules" $H^d := F_{\phi^d}$, with

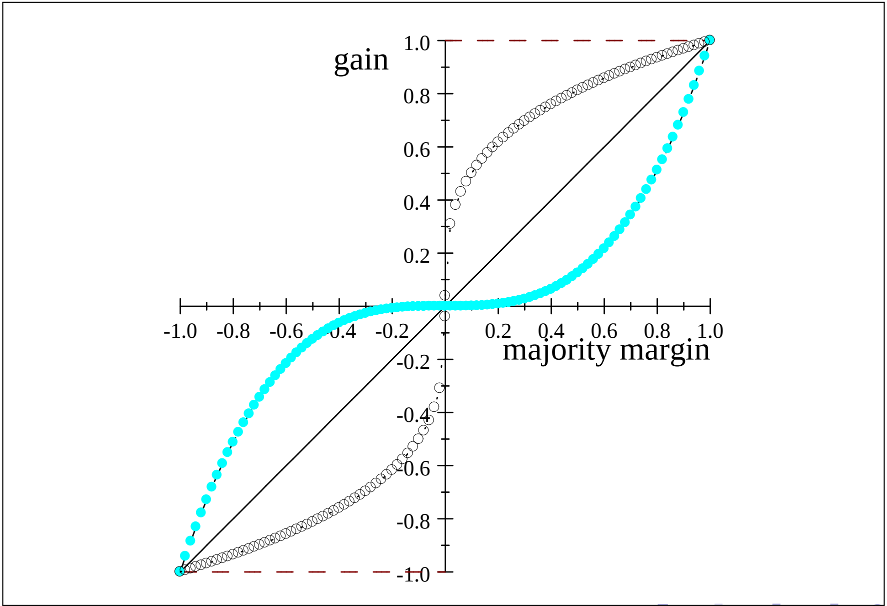
$$\phi^d(r) = sgn(r)|r|^d.$$

-
  - $d = 1$ median rule
  - $d > 1$ inverse-S-shape; *consensus-favorig*
  - $d < 1$ S-shape: *cardinality-favoring*

- One majority of size $2r$ balances 2^d majorities of size r.
  - E.g. with $r = 2$, a 70% supermajority balances 4 60% majorities.

- Personal view: only $d \geq 1$ (consensus favoring) normatively attractive.

Gain Functions - Oddness V



Homogeneous Gain Functions for d=0, 0.3, 1, 3

- other simple rules satisfy SME

Example

(Leximin) $xL_\mu y$ if there exist $\overline{q}$ such that $\gamma_{\mu,x}(q) = \gamma_{\mu,y}(q)$ for all $q > \overline{q}$, and $\gamma_{\mu,x}(q) > \gamma_{\mu,y}(q)$.

$$\begin{aligned} F_{lex\,\min}(X,\mu) \quad &: \quad = \max(X, L_\mu) \\ &: \quad = \{x \in X : \text{for no } y \in X, xL_\mu y\}. \end{aligned}$$

- Looks non-additive, but can be described by allowing ϕ to be hyperreal-valued:

Hyperreal-Valued Gain Functions II

- **hyperreals** ${}^*\mathbb{R}$:
  real closed field
  1. contain $\mathbb{R}$
  2. field: can multiply and divide (usual rules for arithmetic)
  3. linearly ordered
  4. no sups and infs

Hyperreal-Valued Gain Functions III

Example

$\phi_{slat,\psi}(r) := sgn(r) + \epsilon\psi(r)$, where
ϵ denote non-zero infinitesimal, and
ψ be real-valued gain function.

- $F_{\phi_{slat,\psi}}$ applies F_ψ to Slater-maximizes.
- $F_{\phi_{slat,\psi}}$ is SME-refinement of Slater rule.

Example

$F_{\text{lexmin}} = F_{\phi^d}$, with d any infinite hyperreal ι.

- For verification, note that $r > r' > 0$ implies $r^\iota > nr'^\iota$, for any $n \in \mathbb{N}$.

Hyperreal-Valued Gain Functions IV

Definition

A **gain function** ϕ is an odd, increasing function from $[-1,+1]$ to ${}^*\mathbb{R}$.

Definition

An aggregation rule F is an **additive support rule** if there exists a gain function $\phi : [-1,+1] \to {}^*\mathbb{R}$ such that, for all $X \in \mathfrak{X}$ and $\mu \in \Delta(X)$,

$$F_\phi(\mu) = \arg\max_{x \in X} \sum_{k \in K} \phi\left(x_k \widetilde{\mu}_k\right).$$

From SME to Additive Support Rules I

- Need additional normative axiom: Separability
  - Natural setting: domains $\mathfrak{X}$ closed under Cartesian products.

Axiom

(Separability) *For any If* $X_1, X_2 \in \mathfrak{X}$:
$F(X_1 \times X_2, \mu) = F(X_1, marg_1 \mu) \times F(X_2, marg_2 \mu)$

- Interpretation: in the absence of any logical interconnection, the optimal group view can be determined by combining optimal group views in each component problem.
  - "optimal" could mean different things in different context; here "optimal" = "most majoritarian", "most widely supported"

From SME to Additive Support Rules II

We will present two representation theorems

1. Narrow domain: fixed finite population and a fixed judgment space
   - real-valued representation sufficient
2. Wide domains: variable population and variable judgment spaces.
   - the general, hyper-realvalued representation becomes indispensable.

- (1) is key building block for (2).

Separable Extensions

- Let $\langle X \rangle := \bigsqcup_{n \in \mathbb{N}} X^n$,

  $$\text{with } X^n := \underbrace{X \times X \times \ldots \times X}_{(n \text{ times})}$$

  - Interpretation: $\langle X \rangle$ consists of the combination of multiple instances of the same (isomorphic) judgment problem X with different views of the individuals in each instance
  - e.g. preference aggregation over ℓ alternatives.

- Given F on X, there exists unique separable aggregation rule $G = F^*$ on $\langle X \rangle$ such that $G(X, \cdot) = F$
  - F^* is the **separable extension** of F

From SME to Additive Support Rules IV

Fixed Population, Fixed Space

- anonyomous profiles generated from W voters:

$$\Delta_W(X) := \{\frac{1}{N}\sum_{i=1}^{N}\delta_{x_i} : x_i \in X \text{ for all } i\}$$

  - dto. $\Delta_W(\mathfrak{X})$

Theorem

Let X be any judgment space, $N \in \mathbb{N}$ a fixed number of voters, and F be any aggregation rule on $\Delta_N(X)$. Then the separable extension of F is SME if and only if there exists a real-valued gain-function ϕ such that $F \subseteq F_\phi$.

From SME to Additive Support Rules V

Variable Population, Variable Spaces

Theorem

Let $\mathfrak{X}$ be any domain of judgment spaces closed under Cartesian products, and F any separable aggregation rule on $\Delta(\mathfrak{X})$.

1. *F is SME if and only if there exists a hyperrealvalued gain function ϕ such that $F \subseteq F_\phi$.*
   In this case, for every $X \in \mathfrak{X}$, there exists a dense open set $\mathcal{O}_X \subseteq \Delta(X)$ such that, for all $\mu \in \mathcal{O}_X$,
   $$\#F_\phi(X,\mu) = 1, \text{ and thus } F(X,\mu) = F_\phi(X,\mu).$$
2. *If F satisfies in addition OM (uhc), then $F = F_\phi$.*

From SME to Additive Support Rules VI

Axiom

(Overwhelming Majority) *For any* μ *there exists* α' *such that, for all* $\mu' \in \Delta(X)$ *and* $\alpha \geq \alpha'$, $F(\alpha\mu + (1-\alpha)\mu') \subseteq F(\mu)$*;*

equivalently:

(U.h.c.) *For any* $x \in X$*, any* $\mu, \{\mu_n\} \in \Delta(X)$ *such that* $\mu_n \to \mu$,

$$x \in F(\mu) \text{ if } x \in F(\mu_n) \text{ for all } n \in \mathbb{N}.$$

- Even under u.h.c., may need hyperreal-valued co-domain
  - shows that additive representation in Thm. 3.2b) cannot be obtained by infinite-dimensional separation theorem.

Which Gain Function? I

- Part II (Marcus)

- Clearly: Median rule ($\phi = id$) is the benchmark
  - $\phi = id$

- $\boxed{\text{Majoritarianism under Issue Parity} = \text{Median Rule ?}}$

- Considerations consistent with Majoritarianism but potentially conflicting with Median Rule
  1. Robustness
     - "Cloning"
  2. Propositionwise Unanimity
  3. Core Selection

Which Gain Function? II

Core-Selection

Definition

Let $r \in [0,1]$.
$Core_{r,X}(\mu) := \{x \in X : |\widetilde{\mu}_k| > 2r - 1 \text{ implies } x_k \widetilde{\mu}_k = 1.\}$

- The "propositional core" contains all views that contain no proposition k to which any supermajority of size *strictly greater* than r objects to.

- $Core_{r,X}(\mu) \neq \varnothing$ iff μ Condorcet consistent

Which Gain Function? III

Axiom

(Core Selection) *For any* $r \in [0,1]$ *and* $\mu \in \Delta(X)$, $F(\mu) \subseteq Core_{r,X}(\mu)$ *if* $Core_{r,X}(\mu) \neq \varnothing$.

Proposition. For any X, F_{lexmin} satisfies Core Selection.

- (some) converse should hold, too.