Majority Rule in the Absence of a Majority

Klaus Nehring and Marcus Pivato

September 10, 2011
Majoritarianism

- To fix ideas, cursory definition of “Majoritarianism” as normative view of judgement aggregation / social choice:
  - Principle that **the “most widely shared” view should prevail**

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Grounding MAJ requires resolving two types of questions?

1. The Analytical Question:
   What is “the most widely shared” view?
   - on complex issues, there may be none (total indeterminacy), or only a set of views can be identified as more or less predominant (partial indeterminacy)
Majoritarianism

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Grounding MAJ requires resolving two types of questions?

1. **The Analytical Question:**
   What is “the most widely shared” view?
   - on complex issues, there may be none (total indeterminacy), or only a set of views can be identified as more or less predominant (partial indeterminacy)

2. **The Normative Question:**
   Why should the most widely shared view prevail?
   - may invoke principles of democracy, self-governance, political stability etc.
Here we shall focus on analytical question: What is Majority Rule without a Majority?

- stay agnostic about normative question
- in practice, many institutions seem to adopt majoritarian procedures
  - prima facie case for majoritarian commitments,
    - but not clear how deep it is.
standard JA framework: individuals (voters) and the group hold judgments on a set of interdependent issues (“views”)

- $K$ set of issues
- $X \subseteq \{\pm 1\}^K$ set of feasible views
- $x \in X$ particular views (“sets of judgments”) on $x \in X$.

shall describe anonymous profiles of views by measures $\mu \in \Delta (X)$

- allow profiles to be real-valued

$(K, X, \mu)$ “JA problem”
Framework II

- Systematic criteria to select among views in JA problems described by aggregation rules
  - Aggregation rule \( F : (X, \mu) \mapsto F (X, \mu) \subseteq X \).
  - will consider different domains
    - \( X \) frequently fixed
  - leave domain unspecified for now to emphasize single-profile issue: what views are majoritarian in the JA problem \((X, \mu)\)?

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Tally Vectors

- Central role: *tally vector* $\tilde{\mu} \in [0, 1]^K$ given by

$$
\tilde{\mu}_k := \sum_{x \in X} x_k \mu(x)
$$

- E.g.: If 57% affirm proposition $k$ at $\mu$, $\tilde{\mu}_k = 0.14$

- Set of feasible tally vectors:

$$
\{ \tilde{\mu} : \mu \in \Delta(X) \} = \text{conv}(X).
$$

- Aside: a lot of the technical difficulties arise from need to consider general 0-1-polytopes, rather than $[0, 1]^K$
Axiom

(Binary Majoritarianism)

If $\mu(x) > \frac{1}{2}$, then $F(X, \mu) = \{x\}$.

- “Majority Rule in the presence of a majority”
- If reject BM, simply reject Majoritarianism.
- Evident Problem: premise rarely satisfied if $K > 1$. 
Condorcet Consistency I

- \( M(x, \mu) := \{ k \in K : x_k \tilde{\mu}_k \geq 0 \} \)
  - those issues in which \( x \) aligned with majority

- Condorcet Consistency: if majority judgment on each issue is consistent, this is the majority view.
  - \( \text{Maj}(\mu) := \{ x \in \{\pm 1\}^K : M(x, \mu) = K \} \)

### Axiom (Condorcet Consistency)

\( \text{If Maj}(\mu) \cap X \neq \emptyset, \text{ then } F(X, \mu) \subseteq \text{Maj}(\mu). \)

- Obvious Limitation: easily \( \text{Maj}(\mu) \cap X = \emptyset. \)
  - Condorcet Paradox
Condorcet Admissibility I

- Condorcet Admissible Set (NPP 2011):

  \[ x \in \text{Cond}(X, \mu) \text{ iff, for no } y \in X, \ M(v, \mu) \supseteq M(x, \mu). \]

**Axiom**

**Minimal Majoritarianism**

\[ F(X, \mu) \subseteq \text{Cond}(X, \mu). \]

- Claim in NPP 2011: this captures normative implications of Majoritarianism *per se*.

- Problem: except for particular spaces (mainly median spaces), \( \text{Cond}(X, \mu) \) may be set-valued, and often large.
But selection from $\text{Cond}(X, \mu)$ not matter of indifference

- there may be further considerations that favor some Condorcet admissible views over another
- these may not flow from Majoritarianism per se, but appeal to Majoritarian among others.

- here: refine $\text{Cond}$ based on considerations of “parity” among issues.
Premise: Majoritarianism plus Parity

**Parity**: “each issue counts equally”
- sometimes, Parity may be justified by symmetries of judgment space $X$
  - e.g. preference aggregation, equivalence relations
- but Parity has broader applicability
- Parity not always plausible, e.g. truth-functional aggregation
Example: (Preference Aggregation over 3 Alternatives)

- \( A = \{a, b, c\} \)
- \( X = X_A^{pr}; \) (3-Permutahedron)
  - \( K = \{ab, bc, ca\} \)
- \( \mu(a \succ b) = 0.7; \)
- \( \mu(b \succ c) = 0.6; \)
- \( \mu(b \succ a) = 0.55 \)
- \( \text{Cond}(X, \mu) = \{abc, bca, cab\}. \)

- Each Condorcet admissible ordering overrides one majority preference
- *Arguably, the ordering abc is the most widely supported (hence “most majoritarian”) since it overrides the weakest majority*
Argument via “Supermajoritarian Dominance”

- compare $bca$ to $cab$
  
  - $bca$ has advantage over $cab$ on $bc$ (at 0.6);
  - $cab$ has advantage over $bac$ on $bc$ (at 0.55);
  - since $0.6 > 0.55$, $bca$ supermajority-dominates $cab$
Supermajoritarian Efficiency IV

- General idea: $x$ supermajority dominates $y$ at $\mu$ if it sacrifices smaller majorities for larger majorities.
  - assumes that each proposition $k \in K$ counts equally.

- For any threshold $q \in [0, 1]$,
  \[
  \gamma_{\mu,x}(q) := \#\{ k \in K : x_k \tilde{\mu}_k \geq q \}.
  \]

- $x$ supermajority-dominates $y$ at $\mu$ ("$x \triangleright_{\mu} y\"")
  if, for all $q \in [0, 1]$, $\gamma_{\mu,x}(q) \geq \gamma_{\mu,y}(q)$, and,
  for some $q \in [0, 1]$, $\gamma_{\mu,x}(q) > \gamma_{\mu,y}(q)$.

- for economists: note analogy to first-order stochastic dominance.
- $x$ is **supermajoritarian efficient** at $\mu$ (\(\boxed{x \in SME(X, \mu)}\)) if, for no $y \in X$, $y \succ_\mu x$.
  - SM efficiency normatively transparent since *single-profile* criterion
    - “WYSWYG”

- In example: $SME(X, \mu) = \{abc\}$. 
Supermajoritarian Efficiency VI

In 3-permutahedron, this is general situation

- $x$ is **SM equivalent** to $y$ at $\mu$ ("$x \approx_\mu y$")
  
  if, for all $q \in [0, 1]$, $\gamma_{\mu,x}(q) = \gamma_{\mu,y}(q)$.

- $\mu$ is **supermajoritarian determinate** if $x \approx_\mu y$ for any $x, y \in SME(X, \mu)$.
  
  for these profiles, SM efficiency is maximally selective.

- $X$ is **supermajoritarian determinate** if $\mu$ is supermajoritarian determinate for all $\mu \in \Delta(X)$.

**Observation.** If $\#A = 3$, $X_A^{pr}$ is SM determinate.
Supermajoritarian Efficiency VII

- Does not generalize to $\#A > 3$.

SME in 4-Permutahedron:

1. If $\mu$ has top cycle $B$ of size $\leq 3$, then $\mu$ is SM determinate
2. If $\mu$ has top cycle $B$ of size 4, then $\mu$ may be SM determinate.

In case (2), wlog

$$\text{Maj}(\mu) = \{ab, bc, cd, da, ac, bd\}$$

$$\text{Cond}(\mu) = \{abcd, bcda, cdab, dabc\}$$

- $\#M(abcd) = 5$,
- $\#M(bcda) = 4$,
- $\#M(cdab) = 3$,
- $\#M(dabc) = 4$. 
When $abcd \succ_{\mu} bcda$?
- Iff $\tilde{\mu}_{da} \leq \max(\tilde{\mu}_{ab}, \tilde{\mu}_{ac})$

When $bcda \succ_{\mu} abcd$?
- Never, since second-lowest tally of $abcd > 0.5$
  while second-lowest tally of $bcda < 0.5$
- If $\tilde{\mu}_{da} > \max(\tilde{\mu}_{ab}, \tilde{\mu}_{ac})$, then tradeoff between overriding one larger or two smaller majorities
  - this tradeoff not governed by SME

For any $\mu$ satisfying (1), $abcd \in SME(\mu)$.
There exists $\mu$ satisfying (1) such that $Y = SME(\mu)$ if and only if $abcd \in Y$.
- In particular, there exists $\mu$ such that $SME(\mu)$ is issue-wise indeterminate – but much rarer than for $Cond(\mu)$
Which spaces are SM determinate? 1

- \( d(x, y) := \{ k \in K : x_k \neq y_k \} \)
  - “Hamming distance”

- Basic observation: If \( x, y \in SME(X, \mu) \) and not \( x \approx \mu y \), then \( d(x, y) \geq 3 \).

1. Any \( X \) with \( \#K \leq 3 \)
   - parallel argument to 3-Permutahedron

2. Median Spaces
   - these are majoritarian determinate:
     \( x \approx \mu y \) for any \( x, y \in Cond(X, \mu) \).

• \( \{x, y\} \) is an **edge** of the polyhedron \( \text{conv}(X) \) if there exists \( c \in \mathbb{R}^K \) such that, for all \( z \in X \setminus \{x, y\} \)

\[
x \cdot c = y \cdot c > z \cdot c.
\]

• \( \{x, y\} \) is an **internal edge** if \( c \) can be chosen from \( \text{conv}(X) \).

• \( X \) is **proximal** if, for any internal edge \( \{x, y\} \), \( d(x, y) \leq 2 \).

**Theorem**

a) If \( X \) is proximal, it is SM determinate.

b) If \( X \) is SM determinate and \( \text{int}(\text{conv}(X)) \neq \emptyset \), then it is proximal.

• The assumption that \( \text{int}(\text{conv}(X)) \neq \emptyset \) cannot simply be dropped, since any \( X \) with \( \#X = 2 \) is SM determinate.
Example

Committee spaces $X_{I,J} := \{ x \in \{ \pm 1 \}^K : I \leq \# \{ k : x_k = 1 \} \leq J \}$.

- here $K$ : set of candidates

- more general: Resource Allocation spaces
  - e.g. allocation of public good (NPP, LNP)
Majoritarianism plus Parity = SME?
- doubtful, since SME exploits only ordinal tally information; ignores cardinal differences in strength of majorities

To select among SME views, need to make tradeoffs between number and strength of majorities overruled
- systematic tradeoff criterion described by “additive support rules”
Additive Support Rules II

Rules

1. Fixed space $X$
   
   - An **aggregation rule** for $X$ is a correspondence $F : \Delta (X) \rightharpoonup X$

2. Variable spaces $X \in \mathcal{X}$:
   
   - An **aggregation rule** for domain of spaces $\mathcal{X}$ is a correspondence $F : \bigsqcup_{X \in \mathcal{X}} \Delta (X) \rightharpoonup \bigsqcup_{X \in \mathcal{X}} X$ such that $\mu \in \Delta (X)$ implies $F(\mu) \subseteq X$.
     
   - write here $F(X, \mu)$ to highlight underlying space.
key ingredient: **gain function** \( \phi \)

- first cut: \( \phi : [-1, +1] \rightarrow \mathbb{R} \), *increasing*;
- induces **additive support rule** \( F_\phi \) via

\[
F_\phi (\mu) := \arg \max_{x \in X} \sum_{k \in K} \phi (x_k \widetilde{\mu}_k).
\]

- \( x_k \widetilde{\mu}_k \) “majority advantage” for \( x \) on issue \( k \)
- \( \phi (x_k \widetilde{\mu}_k) \) is the support for \( x \) on issue \( k \);
- \( \phi \) measures how much majorities of different sizes *count*.
- \( \sum_{k \in K} \phi (x_k \widetilde{\mu}_k) \) is total support for \( x \)
Gain Functions II

Remark

Since $\phi$ increasing, $F_{\phi} \subseteq SME$.

Example

(Median Rule: $\phi = id$);

$$F_{med}(\mu) := F_{id}(\mu) = \arg \max_{x \in X} \sum_{k \in K} x_k \tilde{\mu}_k$$

- maximizes total number of votes for $x$ over all issues.
  - in preference aggregation: Kemeny rule.
  - widely studied as general-purpose aggregation rule (Barthelemy, Monjardet, Janowitz, ...)

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Majority Rule

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Example

(Slater Rule) \( \phi (r) = \text{sgn}(r); \)

\[
F_{\text{slat}} (\mu) := F_{\text{sgn}} (\mu) = \arg \max_{x \in X} \# \{ k : x_k \tilde{\mu}_k > 0 \}.
\]

- maximizes number of propositions in which there is majority support
- in general, \( F_{\text{slat}} \not\subset \text{SME} \), since \( \phi \) is not increasing
Gain Functions - Oddness I

- Gain functions *are odd* wlog.

- $\phi : [-1, +1] \rightarrow \mathbb{R}$ is odd if $\phi(-r) = -\phi(r)$ for all $r \in [-1, +1]

**Observation.** For any $\phi$, let $\tilde{\phi}$ be given by

$$\tilde{\phi}(r) = \phi(r) - \phi(-r).$$

Then $\tilde{\phi}$ is odd and increasing, and $F_\phi = F_{\tilde{\phi}}$.

- Upshot: positive and negative parts of $\phi$ do not have independently meaningful choice content.
  - $\tilde{\phi}(r)$ describes gain from *realizing rather than overriding* majority of size $r$.
  - Hence: will assume gain-functions to be odd throughout.
Gain Functions - Oddness II

Observation. If \( \phi \) odd, then

\[
F_{\phi}(\mu) = \arg\max_{x \in X} \sum_{k \in K} x_k \phi(\tilde{\mu}_k)
\]

\[
= \arg\max_{x \in X} x \bullet \phi(\tilde{\mu}).
\]

As it were,

\[
F_{\phi}(\mu) = F_{med}(\phi(\tilde{\mu})).
\]  \hspace{1cm} (2)

- \( F_{\phi} \)-maximization linear programming problem with integer constraints.
- But technically (2) false, since feasible gain vectors \( \phi(\tilde{\mu}) \) for \( F_{med} \) given by \( \text{conv}(X) \), and for \( F_{\phi} \) given by \( \phi[\text{conv}(X)] \)
- source of significant technical problems.
Can understand $\phi$ in terms of relative gains for realizing supermajorities of various sizes.

$\phi$ **inverse-S shaped**: large supermajorities count disproportionately "consensus favoring";

$\phi$ **S-shaped**: size of supermaj. less important, cardinality of majority propositions more important.

"cardinality favoring"

contrast well-illustrated with homogeneous rules
“homogeneous rules” \( H^d := F_{\phi^d} \), with

\[
\phi^d (r) = \text{sgn} (r) |r|^d.
\]

- \( d = 1 \) median rule
- \( d > 1 \) inverse-S-shape; consensus-favoring
- \( d < 1 \) S-shape: cardinality-favoring

One majority of size \(2r\) balances \(2^d\) majorities of size \(r\).
- E.g. with \(r = 2\), a 70\% supermajority balances 4 60\% majorities.

Personal view: only \(d \geq 1\) (consensus favoring) normatively attractive.
Gain Functions - Oddness V

Homogeneous Gain Functions for $d=0, 0.3, 1, 3$.
other simple rules satisfy SME

Example

(Leximin) $xL_\mu y$ if there exist $\bar{q}$ such that $\gamma_{\mu,x}(q) = \gamma_{\mu,y}(q)$ for all $q > \bar{q}$, and $\gamma_{\mu,x}(q) > \gamma_{\mu,y}(q)$.

$$F_{\text{lex min}}(X, \mu) : = \max(X, L_\mu)$$

$$: = \{x \in X : \text{for no } y \in X, xL_\mu y\}.$$
Hyperreal-Valued Gain Functions II

- **hyperreals** \( ^* \mathbb{R} \):
  - real closed field
    - contain \( \mathbb{R} \)
    - field: can multiply and divide (usual rules for arithmetic)
    - linearly ordered
    - no sups and infs
Example

\[
\phi_{s\text{lat}, \psi}(r) := \text{sgn}(r) + \epsilon \psi(r), \quad \text{where}
\]
\[
\epsilon \text{ denote non-zero infinitesimal, and}
\]
\[
\psi \text{ be real-valued gain function.}
\]
- \(F_{\phi_{s\text{lat}, \psi}}\) applies \(F_\psi\) to Slater-maximizes.
- \(F_{\phi_{s\text{lat}, \psi}}\) is SME-refinement of Slater rule.

Example

\[F_{\text{lexmin}} = F_{\phi^d}, \quad \text{with } d \text{ any infinite hyperreal } \iota.\]

- For verification, note that \(r > r' > 0\) implies \(r^\iota > nr'^\iota, \quad \text{for any } n \in \mathbb{N}.\)
A **gain function** $\phi$ is an odd, increasing function from $[-1, +1]$ to $\star \mathbb{R}$.

An aggregation rule $F$ is an **additive support rule** if there exists a gain function $\phi : [-1, +1] \rightarrow \star \mathbb{R}$ such that, for all $X \in \mathcal{X}$ and $\mu \in \Delta (X)$,

$$F_{\phi} (\mu) = \arg \max_{x \in X} \sum_{k \in K} \phi (x_k \tilde{\mu}_k).$$
Need additional normative axiom: Separability

- Natural setting: domains $\mathcal{X}$ closed under Cartesian products.

**Axiom**

**Separability** For any $X_1, X_2 \in \mathcal{X}$:
$$F(X_1 \times X_2, \mu) = F(X_1, \text{marg}_1 \mu) \times F(X_2, \text{marg}_2 \mu)$$

Interpretation: in the absence of any logical interconnection, the optimal group view can be determined by combining optimal group views in each component problem.

- “optimal” could mean different things in different context; here “optimal” = “most majoritarian”, “most widely supported”
We will present two representation theorems

1. Narrow domain: fixed finite population and a fixed judgment space
   - real-valued representation sufficient

2. Wide domains: variable population and variable judgment spaces.
   - the general, hyper-realvalued representation becomes indispensable.

(1) is key building block for (2).
Let $\langle X \rangle := \bigcup_{n \in \mathbb{N}} X^n$,

with $X^n := \underbrace{X \times X \times \ldots \times X}_{(n \text{ times})}$

Interpretation: $\langle X \rangle$ consists of the combination of multiple instances of the same (isomorphic) judgment problem $X$ with different views of the individuals in each instance

- e.g. preference aggregation over $\ell$ alternatives.

Given $F$ on $X$, there exists unique separable aggregation rule $G = F^*$ on $\langle X \rangle$ such that $G(X, \cdot) = F$

- $F^*$ is the **separable extension** of $F$
anonymoumous profiles generated from $W$ voters:

$$\Delta_W (X) := \left\{ \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i} : x_i \in X \text{ for all } i \right\}$$

dto. $\Delta_W (x)$

**Theorem**

Let $X$ be any judgment space, $N \in \mathbb{N}$ a fixed number of voters, and $F$ be any aggregation rule on $\Delta_N (X)$. Then the separable extension of $F$ is SME if and only if there exists a real-valued gain-function $\phi$ such that $F \subseteq F_\phi$. 
Theorem

Let $\mathcal{X}$ be any domain of judgment spaces closed under Cartesian products, and $F$ any separable aggregation rule on $\Delta (\mathcal{X})$.  

1. **$F$ is SME if and only if there exists a hyperreal-valued gain function $\phi$ such that $F \subseteq F_\phi$.**
   
   In this case, for every $X \in \mathcal{X}$, there exists a dense open set $O_X \subseteq \Delta (X)$ such that, for all $\mu \in O_X$,  
   
   $$\#F_\phi (X, \mu) = 1,$$
   
   and thus $F (X, \mu) = F_\phi (X, \mu)$.  

2. **If $F$ satisfies in addition OM (uhc), then $F = F_\phi$.**
Axiom

(Overwhelming Majority) For any $\mu$ there exists $\alpha'$ such that, for all $\mu' \in \Delta(X)$ and $\alpha \geq \alpha'$, $F(\alpha \mu + (1 - \alpha) \mu') \subseteq F(\mu)$;
equivalently:

(U.h.c.) For any $x \in X$, any $\mu, \{\mu_n\} \in \Delta(X)$ such that $\mu_n \rightarrow \mu$,

$$x \in F(\mu) \text{ if } x \in F(\mu_n) \text{ for all } n \in \mathbb{N}.$$  

Even under u.h.c., may need hyperreal-valued co-domain

- shows that additive representation in Thm. 3.2b) cannot be obtained by infinite-dimensional separation theorem.
Part II (Marcus)

Clearly: Median rule ($\phi = id$) is the benchmark

$\phi = id$

Majoritarianism under Issue Parity = Median Rule?

Considerations consistent with Majoritarianism but potentially conflicting with Median Rule

1. Robustness
   - “Cloning”

2. Propositionwise Unanimity

3. Core Selection
Core-Selection

**Definition**

Let $r \in [0, 1]$.

$$Core_{r,X} (\mu) := \{x \in X : |\tilde{\mu}_k| > 2r - 1 \text{ implies } x_k\tilde{\mu}_k = 1.\}$$

- The “propositional core” contains all views that contain no proposition $k$ to which any supermajority of size *strictly greater* than $r$ objects to.

- $Core_{r,X} (\mu) \neq \emptyset$ iff $\mu$ Condorcet consistent
Axiom

(Core Selection) For any $r \in [0, 1]$ and $\mu \in \Delta(X)$, $F(\mu) \subseteq Core_{r,X}(\mu)$ if $Core_{r,X}(\mu) \neq \emptyset$.

Proposition. For any $X$, $F_{\text{lexmin}}$ satisfies Core Selection.

• (some) converse should hold, too.