Order Extensions and the Evaluation of Groups

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Abstract

It is important for a team whether it is good or bad for the common weal, when a certain individual joins. This will depend on the aims of that team and the qualities of that individual. In this article criteria will be developed which are concerned with this problem. Based on the evaluations of the welfare and special properties or abilities of individuals, we will develop formal rules for the evaluation of groups built by these individuals.

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1. Introduction

According to basic ideas of neoclassical theory the importance of a group, a society or a team depends on the individuals of whom it consists. Every person contributes his or her share to the significance of the group to which he or she belongs. Generally, every individual will be a member of several groups, and the contribution of the individual to the common weal depends on the aims or the reasons for the existence of these groups. In welfare theory the well-being of a society depends on the utility functions of the group members. This demands that the utility functions of the individuals are cardinal and interpersonally comparable. These problematic assumptions have been extensively discussed in the literature (see Sen (1970), for instance).
In order to avoid this problem, the welfare of a country is usually measured by the real national income. However, this is also problematic in another respect, since national income does not say anything about its distribution. Therefore, other welfare indicators than national income are also used in order to measure the well-being of a people. If persons build up a group, then this does not only mean a mere addition of skills and abilities, but one will get better results by synergetic effects through interpersonal exertion of influence. Usually every member of a group also pays regard to the interests of the other members of it, otherwise the group would easily collapse. Depending on the aims or significance of a group, their members are not only self-interested but they often have various feelings for each other. Already in his "Theory of Moral Sentiments" Adam Smith directed our attention to sympathy which influences interpersonal actions. Sympathy is the capacity "to put yourself into the shoes of others". However, not only sympathy for each other, but also responsibility of the members for the welfare and survival of the whole group is very important. Responsibility of the group members for the success of their common work plays an important role for the success of a firm.

In the following we will investigate the contributions of individuals to the welfare or success of groups in certain situations. Especially, we will investigate the effect on the common weal when an individual joins a group. The evaluation of the groups should satisfy some reasonable rules, which will be examined in this article. These rules pay regard to the individuals′ contribution to the common weal of the group, of which they are or become members.
The formal instruments, which will be used for our analysis, will be chosen from the field of order extensions. This field, which has been pioneered by Gärdenfors (1976) and Kannai and Peleg (1984), has been developed by many articles, meanwhile. We restrict our analysis to the extension rules of Gärdenfors (1976), Kannai and Peleg (1984), Nitzan and Pattanaik (1984). This should not mean that other rules are less important for the evaluation of the common weal or efficiency of groups depending on individual abilities or well-being. However, the rules used here should only be considered as examples. We will realize that the rules of Gärdenfors, Kannai and Peleg are reasonable for the evaluation of groups to some extent. However, we will also realize that measures of welfare like per capita income or national income do not satisfy all of these rules.

Above all, we will realize that the interpretation of these rules on order extensions leads to some exciting results on group evaluations. This demonstrates once more that mathematical tools developed for certain reasons may become a strong instrument in another field, not necessarily a mathematical one.

2. Group Evaluation

We will now investigate an evaluation of groups in certain states by the application of mathematical tools developed in the theory of order extensions. Therefore, let us consider a set of individuals $I = \{1, \ldots, m\}$ and a set of states X. We will assume throughout this paper

$$X \neq \emptyset \qquad \text{and} \qquad X \text{ is finite.}$$

Based on the individuals well-being, skills or benefits, we will compare the common weal of a group in certain states. (x, i) means: being in the position of individual i in social state x. If $x \in \mathbb{R}^m_+$, then x may be a distribution of income among the m members of the group I, for instance. If we interpret (x, i) by the welfare of individual i in state x, then it becomes immediately obvious that welfare must be evaluated by somebody else according to certain rules or norms. This problem has been often discussed in the literature (see Sen (1970), for instance). We can base the evaluation of the position of individual i in state x on the judgement of a special outstanding objective person k. Then

$$(x, i) \; \widetilde{R}_k \; (x, j)$$

means, in the judgement of individual k the position of individual i in state x is at least as good as that one of individual j in the same state. It can be assumed that individual k is able to place himself in the position of every member of the group. Individual k needs not be a member of the group but he or she can be any impartial observer whom the group members accept for certain reasons. We can also replace the impartial observer by objective criteria, as for instance by measuring the income of the individuals in certain states. Then (x, i) means the income of individual i in state x. Based on the abilities of the individuals, we will evaluate welfare, success, effectiveness or productivity of a group according to certain criteria.

Throughout this paper we will denote the preference relation of the impartial observer on $X \times I$ by $\widetilde{R}$. Particularly, we can also interpret $\widetilde{R}$ as ethical norms or an objective standard accepted by everybody of the society.
The judgement on groups should regard certain rules. For this purpose, we will modify rules originally established for extending orders on a set Y to the power set of Y. Firstly, we will modify criteria of Kannai and Peleg (1984). Therefore, let us denote by $\Pi(I)$ the power set of I excluding the empty set. Every element of $\Pi(I)$ denotes a group of individuals. Everybody has more or less influence on the efficiency, behavior, results or stability of the groups which he or she joins. Of course, the character of a group may completely change, when a new member joins. The following rules form minimal requirements on the influence of the individuals on groups.
The position of the individuals will be ranked according to the evaluation of an impartial

observer. More strictly, we could require that $\widetilde{R}$ represents the ranking of the individuals of our society I who have identical preferences, i.e.

$$\forall\, i, j \in I : \widetilde{R}_i = \widetilde{R}_j \ .$$

However, this is a very strong postulate. In the following we will assume that the positions of the individuals are evaluated by an impartial observer with preference relation $\widetilde{R}$.

We will compare individuals i, j only in the same state x. Therefore, we will consider relationships $(x, i)\widetilde{R}(x, j)$ and not $(x, i)\widetilde{R}(y, j)$ for $x \neq y$. However, the following analysis could also be done for different states: the results would not be very different, and we would not win important new insights.

Firstly, we will establish properties of the preference relation $\widetilde{R}$ of the impartial observer.

a) $\widetilde{R}$ is complete on $\{x\} \times I$, for all $x \in X$. This means,

$$\text{for all } i, j \in I : \quad (x, i)\ \widetilde{R}\ (x, j) \ \text{ or } \ (x, j)\ \widetilde{R}\ (x, i) \ .$$

This kind of completeness insures that the impartial observer has complete preferences in state x, but the preferences may not be complete on $X \times I$, i.e., $(x, i)\widetilde{R}(y, j)$ or $(y, j)\widetilde{R}(x, i)$ may not hold.

b) $\widetilde{R}$ is transitive on $\{x\} \times I$, for all $x \in X$, i.e.

$$(x, i)\ \widetilde{R}\ (x, j) \ \wedge\ (x, j)\ \widetilde{R}\ (x, k) \quad \Rightarrow \quad (x, i)\ \widetilde{R}\ (x, k) \qquad \forall\, i, j, k \in I \ .$$

c) $\widetilde{R}$ is antisymmetric on $\{x\} \times I$, for all $x \in X$, i.e.

$$(x, i)\widetilde{R}(x, j) \ \wedge\ (x, j)\widetilde{R}(x, i) \quad \Rightarrow \quad i = j \ .$$

Depending on the judgement of the impartial observer on the welfare of individuals in state x, one can build an evaluation of groups consisting of subsets of I according to certain rules. These rules should represent a minimal frame for judging groups rationally. The evaluation of groups in state x is reflected by a preference relation $\succsim$ on $\{x\} \times \Pi(I)$. For any relation $\succsim$ on $\{x\} \times \Pi(I)$, we denote by $\succ$ the asymmetric part of $\succsim$ and by $\sim$ the symmetric part of $\succsim$.

The first rule we will consider is:

(GP) (i): For all $x \in X$ and for all $J \in \Pi(I)$ and $i \in I \backslash J$:

$$(x, i)\widetilde{R} \max_{\widetilde{R}} \{(x, j)\}_{j \in J} \quad \Rightarrow \quad (x, J \cup \{i\}) \ \succ\ (x, J).$$

$\max_{\widetilde{R}} \{(x, j)\}_{j \in J}$ means: maximum of all $(x, j) \in \{x\} \times J$ with respect to $\widetilde{R}$.

(x, J) means: the position of group J in state x.

(GP) is a modification of the Gärdenfors principle (1976). The interpretation of (GP)(i) is obvious: if, in the judgement of the impartial observer, individual i is in a better position in social state x than everybody from group J, then the common weal or importance of J increases if individual i joins.

We will turn to the next rule now. This is

$$\text{(GP) (ii):} \qquad \min_{\widetilde{R}} \underset{j \in J}{\{(x,j)\}} \widetilde{R}(x,i) \quad \Rightarrow \quad (x,J) \ \succ \ (x, J \cup \{i\})$$

This criterion can be interpreted in the following way: if the worst-off individual of group J in situation x is better-off than individual i, then the group changes for the worse if individual i becomes a member. For instance, if we consider a group J of mountain-climbers and if individual i, which is not belonging to J, is a worse mountain-climber than every member of J, then when climbing a high and dangerous mountain, the danger for the group $J \cup \{i\}$ is higher than the danger for J. Thus, J is better-off than $J \cup \{i\}$ in situation x.

The next rule is a modification of the corresponding principle for order extensions due to Kannai and Peleg (1984).

$$\text{(M):} \qquad \text{Consider } \ J^1, J^2 \subseteq J, \ \text{ and let } \ (x, J^1) \ \succ \ (x, J^2).$$

$$\text{For } \ i \notin J^1 \cup J^2 : (x, J^1 \cup \{i\}) \ \succsim \ (x, J^2 \cup \{i\})\,.$$

This rule can be interpreted in the following way: If group J^1 in social state x is better-off than J^2, then welfare of $J^1 \cup \{i\}$ in social situation x is at least as high as welfare of the group $J^2 \cup \{i\}$ in x.
It seems that this criterion is evident, but one can find situations where (M) is not fulfilled.

An important criterion reflecting judgements on justice of social states is due to Rawls (1971). In this article we present a modified version of this criterion.

Definition 2.1 *For all* $J^1, J^2 \in \Pi(I)$:

$$\begin{array}{lll} a) \ \textit{For any } x \in X: & (x, J^1) \succsim_x^R (x, J^2) & :\Leftrightarrow \ \Big[\exists\, k \in J^2 : \ (\forall\, i \in J^1 : (x,i)\ \widetilde{R}\ (x,k))\Big]\,, \\ b) & J^1 \succsim^R J^2 & :\Leftrightarrow \ \forall\, x \in X: \ J^1 \succsim_x^R J^2\,. \end{array}$$

By $\succsim_x^R$ welfare, justice or objective ethical standards of groups in state x depending on the worst-off individuals can be ranked. The relation $\succsim^R$ means, being better-off or equal-off in all social states. X may consist of social states existing over a time of several years.

We immediately have

Lemma 1 *Depending on the transitivity of* $\widetilde{R}$, i) $\succsim_x^R$ *and* ii) $\succsim^R$ *are transitive.*

Proof. Since i) is a specialization of ii) we will prove the latter assertion only.

Consider $J^1, J^2, J^3 \subseteq J$ such that $J^1 \succsim^R J^2 \wedge J^2 \succsim^R J^3$. Then by definition:

$$\begin{array}{l} \forall\, x \in X: \ \ [\, \exists\, k^1 \in J^2: \ \ (\forall\, i \in J^1: \ \ (x,i)\ \widetilde{R}\ (x,k^1)\ )\ ] \quad \wedge \\ \forall\, x \in X: \ \ [\, \exists\, k^2 \in J^3: \ \ (\forall\, j \in J^2: \ \ (x,j)\ \widetilde{R}\ (x,k^2)\ )\ ]\ . \end{array}$$

Transitivity of $\widetilde{R}$ yields,

$$(x,i)\ \widetilde{R}\ (x,k^2) \qquad \text{for all } i \in J^1\ .$$

Hence,

$$\forall\, x \in X: \ \ [\, \exists\, k \in J^3: \ \ (\forall\, i \in J^1: \ \ (x,i)\ \widetilde{R}\ (x,k)\ )\ ]\ .$$

■

Obviously, completeness of $\widetilde{R}$ does not imply completeness of $\succsim_x^R$ or $\succsim^R$.

In view of the following theorems, it should be remembered that $\widetilde{R}$ is assumed to be transitive, complete and antisymmetric on $\{x\} \times I$ for all $x \in X$. In Theorem 1 to Theorem 3 the latter property can also be removed. In that case more than one individual can be worst-off in I, and therefore the exposition of the proofs needs more space.

In the following we will use the notion of a "linear ordering" for a complete, transitive and antisymmetric relation, and we will use the notion of a "weak ordering" for a complete and transitive relation.

Theorem 1 *Let* $\widetilde{R}$ *be a linear ordering on* $\{x\} \times I$ *for all* x*. Then* $\succsim^R$ *satisfies* (M) *and* (GP)(ii)*, but not* (GP)(i)*.*

Proof. In order to prove (M), let $(x, J^1) \succ_x^R (x, J^2)$, $\forall x \in X$. Then $(x, J^1) \succsim_x^R (x, J^2)$. Now, let us consider an individual $i \in I$, $i \notin J^1 \cup J^2$. Since $\widetilde{R}$ is complete on $\{x\} \times I$ for all x, $(x,j)\widetilde{R}(x,i)$ or $(x,i)\widetilde{R}(x,j)$, for all $j \in J^1 \cup J^2$ follows. Since $(x, J^1) \succsim_x^R (x, J^2)$, the transitivity of $\widetilde{R}$ on $\{x\} \times I$ for all x and the definition of $\succsim^R$ imply, $(x, J^1 \cup \{i\}) \succsim_x^R (x, J^2 \cup \{i\})$ for all x, and thus $J^1 \cup \{i\} \succsim^R J^2 \cup \{i\}$.

The proof for (GP)(ii) follows by similar arguments. However, (GP)(i) does not hold, since in view of the definition of $\succsim^R$, from $(x,i)\ \widetilde{R}\ \max_{\widetilde{R}}\{(x,j)\}_{j \in J}$ for all x, it does not follow that $(J \cup \{i\}) \succ^R J$, but we will only have $(J \cup \{i\}) \sim^R J$. ■

Obviously, by the same arguments it follows that $\succsim_x^R$ does not satisfy (GP)(i).

For the next theorem, remember, if $\succ$ is a transitive relation, then $\succsim$ is called quasitransitive.

Theorem 2 *Let $\widetilde{R}$ be a linear ordering on $\{x\} \times I$ for all x. If for some $x \in X$ conditions* (GP) *and* (M) *are fulfilled, and if $\succsim$ is reflexive and quasitransitive on $\{x\} \times \Pi(I)$, then*

$$(x, J) \quad \sim \quad (x, \{\bar{i}, \bar{j}\})$$

with

$$(x, \bar{i}) := \min_{\widetilde{R}} \{(x, i)\}_{i \in J} \ , \qquad (x, \bar{j}) := \max_{\widetilde{R}} \{(x, j)\}_{j \in J} \ .$$

The assertion of the above theorem demonstrates that under the above restrictions, welfare of a group only depends on the worst-off and best-off individual.

The proof of Theorem 2 can be done analogously to *Lemma* in Kannai and Peleg ((1984), p.174). As already mentioned in Bossert (1989), in *Lemma* "transitivity" of $\succsim$ can be replaced by "quasitransitivity" of $\succsim$.

Although the conditions (GP) and (M) seem to be easily fulfilled, the following theorem states the disappointing result that there does not exist a weak ordering $\succsim$ on $\{x\} \times \Pi(I)$ which satisfies (GP) and (M). The original version of this theorem for order extension is due to Kannai and Peleg (1984).

Theorem 3 *Let $\widetilde{R}$ be a linear ordering on $\{x\} \times I$ for all $x \in X$, and let $|I| \geqq 6$. Then there does not exist a weak ordering $\succsim$ on $\{x\} \times \Pi(I)$ satisfying* (GP) *and* (M).

As already mentioned, Theorem 3 remains true (see Kannai and Peleg (1984), Remark 3, p. 175), if it is only assumed that $\widetilde{R}$ is a weak ordering instead of being a linear ordering. Then the number of individuals ranked according to the linear ordering $\widetilde{P}$, which is the asymmetric part of $\widetilde{R}$, must be at least 6.
By the following example we will illustrate the force of this theorem for the evaluation of groups. We return to per capita income as a welfare measure. By x we mean a distribution of national income among the members of a society I. $(x, i)\widetilde{R}(x, j)$ means, the income of individual i is higher than the income of individual j when distribution x is given and $i \neq j$. Consider $J^1, J^2 \subseteq I$. Let $(x, J^1) \succ (x, J^2)$, i.e., the per capita income of the group J^1 is higher than the per capita income of group J^2 for a given distribution x. As $\succsim$ is a weak ordering, according to Theorem 3, at least one of the postulates (GP) or (M) cannot be fulfilled by this example. We will affirm this result by real numbers. Consider

$J^1 = \{j_1, j_2, j_3\}$ and $J^2 = \{i_1, i_2\}$,
$x_{j_1} = E,\ x_{j_2} = 3E,\ x_{j_3} = 8E,\ x_{i_1} = 2E,\ x_{i_2} = 5E,$
where kE means k units of income.

Since $\frac{8+3+1}{3} > \frac{5+2}{2}$, we obtain $(x, J^1) \succ (x, J^2)$. Now, consider another individual $l \in I \setminus (J^1 \cup J^2)$ having income $10E$. If she or he joins J^1, then the per capita income of $J^1 \cup \{l\}$ is $\frac{11}{2}$. The per capita income of $J^2 \cup \{l\}$ is $\frac{17}{3}$, and therefore

$$(x, J^1 \cup \{l\}) \quad \prec \quad (x, J^2 \cup \{l\}) \ .$$

Hence (M) is not satisfied.
The example demonstrates that condition (M) is stronger than it seems to be. We can modify this example using the aggregate income $\sum_{i \in J} x_i$ instead of the per capita income as a measure of welfare. Then (M), but not (GP)(ii) is satisfied since, in view of the above distribution of income, $(x,\ \min\{j_2, j_3, i_1, i_2\})\widetilde{R}(x, j_1)$ but $(x,\ \{j_1, j_2, j_3, i_1, i_2\}) \succ (x,\ \{j_2, j_3, i_1, i_2\})$.
The above example demonstrates that conditions (M) and (GP) establish a more qualitative than quantitative evaluation of welfare and justice. Since measuring by real numbers always yields a transitive and complete ranking, conditions (M) and (GP) cannot be both satisfied by numerical evaluations. Thus, we immediately obtain the following important conclusion from Theorem 3 and the above examples:

Corollary 3.1 *Assume the conditions of Theorem 3. If $\succsim$ satisfies (M) and (GP), then there does not exist a numerical function, representing the relation $\succsim$, and vice versa. (A numerical function $u : Y \to I\!R$ represents a relation Q on Y if $xQy \Leftrightarrow u(x) \geqq u(y),\ \forall x, y \in Y$).*

In order to construct applications of Theorem 3 still more appealing than the above example, we modify condition (M) by

(M') : Consider $J^1, J^2 \subseteq J$, and let $(x, J^1) \succ (x, J^2)$. For $i, j \notin J^1 \cup J^2$ let $(x, i)\widetilde{I}(x, j)$,

where $\widetilde{I}$ is the symmetric part of $\widetilde{R}$. Then

$$(x, J^1 \cup \{i\}) \succsim (x, J^2 \cup \{j\}).$$

Instead of requiring $\widetilde{R}$ to be a linear ordering we will now assume that $\widetilde{R}$ is a weak ordering (i.e., transitive and complete). We will denote by $[j]$ the equivalence class of j, and define,

$$k \in [j]_x :\Leftrightarrow (x, k)\widetilde{I}(x, j).$$

Then we can reformulate Theorem 3 in the following way:

Theorem 4 *Let $\widetilde{R}$ be a weak ordering on $\{x\} \times I$ for all $x \in X$, and let $|\{[j]_x | j \in I\}| \geqq 6$. Then for every $x \in X$ there does not exist a weak ordering $\succsim$ on $\{x\} \times \Pi(I)$ which satisfies (GP) and (M').*

The proof of Theorem 4 can be done accordingly to Theorem 3 choosing a representant of every equivalence class.

Using the per capita income as a welfare measure again, we will modify our previous example now by requiring that there exist two individuals l_1 and l_2 having income 10E, and thus $(x, l_1)\widetilde{I}(x, l_2)$. As in the previous example, we obtain $(x, J^1 \cup \{l_1\}) \prec (x, J^2 \cup \{l_2\})$, although we have $(x, J^1) \succ (x, J^2)$.

We will now turn to another application of a result on order extensions to group evaluations, which is due to Bossert (1989). We introduce the following rule (NT).

(NT) : For all $J^1, J^2 \in \Pi(I),\ x \in X,$ and for every one-to-one mapping

$$f : J^1 \cup J^2 \to I \text{ and } i \in J^1 \text{ and } j \in J^2 :$$

$$\Big[\,(x,i)\widetilde{R}\,(x,j) \Leftrightarrow (x,f(i))\widetilde{R}\,(x,f(j)) \quad \wedge \quad (x,j)\widetilde{R}\,(x,i) \Leftrightarrow (x,f(j))\widetilde{R}\,(x,f(i))\,\Big]$$

$$\Rightarrow \quad \Big[\,(x,J^1) \succsim (x,J^2) \quad \Leftrightarrow \quad (x,f(J^1)) \succsim (x,f(J^2)) \quad \wedge$$
$$(x,J^2) \succsim (x,J^1) \quad \Leftrightarrow \quad (x,f(J^2)) \succsim (x,f(J^1))\,\Big].$$

Condition (NT) can be interpreted in the following way: if in situation x the impartial observer's ranking of $i \in J^1$ and $j \in J^2$ equals the ranking of the corresponding persons $f(i)$ and $f(j)$ in $f(J^1)$ and $f(J^2)$ respectively, then the ranking of the corresponding groups must preserve this evaluation.
Application of Bossert's result to the theory of groups (1989) yields

Theorem 5 *Let $|I| \geqq 5$ and assume $\widetilde{R}$ be a linear ordering on $\{x\} \times I$ for all $x \in X$, and let $\succsim$ be reflexive and quasitransitive.*
Then for all $J^1, J^2 \in I$ and $x \in X$,

$$(x,J^1) \succsim (x,J^2) \quad \Leftrightarrow \quad \Big[\,\min_{\widetilde{R}}\underset{j \in J^1}{\{(x,j)\}}\ \widetilde{R}\ \min_{\widetilde{R}}\underset{j \in J^2}{\{(x,j)\}} \ \wedge \ \max_{\widetilde{R}}\underset{j \in J^1}{\{(x,j)\}}\ \widetilde{R}\ \max_{\widetilde{R}}\underset{j \in J^2}{\{(x,j)\}}\,\Big]$$

iff $\succsim$ satisfies (GP)*,* (M) *and* (NT)*.*

We thus have seen that a reflexive and quasitransitive ranking of groups or societies, which satisfies (GP), (M) and (NT), only depends on the well-being of the best-off and the worst-off individuals. Conversely, if social preference orderings only depend on the worst-off and best-off individuals in the group, then it also satisfies (GP), (M) and (NT). Since the welfare-measures per capita income and aggregate income pay regard to every one's income and not only to the best-off and the worst-off, they cannot satisfy the criteria (GP), (M) and (NT) altogether.

We will now introduce a further criterion which compares two groups over a couple of social states in the course of time.

Definition 2.2 *Let $\succsim$ be an relation on $\{x\} \times \Pi(I)$ for all $x \in X$.*
For any groups $J^1, J^2 \in \Pi(I)$:

$$J^1 \succsim \# J^2 \quad :\Leftrightarrow \quad |\{x \in X \mid (x,J^1) \succsim (x,J^2)\}| \ \geqq \ |\{y \in X \mid (y,J^2) \succsim (y,J^1)\}|\,.$$

This is a simple majority rule.

Then we have

Lemma 2 *Completeness and transitivity of $\succsim$ on $\{x\} \times \Pi(I)$ for all $x \in X$ do not imply that $\succsim_{\#}$ is transitive.*

Proof. By a counterexample:
Consider three states x, y, z and three groups J^1, J^2, J^3 such that

$$\begin{array}{lll} (x, J^1) \succ (x, J^2), & (x, J^2) \succ (x, J^3), & (x, J^1) \succ (x, J^3)\ ; \\ (y, J^2) \succ (y, J^3), & (y, J^3) \succ (y, J^1), & (y, J^2) \succ (y, J^1)\ ; \\ (z, J^3) \succ (z, J^1), & (z, J^1) \succ (z, J^2), & (z, J^3) \succ (z, J^2)\ . \end{array}$$

By definition of $\succsim_{\#}$, we obtain

$$J^1 \succ_{\#} J^2,\ J^2 \succ_{\#} J^3 \ \text{ and } \ J^3 \succ_{\#} J^1\ .$$

■

This is another version of the Arrow Paradoxon.

3. Median-based Rankings of Groups

In the literature, there are many rules for extending orders. Several of them can be used for ranking groups on the basis of individual welfare, as we already mentioned. For our purpose the following criteria due to Nitzan and Pattanaik (1984) are also interesting, since they yield a surprising interpretation with respect to the ranking of groups.
For simplifying our analysis we will assume again that $\widetilde{R}$ is a linear ordering, and as usual $\widetilde{R}$ can be interpreted as the preferences of an impartial observer or as ethical norms, etc. We will consider a group $J \subseteq I$ in a certain state x, and assume $|J| = n$. The members of J are denoted by j_1 to j_n. Without loss of generality, let the members j_i of J in state x be ordered in the following way: $(x, j_i)\widetilde{R}(x, j_{i+1}),\ 1 \leqq i < n$, and define:

$$\text{med}\ (x, J) \ = \begin{cases} (x, \{j_{\frac{n+1}{2}}\}) & , \quad \text{if } |J| \text{ is odd} \\ (x, \{j_{\frac{n}{2}}, j_{\frac{n}{2}+1}\}) & , \quad \text{if } |J| \text{ is even.} \end{cases}$$

Nitzan and Pattanaik used the notion of a median-based relation on X (1984). We will modify this notion for our purpose in the following way:

Definition 3.1 *A social relation $\succsim$ on $\{x\} \times \Pi(I)$ is called* ***median-based in x*** *iff for all $J \subseteq I$:*

$$(x, J) \ \sim \ \text{med}\ (x, J)\ .$$

The above equivalence means, that in the social state x a group J is as valuable for a firm or a society as the groups $\{j_{\frac{n+1}{2}}\}$ or $\{j_{\frac{n}{2}}, j_{\frac{n}{2}+1}\}$, consisting of the middle-most members, respectively. One can imagine, that this is a problematic conclusion, and we will observe this in reality only in very few cases, as for instance, when the work done by J can also be done by the individual $j_{\frac{n+1}{2}}$ or by the individuals $j_{\frac{n}{2}}$ and $j_{\frac{n}{2}+1}$ alone. Nitzan and Pattanaik base their exposition on four rules. One of them (modified for our purpose) is the following one.

Rule of Weak Independence *(WIND):*
For all $J^1, J^2 \in \Pi(I)$, for all $i, j \in I$, such that $(x,i)\widetilde{P}(x,d)\widetilde{P}(x,j)$ for all $d \in J^1 \cup J^2$:

$$(x, J^1) \succsim (x, J^2) \quad \Rightarrow \quad (x, J^1 \cup \{i,j\}) \succsim (x, J^2 \cup \{i,j\}) \ .$$

Based on the rule (WIND) and under assumption of three further modified criteria (see (WEE), (WN) and (WD) in Nitzan and Pattanaik (1984)), one can show that a group J is as valuable as a subset of J consisting of the middle-most members, i.e. $(x,J) \sim med(x,J)$.

This is a very astonishing result, and we will end this article with this conclusion. It should be emphasized again, that also other rules for order extensions, such as we can find in Fishburn (1984), a.o. can be reformulated for group evaluations depending on the individuals' welfare or capabilities.

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