Multi-Dimensional Social Choice under Frugal Information: Imprecise Bayesian Foundations for the Tukey Median*

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Abstract

We study a voting model with partial information in which the evaluation of social welfare must be based on information about agents' top choices plus qualitative background conditions on preferences. The resulting uncertainty is modeled in terms of the imprecise Bayesian beliefs of an evaluator who adopts an ex-ante Condorcet criterion. We show that for an appropriate class of imprecise beliefs, ex-ante Condorcet winners exist and refine the set of Tukey medians (Tukey, 1975). Tukey medians enjoy notable robustness to belief misspecification, and are distinguished also from a mechanism design perspective.

Keywords: Social choice with imprecise priors; ex-ante Condorcet approach; Tukey median; participatory budgeting.

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1 Introduction

Many economic and political decisions involve the allocation of resources under a budget constraint. Examples are the allocation of public goods, the redistribution across classes of beneficiaries, the allocation of tax burden, the choice of intertemporal expenditure streams, or the macro-allocation between expenditure, tax receipts and net debt. Here we explore the possibility of taking these decisions collectively by voting by an electorate or a representative body. This will be done in a general multi-dimensional setting in which alternatives are elements of a convex subset of a Euclidean space and preferences are convex. The budget allocation problem is the special case in which the set of alternatives is a budget hyperplane.

Standard approaches to preference aggregation and voting assume ordinal or even cardinal preference information as their input. Their application to budget allocation and general multi-dimensional problems poses substantial difficulties. At the foundational level, except for the one-dimensional case with two public goods and single-peaked preferences (Black, 1948; Arrow, 1951/63), one is faced with generic impossibility results under all reasonable domain restrictions (Kalai *et al.*, 1979; Le Breton and Weymark, 2011) just as in spatial voting models (Plott, 1967). In particular, in higher dimensions there is no hope to generally find a Condorcet winner even if all agents have well-behaved (e.g. Euclidean) preferences. Indeed, the indeterminacy of majority voting is generic and can be severe (McKelvey, 1979). Thus, from the point of view of ordinal social choice theory, it is not even conceptually clear what allocations an optimal voting rule should aim at.

From a more applied mechanism design perspective, the task of articulating and communicating a complete ordering over the set of all alternatives for each agent (whether citizen or representative) is hard unless the number of alternatives is small. It seems impossibly hard in the budget allocation problem in which the number of alternatives grows exponentially in the number of dimensions (i.e. alternative uses of the public resource). Clearly, much is to be said for making the task of the voter as easy as possible.

Tops-Only Elicitation

Here, we take an informationally minimalist approach by assuming that only voters' preference tops are individually elicited. This keeps cognitive demands on the voter at a minimum in that it requires every voter only to determine what he must in order to arrive at a choice all by himself.[1] Further, knowledge of at least voters' tops is indeed required to arrive at a well-founded social decision in the most elementary instances of aggregation, namely those of complete unanimity. Lastly, tops-only elicitation stands out also when incentive considerations are introduced. While tops-onliness can obviously not generally ensure strategy-proofness, it avoids the 'built-in' manipulability of sub-top preference information on rich domains of generalized single-peaked (in particular: convex) preferences; see Section 6 for more details on the incentive perspective.

Alluding to the notion of 'fast and frugal heuristics' due to Gigerenzer and Goldstein (1996), we refer to our modeling approach as *frugal* in the sense of being informationally parsimonious while ensuring 'good enough' outcomes at the same time.

The Social Evaluator as an Imprecise Bayesian

Besides the individually elicited tops, relevant information may include background knowledge about the structure of voters preferences; here we shall focus on knowledge of preference convexity. We aim to determine which alternatives are socially optimal given this tightly limited information. This task is framed as the decision problem of a 'social evaluator' who is modeled as imprecise Bayesian whose beliefs are described by a set of admissible probability measures ('priors') over profiles of ordinal preferences.[2]

There are two key ingredients to such a model: assumptions on the evaluator's beliefs and a decision criterion relative to those beliefs. As decision criterion, we shall introduce an *ex-ante* extension of the classical Condorcet criterion to imprecise Bayesian uncertainty. The evaluator's beliefs must respect the available information, elicited individually and qua background. Furthermore, we assume the evaluator to be substantially ignorant beyond what is exactly known. This forces her beliefs to be imprecise (i.e. to consist of a proper, in fact quite large, set of priors).

By specifying an appropriate class of belief models, we find a sweet spot where

[1]Reliance on tops-only elicitation thus addresses a fundamental tension in the standard ordinal aggregation framework, as the elicitation of a complete ordinal ranking requires much more cognitive effort on part of the individuals than would be required for solo decision making, while individual incentives to figure out ones own preferences are greatly reduced due the diluted impact of a voter on the final choice. This theme of 'rational ignorance' goes back to Down's classic treatment (Downs, 1957, pp. 244-246, 266-271).

[2]In contrast to a 'social *planner*' who pursues own goals, our envisioned 'social evaluator' is assumed to act on behalf and in the interest of the group of agents.

ex-ante Condorcet winners do exist and can be tractably characterized and analyzed. Indeed, from such premises, the central result of the paper, Theorem 1, obtains a decision-theoretic foundation of a refinement of the classical multi-dimensional generalization of the median rule, the Tukey median (Tukey, 1975).

We now describe in a bit more detail the decision criterion and the model of beliefs at the heart of this argument.

The Ex-Ante Condorcet Approach

To determine 'ex-ante' optimal social choice with imprecise Bayesian beliefs, we propose a novel ex-ante Condorcet (EAC) approach. The EAC approach relies on ex-ante comparisons between arbitrary pairs of alternatives. These comparisons are based on the interval of expected majority counts consistent with the evaluator's imprecise set of priors. Significantly, these pairwise comparisons can be made on this basis in an arguably canonical manner without reference to subjective attitudes of pessimism vs. optimism, or ambiguity aversion vs. ambiguity seeking. The EAC approach then uses this ex-ante majority relation to select an ex-ante Condorcet winner if it exists, and settles for some Condorcet extension rule – left unspecified here – if not. Remarkably, in the models at the center of this paper, ex-ante Condorcet winners do exist and can be characterized explicitly.

Knowledge of Convexity is not Enough

The most straightforward model of the evaluator's belief set, the *plain convex model*, is to include *all* priors consistent with the available information. While seemingly natural, the plain convex model results in degenerate optimal choices which we attribute to the implied extreme – and excessive – ignorance.

In the one-dimensional case in which convexity is tantamount to single-peakedness of ex-post preferences, the plain convex model is very successful. As the ex-post Condorcet winner is the median of voters' tops, it is known ex-ante and equal to the ex-ante Condorcet winner.[3] But in the multi-dimensional case (at least three competing uses of resources), convexity by itself loses much of its bite. In particular, with tops in general position, convexity does not permit any significant novel inferences

[3]We use the ex-ante vs. ex-post metaphor purely for conceptual purposes in order to describe the epistemic state of the social evaluator, without any assumption of an ex-post stage in real time at which the actual profile of ('ex-post') preferences is observed.

about preferences beyond those available from knowledge the tops; by consequence, all tops are ex-ante Condorcet winners (Proposition 3). This appears quite counterintuitive and unsatisfactory, since any notion of centrality of the putative optimal social choice is lost, in stark contrast to the one-dimensional case and to the 'preference for centrality' built into the preferences of each individual voter qua convexity.

A closer look reveals that this counterintuitive conclusion is driven by the mere possibility of particular ex-post profiles that look very contrived and appear unlikely a priori. In other words, assuming literally *complete ignorance* over preference profiles (beyond what is known from the background and tops) is too permissive and entails a consequential misspecification of plausible evaluator's beliefs. This can be rectified by injecting minimal probabilistic commitments while maintaining substantial ignorance otherwise.

Symmetry of Marginals

To execute this, we adopt a parametric form of convex preferences, namely quadratic preferences. A particular quadratic form describes the substitution-complementation structure of a quadratic preference ordering in terms of the cross-partials of the utility function. Notably, assuming quadraticity does not fix by itself the counterintuitive implications of the plain convex model, for the expected majority counts remain exactly the same as the plain convex model (Fact 4.1).

Within this setting, the counterintuitive conclusions of the plain convex and plain quadratic models can be overcome by assuming that the evaluator's beliefs are symmetric in the sense that, for each admissible prior, the marginal distribution over quadratic forms is the same across voters irrespective of their top. Such 'symmetry of marginals' expresses the idea that the evaluator lacks any grounds to form different probabilistic beliefs about the unknown quadratic preference structure of different voters; in particular, the knowledge of voters' tops does not provide such a ground. Symmetry of marginals is weak in that it does not impose any additional restriction on the empirical joint distribution of the actual preference profile.

Besides symmetry of marginals, we furthermore assume that beliefs about these marginals are completely ignorant (maximally imprecise), just as in the plain convex model. These assumptions define the class of *symmetrically ignorant quadratic (s.i.q.) models* of the evaluator's beliefs. The main result of the paper, Theorem 1, shows that in any s.i.q. model ex-ante Condorcet winners exist and coincide, when unique, with the classical Tukey median. When not unique, the EAC winners are shown to coincide with a well-defined refinement of the set of Tukey medians. The Tukey median is a well studied coordinate-free (affine invariant) generalization of ordinary medians to multiple dimensions (Small, 1990; Rousseeuw and Hubert, 2017).

While our decision-theoretic foundation via Theorem 1 literally assumes knowledge of quadratic preferences, this is not really necessary for the thrust of the result. In Appendix C, we show that the result generalizes to mixtures of the s.i.q. models and the plain convex model – exactly with a continuum of voters, and approximately in the finite case. In this way, the ex-post quadraticity restriction on profiles can be avoided.

A more orthodox modeling strategy would postulate the evaluator's beliefs to be precise (i.e. consist of unique priors). But this approach has limited appeal here, as we argue in Section 5 below. Most importantly, we prove an impossibility result (Theorem 2) showing that, in the present setting, the evaluator's lack of knowledge of the underlying profiles of convex preferences cannot be captured by precise priors since such priors cannot satisfy the pertinent affine invariance requirements.

The Tukey Median as Voting Mechanism

The main contribution of the paper is a characterization of the social optimum under specific informational assumptions which include knowledge of the voters tops. With self-interested agents, such knowledge cannot be taken for granted. While a full exploration of these incentive-issues is beyond the scope of the present paper, we argue in Section 6 that the Tukey median retains significant merits also from the incentive perspective and that it compares favorably to off-the-shelf alternatives in the economic and statistical literature.

While full strategy-proofness on the domain of all convex, or all quadratic, preferences is out of the question, due to well-known impossibility results such as Zhou (1991), one can ask for restricted strategy-proofness properties. The Tukey median indeed satisfies some salient properties of this kind (cf. Proposition 4) – along with other multi-dimensional medians, but in contrast to the mean rule and many similar score-based rules. Among the multi-dimensional medians, the Tukey median stands out in its adaptation to the informational setting. Such adaptation requires in particular *affine invariance* and thus rules out a number of other multi-dimensional medians considered in the literature on strategy-proof social choice (Gershkov *et al.*, 2019, 2020; Freeman *et al.*, 2021), see Section 6 for further discussion.

Finally, the Tukey median receives additional motivation from the connection to the literature on strategy-proof social choice on restricted domains. Indeed a central finding of that literature has been that, on those domains on which strategy-proof social choice rules exist at all, these rules must be tops-only, see Chatterji and Sen (2011) and the references therein. Moreover, if one focuses on anonymous and neutral rules, these must be multi-dimensional medians of sorts ('issue-wise majority voting') (Nehring and Puppe, 2007b). Heuristically, the Tukey median can thus be viewed as a mechanism that imitates salient qualitative features of such mechanisms on an impossibility domain.

Related Literature

To the best of our knowledge, the present EAC approach and its application to the 'frugal aggregation' model of budget allocation are new to the literature. But there are, of course, related approaches in the literature. Indirectly, the Tukey median has been studied in the social choice literature inasmuch as it is equivalent to the outcome of the minimax voting rule in standard spatial voting with Euclidean preferences (Kramer, 1977; Demange, 1982; Caplin and Nalebuff, 1988). This model can be viewed as a degenerate frugal model in which voters preferences conditional on their top are known. (But with this additional, sub-top preference information, the Tukey median is no longer welfare optimal as we argue in Appendix C.2.2.)

Most work of theoretical interest in the problem of incomplete information as studied here has come from the computer science literature, see Boutilier and Rosenschein (2016) for an overview.[4] One strand explores the implications of partial knowledge of complete (ex-post) preference profiles for inferences about the outcome of standard social choice rules and criteria, e.g. via the notions of 'possible' vs. 'necessary' winners (Konczak and Lang, 2005). A rather small strand in the literature adopts a decision-theoretic ex-ante approach as the present paper does. Some papers seek solutions that maximize expected welfare based on some utilitarian welfare criterion and a probability distribution over profiles, frequently uniform. Others argue for the modeling of the social evaluator's epistemic state in terms of a set of possible profiles, as we do, and propose to apply classical criteria of decision making under

[4]We thank Jérôme Lang who pointed us to the pertinent literature.

ignorance such as maximin or minimax regret (Lu and Boutilier, 2011). In the highly complex state spaces associated with the epistemic models studied here, it may be very difficult to execute these approaches if that is possible at all. Significantly, the two quoted strands share the major conceptual limitation of having to rely on an interprofile-comparable standard of aggregate welfare ex post. Thus, they in fact suppose that the Arrovian problems of coherent aggregation and interpersonal non-comparability have been solved or assumed away, e.g. by assuming strong forms of utilitarian aggregation ex post.

By contrast, the EAC approach introduced here rests on an evaluation of decisions in pairs of alternatives taking the full state space (set of possible profiles) into account. In such pairwise comparisons, the majority criterion carries over naturally to the ex-ante stage, without raising new issues of interpersonal comparison, and allowing a tractable characterization in many cases. These pairwise comparisons need then be put together to obtain a coherent rationale for an ex-ante evaluation of complex choices such as budget allocations. At this juncture, Arrovian style issues of coherent aggregation might arise in principle. It is a rather remarkable finding of this paper that, for symmetrically ignorant sets of priors, these problems do not materialize.

With respect to the focal application to the allocation of public budgets, there is an important, lively literature on 'participatory budgeting' with intended application to cities and local communities (Shah, 2007). Participatory budgeting schemes have been put into practice at various scales in many places around the world. The ballots are typically very parsimonious, often taking the form of a set of projects approved.[5] Again, most of the theoretical contributions come from the computer science community, with a focus on indivisibilities and on 'proportionality' considerations to ensure that the interest of different local subcommunities are fairly represented (Aziz and Shah, 2020). By contrast, our focus is on continuous divisible budgets, and on finding allocations that best satisfy the aggregate interest (in line with most of standard voting theory).

[5]See, for instance, the open source project 'Stanford Participatory Budgeting Platform' (https://pbstanford.org) which offers guidance and allows municipalities, cities and other institutions to run participatory budgeting elections online.

Overview of Paper

The remainder of this paper is organized as follows. In Section 2, we introduce the general EAC approach. The subsequent two sections apply it to the budget allocation problem under the assumption that the social evaluator knows the profile of voters' top alternatives (the 'frugal aggregation' model). Section 3 studies the plain convex model which assumes in addition knowledge of convexity of voters' preferences but complete ignorance about anything else. Proposition 3 shows that, generically, the set of ex-ante Condorcet winners coincides with the set of voters' tops. In contrast, by our main result, Theorem 1 in Section 4, in the symmetric quadratic model the ex-ante Condorcet winner coincide with (a refinement of) the Tukey median. Section 5 proves an impossibility result (Theorem 2) showing that the epistemic state of the social evaluator envisaged here cannot be described by a single 'uninformative' prior. Section 6 offers some considerations on the use of the Tukey median as a voting mechanism played by self-interested voters, and Section 7 concludes

Most proofs are gathered in Appendix B. Appendix C discusses the robustness of our analysis with respect to the specific epistemic assumptions about the social evaluator.

2 Condorcet Winners, Ex-Ante

We envisage a social evaluator who has to choose from a universe of alternatives X on behalf of a group of $n \in \mathbb{N}$ voters under uncertainty about their preferences. The social evaluator is modeled as an 'imprecise' Bayesian decision maker, i.e. his epistemic state is described by a set of probability distributions over 'admissible' profiles $\succcurlyeq = (\succcurlyeq_1, ..., \succcurlyeq_n)$ of true ('ex-post') preferences.

Concretely, denote by π a probability measure over profiles $(\succcurlyeq_1, ..., \succcurlyeq_n)$ of complete preference orderings over X, and by Π a non-empty set of admissible such priors.[6] The social evaluator is completely ignorant as to which probability distribution in Π is the most appropriate and therefore needs to take into account all of them.

Often one will be interested in cases in which the priors in Π satisfy specific additional properties. For instance, an important special case in the following will

[6] To make this fully rigorous, one needs to specify a measure space on the set of profiles. For our purposes, the essential property is that, for each agent i and all alternatives x and y, the 'event' that agent i prefers x to y represents a measurable set.

involve $X \subseteq \mathbb{R}^L$ and the assumption that all priors are concentrated over profiles of *convex* preferences.

For all distinct $x, y \in X$, a prior $\pi \in \Pi$ induces an expected support count $m_\pi(x, y)$ of votes for x against y, i.e.

$$m_\pi(x, y) \ := \ E_\pi \left[\#\{i : x \succ_i y\}\right], \tag{2.1}$$

where E_π denotes the expectation operator with respect to the probability distribution π. Thus, a set of priors induces an interval $m_\Pi(x, y)$ of expected support counts in the vote of x against y,

$$m_\Pi(x, y) \ := \ \left[m_\Pi^-(x, y)\,,\, m_\Pi^+(x, y)\right],$$

where

$$m_\Pi^-(x, y) \ := \ \inf_{\pi \in \Pi} m_\pi(x, y), \tag{2.2}$$

$$m_\Pi^+(x, y) \ := \ \sup_{\pi \in \Pi} m_\pi(x, y). \tag{2.3}$$

The family of these intervals will be what matters in our analysis.[7] In deciding ex-ante on a hypothetical choice between x and y, it is natural to base this choice on a comparison of the intervals $m_\Pi(x, y)$ and $m_\Pi(y, x)$. Due to the imprecision of priors, the intervals $m_\Pi(x, y)$ and $m_\Pi(y, x)$ may overlap in general. But due to the additivity of the complementary vote counts for x against y and for y against x, a comparison of the lower and upper expected counts must yield the same result. This evidently holds if preferences are known to be strict ex-post. To guarantee it more generally, the following regularity condition is needed which ensures that possible indifferences play a negligible role; this condition is satisfied in all applications considered in the following, and we maintain it throughout. Say that a set of priors Π is *regular* if for all priors $\pi \in \Pi$ and all pairs $x, y \in X$ of distinct alternatives, there exists a prior π' such that $\pi'(x \sim_i y) = 0$ for all $i = 1, ..., n$, and $m_{\pi'}(x, y) \leq m_\pi(x, y)$. Thus, regularity guarantees that, for any pair $x, y \in X$, the minimal/infimal expected support for x against y is realized by a prior for which all indifferences between x and y have zero

[7]Other approaches are conceivable; for a justification of our modeling choice to base the ex-ante decision between two alternatives x and y on the support counts defined in (2.1), see Appendix A below.

probability.

Proposition 1. *Let* Π *be regular. For all* θ *and all distinct* $x, y \in X$*,*

$$m_\Pi^-(x,y) \geq m_\Pi^-(y,x) \iff m_\Pi^+(x,y) \geq m_\Pi^+(y,x). \tag{2.4}$$

(Proof in appendix.)

By Proposition 1, an *unambiguous* balance of uncertainties ex-ante is possible; in contrast to the classical theory of decision making under ignorance (Luce and Raiffa, 1957), there is no need or even meaningful role for an evaluator's degree of pessimism vs. optimism (ambiguity aversion vs. ambiguity proneness in more modern terminology).[8]

The **ex-ante majority relation** R_Π (for regular Π) is now defined as follows. For all distinct $x, y \in X$,

$$\begin{aligned} xR_\Pi y \;\; &:\iff \;\; m_\Pi^-(x,y) \geq m_\Pi^-(y,x) \\ &\iff \;\; m_\Pi^+(x,y) \geq m_\Pi^+(y,x). \end{aligned} \tag{2.5}$$

The maximal elements with respect to the ex-ante majority relation are referred to as the **ex-ante Condorcet winners**, i.e.

$$\mathrm{CW}(\Pi) \;:=\; \{x \in X \mid xR_\Pi y \text{ for all } y \in X\}.$$

An aggregation rule is called **ex-ante Condorcet consistent** if it selects all ex-ante Condorcet winners (if there are any).

In the following, we will refer to a set of priors Π as a **model** (of the evaluator's epistemic state). Moreover, we will say that two models are *equivalent* if they induce the same expected majority intervals. Note that, trivially, sets of priors with the same convex hull are equivalent, but the converse need not be true. Evidently, two equivalent models induces the same set of ex-ante Condorcet winners, i.e. $\mathrm{CW}(\Pi') = \mathrm{CW}(\Pi)$ whenever Π' and Π are equivalent.

[8] Nor is there a conflict – possibly even threatening an Arrow-like impossibility – between axioms of choice consistency and of independence; see Milnor (1954); Arrow (1960); Nehring (2000, 2009).

3 The Plain Convex Model

In the rest of this paper, we will study the case in which X is a convex subset of $\mathbb{R}^L$ for some $L \in \mathbb{N}$, and all preferences in any profile are convex. For our purposes, the following notion of convex preference will be useful. A weak order $\succcurlyeq$ on $X \subseteq \mathbb{R}^L$ is *convex* if, (i) for all $x, y, z, w \in X$, $y = t \cdot x + (1-t) \cdot z$ for some $0 \leq t \leq 1$, $x \succcurlyeq w$ and $z \succcurlyeq w$ jointly imply $y \succcurlyeq w$, and (ii) for all $x, y, z \in X$, $y = t \cdot x + (1-t) \cdot z$ for some $0 < t < 1$, and $x \succ z$ jointly imply $y \succ z$.[9]

An important economic application is the *budget allocation problem* in which X takes the form of a budget hyperplane. Concretely, consider a group of agents that has to collectively decide on how to allocate a fixed budget, normalized to unity, to a number L of public goods. Assuming given prices, the problem is fully determined by specifying the expenditure shares. The corresponding allocation problem can thus be modeled as the choice of an element of the following $(L-1)$-dimensional polytope:

$$X := \left\{ x \in \mathbb{R}^L \mid \sum_{\ell=1}^{L} x^\ell = 1 \text{ and } x^\ell \geq 0 \text{ for all } \ell = 1, ..., L \right\}, \tag{3.1}$$

where $x = (x^1, ..., x^L)$. Convex preferences are entirely standard in this context.

Other applications include the spatial voting model in which the coordinates represent different issues and alternatives represent political positions on these issues (Downs, 1957), or the collective choice of design of projects positioned in a characteristics space in the sense of Lancaster (1966).

The model of all priors with convex preferences on X without any further restriction is referred to as the **plain convex model** and denoted by Π_{co}.

3.1 Certainty about Tops

To simplify the task of the social evaluator, we assume in the main text that the evaluator knows the top choices of voters (in Appendix C.1, we show that this assumption can be relaxed). Concretely, denote by $\theta = (\theta_1, ..., \theta_n)$ the profile of the voters' top alternatives which we assume to be unique. The epistemic state of the social evaluator will now be denoted by Π^θ_{co} to indicate the knowledge of θ. Here, the

[9] Observe that (ii) is clearly implied by but significantly weaker than strict convexity. For instance, linear preferences satisfy both conditions (i) and (ii) but are not strictly convex.

set $\Pi_{\rm co}^{\theta}$ is assumed to consist only of priors π that are *compatible* with the top profile θ in the sense that every profile $\succcurlyeq = (\succcurlyeq_1, ..., \succcurlyeq_n)$ in the support of π has $\theta = (\theta_1, ..., \theta_n)$ as the corresponding top profile.

3.2 The One-Dimensional Case: Median Voting

In the one-dimensional case, our notion of preference convexity is equivalent to the standard notion of single-peakedness, and the choice of the median top(s) constitutes the unique ex-ante Condorcet consistent aggregation rule; specifically, we have the following result. For every profile $\theta = (\theta_1, ..., \theta_n)$, denote by $\theta_{\rm med}$ the unique median if n is odd, and by $[\theta_{{\rm med}^-}, \theta_{{\rm med}^+}]$ the median interval if the number of voters is even.

Proposition 2. *Suppose that $X \subseteq \mathbb{R}$, and let $\theta = (\theta_1, ..., \theta_n)$ be a profile of tops in X. Then,*

$$\mathrm{CW}(\Pi_{\rm co}^{\theta}) \ = \ \begin{cases} \{\theta_{\rm med}\} & \textit{if } n \textit{ is odd} \\ [\theta_{{\rm med}^-}, \theta_{{\rm med}^+}] & \textit{if } n \textit{ is even} \end{cases} .$$

(Proof in appendix.)

Thus, in the one-dimensional case the ex-post and ex-ante Condorcet criterion give the same result under single-peakedness. The reason is, evidently, that under knowledge of single-peakedness any given top uniquely determines the preference on both sides of the top, and that is all what is needed to apply the Condorcet criterion.

3.3 The Multi-Dimensional Case: Generic Plurality Rule

In the multi-dimensional case, a result similar to Proposition 2 holds if the top profile is contained in a one-dimensional subspace; but in general, in the plain convex model the ex-ante Condorcet winners essentially coincide with the plurality winners.

In the following, we say that a set of points $Y \subseteq \mathbb{R}^L$ is *in general position* if no three elements of Y are collinear. The crucial observation for the plain convex model is that, if θ, x, y are not collinear, then there exist convex preferences $\succcurlyeq$ and $\succcurlyeq'$ with top θ such that $x \succ y$ and $y \succ' x$. This implies the following characterization of the ex-ante Condorcet winners in the plain convex model. For its formulation, it will be useful to identify profiles of individual tops with *type profiles* of tops with different counts. Specifically, we denote by $\theta = (\theta_1; p_1, ..., \theta_m; p_m)$ the anonymous profile in which the fraction p_i of all voters has top θ_i, where $0 < p_i \leq 1$ and $\sum_i p_i = 1$; in that context, we also refer to θ_i as the *type* of voter i and assume without of loss of generality that the θ_i are pairwise distinct.

Proposition 3. *Consider a type profile $(\theta_1; p_1, ..., \theta_m; p_m)$ such that $\{\theta_1, ..., \theta_m\} \subseteq X$ are in general position. If p_{i^*} is maximal among $\{p_1, ..., p_m\}$, then $\theta_{i^*} \in \mathrm{CW}(\Pi^\theta_{\mathrm{co}})$. Moreover, if p_{i^*} is uniquely maximal among $\{p_1, ..., p_m\}$, then*

$$\mathrm{CW}(\Pi^\theta_{\mathrm{co}}) \ = \ \{\theta_{i^*}\}.$$

(Proof in appendix.)

This is somewhat paradoxical. Intuitively it would appear that preference convexity contains substantial information beyond knowledge of the tops but Proposition 3 appears to contradict this. What is amiss?

Example 1. *Consider a set of voters with pairwise distinct tops in a set U. In addition, suppose that two voters are concentrated at a point x outside U (see Figure 1). If all tops in U plus the point x are in general position then, according to Proposition 3, x is the unique ex-ante Condorcet winner. Indeed, for any point $z \neq x$, $m^-_{\Pi^\theta_{\mathrm{co}}}(x, z) = 2$ while $m^-_{\Pi^\theta_{\mathrm{co}}}(z, x) \leq 1$, or equivalently, $m^+_{\Pi^\theta_{\mathrm{co}}}(x, z) \geq n - 1$. Note that the expected majority intervals are extremely wide, and the ex-ante Condorcet winner is left to 'grasp for straws' in picking the optimal alternative that happens to be the top of two voters rather than just of one. Nonetheless, if the epistemic state of the social evaluator is* literally *that of complete ignorance within Π^θ_{co}, then the ex-ante preference for x over any other alternative z seems defensible.*

However, this rationale is not very robust. Consider in particular the comparison of x to y where y is sufficiently close to x and 'between' x and U as shown in Fig. 1. Note that for x to be preferred to y by some voter with top θ_i in U, i's preference must be very special; for instance, geometrically, only rather special ellipses with center at θ_i that include x will not include y.

The conceivable convex preference for x against y of a voter with top in U, on which the conclusion in Example 1 hinges, seems very unlikely a priori. It would therefore be desirable to capture this intuition by an appropriate specification of somewhat more precise evaluator's beliefs. The challenge is to describe these regularized beliefs in a qualitative manner that is weak enough to be acceptable on slim information

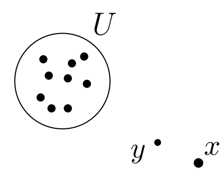

Figure 1: Illustration of Proposition 3

while at the time sufficiently strong to have substantive implications. This task is at the heart of this paper and is taken on in the next section.

4 Symmetrically Ignorant Quadratic Models

Our proposal for modeling the evaluator's beliefs in a more appropriate and specific manner involves two key features: First, we conceptually separate voters' tops from the substitution vs. complementation structure of preferences, and secondly, we assume that this substitution vs. complementation structure is 'ex-ante independent' of the tops. The first feature allows one to model preferences as quadratic; the second feature means that knowledge of the tops is not informative ex-ante for the substitution vs. complementation structure described by the quadratic preferences.

Specifically, say that a preference $\succcurlyeq$ on X is *quadratic* if it can be represented ordinally by a utility function of the form

$$u_{\theta_i}(x) \;=\; -(x-\theta_i)^T \cdot \mathcal{Q}_i \cdot (x-\theta_i), \tag{4.1}$$

for some $\theta_i \in X$ and a positive definite, symmetric $L \times L$ matrix $\mathcal{Q}_i$. Geometrically, the representation in (4.1) means that the indifference curves are ellipsoids generated from circles with center θ_i by a common affine transformation. The special case in which the quadratic form $\mathcal{Q}_i$ is the identity matrix $\mathcal{I}$ corresponds to the case of *Euclidean preferences* which has been extensively studied in the literature on spatial voting (Austen-Smith and Banks, 1999).

The cross-partial derivatives given by $\mathcal{Q}_i$ capture the specific pattern of complementarities and/or substitutabilities between different goods. Quadratic preferences can thus also be viewed as (second-order) Taylor approximations of arbitrary smooth convex preferences. Denote by $\Pi_{\text{quad}} \subseteq \Pi_{\text{co}}$ the model consisting of all sets of priors over profiles of quadratic preferences on X, the *plain quadratic model*. Evidently, for all tops $\theta_i \in X$ and all $x, y \in X$ such that θ_i, x, y are not collinear, there exist quadratic preferences $\succcurlyeq_i, \succcurlyeq_i'$ both with top θ_i such that $x \succ_i y$ and $y \succ_i' x$. By consequence, we have:

Fact 4.1. *The models* Π_{quad} *and* Π_{co} *are equivalent. In particular, the two models induce the same ex-ante majority relation and* $\text{CW}(\Pi_{\text{quad}}) = \text{CW}(\Pi_{\text{co}})$.

Thus, the plain quadratic model can be viewed as a parametrized version of the plain convex model. In particular the 'generic plurality' conundrum posed by Example 1 continues to apply to the plain quadratic model. But the great boon of the quadratic model is that it allows for a clear separation between the preference top and the preference *structure* (described by the quadratic from $\mathcal{Q}_i$). This will be the key in our proposed resolution of the puzzle posed by Example 1.

Specifically, the epistemic state of the evaluator is given by a set of priors Π with state space $X^n \times \boldsymbol{\mathcal{Q}}^n$ where X is the set of all possible tops and $\boldsymbol{\mathcal{Q}}$ the set of all quadratic forms (symmetric and positive definite $L \times L$ matrices). For every prior $\pi \in \Pi$ and all voters $i = 1, ..., n$, denote by $\pi|_{X_i}$ and $\pi|_{\mathcal{Q}_1}$ the marginal distributions induced by π on X_i (the ith copy of X) and $\boldsymbol{\mathcal{Q}}_i$ (the i-th copy of $\boldsymbol{\mathcal{Q}}$), respectively.

In the remainder of this section, we will impose the following conditions on a model Π. For all $x \in X$, denote by δ_x the degenerate probability distribution that puts unit mass on x; similarly, for all $\mathcal{Q} \in \boldsymbol{\mathcal{Q}}$, denote by $\delta_{\mathcal{Q}}$ the degenerate probability distribution that puts unit mass on $\mathcal{Q}$.

1. **Quadratic Preferences.** $\Pi \subseteq \Pi_{\text{quad}}$.

2. **Tops Certainty.** For all $\pi \in \Pi$ and all i, $\pi|_{X_i} = \delta_{\theta_i}$ for some $\theta_i \in X$.

3. **Symmetry of Marginals.** For all $\pi \in \Pi$ and all i, j, $\pi|_{\mathcal{Q}_i} = \pi|_{\mathcal{Q}_j}$.

4. **Complete Ignorance of Marginals.** For all i and all $\mathcal{Q} \in \boldsymbol{\mathcal{Q}}$, there exists $\pi \in \Pi$ such that $\pi|_{\mathcal{Q}_i} = \delta_{\mathcal{Q}}$.

A model Π satisfying Assumptions 1 to 4 will be called **symmetrically ignorant quadratic**, or **s.i.q.** for short. Assumption 2 means that all voters' tops are known; therefore Assumptions 1 and 2 can be summarized as requiring $\Pi \subseteq \Pi^{\theta}_{\text{quad}}$ in our previous notation, where $\theta = (\theta_1, ..., \theta_n)$ is the known profile of voters tops. Assumption

3 means that an individual's top (or any other observable individual characteristic) does not contain any information on the distribution of the individual's preferences by itself. Finally, Assumption 4 assumes in effect complete ignorance about each agent's $\mathcal{Q}_i$. More detailed discussion of these assumptions is provided in Section 5 below and in Appendix C.

The plain quadratic model satisfies all assumptions except Symmetry of Marginals. The 'regularizing' effect of the Symmetry of Marginals assumption can be illustrated in Example 1.

Example 1 (cont.) *Consider again the situation depicted in Fig. 1 above, but now suppose that the epistemic state of the social evaluator is described by a symmetric quadratic model* Π *rather than by the plain convex model. The minimal expected majority count for* x *against* y *is still* 2*, since it is evidently possible to find a symmetric prior such that all voters in* U *prefer* y *to* x*, i.e.* $m_{\Pi}^{-}(x,y)=2$*. For example, one may take the prior that assumes with certainty that all preferences are Euclidean. What about* $m_{\Pi}^{-}(y,x)$*? As before, one can assign quadratic forms* $(\mathcal{Q}_1,...,\mathcal{Q}_n)$ *to the tops such that all voters with top in* U *prefer* x *to* y*. But, as is evident from Fig. 1, these quadratic forms generally have to be distinct for different voters; the prior assuming this profile with certainty is therefore not symmetric. Any symmetric prior must thus be properly probabilistic; for example, a symmetric prior might assign equal probability* $1/n!$ *to each of the permutations of the profile* $(\mathcal{Q}_1,...,\mathcal{Q}_n)$*. But for any such prior the expected majority count for* y *against* x *will be at least* 3*, i.e.* $m_{\Pi}^{-}(y,x)\geq 3$*, as a key argument in the proof of our main result shows; for the geometric intuition behind this argument, see Figure 3 below. Hence, for any symmetric quadratic model* Π *we obtain* $m_{\Pi}^{-}(y,x)>m_{\Pi}^{-}(x,y)$*, and thus* $yP_{\Pi}x$*, where* P_{Π} *denotes the asymmetric part of the ex-ante majority relation* R_{Π}*; in other words,* x *is not an ex-ante Condorcet winner.*

At one extreme, there exists a unique largest (most imprecise) s.i.q. model Π_{quad}^{sym} consisting of *all* symmetric priors. Note that it does not impose any additional knowledge, i.e. probability one restrictions, beyond the plain quadratic model; it can thus be viewed as a regularized version of that model.

At the other extreme, there is also a unique smallest (most precise) s.i.q. model, as follows. Call a prior *uniform* if all voters have the same quadratic form with probability one (while there may be uncertainty about what the common quadratic form is), and denote by Π_{unif} the **uniform (quadratic) model** consisting of all uniform priors. Moreover, denote by Π_{exunif} the **extremal uniform model** consisting of all uniform priors of the form $\delta_{(\mathcal{Q},...,\mathcal{Q})}$, i.e. all priors that put unit mass on some single profile of the form $(\mathcal{Q}, ..., \mathcal{Q})$. Evidently, the extremal uniform model satisfies Assumptions 1 to 4; conversely, combining Assumptions 3 and 4 also shows that any s.i.q. model contains the extremal uniform model.[10]

There is a wide range of intermediate specifications. For example, the quadratic forms $\mathcal{Q}_i$ can be assumed to be drawn i.i.d. from some unknown distribution. In the subjectivist tradition, this is captured (and slightly generalized in the finite case) by assuming that Π consists of all exchangeable priors in the sense of de Finetti (1931).[11] Finally, a s.i.q. model Π may also incorporate beliefs about possibly learnable correlations between the tops and the quadratic forms.

It turns out that ex-ante Condorcet winners in the s.i.q. models exist, and that they are *Tukey medians* (Tukey, 1975) of a particular kind. For all $x \in X$, denote by $\mathcal{H}_x$ the family of all Euclidean half-spaces that contain x (i.e. the family of all sets of the form $\{y \in X : a \cdot y \geq a \cdot x\}$ for some non-zero vector $a \in \mathbb{R}^L$). For all profiles $\theta = (\theta_1, ..., \theta_n)$ and all half-spaces H, denote $\theta(H) := \#\{i : \theta_i \in H\}$, and define the *Tukey depth* of x at the profile θ by

$$\mathfrak{d}(x; \theta) := \min_{H \in \mathcal{H}_x} \theta(H).$$

Intuitively, the Tukey depth measures the 'centrality' of x with respect to the profile of tops: the larger $\mathfrak{d}(x; \theta)$ the more tops θ_i are guaranteed to lie in every direction viewed from x, and $\mathfrak{d}(x; \theta) = 0$ means that x can be separated from the entire set of tops θ by a hyperplane. Denote by $\mathfrak{d}^*(\theta) := \max_{x \in X} \mathfrak{d}(x; \theta)$ the maximal Tukey depth over X. The *Tukey median rule* selects, for every profile θ, the alternatives that attain this maximal depth:

$$T(\theta) \;:=\; \arg\max_{x \in X} \mathfrak{d}(x; \theta) \;=\; \{x \in X \mid \mathfrak{d}(x; \theta) = \mathfrak{d}^*(\theta)\}. \tag{4.2}$$

Our main result involves the following refinement. For all profiles θ and all x, denote by $\mathcal{H}_x^* := \{H \ni x : \theta(H) = \mathfrak{d}^*(\theta)\}$. A Tukey median $x \in T(\theta)$ is *strict* if, for

[10]By the preceding observation, the class of all s.i.q. models forms a bounded lattice partially ordered by set inclusion.

[11]Specifically, in our context a prior π is *exchangeable* if, for all events $E \subseteq \boldsymbol{\mathcal{Q}}_1 \times \ldots \times \boldsymbol{\mathcal{Q}}_n$ and all permutations σ of agents, $\pi(E) = \pi(\sigma(E))$, where $\sigma(E)$ is the event obtained from E by applying σ.

no $y \in T(\theta)$, $\mathcal{H}_y^* \subsetneq \mathcal{H}_x^*$. The set of **strict Tukey medians** is denoted by $T^*(\theta)$.

Theorem 1. *For all profiles θ and every symmetrically ignorant quadratic model $\Pi \subseteq \Pi^\theta_{\text{quad}}$, $\mathrm{CW}(\Pi)$ is non-empty. Moreover,*

$$\mathrm{CW}(\Pi) \;=\; T^*(\theta).$$

The proof of Theorem 1 (provided in the appendix) proceeds in a series of steps. First, it is shown that all s.i.q. models are equivalent. The argument relies crucially on both the symmetry assumption and the EAC solution concept, in particular on its definition based on the expected support counts (2.1). This allows us to focus on the characterization of the analytically convenient uniform model, thus simplifying matters greatly since the uniform model is characterized by strong ex-post restrictions on profiles. In particular, profiles of preferences with a common quadratic form are intermediate preferences in the sense of Grandmont (1978). More specifically, for any two alternatives x and y, the tops in a profile of preferences with a common quadratic form $\mathcal{Q}$ that prefer x to y are separated from those preferring y to x by a hyperplane through the midpoint between x and y, see Lemma B.2 in the appendix; Figure 2 shows these hyperplanes for selected common quadratic forms (for the identity matrix $\mathcal{Q} = \mathcal{I}$, the hyperplane is perpendicular to the straight line through x and y).

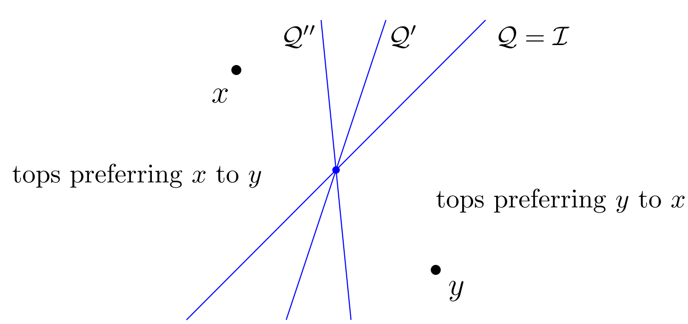

Figure 2: Intermediate preferences with separating hyperplane

From this one can show that the ex-ante majority relation of the uniform model coincides locally with the comparison of alternatives in terms of their *relative Tukey* *depth*: for all distinct $x, y \in X$, let

$$xR_{\mathfrak{d}}y \;:\Longleftrightarrow\; \min_{H\in\mathcal{H}_x,\, y\notin H}\theta(H) \geq \min_{H\in\mathcal{H}_y,\, x\notin H}\theta(H). \tag{4.3}$$

The ex-ante majority relation does not coincide globally with the relative Tukey depth relation (4.3) since the half-spaces separating the underlying tops in the quadratic model must go through the midpoint between x and y. Nonetheless, the set of local maxima of this relation is shown to coincide with the set of global maxima, which in turn coincides with the set of strict Tukey medians. Finally, the existence of strict Tukey medians is shown by an appeal to the Hausdorff maximal principle.

Example 1 (cont.) *In Example 1, the Tukey depth of x relative to y is evidently $\min_{H\in\mathcal{H}_x,\, y\notin H}\theta(H) = 2$. Conversely, the Tukey depth of y relative to x is obtained by looking at the straight line ∂H through x and y: the tops that support y against x must at least contain the tops in $U \cap H$, or the tops in $U \cap H^c$. As can be inferred from Fig. 3, we therefore have $\min_{H\in\mathcal{H}_y,\, x\notin H}\theta(H) = 3$, and hence $yP_{\mathfrak{d}}x$, where $P_{\mathfrak{d}}$ denotes the asymmetric part of $R_{\mathfrak{d}}$. It follows from the arguments provided in the proof of Theorem 1 in the appendix that we thus also obtain $yP_{\Pi}x$ for any s.i.q. model Π, as claimed above.*

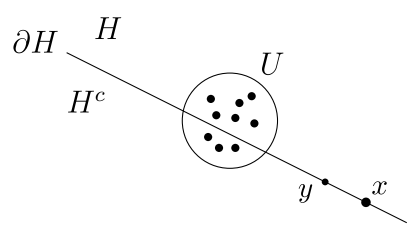

Figure 3: The Tukey depth of y relative to x is equal to 3

To illustrate the relation between absolute and relative Tukey depth, consider the following example.

Example 2. *Suppose that there are five voters whose tops $\theta_1, ..., \theta_5$ form a pentagon as shown in Figure 4. The (strict) Tukey median (and hence by Theorem 1 also the ex-ante Condorcet winners of any s.i.q. model) is given by the points in the inner convex pentagon marked in red.*[12] *Fig. 4 also shows a point y and its associated upper*

[12]This can be verified from the following observations. First, any line passing through the inner

contour set with respect to the relative Tukey depth relation given by (4.3) in blue color. Note in particular that the points x and y have the same (absolute) Tukey depth but different relative depth, to wit $xP_{\mathfrak{d}}y$.

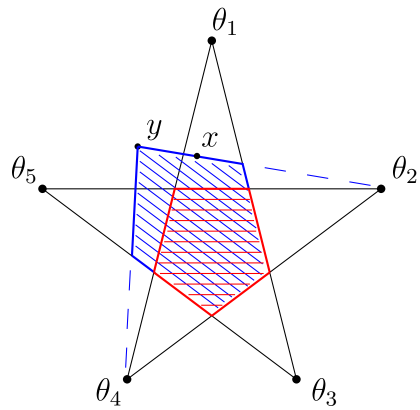

Figure 4: Absolute versus relative Tukey depth

While every Tukey median is strict in Example 2, it is an open question if this is the case generally. It must be the case whenever Tukey medians are unique (because strict Tukey medians always exist). Demange (1982) has in fact shown such uniqueness whenever voters' tops are continuously distributed with a convex support. Using this result, we obtain the following 'continuous' version of Theorem 1.

Theorem 1′ *Suppose that voters' tops are distributed according to a continuous measure θ with convex support. Then, the strict Tukey median set $T^*(\theta)$ consists of a single point, and for every s.i.q. model $\Pi \subseteq \Pi^{\theta}_{\text{quad}}$,*

$$\text{CW}(\Pi) \;=\; T(\theta) \;=\; T^*(\theta).$$

(Proof in appendix.)

Appendix C discusses in detail the robustness of Theorems 1 and 1′ with respect to our assumptions on the evaluator's beliefs. Section C.1 shows that Assumption 2 (Tops Certainty) can be relaxed, while Section C.2 considers different specifications of beliefs corresponding to various relaxations of Assumptions 1,3 and 4. In particular,

red pentagon has at least two tops on either side; on the other hand, for any point outside the inner pentagon there is a Euclidean half-space containing that point and at most one top. In particular, the maximal Tukey depth is $\mathfrak{d}^*(\theta) = 2$; all points in the convex hull of the tops that are not in this inner pentagon have depth one, and all points outside the convex hull of the tops have depth zero.

Assumptions 1 (Quadratic Preferences) and 4 (Complete Ignorance of Marginals) show substantial robustness with respect to smaller and greater precision, respectively.

5 Why not Bayesian?

An alternative, more standard modeling strategy would postulate a Bayesian social evaluator characterized by one single prior. But this approach has limited appeal here. First of all, the precise Bayesian approach seems difficult to execute since under its premises ex-ante Condorcet winners will frequently fail to exist, just as in the standard setting. This entails at least two drawbacks. First, one need to settle – with more or less well-founded arguments – for some (ex-ante) Condorcet extension. Moreover, even when this is resolved, one is likely going to lose much of the analytical tractability of the present analysis, including its ability to characterize frugal optima.

Introducing probabilistic subjectivity one also loses a deeper potential of the frugal aggregation framework. Indeed, this framework not only enables frugality on part of the voters by parsimonious tops-only elicitation, but also on the side of the social evaluator. Indeed, *whose* subjective probability is supposed to serve as the basis of the evaluation? If the social evaluator was understood as a social planner ('bureaucrat'), one may think of the required judgmental input as reflecting the planner's expertise; but in a group decision context, the social evaluator is naturally viewed as representing 'the group' at a constitutional stage at which individual preference profiles are unknown. Frugality in this more encompassing sense requires that the evaluator's beliefs be *grounded* in the (sparse) available information, not just *consistent with* this information.

Such grounding can be captured formally by taking beliefs to be functions of the situation-specific information (the profile of tops), i.e. *prior mappings* $\theta \mapsto \Pi(\theta)$. To ensure that the prior mapping does not 'smuggle in' implicit information, we shall impose two minimal *uninformativeness* conditions. We shall show that while these conditions are satisfied by the imprecise models in Sections 3 and 4, they cannot be satisfied by any precise model. Impossibilities of this kind are classic themes in both the ancient and contemporary discussion of the foundations of probability. For example, Bernardo and Smith summarize in their authoritative treatment of Bayesian theory that "in continuous multiparameter situations there is no hope for a single, unique, 'non-informative prior' [...]" (Bernardo and Smith, 1994, p. 366).

The first condition of uninformativeness is a requirement of description invariance akin to those invoked in this literature. It requires that the admissible prior mappings be invariant with respect to all structure-preserving bijections on the space of alternatives. In the case of convex or quadratic preferences, the structure-preserving mappings are the affine transformations. Formally, we will thus require affine invariance of the social evaluator's beliefs, as follows.[13]

Denote by $\mathcal{C}$ the set of all profiles of convex preference orderings on $\mathbb{R}^L$, and by $\mathcal{C}^n$ the set of all profiles of all convex preference orderings with n voters. For any (measurable) event $E \subseteq \mathcal{C}^n$ and any affine transformation $t : \mathbb{R}^L \longrightarrow \mathbb{R}^L$ denote by $t(E) := \{(\succ_1^t, ..., \succ_n^t) : (\succ_1, ..., \succ_n) \in E\}$ the transformed event where, for all i, $\succ_i^t$ is the preference ordering defined by $y \succ_i^t z :\Leftrightarrow t(y) \succ_i t(z)$.

Affine Invariance. The set of priors $\Pi(\theta)$ is *affine invariant*, if for all events E and all affine transformations t,

$$\Pi(\theta)(E) \;=\; \Pi(t(\theta))(t(E)),$$

where $\Pi(\theta)(E) := \{\pi(\theta)(E) : \pi(\theta) \in \Pi(\theta)\}$, $\pi(\theta)(E)$ is the probability assigned by $\pi(\theta)$ to the event E, and $t(\theta)$ is the transformed profile of tops $(t(\theta_1), ..., t(\theta_n))$.

The next condition reflects the assumption that information about *other* voters' tops does not convey information about the sub-top preferences of any given voter.

Top/Sub-top Independence. The set of priors $\Pi(\theta)$ is *top/sub-top independent* if for all i and all i-marginal events $E \subseteq \mathcal{C}_{[i]}$ – the i-th copy of $\mathcal{C}$ – and all profiles θ, θ' such that $\theta_i = \theta'_i$,

$$\Pi(\theta)(E) \;=\; \Pi(\theta')(E).$$

The case for Top/Sub-top Independence is especially transparent in the quadratic submodel. Here, the quadratic form describes the substitution/complementation preference of voters for alternatives below the top, a dimension of preference distinct in kind from the location of their tops.

These two uninformativeness requirements cannot be satisfied by precise prior mappings, indeed not even by mappings with precise prior marginals.

[13] For simplicity, we assume $X = \mathbb{R}^L$ in this and the next section; the general case of a convex subset $X \subseteq \mathbb{R}^L$ can be treated similarly, albeit with some additional work.

Marginal Precision. The set of priors $\Pi(\theta)$ is *marginally precise* if, for all $\pi, \pi' \in \Pi(\theta)$ and all i, $\pi|_{\mathcal{C}_{[i]}} = \pi'|_{\mathcal{C}_{[i]}}$.

Theorem 2. **a)** *The plain convex model* Π_{co} *and the s.i.q. model* Π_{quad}^{sym} *consisting of all symmetric priors (understood as prior mappings) satisfy affine invariance and top/sub-top independence.*

b) *If* $L \geq 2$*, any prior mapping* Π *that satisfies affine invariance and top/sub-top independence violates marginal precision.*

(Proof in appendix.)

As a corollary we also obtain that there does not exist a precise set of priors satisfying affine invariance and top/sub-top independence over the set of all profiles of quadratic preference orderings, since such prior would satisfy the same conditions on the set of profiles of convex preference orderings.

The asserted impossibility is a natural consequence of the richness of the group of affine transformations, which in turn reflects the richness and paucity of structure of the domain of convex preferences and thus, in turn again, the paucity of information available for the formation of beliefs. Mathematically, affine invariance precludes the use of geometric information about angles and lengths of line segments. By contrast, smaller domains such as the domain of Euclidean preferences are not invariant with respect to all affine transformations but only to the subclass of all *isometric* transformations, and top-subtop independent priors that are invariant with respect to the latter clearly exist.

The main observation in the proof of Theorem 2 is that invariance with respect to independent rescaling of different coordinates (an affine transformation) conflicts with convexity plus completeness of preferences. And indeed a statement similar to that of Theorem 2 does not hold if one allows for incomplete preferences, as follows. For all $x \in X$, denote by $\succ_x^0$ the partial order with top x that only compares alternatives along rays emanating from x; then for instance, the prior such that the marginals put probability one on the profile that agrees with $\succ_{\theta_i}^0$ for each i satisfies all conditions of Theorem 2.

Also observe that the result does not hold for $L = 1$. As a simple example, for each $x \in \mathbb{R}$, consider the marginals that put full probability on the metric preference ordering $\succ_x^d$ with top x (i.e. $y \succcurlyeq_x^d z \Leftrightarrow d(y, x) \leq d(z, x)$), where d is the Euclidean distance; then, the prior such that the marginals put probability one on the profile that agrees with $\succ^d_{\theta_i}$ for each i satisfies all conditions of Theorem 2.

On the other hand, if $L = 1$, description invariance implies more than just affine invariance since preference convexity is then preserved under all *monotone* transformations. And in fact a result akin to Theorem 2 holds also for $L = 1$ if the invariance condition is strengthened accordingly in this case, demonstrating its conceptual robustness.

6 The Tukey Median as a Voting Mechanism

We have argued for the strict Tukey median as the normative welfare optimum under restricted ('frugal') information and an appropriate class of s.i.q. beliefs of the social evaluator; in fact, in Appendix C, we demonstrate that this finding is robust with respect to the specification of beliefs: it suffices that the evaluator puts some small positive probability on preferences being quadratic on top of being convex. One can understand this normative optimum as constituting an *ideal* voting mechanism in which voters always report their true tops.

In actual voting situations, one needs to take into account the possibility of strategic voting. Indeed, strategic voting will necessarily entail misrepresentation at some profiles. This follows for the domain of all convex as well as for the the domain of all quadratic preferences from well-known impossibility results, see Zhou (1991). Hence the question arises whether the extent and/or frequency of misrepresentation are sufficiently limited so that the Tukey mechanism can be viewed as implementing the Tukey median as normative ideal 'approximately.' There is no room here to enter in detail into the various game-theoretic issues that arise. The following heuristic considerations aim to show that, indeed, the Tukey median holds significant promise as a voting mechanism implementing the Tukey ideal approximately.[14] If successful, such approximation is good enough for our purposes. Indeed, if on top of that, other mechanisms could be shown to be superior in some convincing sense – a demonstration that seems a long way off – this would be interesting and further support the approximability of the Tukey ideal.

[14]For simplicity, we neglect the difference between the strict and non-strict versions of the Tukey median in the following.

Bounded Manipulability

So, how vulnerable is the Tukey median against unilateral manipulations by single individuals and coalitions of voters? We will now show that both the extent and the potential welfare loss of such manipulations are quite limited. To formulate this precisely, denote by $O_J(\theta)$ the *option set* of the coalition $J \subseteq \{1, ..., n\}$ at the profile $\theta = (\theta_1, ..., \theta_n)$, i.e.

$$O_J(\theta) \;:=\; \{x \in X \,|\, x \in T^*(\hat{\theta}) \text{ for some } \hat{\theta} \text{ with } \hat{\theta}_i = \theta_i \text{ for all } i \notin J\};$$

moreover, consider the Tukey depth level sets, i.e. the upper contour sets of the Tukey depth function. For all profiles θ and all $m > 0$, let

$$T^{[m]}(\theta) \;:=\; \{x \in X \,|\, \mathfrak{d}(x;\theta) \geq m\}, \tag{6.4}$$

so that $T^*(\theta) = T^{[\mathfrak{d}^*(\theta)]}(\theta)$. It is easily seen that $\{T^{[m]}(\theta)\}_{m\in\mathbb{N}}$ form a nested family of convex sets contained in the convex hull of θ for all $m > 0$; these properties suggest the following result.

Proposition 4. *For all profiles θ and all coalitions $J \subseteq \{1, ..., n\}$,*

$$O_J(\theta) \;\subseteq\; T^{[\mathfrak{d}^*(\theta)-2\#J]}(\theta).$$

Thus, any manipulation of a coalition of size k can reduce the resulting Tukey depth by at most $2k$. The substantive interest in Proposition 4 stems from the nestedness, boundedness and convexity of the sets $T^{[m]}(\theta)$ for $m > 0$ which suggests that the Tukey level sets stay small if m is close to $\mathfrak{d}^*(\theta)$. How close the outcome resulting from manipulations stay to the true Tukey median, of course, depends on the specifics of the underlying distribution. For instance, if θ is a joint normal distribution the Tukey depth level sets correspond to the upper contour sets of the density function (spheres in the i.i.d. case).

The proof of Proposition 4 is simple. It is easy to see that any alternative in the Tukey median set after misrepresentation by a single voter loses at most 1 count of Tukey depth at the true profile, and any alternative outside the Tukey median set can gain at most 1 count. This holds for any member of a coalition which immediately proves the claim. Mathematically, Proposition 4 only provides a rough upper bound;

better bounds can be obtained in special cases. For instance, in the one-dimensional case, it is straightforward to verify that $O_J(\theta) \subseteq T^{[\mathfrak{d}^*(\theta)-\#J]}(\theta)$, i.e. a manipulation by a coalition of size k can reduce the resulting Tukey depth by at most k. In two or more dimensions this is no longer true. This is easily verified in the setting of Example 2 by adding a voter with top in the inner pentagon.[15]

Proposition 4 shows that manipulations by single voters have little impact, and this has also implications for Nash equilibria. If – for whatever reasons – only a moderate fraction of k voters manipulate, the outcome stays within $T^{[\mathfrak{d}^*(\theta)-2k]}(\theta)$, thus close to the true optimum. And there may indeed be several reasons for limited manipulation. First, it may well be that only few voters have an incentive to deviate. Secondly, even if they have such incentive in principle, cognitive costs may exceed the expected benefits; this would be particularly relevant for voters who understand that, in view of Proposition 4, they have limited influence on the outcome. Outside the purview of Proposition 4, note that even if a large fraction of voters manipulate, these manipulations could largely cancel each other out so that the equilibrium again stays close to the true optimum.

Proposition 4 is also interesting from the perspective of coordinated coalitional manipulation. To put its assertion into proper context, two observations seem particularly relevant. First, coordination of coalitions is particularly demanding in our setting with limited information. Secondly, if successful coordinated coalitional manipulation may generally be expected to have a disproportionate impact (as, e.g., in those cases in which strategy-proofness does not imply coalitional strategy-proofness, see, among others, Nehring and Puppe 2007a); but here the impact exhibits a degree of proportionality by Proposition 4.

Non-trivial robustness properties such as the one described by Proposition 4 generally distinguish the class of *multi-dimensional medians* that extend the univariate median to higher dimensions with the main aim of ensuring resistance to outliers, see Small (1990); Rousseeuw and Hubert (2017) for overviews.

A straightforward example of a tops-only mechanism that lacks such robustness is the mean, for which a single voter can achieve *any* outcome by an appropriate

[15]If the true top θ_6 of an additional sixth voter is contained in the inner pentagon in Example 2, the resulting (unique) Tukey median is θ_6 with a Tukey depth of 3; if this voter misrepresents a top $\hat{\theta}_6$ outside the inner pentagon, the alternative θ_6 looses one depth unit and the alternative $\hat{\theta}_6$ gains one unit. If $\hat{\theta}_6$ is in the convex hull of the other voters' (true) tops, it receives a Tukey depth of 2 and will therefore be among the Tukey medians under the unilateral manipulation.

manipulation (in the full Euclidean space). If we consider, somewhat more generally, mechanisms that minimize $\sum_i d(x, \theta_i)^\alpha$ for the Euclidean metric d, the same applies to all $\alpha > 1$ (note that the mean corresponds to $\alpha = 2$).[16]

Neutrality

In analogy to Section 5, a natural requirement from a frugal mechanism perspective is a neutrality condition reflecting the 'description invariance' of the underlying problem. Specifically, say that a choice rule is *neutral* if it commutes with all structure-preserving bijections on the space of alternatives. Since in the case of convex or quadratic preferences, the structure-preserving mappings are the affine transformations, the neutrality requirement thus translates into invariance of the choice rule with respect to all affine transformations. Evidently, the Tukey median satisfies affine invariance; since non-invariant mechanisms will tend to induce non-neutral (description-dependent) outcomes, it is hard to see how such mechanisms could robustly 'approximate' the affine invariant Tukey median ideal.

While most of the multi-dimensional medians discussed in the economic literature on strategy-proofness are not affine invariant,[17] the statistical literature has explored a number of multi-dimensional medians that are affine invariant. Historically, the Tukey median (Tukey, 1975) was the first of these followed, among others, by the Oja median (Oja, 1983), the simplicial median (Liu, 1990), and the projective median(s) (Zuo, 2003).

[16] If $\alpha < 1$, the minimand is not convex and the minimizer not continuous; if α tends to zero, we obtain plurality rule which suffers from similar drawbacks. Within this class, the case $\alpha = 1$ is singled out and defines a classical multi-dimensional median, the 'geometric' or 'spatial' median (Weber, 1909).

[17] Examples are the coordinate-wise median, the geometric median, and more generally all medians based on the minimization of the sum of the L_p-distances to the voters' tops for $p \geq 1$. Some of these can be motivated by their desirable properties on rather tightly restricted subdomains of convex preferences. For instance, Gershkov *et al.* (2019, 2020) have argued for the coordinate-wise ('marginal') median in models in which voters' preferences are induced by norms with the coordinates chosen endogenously depending on the norms; and Freeman *et al.* (2021) have shown that the L_1-median (i.e. the minimizer of the sum of the L_1-distances to the voters' tops) is strategy-proof if voters L_1-metric preferences and tops on a budget hyperplane. Thus, different domain restrictions may motivate different multi-dimensional medians. But due to their lack of affine invariance, such mechanisms are not well-suited to the much larger domain of all convex preferences, or to the domain of all quadratic preferences.

Voting by Issues

Among the affine invariant multi-dimensional medians, the Tukey median is distinguished by its representation in terms of half-spaces. These can be interpreted as 'issues' in the sense of the literature on strategy-proof social choice on rich domains of generalized single-peaked preferences (Barberà *et al.*, 1991, 1997; Nehring and Puppe, 2007b, 2010), lending further support to the Tukey median from a mechanism design perspective. Indeed, the latter two papers characterize strategy-proof social choice rules on these domains as *independent* and consistent, tops-only 'voting by issues.' In the anonymous and neutral case these rules take the form of issue-wise majority voting.

The domains of convex and that of quadratic preferences are both instances of rich single-peaked domains with respect to the Euclidean betweenness relation according to which a point is between two others if and only if it lies on the line segment spanned by them. The relevant issues $\{H, H^c\}$ are given by the Euclidean half-spaces and their complements. Consistency cannot be achieved non-dictatorially by independent issue-wise aggregation unless there is an alternative with Tukey depth exceeding $\frac{n}{2}$; by consequence, strategy-proofness cannot be generally be obtained in a non-dictatorial way.

The Tukey median achieves consistency among issue-wise majorities by prioritizing the larger ones according to (4.2). If a half-space H receives large enough support (specifically, if $\theta(H) \geq 1 - 1/(L+1)$), the Tukey median will choose in H. Note that this holds independently of the location of tops within H and its complement H^c. In this way, the Tukey median approximates the unifying structure of strategy-proof social choice rules rather tightly.

Table 1 summarizes the basic properties satisfied by different tops-only mechanisms.[18]

[18]Legend to Table 1: The 'L_p-medians' are the minimizers of the sum of the L_p-distances to voters' tops; the bounded manipulability properties of the simplicial and Oja medians follow from their respective 'breakdown values' which are inferior to that of the Tukey median, by contrast the projective median has an optimal breakdown value (see Rousseeuw and Hubert (2017)); finally, plurality rule can be construed as 'voting by issues' using the particular issues of 'being /not being identical to' any given alternative.

	bounded manipulability	affine invariance	voting by issues
mean rule	✗	✓	✗
plurality rule	✗	✓	✓
L_p-median, $p > 1$	✓	✗	✗
coordinate-wise median	✓	✓	✗
simplicial, Oja, and projective median	✓	✗	✓
Tukey median	✓	✓	✓

Table 1: Properties of voting mechanisms

7 Concluding Remarks: A Middle Ground?

In this paper, we have proposed a minimalist ('frugal') approach to voting in multi-dimensional settings and singled out a particular choice rule, the Tukey median. A natural follow-up challenge is to explore the middle ground between such minimalism and the standard maximalist assumption of knowing the voters' complete preference profile.

Such middle ground is easy to identify in the absence of any background information in which plurality is the canonical frugal choice rule if only for want of something better (Goodin and List, 2006). In that setting, it is conceptually and pragmatically quite straightforward to obtain and utilize additional information. Two popular ways of doing so is to ask for a top set of alternatives (approval voting) or to rank the top $k > 1$ alternatives associated with a variety of salient methods of aggregation.

Yet, in multi-dimensional settings with a background knowledge of preference convexity, it seems a lot harder to determine a viable middle ground. Ranking the top k alternatives is not well-defined, so this route seems precluded. On the other hand, it is straightforward conceptually to ask voters for a top set of alternatives in analogy to approval voting – here, one would presumably be looking for an extension of the Tukey median to (convex) set-valued inputs. Note, however, that this is cognitively highly demanding and lacking in parsimony.

Other types of input may be elicited quite easily, but might be challenging to utilize in a cogent manner. Suppose, for example, the evaluator has obtained additional preference comparisons to a set of other alternatives (which might itself be endogeneous, e.g. chosen based on the profile of tops). While it is easy to incorporate this into the knowledge base of the evaluator, how should it change her beliefs? Recall that the step from knowledge to beliefs has been crucial in our argument how to properly exploit the background information of preference convexity.

These are some of the interesting challenges for future research. They might lead to a variety of novel mechanisms based on alternative input specifications. On the other hand, it may well prove difficult to enhance the minimalist tops-only plus preference-convexity specification assumed here while preserving both parsimony of the input and cogency of the use of that information by the mechanism.

Appendix A: Why expected support counts?

In our EAC approach, the ex-ante decision between two alternatives is based on the support count as defined in (2.1). More generally, one could assume that the social evaluator evaluates majority margins via the expected *utility* of the majority margin, i.e. via

$$E_\pi[u(\#\{i : x \succ_i y\}] \tag{A.1}$$

for some utility function $u(\cdot)$ and each prior π. However, this more general approach violates fundamental axioms unless u is in fact linear as in (2.1).

To illustrate, consider the extreme case in which $u(k) = 0$ if $k < n/2$ and $u(k) = 1$ if $k > n/2$ so that (A.1) simply corresponds to the probability that x wins by a majority against y under π. For simplicity, suppose that there are three voters with tops $\theta_i \in \{x, y\}$, and that the evaluator's prior is concentrated on four profiles with the probabilities specified in the first column of Table 2 (columns 2 - 4 represent voters, rows represent profiles). Now, given the evaluator's belief, alternative x wins by a majority against y with probability $3/5$, while each voter ex-ante prefers y to x. Thus, this specification of u leads to a violation of an attractive ex-ante Pareto criterion. As can be shown this holds more generally whenever $u(\cdot)$ is not linear as in (2.1).

probability	θ_1	θ_2	θ_3
1/5	x	x	y
1/5	x	y	x
1/5	y	x	x
2/5	y	y	y

Table 2: Three voters and four profiles over two alternatives

Appendix B: Proofs

Proof of Proposition 1. Given the pair $x, y \in X$, let $\Pi' \subseteq \Pi$ be the subset of all priors $\pi \in \Pi$ such that, for all $i = 1, ..., n$, $\pi(x \sim_i y) = 0$. By the regularity assumption, we have $m^-_\Pi(x, y) = m^-_{\Pi'}(x, y)$ and $m^-_\Pi(y, x) = m^-_{\Pi'}(y, x)$, and therefore also $m^+_\Pi(x, y) = m^+_{\Pi'}(x, y)$ and $m^+_\Pi(y, x) = m^+_{\Pi'}(y, x)$. By construction, we have for all $\pi \in \Pi'$,

$$m_\pi(x, y) + m_\pi(y, x) \;=\; n.$$

This implies

$$\begin{aligned} m^-_{\Pi'}(x, y) + m^+_{\Pi'}(y, x) &= n \text{ and} \\ m^-_{\Pi'}(y, x) + m^+_{\Pi'}(x, y) &= n, \end{aligned}$$

therefore

$$m^-_{\Pi'}(x, y) + m^+_{\Pi'}(y, x) \;=\; m^-_{\Pi'}(y, x) + m^+_{\Pi'}(x, y),$$

and hence (2.4). □

For the following proofs, the following observation will be useful. Denote by Π_{exco} the 'extremal' convex model, i.e. the submodel of the plain convex model which only contains sets of priors that put all mass on a *single* profile of convex preferences.

Lemma B.1. *The two models* Π_{exco} *and* Π_{co} *are equivalent.*

Proof. Consider any pair of distinct alternatives $x, y \in X$. Let π^0 be a minimizer of the support count for x against y under the model Π_{co}, i.e. $m^-_{\Pi_{\text{co}}}(x, y) = m_{\pi^0}(x, y)$. Furthermore, let $\succcurlyeq^0$ be a profile of convex preferences in the support of π^0 such that $\#\{i : x \succ_i y\}$ is minimal among all profiles in the support of π^0. Let $\delta_{\succcurlyeq^0}$ be the prior that puts all mass on $\succcurlyeq^0$; then, $m_{\delta_{\succcurlyeq^0}}(x, y) \leq m_{\pi^0}(x, y)$. But since $\delta_{\succcurlyeq^0}$ is an admissible prior under the model Π_{co}, we have in fact $m_{\delta_{\succsim^0}}(x, y) = m_{\pi^0}(x, y)$; this implies the desired result. $\square$

Proof of Proposition 2. Using Lemma B.1, the proof is straightforward from well-known properties of single-peaked preferences. $\square$

Proof of Proposition 3. Again using Lemma B.1, it is sufficient to prove the statement for the model $\Pi_{\text{exco}} \subseteq \Pi_{\text{co}}^{\theta}$. Consider any $x \neq \theta_{i^*}$; by assumption, there is at most one $\theta_j \neq \theta_{i^*}$ on the straight line through x and θ_{i^*}. Since preference convexity entails no restriction in the comparison of x and θ_{i^*} for tops outside that straight line, and since θ_{i^*} has largest popular support, this implies $m^-_{\Pi_{\text{exco}}}(\theta_{i^*}, x) \geq m^-_{\Pi_{\text{exco}}}(x, \theta_{i^*})$, i.e. θ_{i^*} is an ex-ante majority winner against x; if θ_{i^*} has uniquely largest popular support, we even have $m^-_{\Pi_{\text{exco}}}(\theta_{i^*}, x) > m^-_{\Pi_{\text{exco}}}(x, \theta_{i^*})$. Since x was chosen arbitrarily, the result follows. $\square$

Proof of Theorem 1

The proof of Theorem 1 is given by means of a series of auxiliary results. First, Proposition 5 shows that all s.i.q. models are equivalent to the uniform quadratic model hence, by an argument analogous to that given in the proof of Lemma B.1, also to the extremal uniform model Π_{exunif}. The main subsequent steps are summarized in two further propositions: Proposition 6 shows that the ex-ante Condorcet winners of the extremal uniform model coincide with the maximizers of the relative Tukey depth. Finally, Proposition 7 demonstrates that the set of maximizers of the relative Tukey depth is non-empty and coincides with the strict Tukey median.

A key fact about the s.i.q. models is that they are all equivalent; specifically, we have the following result.

Proposition 5. *All symmetrically ignorant quadratic models are equivalent.*

Proof. We show that any s.i.q. model Π is equivalent to the uniform quadratic model Π_{unif}. Consider a fixed pair $x, y \in X$ of distinct alternatives, and a fixed profile θ of tops. Let π be any symmetric prior and consider any fixed voter $h = 1, ..., n$. Denote by $\tilde{\pi} \in \Pi_{\text{unif}}$ be the unique prior that is concentrated on uniform profiles and satisfies $\tilde{\pi}_{\mathcal{Q}_h} = \pi|_{\mathcal{Q}_h}$. By symmetry of π, we have $\pi|_{\mathcal{Q}_i} = \pi|_{\mathcal{Q}_h}$ for all $i = 1, ..., n$, and by construction, $\tilde{\pi}_{\mathcal{Q}_i} = \tilde{\pi}_{\mathcal{Q}_h}$ for all $i = 1, ..., n$; hence, $\tilde{\pi}_{\mathcal{Q}_h} = \pi|_{\mathcal{Q}_h}$ for all $i = 1, ...n$. This implies

$$m_{\tilde{\pi}}(x,y) \;=\; E_{\tilde{\pi}}[\#\{i : x \succ_i y\}] \;=\; \sum_{i=1}^{n} E_{\tilde{\pi}_{\mathcal{Q}_i}}[x \succ_i y] \;=\; \sum_{i=1}^{n} E_{\pi_{\mathcal{Q}_i}}[x \succ_i y] \;=\; E_{\pi}[\#\{i : x \succ_i y\}] \;=\; m_{\pi}(x,y).$$

In other words, for every prior $\pi \in \Pi$ there exists a uniform prior $\tilde{\pi} \in \Pi_{\text{unif}}$ that induces the same expected majority count for x against y. This implies

$$m^{-}_{\Pi_{\text{unif}}}(x,y) \;\leq\; m^{-}_{\Pi}(x,y). \tag{B.1}$$

On the other hand, by Assumptions 3 (Symmetry of Marginals) and 4 (Complete Ignorance of Marginals), every s.i.q. model contains the extremal uniform model Π_{exunif}, hence

$$m^{-}_{\Pi}(x,y) \;\leq\; m^{-}_{\Pi_{\text{exunif}}}(x,y). \tag{B.2}$$

Finally, by an argument completely analogous to the argument in the proof of Lemma B.1, we have

$$m^{-}_{\Pi_{\text{exunif}}}(x,y) \;=\; m^{-}_{\Pi_{\text{unif}}}(x,y). \tag{B.3}$$

Combining (B.1), (B.2) and (B.3), we obtain that the arbitrary s.i.q. model Π induces the same intervals of expected majority counts as the uniform quadratic model Π_{unif}. □

In the following, it will be useful to denote the relative Tukey depth of x with respect to y by

$$\mathfrak{d}(x,y;\theta) := \min_{H \in \mathcal{H}_x,\, y \notin H} \theta(H),$$

so that $xR_{\mathfrak{d}}y :\Leftrightarrow \mathfrak{d}(x,y;\theta) \geq \mathfrak{d}(y,x;\theta)$ (cf. (4.3)), as well as

$$S(\theta) \;:=\; \{x \in X |\text{ for no } y \in X,\; yP_{\mathfrak{d}}x\}.$$

Due to Proposition 5 and Lemma B.1, we can concentrate in the remainder of the proof of Theorem 1 on the extremal uniform model Π_{exunif} (with the fixed top profile θ). Our goal is to prove the following result.

Proposition 6. *For all profiles θ and $\Pi_{\text{exunif}} \subseteq \Pi^{\theta}_{\text{quad}}$,*

$$\text{CW}(\Pi_{\text{exunif}}) \;=\; S(\theta).$$

One difficulty in showing this is that the ex-ante majority relation of the extremal uniform model does in fact *not* coincide with the relative Tukey depth, as noted in Example 2 above. Nevertheless, their maximal elements coincide.

Again, we need a preliminary result. Observe that, since a preference is quadratic if and only if is obtained from a Euclidean preference (with circles as indifference curves) by an affine transformation, we have the following result.

Lemma B.2. *Let $x, y \in X$ be any two distinct alternatives, and $\succcurlyeq = (\succcurlyeq_1, ..., \succcurlyeq_n)$ a uniform profile of quadratic preferences with tops $\theta = (\theta_1, ..., \theta_n)$. Then, there exists a (Euclidean) half-space $H \subseteq \mathbb{R}^L$ such that the hyperplane ∂H passes through the midpoint between x and y, and*

$$\{\theta_i \,|\, x \succ_i y\} \;\subseteq int(H) \;\;\; and \;\;\; \{\theta_i \,|\, y \succ_i x\} \;\subseteq int(H^c), \tag{B.4}$$

where H^c is the complement of H in $\mathbb{R}^L$. Conversely, for any (Euclidean) half-space H that separates x from y such that ∂H passes through the midpoint between x and y, there exists a uniform profile of quadratic preferences that satisfies (B.4).

Proof of Proposition 6. Let $x^* \in \text{CW}(\Pi_{\text{exunif}})$, i.e. $x^* R_{\Pi_{\text{exunif}}} y$ for all $y \in X$. By contradiction, assume that $x^* \notin S(\theta)$. Then, $y P_{\mathfrak{d}} x^*$ for some $y \in X$, i.e.

$$\mathfrak{d}(x^*, y; \theta) < \mathfrak{d}(y, x^*; \theta). \tag{B.5}$$

Let $H^0 \in \mathcal{H}_{x^*}$ be a Euclidean half-space that separates x^* from y and that minimizes the measure $\theta(H)$ among all such half-spaces. Without loss of generality, we may assume that $x^* \in \partial H^0$ and that $\partial H^0 \cap \{\theta\}_{i=1}^n \subseteq \{x^*\}$ (the latter by the fact that $\{\theta\}_{i=1}^n$ is a discrete set). Therefore, we my shift H^0 slightly towards y to $\widetilde{H}^0$ while keeping the mass with respect to θ constant, i.e. such that $\theta(H^0) = \theta(\widetilde{H}^0) = \mathfrak{d}(x^*, y; \theta)$. Consider the intersection point w of the straight line L connecting y and x^* with $\partial \tilde{H}^0$, and the point z on L such that w is the midpoint between w and x^* (see Figure 7). By Lemma B.2 we have $m^-_{\Pi_{\text{exunif}}}(x^*, z) = \theta(\widetilde{H}^0) = \mathfrak{d}(x^*, y; \theta)$. Moroever, we evidently also have $\mathfrak{d}(x^*, y; \theta) = \mathfrak{d}(x^*, z; \theta)$, and $\mathfrak{d}(z, x^*; \theta) \geq \mathfrak{d}(y, x^*; \theta)$. Thus, using (B.5) and the fact that, for all $w, v \in X$, $m^-_{\Pi_{\text{exunif}}}(w,v) \geq \mathfrak{d}(w,v;\theta)$, we obtain,

$$m^-_{\Pi_{\text{exunif}}}(z,x^*) \geq \mathfrak{d}(z,x^*;\theta) \geq \mathfrak{d}(y,x^*;\theta) > \mathfrak{d}(x^*,y;\theta) = \mathfrak{d}(x^*,z;\theta) = m^-_{\Pi_{\text{exunif}}}(x^*,z).$$

i.e. $zP_{\Pi_{\text{exunif}}}x^*$ in contradiction to the initial assumption that $x^* \in \mathrm{CW}(\Pi_{\text{exunif}})$.

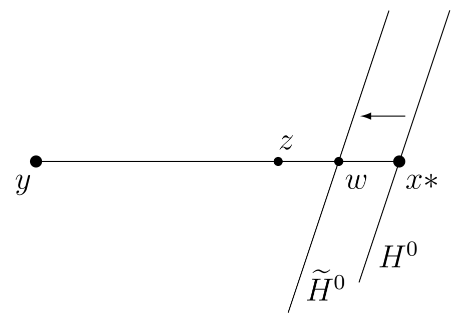

Figure 5: ‘Localization’ argument

Conversely, let $x^* \in S(\theta)$, i.e. $x^* R_{\mathfrak{d}} x$ for all $x \in X$. Consider any fixed $y \in X$ distinct from x^*, and let w denote the midpoint of the line segment connecting x^* and y. Let H^1 be a half-space with $x^* \in H^1$, $w \in \partial(H^1)$ such that $\theta(H^1)$ is minimal among all half-spaces with these two properties. Then, by Lemma B.2,

$$m^-_{\Pi_{\text{exunif}}}(x^*,y) = \theta(H^1). \tag{B.6}$$

Since x^* is in the interior of H^1 and w on its boundary, we have

$$\theta(H^1) \geq \mathfrak{d}(x^*,w;\theta). \tag{B.7}$$

By the assumption $x^* \in S(\theta)$, we have

$$\mathfrak{d}(x^*,w;\theta) \geq \mathfrak{d}(w,x^*;\theta), \tag{B.8}$$

and again by Lemma B.2,

$$\mathfrak{d}(w,x^*;\theta) = m^-_{\Pi_{\text{exunif}}}(y,x^*). \tag{B.9}$$

Combining (B.6) - (B.9), we thus obtain,

$$m^{-}_{\Pi_{\text{exunif}}}(x^*, y) \;\geq\; m^{-}_{\Pi_{\text{exunif}}}(y, x^*),$$

i.e. $x^* R_{\Pi_{\text{exunif}}} y$. Since y was arbitrarily chosen, we thus obtain $x^* \in \mathrm{CW}(\Pi_{\text{exunif}})$ as desired. □

It remains to be shown that $S(\theta)$ coincides with the strict Tukey median, and that these two sets are indeed non-empty.

Proposition 7. *For all θ, the strict Tukey median $T^*(\theta)$ is non-empty and*

$$S(\theta) \;=\; T^*(\theta).$$

The proof of Proposition 7 is given through a series of lemmata.

Lemma B.3. *For all $x, y \in X$, $\mathfrak{d}(x;\theta) > \mathfrak{d}(y;\theta)$ implies $x P_{\mathfrak{d}} y$.*

Proof. By assumption there exists a half-space H containing y with $\theta(H) < \mathfrak{d}(x;\theta)$, hence in particular $x \notin H$. Thus,

$$\mathfrak{d}(x, y;\theta) \;\geq\; \mathfrak{d}(x;\theta) \;>\; \theta(H) \;\geq\; \mathfrak{d}(y, x;\theta).$$

□

Note that Lemma B.3 implies $S(\theta) \subseteq T(\theta)$.

Lemma B.4. *For all distinct $x, y \in X$ such that $\mathfrak{d}(x;\theta) = \mathfrak{d}(y;\theta) =: \alpha$, one has $\mathfrak{d}(x, y;\theta) = \alpha$ or $\mathfrak{d}(y, x;\theta) = \alpha$.*

Proof. Let $H \ni x$ be such that $\theta(H) = \alpha$; without loss of generality, we may assume that x is on the boundary $\partial(H)$ of H (otherwise, one may shift the boundary of H to x without increasing $\theta(H)$). For any such H, $\theta(\partial H \setminus \{x\}) = 0$. Indeed, if $\partial H \setminus \{x\}$ contained some voters' tops, an appropriate slight rotation around x to H' would eliminate some of them without including additional ones (by the finiteness of the set $\{\theta_i\}_{i=1}^n$); but this would entail $\mathfrak{d}(x;\theta) < \alpha$, a contradiction.

If $y \notin H$, then $\mathfrak{d}(x, y;\theta) = \alpha$. If $y \in H$, since x is on the boundary of H, H could be changed slightly (by appropriate shift plus slight rotation) to H' with $y \in H'$

eliminating x without including any additional tops (by the argument above). Thus, $\alpha \leq \theta(H') \leq \theta(H) \leq \alpha$, in particular $\theta(H') = \alpha$. In other words, we have constructed H' such that $\theta(H') = \alpha$, $y \in H'$ and $x \notin H'$, hence $\mathfrak{d}(y, x; \theta) = \alpha$. □

Lemma B.5. *For all distinct $x, y \in X$ with $\mathfrak{d}(x; \theta) = \mathfrak{d}(y; \theta) =: \alpha$,*

$$xP_{\mathfrak{d}}y \iff \mathfrak{d}(x, y; \theta) > \alpha \iff \mathcal{H}_x^\alpha \subsetneq \mathcal{H}_y^\alpha, \tag{B.10}$$

where $\mathcal{H}_x^\alpha := \{H \in \mathcal{H} \mid x \in H \text{ and } \theta(H) = \alpha\}$. In particular, the relation $P_{\mathfrak{d}}$ is a strict partial order and

$$S(\theta) \;=\; T^*(\theta). \tag{B.11}$$

Proof. The first biconditional in (B.10) follows from Lemma B.4 since $\mathfrak{d}(y, x; \theta) \geq \mathfrak{d}(y; \theta) = \alpha$. Thus, we only need to show that $\mathfrak{d}(x, y; \theta) > \alpha \iff \mathcal{H}_x^\alpha \subsetneq \mathcal{H}_y^\alpha$. If $\mathfrak{d}(x, y; \theta) > \alpha$, there does not exist a half-space H such that $x \in H$, $y \notin H$ and $\theta(H) = \alpha$. Moreover, by Lemma B.4, $\mathfrak{d}(y, x; \theta) = \alpha$, i.e. there exists a half-space H such that $y \in H$, $x \notin H$ and $\theta(H) = \alpha$; hence in fact $\mathcal{H}_x^\alpha \subsetneq \mathcal{H}_y^\alpha$.

Conversely, if $\mathcal{H}_x^\alpha \subsetneq \mathcal{H}_y^\alpha$, there does not exist a half-space H such that $x \in H$, $y \notin H$ and $\theta(H) = \alpha$, hence $\mathfrak{d}(x, y; \theta) > \alpha$.

The equality stated in (B.11) now follows from the definition of the strict Tukey median. □

We now show that the sets $T^*(\theta)$ and hence $S(\theta)$ are indeed non-empty. To this end, consider the Tukey median set $T(\theta)$, i.e. the depth level set with maximal depth and denote, for all $x \in T(\theta)$, by $\widetilde{L}_x(\theta) := \{y \in T(\theta) \mid xR_{\mathfrak{d}}y\} \setminus \{x\}$ (i.e. the lower contour set of x with respect to $R_{\mathfrak{d}}$ minus the alternative x itself). Moreover, denote the complement of $\widetilde{L}_x(\theta)$ in $T(\theta)$ by $\widetilde{U}_x(\theta)$, i.e.

$$\widetilde{U}_x(\theta) = \{y \in T(\theta) \mid yP_{\mathfrak{d}}x\} \cup \{x\}$$

(this is the upper contour set of x with respect to $P_{\mathfrak{d}}$ plus the alternative x itself).

Lemma B.6. *For all $x \in T(\theta)$, the sets $\widetilde{U}_x(\theta)$ are relative closed in $T(\theta)$.*

Proof. We show that the complementary sets $\widetilde{L}_x(\theta)$ are relative open in $T(\theta)$. Consider any pair $x, y \in T(\theta)$ such that $xR_{\mathfrak{d}}y$ and $x \neq y$, and let α^* be the maximal Tukey depth. We have $\mathfrak{d}(x; \theta) = \mathfrak{d}(y; \theta) = \alpha^*$, and by Lemmas B.4 and B.5, we have $\mathfrak{d}(y, x; \theta) = \alpha^*$. Thus, there exists a half-space H with $\theta(H) = \alpha^*$, $y \in H$, $x \notin H$. Since the voters' tops form a discrete set, we can move the boundary ∂H slightly towards x in a parallel fashion to obtain a half-space H' such that $H \subseteq H'$, $\theta(H') = \theta(H) = \alpha^*$ and $x \notin H'$. Thus, $\mathfrak{d}(y', x; \theta) = \alpha^*$, and hence again by Lemma B.5, $xR_{\mathfrak{d}}y'$, for all y' in a small neighborhood of y. This shows that $\widetilde{L}_x(\theta)$ is relative open in $T(\theta)$. □

Lemma B.7. *For all profiles θ, $S(\theta)$ is non-empty.*

Proof. Consider chains of upper contour sets, i.e. subsets $\mathcal{C} \subseteq \{\widetilde{U}_x(\theta) \mid x \in T(\theta)\}$ totally ordered by set inclusion, and denote by $\mathcal{U}$ the family of all such chains partially ordered by set inclusion. By Zorn's Lemma, there exists a maximal element in $\mathcal{U}$, i.e. a maximal chain $\mathcal{C}^*$.

The function $x \mapsto \mathfrak{d}(x; \theta)$ is upper semicontinuous, hence the set $T(\theta)$ of its maximizers is non-empty and closed. Hence, since $T(\theta)$ is clearly also bounded, $T(\theta)$ is a compact set. By Lemma B.6, the elements of $\mathcal{C}^*$ are relative closed in $T(\theta)$, hence as relative closed subsets of the compact set $T(\theta)$ themselves compact.

Consider the directed net $(Z, \geq)$ where $Z := \{x \in T(\theta) \mid \widetilde{U}_x(\theta) \in \mathcal{C}^*\}$ and

$$x \geq y \;:\Longleftrightarrow\; \widetilde{U}_x(\theta) \subseteq \widetilde{U}_y(\theta).$$

By the compactness of $T(\theta)$, the net $(Z, \leq)$ contains a convergent subnet in Z; let x^* denote its limit. By the orderedness of the chain $\mathcal{C}^*$, $x \geq y$ implies $x \in \widetilde{U}_y(\theta)$; hence by the closedness of $\widetilde{U}_y(\theta)$, we have $x^* \in \widetilde{U}_y(\theta)$ for all $y \in Z$, and therefore $x^* \in \cap\mathcal{C}^*$.

By Lemma B.5, the relation $P_{\mathfrak{d}}$ is transitive on $T(\theta)$, hence $\widetilde{U}_{x^*}(\theta) \subseteq \widetilde{U}_y(\theta)$ for all $y \in Z$, and therefore $\widetilde{U}_{x^*}(\theta) \subseteq \cap\mathcal{C}^*$. By the maximality of $\mathcal{C}^*$, $\widetilde{U}_{x^*}(\theta) = \{x^*\}$. By the definition of $\widetilde{U}_{x^*}(\theta)$, $x^* \in S(\theta)$, in particular $S(\theta) = T^*(\theta)$ is non-empty. □

Proof of Proposition 7. By Lemma B.5, we have $S(\theta) = T^*(\theta)$, and by Lemma B.7 $S(\theta)$ is non-empty. This completes the proof of Proposition 7. □

Proof of Theorem 1. The proof follows from combining Propositions 5, 6 and 7. □

Remaining Proofs

Proof of Theorem 1′. The first steps in the proof follow closely the proof of Theorem 1. Indeed, many intermediate steps and arguments hold without change in the case of a continuous distribution θ of tops. In particular, Proposition 5 can be shown in an analogous manner, and Lemmas B.2 and B.3 hold without change. Next, we show that

$$\mathrm{CW}(\Pi_{\mathrm{exunif}}) \;=\; S(\theta) \tag{B.12}$$

(cf. Proposition 6). As in the proof of Proposition 6, suppose that $x^* \in \mathrm{CW}(\Pi_{\mathrm{exunif}})$ and, by contradiction, $x^* \notin S(\theta)$, i.e. (B.5) for some $y \in X$. As in the proof of Proposition 6, we choose H^0 such that $y \notin H^0$, $x^* \in \partial(H^0)$ and $\theta(H^0) = \mathfrak{d}(x^*, y; \theta)$. Since θ is continuously distributed, for any positive ε, we can shift H^0 slightly towards y to $\widetilde{H}^0$ as in Fig. 4 above such that $\theta(\widetilde{H}^0) < \theta(H^0) + \varepsilon$. If ε is sufficiently small, we obtain

$$\begin{aligned} m^-_{\Pi_{\mathrm{exunif}}}(z, x^*) \geq \; \mathfrak{d}(z, x^*; \theta) \;\geq\; \mathfrak{d}(y, x^*; \theta) \;> \\ \mathfrak{d}(x^*, y; \theta) + \varepsilon \;>\; \theta(\widetilde{H}^0) \;=\; m^-_{\Pi_{\mathrm{exunif}}}(x^*, z). \end{aligned}$$

i.e. $z P_{\Pi_{\mathrm{exunif}}} x^*$ in contradiction to the initial assumption that $x^* \in \mathrm{CW}(\Pi_{\mathrm{exunif}})$.

The converse statement $S(\theta) \subseteq \mathrm{CW}(\Pi_{\mathrm{exunif}})$ follows exactly as in the proof of Proposition 6 above.

By (Demange, 1982, Sect. 2.4.(ii)), the Tukey median set $T(\theta)$ consists of a unique point x^*. In particular, $\mathfrak{d}(x^*, \theta) > \mathfrak{d}(y, \theta)$ for all $y \in X \setminus \{x^*\}$; hence by Lemma B.3, $S(\theta) = T(\theta) = \{x^*\}$. Thus, by (B.12) also $\mathrm{CW}(\Pi_{\mathrm{exunif}}) = \{x^*\} = T(\theta)$. □

Proof of Theorem 2. Part a) is straightforward. Part b) is shown by contradiction. Thus assume that $\Pi(\theta)$ satisfies all three stated conditions, and let $\pi|_{\mathcal{C}_{[i]}}$ the unique marginal prior corresponding to voter i. By affine invariance, we may assume that $\theta_i = 0$, and write μ instead of $\pi|_{\mathcal{C}_{[i]}}$ for notational simplicity. Consider $y = (1, 0, ..., 0) \in \mathbb{R}^L$, $z = (0, 1, 0, ..., 0) \in \mathbb{R}^L$ and, for all $k \in \mathbb{N}$ the sequence $\varepsilon_k = 1/k$. Since μ is invariant to the (linear) transformation $t_k : \mathbb{R}^L \longrightarrow \mathbb{R}^L$ that maps every $(a^1, ..., a^L)$ to $(a^1, \varepsilon_k \cdot a^2, a^3, ..., a^L)$, we obtain

$$\mu(E_{z \succ y}) \;=\; \mu(E_{\varepsilon_k \cdot z \succ y}) \tag{B.13}$$

for all k, where $E_{z \succ y}$ is the event that z is preferred to y under μ. Denote by E_k the event that ($\varepsilon_k \cdot z \succ y$ and $y \not\succcurlyeq z$). By completeness of preferences, we have

$$E_k \;=\; E_{\varepsilon_k \cdot z \succ y} \setminus E_{z \succ y}, \tag{B.14}$$

and by convexity, for all k, $E_{z \succ y} \subseteq E_{\varepsilon_k \cdot z \succ y}$. From (B.13) and (B.14), we thus obtain for all k, $\mu(E_k) = 0$. Hence, by countable additivity of μ, $\mu(\cup_{k \in \mathbb{N}} E_k) = 0$. But by continuity and the fact that 0 is the unique top of all preferences considered, we have $\cup_{k \in \mathbb{N}} E_k = E_{y \succcurlyeq z}$, and hence $\mu(E_{y \succcurlyeq z}) = 0$.

By the same argument, reversing the roles of y and z, we also obtain $\mu(E_{z \succcurlyeq y}) = 0$ contradicting the completeness of preferences. □

Appendix C: Robustness against Alternative Specifications of Beliefs

C.1 Uncertainty about Tops

We have so far assumed that the only individuating information about individual preferences concerns their tops, and that this information is perfect (tops assumed to be known by the social evaluator). We now extend the frugal aggregation approach maintaining the first assumption while abandoning the second. Formally, we now assume that the evaluator has a precise prior over the top of each voter. Such models may be of interest when individual tops are elicited by a vote or a poll, and when there are doubts whether they should be taken at face value, for instance for incentive reasons. Obviously, for concrete applications this needs to be developed further by specifying how the evaluator's probabilistic beliefs over tops are themselves formed.

Specifically, we adapt our assumptions on the epistemic state Π of the evaluator as follows.

1. **Concentration on Quadratic Preferences.** $\Pi \subseteq \Pi_{\text{quad}}$.

2a. **Tops Probabilism.** For all $\pi, \pi' \in \Pi$ and all i, $\pi|_{X_i} = \pi'|_{X_i} =: \mu_i$.

2b. **Independence.** For all i and all $\theta_i, \theta_i' \in \operatorname{supp} \mu_i$, $\pi|_{\mathcal{Q}_{\theta_i}} = \pi|_{\mathcal{Q}_{\theta_i'}}$.

3. **Symmetry of Marginals.** For all $\pi \in \Pi$ and all i, j, $\pi|_{\mathcal{Q}_i} = \pi|_{\mathcal{Q}_j}$.

4. **Complete Ignorance of Marginals.** For all i, $\theta_i \in \operatorname{supp} \mu_i$ and all $\mathcal{Q} \in \boldsymbol{\mathcal{Q}}$, there exists $\pi \in \Pi$ such that $\pi|_{\mathcal{Q}_{\theta_i}} = \delta_{\mathcal{Q}}$.

Assumption 2a says that all priors in Π agree on the distribution of tops, i.e. the uncertainty about tops is probabilistic rather than imprecise. Assumption 2b adds that any top θ_i in the support of the marginal top distribution μ_i induces the same marginal distribution $\pi|_{\mathcal{Q}_{\theta_i}}$ over quadratic forms. The marginal top distributions can take the form of finite or continuous measures; in the latter case, in order to apply Theorem 1′, we need to assume that all μ_i have a common convex support.[19] Denoting by $\mu = (\mu_1, ..., \mu_n)$ the profile of the marginal distributions over tops, and by Π^{μ}_{quad} the set of all quadratic priors that induce the marginal distribution profile μ, we can summarize Assumptions 1 and 2a by requiring $\Pi \subseteq \Pi^{\mu}_{\text{quad}}$. With slight abuse of terminology, we continue calling a model satisfying these modified assumptions *symmetrically ignorant quadratic (s.i.q.)* since no confusion can arise.

To adapt our main result to the situation in which the social evaluator is uncertain about the voters' tops but has a unique prior μ over the profile of the distribution of tops, denote by $\overline{\mu}$ the average distribution of tops defined by

$$\overline{\mu} \ := \ \sum_{i=1}^{n} \frac{1}{n} \cdot \mu_i.$$

Associate with each i an 'ex-ante subpopulation' with distribution of tops μ_i and relative size $1/n$; these combine to a total ex-ante population with distribution of tops $\overline{\mu}$ and quadratic forms still unknown as in Theorem 1. Independence (Assumption 2b) ensures symmetry within each subpopulation, while Symmetry of Marginals (Assumption 3) ensures symmetry across subpopulations. Therefore, the argument of Theorem 1 applies and yields the strict Tukey median with respect to $\overline{\mu}$ as the ex-ante Condorcet winners; note that, in the following result, Tops Probabilism (Assumption 2a) is indispensable as it is necessary to even define the characterized set $T^*(\overline{\mu})$.

Theorem 3. *For all profiles $\mu = (\mu_1, ..., \mu_n)$ such that the μ_i are either finite or continuously distributed with a common convex support, and for every symmetrically ignorant quadratic model $\Pi \subseteq \Pi^{\mu}_{\text{quad}}$, $\text{CW}(\Pi)$ is non-empty. Moreover,*

$$\text{CW}(\Pi) \ = \ T^*(\overline{\mu}).$$

Proof. Consider any s.i.q. model $\Pi \subseteq \Pi^{\mu}_{\text{quad}}$. Any prior $\pi \in \Pi$ corresponds to a unique symmetric prior $\overline{\pi} \in \Pi^{\overline{\mu}}_{\text{quad}}$ such that, for all $\theta \in \text{supp}\,\mu$, $\overline{\pi}_{\mathcal{Q}_\theta} = \pi_{\mathcal{Q}_\theta}$. Denote by $\overline{\Pi}$ the set of all such priors $\overline{\pi}$, i.e. $\overline{\Pi} := \{\overline{\pi} : \pi \in \Pi\}$. As is easily verified, $\overline{\Pi}$

[19] It might be possible to generalize the result to arbitrary or arbitrary continuous probability distributions, but this would require additional arguments.

satisfies Assumptions 1-4 of Section 4 (in particular, Tops Certainty) with respect to the distribution $\overline{\mu}$. Moreover, for all distinct $x, y \in X$, we have

$$m_\pi(x,y) \;=\; n \cdot m_{\overline{\pi}}(x,y),$$

hence $m^-_\Pi(x,y) = n \cdot m^-_{\overline{\Pi}}(x,y)$; therefore, the ex-ante majority relations corresponding to Π and $\overline{\Pi}$ coincide and we have $\mathrm{CW}(\Pi) = \mathrm{CW}(\overline{\Pi})$. Since $\overline{\Pi}$ satisfies the assumptions required for Theorem 1 in the case of a finite distribution, and for Theorem 1′ in the case of a continuous distribution with convex support, we can conclude that $\mathrm{CW}(\overline{\Pi}) = T^*(\overline{\mu})$. Together with the preceding observation we thus obtain, $\mathrm{CW}(\Pi) = \mathrm{CW}(\overline{\Pi}) = T^*(\overline{\mu})$. □

C.2 Different Beliefs

In our context of frugal aggregation under convex preferences, two fundamental requirements on a solution are (i) that it selects the standard median if voters' tops are contained in a one-dimensional subspace, and (ii) that it be invariant with respect to affine transformations. The first condition is justified by the observation that, under convex preferences, the standard one-dimensional median is the Condorcet winner for *every* specification of the underlying (but unknown) complete preferences; the second requirement is deduced from the fact that convexity of preferences itself is an invariant property under affine transformations. The (strict) Tukey median clearly satisfies these two axiomatic requirements, but there are a number of other affinely invariant generalizations of the standard median to multi-dimensions (Rousseeuw and Hubert, 2017).[20] In fact, the statistical literature seems to be undecided as to which generalization is the 'right' or most natural one.

Here, we have singled out the strict Tukey median as the ex-ante Condorcet winners under symmetrically ignorant and quadratic (s.i.q.) beliefs, and these epistemic assumptions are clearly crucial to obtain the result. What happens with different beliefs? Or, in other words, how robust is our justification of the Tukey median? First note that, unless beliefs have a particular structure, it is unlikely that ex-ante Condorcet winners exist at all. Thus, under many other specifications of beliefs, one would have to resort to some Condorcet extension rule (Copeland, Kemeny-Young,

[20]Observe that, e.g., the minimization of the sum of L_p-distances ($p \geq 1$) to the voters' tops is *not* affinely invariant.

minmax, etc.); such extension rules may be exceedingly difficult to analyze and to compute, and therefore one may want to adopt the s.i.q. models as 'false-but-useful' models.

But beyond this mere appeal to pragmatism, one would like to assess the potential risk from epistemic model misspecification more precisely. This is what we address in this section.

C.2.1 Less Informative Beliefs: Hedging Quadraticity

If quadratic preferences or symmetry cannot be taken for granted, it seems sensible for the social evaluator to hedge the commitment to the s.i.q. model by mixing it with the plain convex model. Formally, assume that the epistemic state of the social evaluator is described by a 'mixture' of models, as follows.

For $\beta \in [0,1]$, define the *mixture* of the models Π and Π' by

$$\beta\Pi + (1-\beta)\Pi' \;:=\; \{\beta\pi + (1-\beta)\pi' \mid \pi \in \Pi \text{ and } \pi' \in \Pi'\}\,.$$

Concretely, consider a distribution θ of tops, any s.i.q. model $\Pi \subseteq \Pi^{\theta}_{\text{quad}}$ and the mixture $\beta\Pi + (1-\beta)\Pi^{\theta}_{\text{co}}$. With continuously distributed tops, the effect of this mixing on the outcome selection is clearcut and striking: as long as $\beta > 0$, there is none, i.e. the Tukey median continues to be the normative optimum, as follows.

Proposition 8. *Suppose that θ is continuously distributed with convex support, and let $\Pi \subseteq \Pi^{\theta}_{\text{quad}}$ be any symmetrically ignorant quadratic model. Then, for all $\beta > 0$,*

$$\text{CW}(\beta\Pi + (1-\beta)\Pi^{\theta}_{\text{co}}) \;=\; T^*(\theta) \;=\; T(\theta).$$

To see this, consider two distinct alternatives x and y. The tops of voters who prefer x to y with probability one under the plain convex model are all located on the line through x and y and therefore have mass zero under a continuous distribution; in other words $m^{-}_{\Pi^{\theta}_{\text{co}}}(x,y) = 0$. Thus,

$$\begin{aligned} m^{-}_{\beta\Pi+(1-\beta)\Pi^{\theta}_{\text{co}}}(x,y) &= \beta \cdot m^{-}_{\Pi}(x,y) + (1-\beta)\cdot m^{-}_{\Pi^{\theta}_{\text{co}}}(x,y) \\ &= \beta \cdot m^{-}_{\Pi}(x,y). \end{aligned} \tag{C.1}$$

By (C.1), the mixed model $\beta\Pi+(1-\beta)\Pi^{\theta}_{\text{co}}$ induces the same ex-ante majority relation as the s.i.q model Π, hence also the same ex-ante Condorcet winner, for all $\beta > 0$. Finally, the identity of the Tukey median and the strict Tukey median follows from Theorem $1'$.

In the finite case, an approximate version of Proposition 8 remains valid for tops in general position: Again, for any given x and y, the tops of those voters who prefer x to y with probability one under the convex model are located on the line through x and y. But if tops are in general position, there can be at most two such voters. Thus, under the mixed model, the relative majority margins can change by at most $(1-\beta)\cdot\frac{2}{n}$ as compared to any s.i.q. model. In particular, one needs to add only a small fraction of voters (with appropriate preferences) to a given profile in order to make the Tukey median an ex-ante Condorcet winner in the mixed model. Thus, the Tukey median remains an almost ex-ante Condorcet winner. We will further elaborate on this 'Condorcet gap' criterion in Subsection 5.3 below.

C.2.2 Knowledge of Marginals

Consider now the polar opposite of the Complete Ignorance of Marginals condition above, namely certain knowledge of the individual quadratic forms $\mathcal{Q}_i$ which, by Symmetry of Marginals, must then coincide with some common quadratic form $\mathcal{Q}$. This is a limiting case of our frugal aggregation framework in which the top reveals the entire preference ordering; in fact, the quest for a frugal optimum boils down to a question of standard ordinal aggregation of complete preferences on a restricted domain. If $\mathcal{Q}$ is the unit matrix, we are in the classical spatial model in which preferences are assumed to be Euclidean. (Note that the aggregation problems for general $\mathcal{Q}$ can be reduced to a Euclidean aggregation problem by a change of coordinates via an appropriate affine transformation of the space of alternatives).

In the case of a known common quadratic form $\mathcal{Q}$, welfare optima are naturally obtained as the maxima of the program

$$\arg\max_{x\in X}\sum_{i=1}^{n} f(u_i(x)), \tag{C.2}$$

where f is a common transformation and the $u_i(\cdot)$ are given as in (4.1) with the common quadratic form $\mathcal{Q}$. The common transform f is pinned down by the Condorcet principle adopted here: While it is well-known that Condorcet winners do not exist generically in this setting (McKelvey, 1979), they do exist if all tops are collinear in which case the Condorcet winner coincides with the standard median on a line. This forces f to be the square root function. In the case of Euclidean preferences, this means that the welfare optima minimize the sum of the Euclidean distances to the tops. In general, the utilitarian welfare optimum (C.2) is given by the 'geometric median' with respect to the quadratic form $\mathcal{Q}$. Concretely, for all profiles θ and all quadratic forms $\mathcal{Q}$, let

$$\mathrm{Med}_{\mathcal{Q}}(\theta) \;:=\; \arg\max_{x\in X} \sum_{i=1}^{n} -\sqrt{(x-\theta_i)^T \cdot \mathcal{Q} \cdot (x-\theta_i)}. \tag{C.3}$$

which we refer to as the *geometric $\mathcal{Q}$-median.* The geometric median is another classic multi-dimensional median; see, e.g., Vardi and Zhang (2000) for its basic properties.[21]

The literature has approached the spatial model as an instance of general-purpose ordinal aggregation rules applied to a specific domain of profiles, focusing on different standard Condorcet extension rules. Most prominent among them is the minmax ('Simpson-Kramer') solution, see Kramer (1977); Demange (1982); Caplin and Nalebuff (1988). Remarkably, the minmax solution under Euclidean preferences coincides with the Tukey median, and this equality generalizes to all uniform profiles of quadratic preferences. By consequence, the minmax rule ignores the available non-top preference information entirely (even though it is available) and thus fails to exploit the metric structure of Euclidean resp. uniformly quadratic preference profiles. This neglect of sub-top information arguably reveals a normative deficit of the minmax criterion. At the same time, the agreement of the Tukey median with the minmax criterion in the Euclidean preference case underlines the robustness of the Tukey median as ex-ante solution, in that even with maximally sharpened beliefs (maintaining symmetry), the chosen alternative(s) are still 'respectable' from the conventional perspective if not fully optimal.

C.2.3 Informative Beliefs

Finally, we ask what happens if we relax the Complete Ignorance of Marginals (Assumption 4) only partly but not completely by assuming full knowledge as in the previous subsection. Concretely, in this subsection we consider sets of priors Π such

[21] For three tops in general position, the geometric median coincides with the so-called Fermat-Toricelli point (Krarup and Vajda, 1997).

that the induced set of marginals over quadratic forms is strictly contained in $\Delta(\boldsymbol{\mathcal{Q}})$. In the extreme case, the prior could even be precise, i.e. Π could be a singleton.

Let Π be a set of priors and $x \in X$ any alternative. The **(ex-ante) Condorcet gap** of x at Π is the minimum fraction of voters that need to be added with a suitably extended Π' (associated with suitable known tops) such that x becomes an ex-ante Condorcet winner. The Condorcet gap is a natural measure to assess the risk of misspecification of beliefs from the present Condorcetian perspective. Note that by Theorem 1, the Tukey medians have a Condorcet gap of zero under any symmetrically ignorant quadratic model. Note that in determining the Condorcet gap of an alternative, one can in effect restrict attention to adding voters with top x; the details of the evaluator's beliefs about their preferences do not matter. This observation immediately implies the following result.

Fact C.1. *For all Π satisfying Tops Certainty (Assumption 2) and all $x \in X$, the ex-ante Condorcet gap of x at Π is given by*

$$\max\left\{0, \sup_{y \in X}[m_{\Pi}^{-}(y,x) - m_{\Pi}^{-}(x,y)]\right\}.$$

An immediate consequence of Fact C.1 is that, if the complete preference profile is known, the Condorcet gap of any x is simply given by the maximal opposition against x. Thus, the standard minmax solution minimizes the Condorcet gap in this case. Since the Tukey median coincides with the minmax solution on all uniform profiles of quadratic preferences, the Tukey median also minimizes the Condorcet gap in these cases. More generally, we have the following result which holds for general probability measures θ, discrete or continuous. Note that, due to the involved normalization, all magnitudes have then to be interpreted as fractional magnitudes; for instance, $\mathfrak{d}(x;\theta)$ is the minimal *fractional* mass of all half-spaces containing x, etc. For convenience, and since no confusion can arise, we do not distinguish this interpretation notationally.

Proposition 9. *For any probability measure θ, and all Π satisfying Assumptions 1 – 3 of Section 4, the Condorcet gap of a Tukey median is at most*

$$1 - 2 \cdot \mathfrak{d}^*(\theta).$$

Proof. Consider any symmetric prior π of profiles of quadratic preferences. By symmetry, we can express the prior as a product measure $\mu_{\mathcal{Q}} \times \mu_\theta$. Consider a Tukey median x and any other alternative y. Using Lemma B.2 and the fact that x maximizes the Tukey depth, for any quadratic form $\mathcal{Q}$, the support of x against y is at least $\mathfrak{d}^*(\theta)$, thus the maximal opposition against x is $1-\mathfrak{d}^*(\theta)$ if $\mathcal{Q}_i = \mathcal{Q}$ for all voters i. This shows that the Condorcet gap of x is at most $1 - 2\mathfrak{d}^*(\theta)$ whenever $\mathcal{Q}_i = \mathcal{Q}$ for all voters i. Now let

$$\mu'_\theta := (1 - 2\mathfrak{d}^*(\theta)) \cdot \delta_x + \mu_\theta,$$

where δ_x is the Dirac measure putting all mass on the alternative x, as well as $\mu' := \mu_{\mathcal{Q}} \times \mu'_\theta$. For any $\mathcal{Q}$ in the support of $\mu_{\mathcal{Q}}$, x is a (weak) Condorcet winner at $\delta_{\mathcal{Q}} \times \mu'_\theta$, hence by reinforcement x is a weak Condorcet winner at μ'. Since this holds for all symmetric priors $\pi \in \Pi$, the Condorcet gap of x at Π is also bounded by $1-2\mathfrak{d}^*(\theta)$. □

By Proposition 9, the Tukey median remains an approximate optimum also without the Complete Ignorance assumption in the sense that often only a 'few' voters (with appropriate preferences) have to be added to a profile in order to make the Tukey median an ex-ante Condorcet winner. In general, the value of $\mathfrak{d}^*(\theta)$ is bounded below by $1/(L+1)$ (Donoho and Gasko, 1992), hence the for symmetric priors of quadratic preferences the Condorcet gap is always at most $1 - \frac{2}{L+1}$. However, many multi-variate distributions used in applications have additional structure that improves this bound considerably. In particular, as observed by Caplin and Nalebuff (1988), the maximal Tukey depth of the large class of log-concave distributions is bounded below by $1/e$. Thus, if voters' tops are sufficiently 'bunched together' in the sense that θ is log-concave, then the Condorcet gap of a Tukey median at any set of symmetric priors of quadratic preferences is at most $1 - 2/e \approx 0.264$.

The Tukey median is also robust in the sense of the Condorcet gap criterion if one adds a little 'precision' to the evaluators beliefs: for any s.i.q. model Π and every precise prior π the Condorcet gap of the Tukey median at the mixed model $\alpha\Pi + (1-\alpha)\pi$ is at most $(1-\alpha)(1-2\mathfrak{d}^*(\theta))$.

While the Tukey median is thus remarkably robust against misspecification of beliefs, we finally note that this does not generally hold for other choice rules. For instance, the Condorcet gap of the plurality winner at a s.i.q. model Π can be arbitrarily close to one (as can be verified by looking at situations of sort considered in Example 1). It is thus not the ex-ante Condorcet approach as such that delivers the robustness, but the ex-ante Condorcet approach *in combination* with a specific belief model.

References

Arrow, K. J. (1951/63). *Social Choice and Individual Values.* New York: Wiley.

— (1960). Decision theory and the choice of a level of significance for the t-test. In I. Olkin, S. G. Churye, W. Hoeffding, W. Madow and H. B. Mann (eds.), *Contributions to Probability and Statistics: Essays in Honor of Harold Hotelling Theories*, Stanford University Press, pp. 70–78.

Austen-Smith, D. and Banks, J. (1999). *Positive Political Theory I: Collective Preference*. Ann Arbor: Michigan University Press.

Aziz, H. and Shah, N. (2020). Participatory budgeting: Models and approaches. In Rudas and Gabor (eds.), *Pathways between Social Sciences and Computational Social Science: Theories Methods and Interpretations*, Springer.

Barberà, S., Massó, J. and Neme, A. (1997). Voting under constraints. *Journal of Economic Theory*, **76**, 298–321.

—, Sonnenschein, H. and Zhou, L. (1991). Voting by committees. *Econometrica*, **59**, 595–609.

Bernardo, J. and Smith, A. (1994). *Bayesian Theory.* Chichester: Wiley.

Black, D. S. (1948). On the rationale of group decision-making. *J. Political Economy*, **56**, 23–34.

Boutilier, C. and Rosenschein, J. S. (2016). Incomplete information and communication in voting. In F. Brandt, V. Conitzer, U. Endriss, J. Lang and A. Procaccia (eds.), *Handbook of Computational Social Choice*, *10*, Cambridge: Cambridge University Press, pp. 223–257.

Caplin, A. and Nalebuff, B. (1988). On 64% majority rule. *Econometrica*, **56**, 787–814.

Chatterji, S. and Sen, A. (2011). Tops-only domains. *Economic Theory*, **46**, 255–282.

DE FINETTI, B. (1931). Funzione caratteristica di un fenomeno aleatoria. *Atti della R. Academia Nazionale dei Lincei, Serie 6. Memorie, Classe die Scienze Fisiche, Mathematice e Naturale*, **4**, 251–299.

DEMANGE, G. (1982). A limit theorem on the minmax set. *Journal of Mathematical Economics*, **9**, 145–164.

DONOHO, D. L. and GASKO, M. (1992). Breakdown properties of location estimates based on halfspace depth and projected outlyingness. *Annals of Statistics*, **20**, 1803–1827.

DOWNS, A. (1957). *An Economic Theory of Democracy.* New York: Harper.

FREEMAN, R., PENNOCK, D. M., PETERS, D. and VAUGHAN, J. W. (2021). Truthful aggregation of budget proposals. *Journal of Economic Theory*, **193**, published online.

GERSHKOV, A., MOLDOVANU, B. and SHI, X. (2019). Voting on multiple issues: What to put on the ballot. *Theoretical Economics*, **14**, 555–596.

—, — and — (2020). Monotonic norms and orthogonal issues in multidimensional voting. *Journal of Economic Theory*, **189**, 105103.

GIGERENZER, G. and GOLDSTEIN, D. G. (1996). Reasoning the fast and frugal way: Models of bounded rationality. *Psychological Review*, **104**, 650–669.

GOODIN, R. and LIST, C. (2006). A conditional defense of plurality rule: Generalizing May's theorem in a restricted informational environment. *American Journal of Political Science*, **50**, 940–949.

GRANDMONT, J.-M. (1978). Intermediate preferences and the majority rule. *Econometrica*, **46**, 317–330.

KALAI, E., MULLER, E. and SATTERTHWAITE, M. A. (1979). Social welfare functions when preferences are convex, strictly monotonic, and continuous. *Public Choice*, **34**, 87–97.

KONCZAK, K. and LANG, J. (2005). Voting procedures with incomplete preferences. In *Proceedings of the Multidisciplinary IJCAI-05 Workshop on Advances in Preference Handling*, Palo Alto, CA: AAAI, pp. 124–129.

KRAMER, G. H. (1977). A dynamical model of political equilibrium. *Journal of Economic Theory*, **16**, 310–334.

KRARUP, J. and VAJDA, S. (1997). On Torricelli's geometrical solution to a problem of Fermat. *Journal of Management Mathematics*, **8**, 215–224.

LANCASTER, K. J. (1966). A new approach to consumer theory. *Journal of Political Economy*, **74**, 132–157.

LE BRETON, M. and WEYMARK, J. A. (2011). Arrovian social choice theory on economic domains. In K. J. Arrow, A. Sen and K. Suzumura (eds.), *Handbook of Social Choice and Welfare, Volume 2*, Amsterdam: North-Holland, pp. 191–299.

LIU, R. (1990). On a notion of data depth based on random simplices. *Annals of Statistics*, **18**, 405–414.

LU, T. and BOUTILIER, C. (2011). Robust approximation and incremental elicitation in voting protocols. In *Proceedings of the 22nd IJCAI*, Palo Alto, CA: AAAI, pp. 287–293.

LUCE, R. and RAIFFA, H. (1957). *Games and Decisions*. New York: Wiley.

MCKELVEY, R. (1979). General conditions for global intransitivities in formal voting models. *Econometrica*, **47** (5), 1085–1112.

MILNOR, J. (1954). Games against nature. In R. M. Thrall, C. H. Coombs and R. L. Davis (eds.), *Decision Processes*, Wiley, pp. 49–59.

NEHRING, K. (2000). A theory of rational choice under ignorance. *Theory and Decision*, **48**, 205–240.

— (2009). Imprecise probabilistic beliefs as a context for decision-making under ambiguity. *Journal of Economic Theory*, **144**, 1054–1091.

— and PUPPE, C. (2007a). Efficient and strategy-proof voting rules: A characterization. *Games and Economic Behavior*, **59**, 132–153.

— and — (2007b). The structure of strategy-proof social choice I: General characterization and possibility results on median spaces. *J.Econ.Theory*, **135**, 269–305.

— and — (2010). Abstract Arrowian aggregation. *J.Econ.Theory*, **145**, 467–494.

— and — (2022). Condorcet solutions in frugal models of budget allocation. *KIT Working Paper Series in Economics # 156*.

Oja, H. (1983). Descriptive statistics for multivariate distributions. *Statistics and Probability Letters*, **1**, 327–332.

Plott, C. (1967). A notion of equilibrium and its possibility under majority rule. *American Economic Review*, **57**, 787–806.

Rousseeuw, P. J. and Hubert, M. (2017). Computation of robust statistics: Depth, median, and related measures. In *Handbook of Discrete and Computational Geometry*, Boca Raton: CRC Press.

Shah, A. (2007). *Participatory Budgeting*. Washington D.C.: The World Bank.

Small, C. G. (1990). A survey of multidimensional medians. *International Statistical Review*, **58**, 263–277.

Tukey, J. W. (1975). Mathematics and the picturing of data. In *Proceedings of the ICM*, Vancouver: Canadian Log Builder Assoc., pp. 523–531.

Vardi, Y. and Zhang, C.-H. (2000). The multivariate l_1–median and associated data depth. *Proc. Nat. Acad. Sci.*, **97**, 1423–1426.

Weber, A. (1909). *Über den Standort der Industrien. Erster Teil: Reine Theorie des Standorts*. Tübingen: Mohr.

Zhou, L. (1991). Impossibility of strategy-proof mechanisms for economies with pure public goods. *Review of Economic Studies*, **58**, 107–119.

Zuo, Y. (2003). Projection based depth functions and associated medians. *Annals of Statistics*, **31**, 1460–1490.