Condorcet Domains

Clemens Puppe Arkadii Slinko

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1 Introduction

Social Choice Theory investigates how collective decisions are, and ought to be, taken if the members of a group, or a society, have conflicting tastes and preferences. The two cornerstone results, Arrow's Impossibility Theorem and the Gibbard–Satterthwaite Theorem are both negative.

Arrows Impossibility Theorem (Arrow, 1951) shows a fundamental difficulty in aggregating individual preferences into a collective ranking. Under a set of seemingly reasonable conditions – such as unrestricted domain of preferences, unanimity, and independence of irrelevant alternatives – no voting rule can always produce a consistent social ranking while avoiding dictatorship. The theorem highlights an inherent tension between individual rationality and collective decision-making. The related Gibbard–Satterthwaite Theorem (Gibbard, 1973; Satterthwaite, 1975) reaches a similarly striking conclusion for voting rules: when there are at least three alternatives, any non-dictatorial voting rule must be manipulable on an unrestricted domain of preferences. In particular, there always exist preference constellations in which some voters have an incentive to misrepresent their true preferences.

These impossibility results raise a natural question: can the negative conclusions be avoided by restricting the set of preference profiles under consideration? Condorcet domains address precisely this question. A Condorcet domain is a collection of preference orders for which pairwise majority voting is guaranteed to produce a transitive social ordering, thereby avoiding the majority cycles that can arise under unrestricted preferences. Pairwise majority voting satisfies Arrow's other two conditions (unanimity and independence of irrelevant alternatives) and thus provides an escape from the impossibility if all voters' preferences come from a Condorcet domain. Moreover, pairwise majority voting represents a non-dictatorial and non-manipulable ('strategy-proof') voting rule on any Condorcet domain.

Studying Condorcet domains therefore provides a way to identify conditions under which collective choice can remain coherent, while still allowing for meaningful variation in individual preferences. In the following, we highlight some interesting facts and results; for a full treatment of the topic, see Puppe and Slinko (2026).

2 Basic Facts on Condorcet Domains

It is usually assumed that a voter in an election can order candidates from best to worst, mathematically, this is modeled as a linear order. For simplicity we assume that the number of voters is odd. In such a case if the society is offered to compare two candidates, say a and b, either one or the other will be preferred by a majority of voters. If a is collectively preferred to b we denote this $a \succ b$ and call $\succ$ the *majority relation*.

The so-called 'Condorcet paradox' is the observation that the majority relation may not be a linear order since the condition of transitivity may fail. On the following picture, we present two groups of three voters whose preferences are shown as columns (the higher the candidate the more she is preferred)

1	2	3
a	b	c
b	c	a
c	a	b

1	2	3
a	b	c
c	a	b
b	c	a

For the first group the majority relation is

$$a \succ b \succ c \succ a$$

and

$$a \succ c \succ b \succ a$$

for the second. In particular, neither of the two groups will be able to identify the best candidate for the group according to majority rule. A candidate who will defeat any other candidate in a race between the two of them is said to be a *Condorcet winner*. As we just saw such a candidate does not always exist.

The set of linear orders belonging to voters of a particular society is called the *preference profile* of this society. Let A be a finite set of alternatives and $\mathcal{L}(A)$ be the set of all linear orders on A. It is often the case that the set of potential preferences of the society $\mathcal{D}$ is structured in some way so that some opinions cannot be expressed (examples will be presented later). This means $\mathcal{D}$ can be a proper subset of $\mathcal{L}(A)$. A subset $\mathcal{D} \subseteq \mathcal{L}(A)$ is called a *Condorcet domain* if every profile composed from preferences belonging to $\mathcal{D}$ has an acyclic majority relation, and hence a Condorcet winner if the number of voters is odd. A Condorcet domain $\mathcal{D}$ is *closed* if it contains the majority relation corresponding to any profile composed of an odd number of elements of $\mathcal{D}$. A Condorcet domain $\mathcal{D}$ is *maximal* if it is not contained in a larger Condorcet domain.

As we saw the two groups of three voters have all six linear orders on the set of alternatives $A = \{a, b, c\}$. If we want to extract a Condorcet domain from $\mathcal{L}(\{a, b, c\})$, we have to remove one linear order from each group. There are three essentially different ways to do so:

$$\begin{aligned} \mathcal{D}_1 &= \{abc,\ acb,\ cab,\ cba\}, \\ \mathcal{D}_2 &= \{abc,\ acb,\ bca,\ cba\}, \\ \mathcal{D}_3 &= \{abc,\ bac,\ bca,\ cba\}. \end{aligned}$$

It can be shown that these three are indeed Condorcet domains. To understand the difference between them we must introduce a bit of geometry.

A linear order v is *between* v' and v'' (Kemeny, 1959) if $v \supseteq v' \cap v''$, i.e., whenever v' and v'' agree on a ranking of a certain pair of alternatives, v also agrees with them. Given a domain $\mathcal{D}$ and two linear orders $v, v' \in \mathcal{D}$, the set $[v', v'']_{\mathcal{D}}$ of all linear orders from $\mathcal{D}$ which are between v' and v'' is called an *interval*. Two orders v' and v'' from a domain $\mathcal{D}$ are *neighbours* if there is nothing between them, i.e.,

$$[v', v'']_{\mathcal{D}} = \{v', v''\}.$$

Definition 1. *The* graph $\Gamma_{\mathcal{D}}$ associated with $\mathcal{D}$ *is the graph whose vertices are linear orders from* $\mathcal{D}$ *and edges are drawn between all pairs of neighbours.*

The graph associated with the universal domain is the edge-graph of the polytope called *permutahedron*. In particular, the graph associated with the universal domain $\mathcal{L}(A)$ for $|A| = 3$, it is a hexagon. The graphs of the three maximal Condorcet domains $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ on A are, respectively:

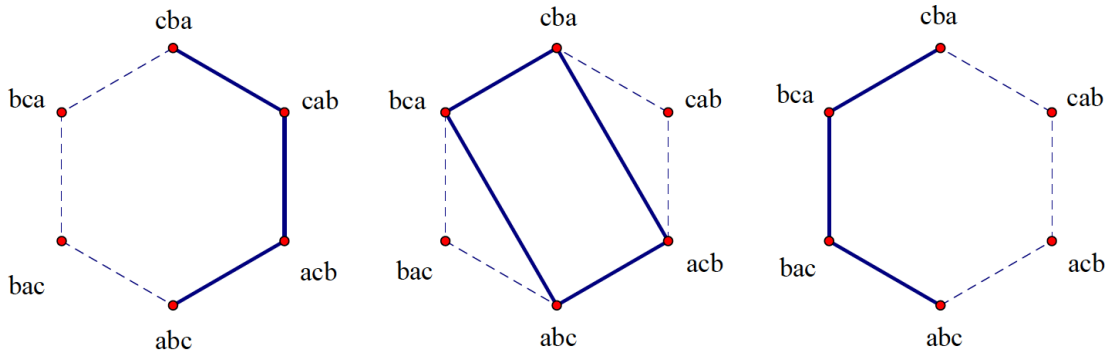


Figure 1: Graphs of $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$

$\Gamma_{\mathcal{D}_1}$ and $\Gamma_{\mathcal{D}_3}$ are subgraphs of the permutahedron; such domains are said to be *connected in the permutahedron*, or simply *connected*. Note that $\Gamma_{\mathcal{D}_2}$ is not connected in this sense.

Now we have to define *median graphs* (Mulder, 1978) which sit at the intersection of graph theory, topology, and metric geometry. The definition of a median graph is deceptively simple. A graph is *median* if for any three vertices a, b, c, there exists exactly one vertex m that lies simultaneously on a shortest path between a and b, between b and c, and between c and a. This vertex m is the 'median' of a, b, c, serving as the graph-theoretic analogue of a geometric centre.

Intuitively, by looking at median graphs, one is studying the *geometry of consensus*, and it is no coincidence that median graphs play an important role in the theory of Condorcet domains (Puppe and Slinko, 2019).

Theorem 2.1. **(a)** *For any closed Condorcet domain* $\mathcal{D}$ *the associated graph* $\Gamma_{\mathcal{D}}$ *is a median graph. Moreover, the majority relation of any triple of orders in* $\mathcal{D}$ *is given by their median with respect to* $\Gamma_{\mathcal{D}}$.

(b) *Every median graph can be obtained as $\Gamma_{\mathcal{D}}$ for some Condorcet domain $\mathcal{D}$.*

Let us now turn our attention to necessary and sufficient conditions that help us to recognise Condorcet domains. Observing our three domains above we note:

- $\mathcal{D}_1$ contains all linear orders on a, b, c where b is never ranked first,
- $\mathcal{D}_2$ contains all linear orders on a, b, c where a is never ranked second, and
- $\mathcal{D}_3$ contains all linear orders where b is never ranked last.

Following Monjardet (2009)we denote these conditions for a triple $\{a, b, c\}$ as $bN_{\{a,b,c\}}1$, $aN_{\{a,b,c\}}2$ and $bN_{\{a,b,c\}}3$, and call them *never-top*, *never-middle* and *never-bottom* conditions, respectively (Fishburn, 1996). They are equivalent to Sen's value restrictions (Sen, 1966) or Ward's 'absence of a Latin square' (Ward, 1965).

Theorem 2.2. *A domain $\mathcal{D} \subseteq \mathcal{L}(A)$ is Condorcet if and only if for every triple $a, b, c \in A$ at least one of the never-conditions is satisfied.*

3 Types of Condorcet domains

3.1 Peak-pit domains

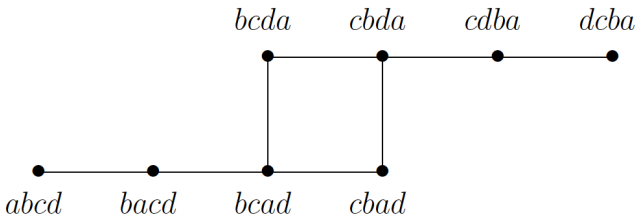
Black (1948) introduced the domain of 'single-peaked preferences', as follows. A preference $\succ$ is *single-peaked* with respect to a linear order $>$ on A if there exists an alternative a^* (the 'peak' of $\succ$) such that

$$\begin{aligned} b < c \leq a^* &\Rightarrow c \succ b, \\ a^* \geq b > c &\Rightarrow b \succ c. \end{aligned}$$

A domain $\mathcal{D} \subseteq \mathcal{L}(A)$ of preferences on A is called a *single-peaked domain* if there exists a linear order $>$ on A such that all preferences in $\mathcal{D}$ are single-peaked with respect to $>$. Such domains arise naturally, for example, when modelling political preferences on the traditional political left-right spectrum.

Theorem 3.1 (Black, 1948)**.** *Any maximal domain of single-peaked preferences is a maximal Condorcet domain.*

Here is the graph of the domain of single-peaked preferences on the set of alternatives $A = \{a, b, c, d\}$.



Among $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ only $\mathcal{D}_3$ is single-peaked.

Another frequently useful sufficient condition for transitivity of the majority relation is the single-crossing property. A domain is said to be *single-crossing* if its linear orders can be put in a sequence so that along this sequence the relative positions of any pair of alternatives is reversed at most once.

Roberts (1977) and Gans and Smart (1996) provide a number of economic applications of single-crossingness. For example, in Roberts' seminal paper voters' preferences on the level of taxation depend solely on their income: the lower the income the higher taxation this individual prefers. While preferences in this setting can fail to be single-peaked, Roberts showed they satisfy the single-crossing property, which restores the median voter result.

Example 1. *Domain $\mathcal{D} = \{abc, bac, bca, cba\}$ is single-crossing since we can put these orders in a sequence as shown on Figure 2.*

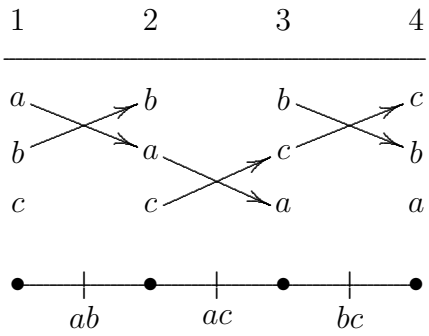


Figure 2: An example of a single-crossing domain

The change in positions of a and b occurs when moving from the first order to the second, then we have the change of positions of a and c and finally the change in positions of b and c.

The graph of a single-crossing domain is especially simple: it is a path.

We say that a domain $\mathcal{D}$ has the *representative voter property* if any odd profile composed of linear orders from $\mathcal{D}$ contains a representative voter whose preferences coincide with the majority relation. Single-crossingness is sufficient and 'almost' necessary for the representative voter property to hold.

Theorem 3.2. *Let $\mathcal{D} \subseteq \mathcal{L}(A)$ be a domain of linear orders on A. Then, $\mathcal{D}$ has the representative voter property if and only if $\mathcal{D}$ is either a single-crossing domain, or $\mathcal{D}$ is a closed Condorcet domain with exactly four elements such that the associated graph $\Gamma_{\mathcal{D}}$ is a 4-cycle.*

Both single-peaked and single-crossing domains are the so-called *peak-pit domains* which means that they can be defined by means of never-top and never-bottom conditions only.

Proposition 3.1. *All single-peaked and single-crossing domains are 'peak-pit' domains.*

Besides the single-peaked and single-crossing domains, the best-known peak-pit domains are the so-called 'Fishburn domains.' They are defined by the so-called alternating scheme of never conditions. There are two types of them.

Definition 2. *Let* $A = \{1, 2, \ldots, n\}$. *A domain is said to satisfy the* alternating scheme *of never conditions if,*

(i) for all $1 \leq i < j < k \leq n$, $jN_{\{i,j,k\}}1$, *if* j *is even, and* $jN_{\{i,j,k\}}3$, *if* j *is odd, or*

(ii) for all $1 \leq i < j < k \leq n$, $jN_{\{i,j,k\}}3$, *if* j *is even, and* $jN_{\{i,j,k\}}1$, *if* j *is odd .*

Following Galambos and Reiner (2008) we denote these domains as F_n and $\overline{F_n}$, respectively. The second domain is flip-isomorphic to the first which means that all orders of F_n are reversed. In particular, $F_2 = \{12, 21\}$, $F_3 = \{123, 132, 312, 321\}$ and

$$F_4 = \{1234, 1324, 3124, 1342, 3142, 3412, 3421, 4312, 4321\}.$$

The following picture shows how this domain is located in the permutahedron.

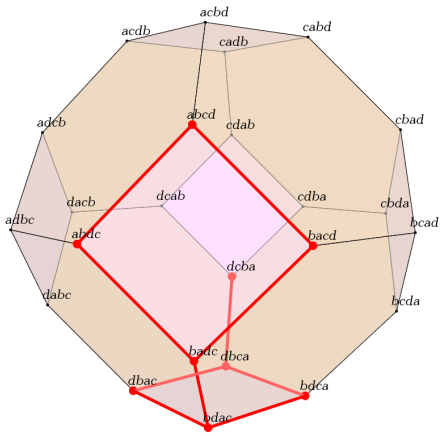


Figure 3: Fishburn's domain F_4 embedded in the permutahedron

The latter has the following median graph associated with it:

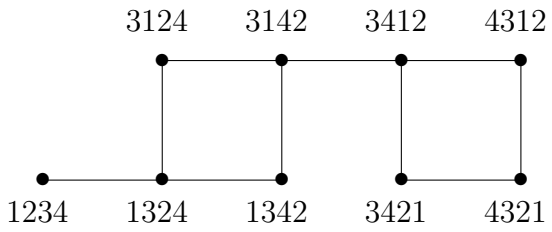


Figure 4: The graph of Fishburn's domain F_4 on four alternatives

Note that F_4 has cardinality 9 and is in fact the (uniquely) largest Condorcet domain on a set of four alternatives (Raynaud, 1982). Fishburn domains are indeed among the largest for an arbitrary number of alternatives. Galambos and Reiner (2008) gave the exact formula for the cardinality of F_n:

$$|F_n| = (n+3)2^{n-3} - \begin{cases} (n-\frac{3}{2})\binom{n-2}{\frac{n}{2}-1} & \text{for even } n; \\ (\frac{n-1}{2})\binom{n-1}{\frac{n-1}{2}} & \text{for odd } n. \end{cases}$$

The following table lists the first 16 values of $|F_n|$:

2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	4	9	20	45	100	222	488	1069	2324	5034	10840	23226	49704	105884

Well-known Condorcet domains which are not peak-pit domains are the so-called group separable domain. They belong to the class of the 'never-middle' domains since they can be defined using never-middle condition only.

3.2 Never-middle or symmetric domains

Maximal never-middle domains are symmetric which means that if a linear order belongs to such a domain then the reversed order also belongs to it. Historically the first example of never-middle domains were introduced by Inada (1964).

Definition 3. *A domain $\mathcal{D}$ on the set A of alternatives is called* group separable *if every subset $B \subseteq A$ with at least two elements can be partitioned into two non-empty subsets B' and B'' such that every order in $\mathcal{D}$ either (i) ranks all elements of B' above all elements in B'', or (ii) ranks all elements of B'' above all elements in B'.*

Let $\mathcal{D}_1, \mathcal{D}_2$ be two Condorcet domains on two disjoint sets of alternatives. Danilov and Koshevoy (2013) introduced an operation of composition the operation

$$\mathcal{D}_1 \star \mathcal{D}_2 \ := \ \{\mathbf{u}_1\mathbf{u}_2 \mid \mathbf{u}_1 \in \mathcal{D}_1, \ \mathbf{u}_2 \in \mathcal{D}_2\} \cup \{\mathbf{u}_2\mathbf{u}_1 \mid \mathbf{u}_1 \in \mathcal{D}_1, \ \mathbf{u}_2 \in \mathcal{D}_2\}$$

We note that the bipartition of a domain $\mathcal{D}$ in the sense of Raynaud (1981) is a representation of $\mathcal{D}$ as a composition of two domains $\mathcal{D} = \mathcal{D}_1 \star \mathcal{D}_2$. This composition allowed

Danilov and Koshevoy to construct a series of never-middle maximal Condorcet domains, namely,

$$a_1 \star a_2 \star a_3 \star \ldots \star a_n$$

with some parenthesisation, where a_i is identified with the trivial domain on a single alternative a_i. These are exactly the group separable domains.

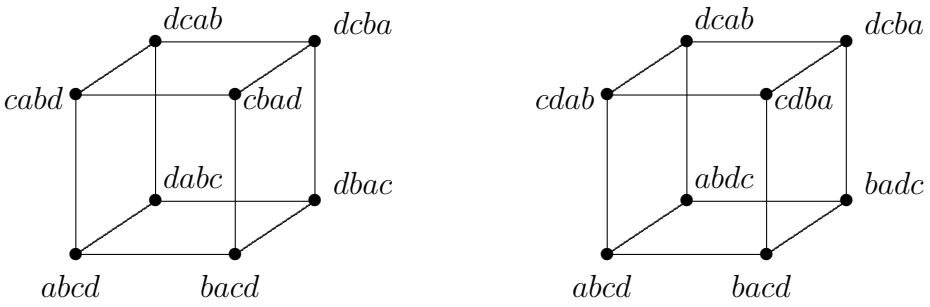


Figure 5: The graphs of $((a \star b) \star c) \star d$ and $(a \star b) \star (c \star d)$ are the same although the domain are not isomorphic.

Danilov and Koshevoy also discovered a surprising family of maximal never-middle Condorcet domains of cardinality just four! We call them *Raynaud domains* as Raynaud (1981) was the first who discovered such a domain in case of four alternatives

$$K = \{1234, 4321, 2413, 3142\}$$

He called it *configuration K*. Danilov and Koshevoy suggested such domains for any number of alternatives. Let $e = 12 \ldots n$ and $x = x_1 x_2 \ldots x_n$ be an arbitrary order. Then $K(x) = \{e, \bar{e}, x, \bar{x}\}$, where $\bar{e}$ and $\bar{x}$ are the reverses of e and x, respectively, is a Condorcet domain (but, of course, not necessarily a maximal one).

Example 2 (Danilov and Koshevoy (2013))**.** *Let* $A = \{1, 2, \ldots, m\}$ *and define*

$$d_m = 24 \cdots (2k)1(2k \pm 1) \cdots 53, \tag{1}$$

where $2k \pm 1$ *is equal to* $2k + 1 = m$*, if* m *is odd, and* $2k - 1 = m - 1$ *if* m *is even. Then* $K(d_m)$ *is a maximal Condorcet domain. For example,* $d_4 = 2413$*,* $d_5 = 24153$*,* $d_6 = 246153$ *and* $d_7 = 2461753$*.*

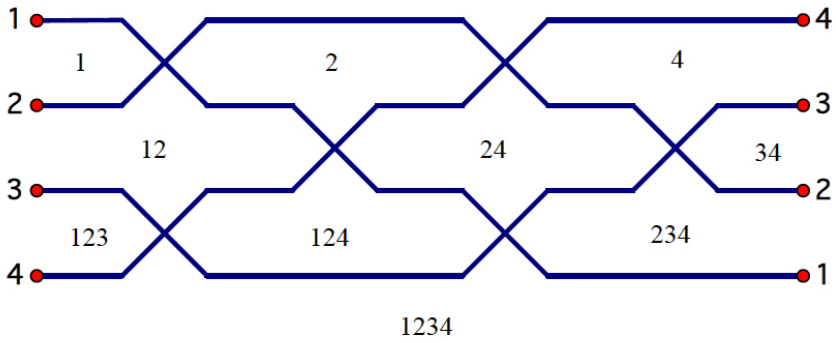
Karpov and Slinko (2023) characterise Raynaud domains by means of simple permutations which is a well-known object in combinatorics. We hypothesise that Raynaud domains are the only non-trivial symmetric indecomposable maximal domains.

4 Combinatorial Representation of peak-pit domains

A *pseudoline* is the graph of a continuous function $y = f(x)$ defined for all real numbers x. By itself it is not an interesting object but several pseudolines taken together in what is

called an 'arrangement of pseudolines' are. The golden rule is: any two pseudolines intersect exactly once. Then they allow us to represent pure combinatorial crossing patterns (Levi, 1926; Grunbaum, 1972).

Fishburn's domain can be represented by an arrangement of numbered pseudolines:



As we can observe, intersecting pseudolines create *chambers*. These chambers are given labels which indicate the lines that go above that chamber. The upper infinite chamber has no label since no line goes above it; the lower infinite chamber has all lines above it. If we walk from the upper infinite chamber to the lower one intersecting each pseudoline exactly once and noticing the labels of chambers which we passed, we obtain an increasing sequence of labels, e.g., $\{4\} \subset \{2,4\} \subset \{1,2,4\} \subset \{1,2,3,4\}$; any such sequence corresponds to a linear order – in our case to 4213. All orders of Fishburn's domain can be obtained in this way.

Another method of representing Condorcet domains is by using rhombus tilings of a regular polygon (Danilov et al., 2012). A *rhombus tiling* (or simply a tiling) is a subdivision into rhombic tiles of a regular $2n$-gon formed by the points $\sum_i a_i\psi_i$, where $0 \le a_i \le 1$ and $\psi_1, \ldots, \psi_n$ are the unit vectors in the upper half-plane. This centre-symmetric $2n$-gon has its bottom vertex b at the origin and the top vertex $t = \psi_1 + \ldots + \psi_n$. An ij-tile is a rhombus congruent to the one formed by the points $\lambda\psi_i + \mu\psi_j$, where $0 \le \lambda, \mu \le 1$. A *snake* is a path from t to b along the boundaries of the tiles which for each $i = 1, \ldots, n$ contains a unique segment parallel to ψ_i. Each snake corresponds to a linear order on $\{1, \ldots, n\}$ in the following way. If a point traveling from t to b passes segments parallel to $\psi_{i_1}, \psi_{i_2} \ldots, \psi_{i_n}$, then the corresponding linear order will be $i_1 i_2 \ldots i_n$. The set of snakes of a rhombus tiling, thus, defines a domain which is called a *tiling domain*.

For $n = 3$ we have a hexagon and it can be split into rhombus tiles in two different ways shown on a Figure 6.

These lead to the following domains:

$$\{123, 213, 231, 321\}, \quad \{123, 132, 312, 321\}$$

which are the familiar to us single-peaked and single-dipped domains, respectively.

Tilings and arrangements of pseudolines are dual objects and there is a one-to-one correspondence between them. Given an arrangement we mark a vertex in each chamber (including both unbounded ones). Two vertices are connected by an edge if their

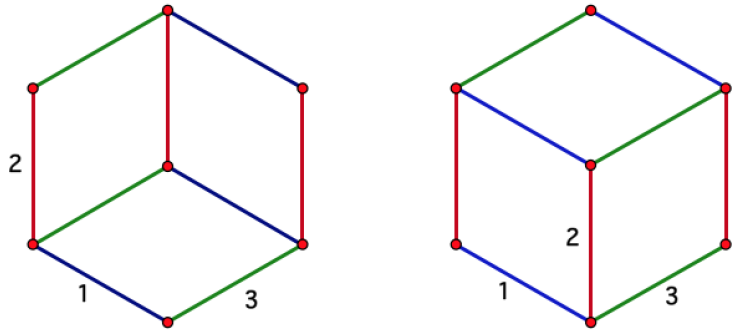


Figure 6: Two tiling domains

corresponding chambers have a common piece of pseudoline. The emerging graph can be shown to be transformed into a rhombus tiling (Elnitsky, 1997; Felsner, 2012). For example, starting from an arrangement of pseudolines corresponding to $\bar{F}_4$ on Figure 7 (one of Fishburn's domains) we obtain a rhombus tiling for this domain.

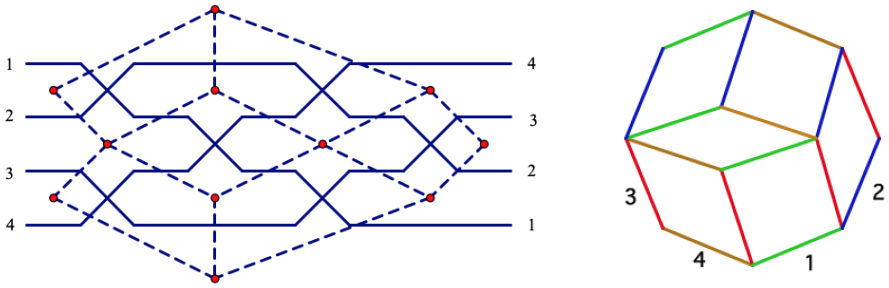


Figure 7: An arrangement of pseudolines and its dual tiling

It is not hard to see that the snakes of a tiling domain correspond to the sequences of chambers in the corresponding arrangement.

A domain is said to have *maximal width* if it contains two reversed orders, e.g., $12\ldots n$ and $n\ldots 21$.

Theorem 4.1. *Peak-pit domains of maximal width are exactly the maximal Condorcet domains that can be represented by rhombus tilings or arrangements of pseudolines.*

5 Arrovian aggregation and strategy-proof social choice on Condorcet domains

Let us demonstrate the usefulness of Condorcet domains in economic applications. We present a characterisation of the class of all monotone Arrovian aggregators on any Con-

dorcet domain, and show that every Condorcet domain admits a large class of strategy-proof social choice functions beyond the choice of the Condorcet winner (Puppe and Slinko, 2019).

Definition 4. *A mapping $f : \mathcal{D}^n \longrightarrow \mathcal{D}$ is called an* Arrovian aggregator *on $\mathcal{D}$ if it satisfies the following conditions.*

(i) (Unanimity) For all $\succ \in \mathcal{D}$, $f(\succ, ..., \succ) = \succ$, and

(ii) (Independence) For any pair of profiles $(\succ_1, ..., \succ_n), (\succ'_1, ..., \succ'_n)$ from $\mathcal{D}$ and all $x, y \in A$, if $x \succ y$ and, for all $i = 1, .., n$, $[x \succ_i y \Leftrightarrow x \succ'_i y]$, then $x \succ' y$, where $\succ = f(\succ_1, ..., \succ_n)$ and $\succ' = f(\succ'_1, ..., \succ'_n)$.

A mapping is called a monotone *Arrovian aggregator if it satisfies in addition the following condition.*

(iii) (Monotonicity) For all $\succ_1, ..., \succ_n, \succ'_i \in \mathcal{D}$ and all $x, y \in A$, if $x \succ' y$ where $\succ' = f(\succ_1, ..., \succ'_i, ..., \succ_n)$ and $x \succ_i y$, then $x \succ y$ where $\succ = f(\succ_1, ..., \succ_i, ..., \succ_n)$.

In the following, let $N := \{1, ..., n\}$ denote the set of voters. For every ordered pair $(x, y) \in A \times A$ of distinct alternatives, let $\mathcal{W}_{xy}$ be a non-empty collection of non-empty subsets $W \subseteq N$ of voters (the 'winning coalitions for x against y) satisfying $[W \in \mathcal{W}_{xy}$ & $W' \supseteq W] \Rightarrow W' \in \mathcal{W}_{xy}$.

Definition 5. *A collection $\mathcal{W} = \{\mathcal{W}_{xy} \mid (x, y) \in A \times A, x \neq y\}$ is called a* structure of winning coalitions *if, for all $x \neq y$, $W \in \mathcal{W}_{xy} \Leftrightarrow (N \setminus W) \notin \mathcal{W}_{yx}$. A structure of winning coaltions is called* order preserving *if, for all distinct pairs $x, y \in A$ and all distinct pairs $z, w \in A$,*

$$\{\succ \in \mathcal{D} \mid x \succ y\} \subseteq \{\succ \in \mathcal{D} \mid z \succ w\} \implies \mathcal{W}_{xy} \subseteq \mathcal{W}_{zw}.$$

In other words, order preservation requires that, if every potential voter who ranks x above y also ranks z above w, then every winning coalition for x over y must also be winning for z over w.

Based on a general possibility result on aggregation on median spaces Nehring and Puppe (2007), the following characterisation can be obtained.

Theorem 5.1. *Let $\mathcal{D}$ be a closed Condorcet domain and let $\mathcal{W}$ by any order preserving structure of winning coalitions. For all profiles $(\succ_1, ..., \succ_n) \in \mathcal{D}^n$ there exists a unique order $\succ^{\mathcal{W}} \in \mathcal{D}$ such that, for all $x, y \in A$,*

$$x \succ^{\mathcal{W}} y \iff \{i \in N \mid x \succ_i y\} \in \mathcal{W}_{xy}. \tag{2}$$

This aggregator is a monotone Arrovian aggregator. Conversely, every monotone Arrovian aggregator takes this form for some order preserving structure of winning coalitions $\mathcal{W}$.

Observe that pairwise majority voting with an odd number n of voters corresponds to the structure of winning coalitions for which all $\mathcal{W}_{xy}$ consist of all coalitions of at least half of the voters. Since in that case $\mathcal{W}_{xy} = \mathcal{W}_{zw}$ for any two pairs (x, y) and (z, w) of distinct alternatives, this structure of winning coalitions is clearly order preserving. However, there is large class of further order preserving structures of winning coalitions different from pairwise majority voting.

Theorem 5.1 allows one to define a large class of strategy-proof social choice functions on every closed (hence also on every maximal) Condorcet domain in a very natural way, as follows. A social choice function $F : \mathcal{D}^n \longrightarrow A$ is *strategy-proof on* $\mathcal{D}$ if, for all voters i, all profiles $(\succ_1, ..., \succ_n) \in \mathcal{D}^n$ and all $\succ_i' \in \mathcal{D}$,

$$F(\succ_1, ..., \succ_i, ..., \succ_n) \ \succeq_i \ F(\succ_1, ..., \succ_i', ..., \succ_n).$$

For each order $\succ \in \mathcal{L}(A)$ denote by $\tau(\succ) \in A$ the unique top element of $\succ$.

Theorem 5.2. *Let $\mathcal{D}$ be a closed Condorcet domain. For every order preserving structure of winning coalitions $\mathcal{W}$, the social choice function defined by*

$$F^{\mathcal{W}}(\succ_1, ..., \succ_n) \ := \ \tau(\succ^{\mathcal{W}}) \tag{3}$$

is strategy-proof on $\mathcal{D}$ where $\succ^{\mathcal{W}}$ is the order given by (2).

In the case of the single-peaked domains, the anonymous social choice functions defined by (3) are exactly the well-known 'generalized medians' identified by Moulin (1983); and in the case of single-crossing domains they coincide with the strategy-proof social choice functions described in Saporiti (2009). Thus, in the two well-studied cases of single-peakedness and single-crossingness, the anonymous social choice functions of the form (3) *exhaust* the class of all anonymous and strategy-proof social choice functions. It is an open and worthwhile problem to determine if this holds more generally.

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