Abstract Arrowian Aggregation*

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Abstract *In a general framework of abstract binary aggregation, we characterize aggregation problems in terms of the monotone Arrowian aggregators they admit. Specifically, we characterize the problems that admit non-dictatorial, locally non-dictatorial, anonymous, and neutral monotone Arrowian aggregation, respectively. As a consequence of these characterizations, we also obtain new results on the possibility of strategy-proof social choice and the "concrete Arrowian" aggregation of preferences into a social ordering on generalized single-peaked domains.*

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1 Introduction

The model of abstract binary Arrowian aggregation introduced by Wilson (1975), and further developed by Rubinstein and Fishburn (1986), provides a general framework for studying the problem of aggregating sets of logically interconnected propositions, a problem that has recently received some attention in the literature on judgement aggregation, following List and Pettit (2002). A great variety of aggregation problems can be analyzed in this framework, among them the classical preference aggregation problem and the problem of strategy-proof social choice on generalized single-peaked domains.

In this paper, we shall adopt a "property space" formulation of the abstract aggregation framework. An aggregation problem is characterized by a set of evaluations which are described in terms of a family of binary properties, or equivalently, in terms of a family of yes/no-issues. Each evaluation corresponds to a unique combination of properties, or yes/no-evaluations. The property space formulation is characterized by an extensional view of properties as *sets* of evaluations. Crucially, the issues are logically interrelated so that some combinations of properties are inconsistent. An aggregator maps profiles of evaluations to "social" evaluations. An aggregator is called Arrowian if it satisfies a property space analogue of the familiar independence condition and respects unanimity. To enable a unifying characterization of the class of aggregators, we assume in addition that they be *monotone*, i.e. that they respond non-negatively to the individual evaluations. For Arrowian (i.e. independent) aggregators, monotonicity is extremely natural, and it is hard to see how non-monotone Arrowian aggregators could be of interest in practice.[1]

The program of the paper is to characterize those problems (property spaces) that admit monotone Arrowian aggregators satisfying various additional properties of interest, such as non-dictatorship, local non-dictatorship, anonymity and neutrality.

Our first result, Theorem 1, characterizes those problems that admit only dictatorial Arrowian aggregators in terms of a condition called "total blockedness." Many, but by far not all, interesting aggregation problems are totally blocked.

[1]Nonetheless, not least in view of the fact that monotonicity plays no role in Arrow's original theorem, there is an obvious technical interest in studying non-monotone Arrowian aggregators; see in particular Dokow and Holzman (2010a) and Dietrich and List (2007).

While this result ensures that if an aggregation problem is not totally blocked non-dictatorial monotone Arrowian aggregators exist, those may still be "almost dictatorial" by giving almost all decision power to a single agent, or by giving all decision power on some issues to one agent and all decision power on all other issues to another agent. Thus, the negation of total blockedness cannot be viewed as securing genuine possibility results. The second main result of the paper, Theorem 2, therefore characterizes those problems that admit anonymous and monotone Arrowian aggregators, ensuring that all agents have equal influence on the chosen outcome. It turns out that the problems that admit anonymous aggregators are exactly those that admit locally non-dictatorial aggregators.

As illustrated by an example, the characterizing condition for the existence of anonymous monotone Arrowian aggregators is rather intricate. The intricacy derives from the existence of contrived cases in which anonymous aggregation rules exist only for an odd number of agents. A simpler and more satisfying characterization is obtained for problems admitting anonymous monotone Arrowian aggregators for an arbitrary number of agents (Theorem 3).

While anonymous aggregation rules treat agents symmetrically, they typically treat social states asymmetrically, for instance by applying different quotas to different issues. We therefore finally characterize the circumstances under which monotone Arrowian aggregation is compatible with different notions of neutrality, i.e. symmetric treatment of social states (Theorem 4).

The remainder of this paper is organized as follows. The following Section 2 introduces our framework and notation; it also presents the characterization of all monotone Arrowian aggregators in terms of the Intersection Property obtained in Nehring and Puppe (2007). Section 3 contains the main results.

Section 4 presents two applications to the aggregation of preferences on restricted domains.[2] The first is to the possibility of strategy-proof social choice on domains of generalized single-peaked preferences. Using the framework developed in Nehring and Puppe (2007) we show how the characterization results of the present paper entail corresponding characterizations of the domains of generalized single-peaked preferences that admit non-dictatorial, locally non-dictatorial, anonymous and neutral strategy-proof social choice functions, respectively.

We conclude by presenting an application to classical "concrete" Arrowian aggregation on restricted domains. Specifically, we consider the aggregation of single-peaked orderings on an arbitrary connected graph into a social ordering. We show that all monotone Arrowian aggregators are dictatorial if the graph is *bi-connected*, i.e. if it remains connected even after removing any single vertex (and all edges connecting it); moreover, there exist locally non-dictatorial monotone Arrowian aggregators if and only if the underlying graph is a line (Theorem 6). Thus, in the context of aggregating single-peaked orderings on a graph into a linear social ordering possibility results are confined to the simple (and well-studied) case of a linear graph.

All proofs not provided in the text are collected in the appendix.

[2] A number of other applications are discussed throughout the text, for further examples see the survey article by List and Puppe (2009).

2 Framework and Basic Results

2.1 Property Spaces: Definition

A *property space* is a pair $(X, \mathcal{H})$, where X is a non-empty and finite set of objects ("evaluations"), and $\mathcal{H}$ is a collection of subsets of X satisfying

H1 $H \in \mathcal{H} \Rightarrow H \neq \emptyset$,
H2 $H \in \mathcal{H} \Rightarrow H^c \in \mathcal{H}$,
H3 for all $x \neq y$ there exists $H \in \mathcal{H}$ such that $x \in H$ and $y \notin H$,

where, for any $S \subseteq X$, $S^c := X \setminus S$ denotes the complement of S in X. The elements $H \in \mathcal{H}$ are referred to as *properties.* Condition H1 is a simple non-triviality condition, while condition H2 ensures that a property space is closed under negation. A pair (H, H^c) as also referred to as an *issue.* Condition H3 requires that two evaluations are distinguished by at least one property.[3] It implies that each evaluation $x \in X$ is identified by the family of its constituent properties $\mathcal{H}_x := \{H \in \mathcal{H} : x \in H\}$ in the sense that, for all $x \in X$,

$$\{x\} = \bigcap \mathcal{H}_x.$$

Property spaces can be identified with particular subsets of the hypercube $\{0,1\}^K$ for a suitable number K; thus, there is a natural interrelation between the property space approach and the abstract aggregation framework introduced by Wilson (1975). Specifically, any property space $(X, \mathcal{H})$ with $\mathcal{H} = \{H_1, H_1^c, H_2, H_2^c, ..., H_K, H_K^c\}$ naturally defines a subset $Z_{(X,\mathcal{H})}$ of $\{0,1\}^K$ as follows. For each $x \in X$, define $z(x) \in \{0,1\}^K$ by $z(x)^k = 1$ if $x \in H_k$ and $z(x)^k = 0$ if $x \in H_k^c$. Then let $Z_{(X,\mathcal{H})} := \{z(x) \in \{0,1\}^K : x \in X\}$. Conversely, for any non-empty $Z \subseteq \{0,1\}^K$ and each $k = 1, ..., K$, define $H_k := \{z \in Z : z^k = 1\}$ and $H_k^c := \{z \in Z : z^k = 0\}$, and let $\mathcal{H} := \{H_1, H_1^c, H_2, H_2^c, ..., H_K, H_K^c\}$. Then $(Z, \mathcal{H})$ is a property space provided that each H_k and H_k^c is non-empty.

Recently, the abstract aggregation framework has been adopted by Dokow and Holzman (2010a,b) who use Wilson's original formulation in terms of subsets of $\{0,1\}^K$. An advantage of a property space formulation is its emphasis on the fact that the same collection of objects may be endowed with *different* structure via different lists of properties (corresponding to different embeddings in the hypercube). An illustrative example is given at the end of Section 2.3 below.

The notion of a property space is also closely related to the notion of an *agenda* in the literature on judgement aggregation, see List and Puppe (2009) for a recent survey of that literature. Specifically, an issue (H, H^c) can be identified with a proposition/negation pair, and the elements of X with complete and consistent judgements on these; equivalently, the elements of X can be identified with the consistent truth-value assignments on the propositions.

2.2 Arrowian Aggregation on Property Spaces

Let $N = \{1, ..., n\}$ be a set of individuals with $n \geq 2$. An *aggregator* is a mapping $f : X^n \to X$. The following conditions on such mappings play a fundamental role in our analysis.

[3]Effectively, condition H3 is without loss of generality by considering the quotient space $X/_\approx$ with respect to the equivalence relation $x \approx y :\Leftrightarrow$ [for all $H \in \mathcal{H} : x \in H \Leftrightarrow y \in H$].

Unanimity $f(x, ..., x) = x$, for all $x \in X$.

Independence If $f(x_1, ..., x_n) \in H$ and, for all $i \in N$, $[x_i \in H \Leftrightarrow y_i \in H]$, then $f(y_1, ..., y_n) \in H$.

An aggregator is called *Arrowian* if it satisfies unanimity and independence. While evidently demanding, independence is motivated by a number considerations. In no small part, the widespread interest in aggregation procedures that satisfy this condition derives from the prevalence of binary voting mechanisms in practice. Independence has evident advantages of informational parsimony. In many contexts, independence reflects informational *robustness*. For example, in the context of preference aggregation, Arrow (1951) motivated his independence assumption philosophically by reference to an "ordinalist" view of individual preferences on which the use or imputation of utility differences (as in the Borda rule, for example) is not behaviorally meaningful. Likewise, in the context of aggregating evaluations on a line, the median – a classical positive example of an Arrowian aggregator – is well-defined in terms of the ordinal geometric structure of the line only, while the mean, for example, relies on additional cardinal distance information.

Furthermore, independence guarantees robustness with respect to the agenda and insensitivity with respect to the sequence in which the issues are elicited ("path-independence"); see, among others, List (2004).

In this paper, we will be concerned with Arrowian aggregators that satisfy in addition the following monotonicity condition.

Monotonicity If $f(x_1, ..., x_i, ..., x_n) \in H$ and $y_i \in H$, then $f(x_1, ..., y_i, ..., x_n) \in H$.

In the presence of independence, monotonicity is an extremely compelling condition. The conjunction of independence and monotonicity is equivalent to the following single condition.

Monotone Independence If $f(x_1, ..., x_n) \in H$ and, for all $i \in N$, $[x_i \in H \Rightarrow y_i \in H]$, then $f(y_1, ..., y_n) \in H$.

Note that under monotonicity, unanimity can be deduced from the weaker requirement that the aggregator is onto, i.e. that any element of X is in the range of f.

Besides its evident appeal as a condition on satisfactory aggregation, a crucial advantage of assuming monotonicity for the purpose of the present paper is the existence of a unified characterization of all monotone Arrowian aggregators, as described in Section 2.4 below. Moreover, strategy-proofness on rich domains of preferences requires not only independence but even monotone independence, as detailed in Section 4 below.

2.3 Application: Aggregation of Preferences

It is well-known that the classical problem of preference aggregation is a special case of the binary aggregation framework considered here. Specifically, let $A = \{a, b, ...\}$ be a finite set of alternatives and let $\mathcal{R}$ be a family of binary relations on A. For each pair $a, b \in A$ let

$$H_{(a,b)} := \{R \in \mathcal{R} : aRb\},$$

and denote by $\mathcal{H}_{\mathcal{R}}$ the family of all such properties and their complements. A binary relation on A can thus be identified with a certain combination of properties, and the family $\mathcal{R}$ can thus be naturally endowed with the structure of property space $(\mathcal{R}, \mathcal{H}_{\mathcal{R}})$. Different requirements on the members of the family $\mathcal{R}$ give rise to different property spaces. For instance, transitivity of the binary relations in $\mathcal{R}$ implies that $\{H_{(a,b)}, H_{(b,c)}, H^c_{(a,c)}\}$ is an inconsistent combination of properties. As an example, consider the set $\mathcal{L}in(A)$ of all strict linear orderings on A. In the case of linear orderings $\succ$ on A, the property $H^c_{(a,b)}$ can be identified with $H_{(b,a)}$, for all distinct a, b. The space $(\mathcal{L}in(A), \mathcal{H}_{\mathcal{R}})$ can thus be embedded in the $\frac{\#A\cdot(\#A-1)}{2}$-dimensional hypercube, see Figure 1 for the case $\#A = 3$.

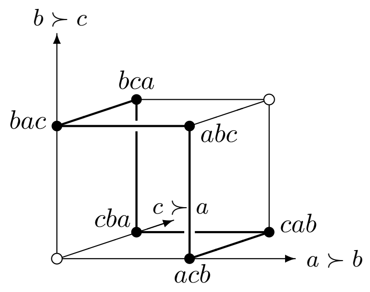


Figure 1: The property space $(\mathcal{L}in(A), \mathcal{H}_{\mathcal{R}})$ *for* $\#A = 3$

Note that the independence condition on $(\mathcal{L}in(A), \mathcal{H}_{\mathcal{R}})$ takes the usual Arrowian "independence of irrelevant alternatives" format according to which the social ranking between two alternatives only depends on the individual ranking between these two alternatives. In particular, by Arrow's theorem, all Arrowian aggregators on $(\mathcal{L}in(A), \mathcal{H}_{\mathcal{R}})$ are dictatorial.

It bears emphasizing that an aggregation problem is well-specified only once a particular family of sets $\mathcal{H}$ satisfying conditions H1 – H3 is pinned down. The set X is thus *endowed* with the structure of a property space, just as one endows a set with a topology. Just as it is meaningless to ask whether a function is continuous in the absence of a specified topology, it is meaningless to ask whether Arrowian aggregation is possible *tout court*. The mathematical structure of the set X may be suggestive of a particular property space structure $\mathcal{H}$ but cannot by itself determine it.

Consider for instance the set of weak orderings (reflexive, transitive and complete binary relations) $\mathcal{W}eak(A)$ on a ground set A. While the above framing of the properties $H_{a,b}$ in terms of instances of the relation "is weakly preferred to" is natural, other framings are possible and not necessarily absurd.[4]

For example, for each non-empty subset $L \subseteq A$ and each weak ordering $\succeq$ on A, say that L is a *lower contour set* of $\succeq$ if $L = \{a \in A : b \succeq a\}$ for some $b \in A$. Moreover, for each non-empty subset $L \subseteq A$ consider the pair of properties

$$H_L := \{\succeq \in \mathcal{W}eak(A) : L \text{ is a lower contour set of } \succeq\}$$

and its complement $H^c_L := \mathcal{W}eak(A) \setminus H_L$, and denote by $\mathcal{H}_{\mathcal{L}} := \{H_L, H^c_L\}_{\emptyset \neq L \subseteq A}$ the family of all such properties. Evidently, each weak ordering is uniquely identified by the family of its lower contour sets; in particular, $\mathcal{H}_{\mathcal{L}}$ satisfies H3. Note also that the

[4] A "pseudo-Arrowian" theorem for $(\mathcal{W}eak(A), \mathcal{H}_{\mathcal{R}})$ has been derived from Theorem 1 below in Nehring (2003). It is only *pseudo*-Arrowian since the implied independence condition is evidently stronger than the usual binary IIA condition for domains in which indifferences are permitted. For approaches that use Arrow's original binary IIA condition in the case of weak orderings within the judgement aggregation framework, see Dietrich and List (2007) and Dokow and Holzman (2010b).

family of lower contour sets of each weak ordering is totally ordered by set inclusion, i.e. forms a chain. Conversely, each chain of non-empty subsets that contains the set A corresponds to a (unique) weak ordering with the given chain as the family of its lower contour sets.

Intriguingly, while $(\mathcal{W}eak(A), \mathcal{H}_{\mathcal{R}})$ is dictatorial, in $(\mathcal{W}eak(A), \mathcal{H}_{\mathcal{L}})$ even issue-by-issue majority voting is consistent as shown in Section 3.3 below. We leave it to the reader to judge whether this 'solves' Arrow's impossibility problem, and if not, why not.

2.4 Characterization of All Monotone Arrowian Aggregators: The Intersection Property

In this subsection, we review the characterization of all monotone Arrowian aggregators obtained in Nehring and Puppe (2007). A family of *winning coalitions* is a non-empty family $\mathcal{W}$ of non-empty subsets of the set N of all individuals satisfying $[W \in \mathcal{W}$ and $W' \supseteq W] \Rightarrow W' \in \mathcal{W}$. A *structure of winning coalitions* on $(X, \mathcal{H})$ assigns a family of winning coalitions $\mathcal{W}_H$ to each property satisfying the following condition,

$$W \in \mathcal{W}_H \Leftrightarrow (N \setminus W) \notin \mathcal{W}_{H^c}. \tag{2.1}$$

In words, a coalition is winning for H if and only if its complement is not winning for the negation of H. Using (2.1) and the fact that families of winning coalitions are closed under taking supersets, we obtain

$$\mathcal{W}_{H^c} = \{W \subseteq N : W \cap W' \neq \emptyset \text{ for all } W' \in \mathcal{W}_H\}. \tag{2.2}$$

To derive (2.2) from (2.1), let W be such that $W \cap W' \neq \emptyset$ for all $W' \in \mathcal{W}_H$. Then, in particular $N \setminus W \notin \mathcal{W}_H$, hence by (2.1), $W \in \mathcal{W}_{H^c}$. Conversely, let $W \in \mathcal{W}_{H^c}$ and $W' \in \mathcal{W}_H$; by contradiction, suppose that $W \cap W' = \emptyset$, then $W' \subseteq (N \setminus W)$, hence by monotonicity, $N \setminus W \in \mathcal{W}_H$ in violation of (2.1).

Each structure of winning coalitions $\mathcal{W}$ determines a correspondence $f_{\mathcal{W}}$ given by

$$f_{\mathcal{W}}(x_1, ...x_n) := \cap\{H : \{i : x_i \in H\} \in \mathcal{W}_H\},$$

which we will refer to as "voting by issues" associated with $\mathcal{W}$. Clearly, by H3, $f_{\mathcal{W}}(x_1, ...x_n)$ contains at most one point. It may however be empty reflecting inconsistency of the outcome of the issue-wise votes. Indeed, it is easily seen that f is a monotone Arrowian aggregator if and only if it is consistent (non-empty-valued) voting by issues for some structure of winning coalitions $\mathcal{W}$.

Characterizing the class of monotone Arrowian aggregators thus amounts to characterizing when voting by issues is consistent. This can be done by focusing on the minimal ("critical") inconsistencies that characterize the property space. These are given by the "critical families" of $(X, \mathcal{H})$ as follows.

Say that a family $\mathcal{G}$ is *inconsistent* if $\cap\mathcal{G} = \emptyset$. $\mathcal{G}$ is *critical* if it is minimally inconsistent, i.e. if it is inconsistent and for all $G \in \mathcal{G}$, $\cap(\mathcal{G} \setminus \{G\}) \neq \emptyset$. Observe that all pairs $\{H, H^c\}$ of complementary properties are critical; they are referred to as the *trivial* critical families. An example of a non-trivial critical family in the case of the aggregation of linear orderings is the combination of properties $\{H_{(a,b)}, H_{(b,c)}, H^c_{(a,c)}\}$ since $a \succ b$, $b \succ c$ and (not $a \succ c$) are jointly inconsistent by transitivity, while any two of these preference judgements are mutually consistent.

Definition (Intersection Property) A structure of winning coalitions $\mathcal{W}$ satisfies the *Intersection Property* if for any critical family $\{G_1, ..., G_l\} \subseteq \mathcal{H}$, and any selection $W_j \in \mathcal{W}_{G_j}$,

$$\bigcap_{j=1}^{l} W_j \neq \emptyset.$$

The following result is proved in Nehring and Puppe (2007, Prop. 3.4) and Nehring and Puppe (2002, Prop. 3.5). We include the proof here again to make the exposition self-contained.

Proposition 2.1 *Let $(X, \mathcal{H})$ be a property space. A mapping $f : X^n \to X$ is a monotone Arrowian aggregator on $(X, \mathcal{H})$ if and only if it is voting by issues $f_{\mathcal{W}}$ with $\mathcal{W}$ satisfying the Intersection Property.*

Proof For sufficiency, let $f = f_{\mathcal{W}}$ with $\mathcal{W}$ satisfying the Intersection Property. Suppose, by way of contradiction, that there exists a profile $(x_1, ...x_n)$ at which voting by issues is inconsistent, i.e. such that $f_{\mathcal{W}}(x_1, ...x_n) = \cap\{H : \{i : x_i \in H\} \in \mathcal{W}_H\} = \emptyset$. Take a minimally inconsistent subset of these properties, i.e. a critical family $\mathcal{G}$ such that $\{i : x_i \in H\} \in \mathcal{W}_H$ for all $H \in \mathcal{G}$. By the Intersection Property, these winning coalitions $W_H = \{i : x_i \in H\}$ must have at least one individual in common; but this individual must affirm all properties in $\mathcal{G}$, i.e. for some i, $x_i \in \cap_{H \in \mathcal{G}} H$, contradicting the criticality of $\mathcal{G}$.

For necessity, suppose that there exists a critical family $\mathcal{G}$ and a selection of winning coalitions $W_j \in \mathcal{W}_{G_j}$ with empty intersection. We show that $f_{\mathcal{W}}$ is inconsistent by constructing a profile $(x_1, ..., x_n)$ at which each of the properties in $\mathcal{G}$ is affirmed.

Since $\cap W_j = \emptyset$, for each i there exists j_i such that $i \notin W_{j_i}$. That is, i is not required for a positive vote on G_{j_i}. On the other hand, by the criticality (that is: *minimal* inconsistency) of $\mathcal{G}$, $\cap_{j:j \neq j_i} G_j \neq \emptyset$. Assign i an evaluation $x_i \in \cap_{j:j \neq j_i} G_j$. This ensures that i lends his support to all other properties in $\mathcal{G}$ where his support may be required.

Indeed, we will verify that $f_{\mathcal{W}}(x_1, ..., x_n) = \emptyset$, the desired inconsistency. To see this, note that for any j and any $i \in W_j$, $j_i \neq j$. By construction, this implies that $x_i \in G_j$. In other words, $W_j \subseteq \{i : x_i \in G_j\}$. By monotonicity of $\mathcal{W}$, it follows that $\{i : x_i \in G_j\}$ is itself a winning coalition. And thus, by the criticality of $\mathcal{G}$, obviously $f_{\mathcal{W}}(x_1, ..., x_n) \subseteq \cap \mathcal{G} = \emptyset$, as claimed.

Note that the proof of necessity of the Intersection Property relies on the assumed monotonicity. A comparably simple characterization of all Arrowian aggregators without monotonicity is not known.

The Anonymous Case

The Intersection Property takes a particularly simple form in the anonymous case. An aggregator $f : X^n \to X$ is *anonymous* if it is invariant with respect to permutations of agents. Under anonymity, voting by issues is characterized by a family $\{m_H : H \in \mathcal{H}\}$ of absolute quotas, where $m_H := \min\{\#W : W \in \mathcal{W}_H\}$. Note that, by (2.1), $m_H + m_{H^c} = n + 1$. It is easily verified that the Intersection Property on an anonymous structure of winning coalitions is equivalent to the following system of linear (in)equalities on the absolute quotas m_H,

$$\text{for all } H \in \mathcal{H} \quad : \quad m_H + m_{H^c} = n + 1 \tag{2.3}$$

$$\text{for all critical families } \mathcal{G} \quad : \quad \sum_{H \in \mathcal{G}} (n - m_H) < n. \tag{2.4}$$

The existence of an anonymous monotone Arrowian aggregator is thus described by an integer programming problem. This can be restated and simplified into a linear programming problem by considering relative quotas, as follows. For each $q \in [0, 1]$ denote by $\mathcal{W}_q := \{W \subseteq N : \#W > q \cdot n\}$.

Proposition 2.2 *Let $(X, \mathcal{H})$ be a property space and let $\{q_H : H \in \mathcal{H}\}$ be a system of relative quotas such that, for all $H \in \mathcal{H}$, $q_H + q_{H^c} = 1$ and $q_H \cdot n$ is not an integer other than 0 or n. If, for every critical family $\mathcal{G}$,*

$$\sum_{H \in \mathcal{G}} (1 - q_H) \leq 1, \tag{2.5}$$

then the structure $\{\mathcal{W}_{q_H} : H \in \mathcal{H}\}$ of winning coalitions defines an anonymous and monotone Arrowian aggregator. Conversely, any anonymous and monotone Arrowian aggregator can be described by relative quotas $\{q_H : H \in \mathcal{H}\}$ such that (i) for all $H \in \mathcal{H}$, $q_H + q_{H^c} = 1$, (ii) for all $H \in \mathcal{H}$, $q_H \cdot n$ is not an integer other than 0 or n, and (iii) for every critical family $\mathcal{G}$, (2.5) is satisfied.

The role of the integer condition (ii) is to ensure that the families $\mathcal{W}_{1-q_H}$ and $\mathcal{W}_{q_H}$ are adjoint in the sense of condition (2.1). This becomes important in situations in which all anonymous social choice functions require some quota q_H to be equal to $\frac{1}{2}$, i.e. majority voting on H; clause (ii) implies in this case that n must be odd, which makes intuitive sense since majority voting is well-defined only for an odd number of individuals. The second part of Proposition 2.2 relies on the observation that if the absolute quotas m_H satisfy conditions (2.3) and (2.4), the relative quotas q_H defined by $q_H := \frac{m_H - 1}{n-1}$ satisfy conditions (i) and (2.5).

Proposition 2.1 immediately entails a characterization of those property spaces that admit issue-by-issue majority voting as an Arrowian aggregator. *Issue-by-issue majority voting* is voting by issues $f_{\mathcal{W}}$ with $\mathcal{W}_H = \mathcal{W}_{\frac{1}{2}}$ for all $H \in \mathcal{H}$.

Definition (Median Space) A property space $(X, \mathcal{H})$ is called a *median space* if all critical families have cardinality two.[5]

Proposition 2.3 *Issue-by-issue majority voting is consistent on $(X, \mathcal{H})$ if and only if $(X, \mathcal{H})$ is a median space and n is odd.*

As stated, this result is due to Nehring and Puppe (2007, Corollary 5) who also discuss the related literature on median spaces in combinatorial mathematics (see, e.g., van de Vel, 1993). This literature contains results that can be viewed as broadly equivalent to the sufficiency part of the proposition (see in particular McMorris, Mulder and Powers (2000, Theorem 4)). There does not appear a formal counterpart to the necessity claim of the Proposition 2.3 – perhaps because the mathematical literature on consensus lacks the general notion of a property space. Applying a lesser standard of formal rigor, the gist of Proposition 2.3, especially on the side of necessity, might be attributed to the insightful early paper by Gilbaud (1952).

[5] In Nehring and Puppe (2007), this is shown to be equivalent to the standard definition (adopted there) in terms of a ternary betweenness relation.

2.5 Example: Comprehensive Subsets of the Hypercube

The power of the Intersection Property can be further illustrated in the case of "comprehensive" subsets of the hypercube, as follows. Say that $X \subseteq \{0,1\}^K$ is *comprehensive* if $x \in X$ implies $\hat{x} \in X$ whenever $x^k = 1 \Rightarrow \hat{x}^k = 1$. Thus, the point $(1,1,...,1)$ is the "default " evaluation, and if an evaluation x is feasible so is any evaluation $\hat{x}$ that is closer to the default.

Electing Candidates

A possible interpretation is in terms of electing members of a committee; another interpretation in terms of acyclic relations is discussed shortly. In the case of committees, each coordinate corresponds to a candidate for a position in a committee, with $x^k = 1$ (resp. $x^k = 0$) denoting election (resp. non-election) of candidate k, and the set X describes the feasible committees. Denote, for all k, by H_1^k the committees that contain candidate k, and by H_0^k the committees that do not contain candidate k.

It is clear that any inconsistency results from the joint exclusion of certain candidates. Any *minimal* inconsistency results thus from the joint exclusion of certain candidates *at least one* of whom needs to be elected. Critical families are thus contained in $\{H_0^1, ..., H_0^K\}$. Consider the case of anonymous aggregators f_q with a uniform quota $q_{H_0^k} = q$ for all k. Clearly, consistency is achievable by making it sufficiently hard for any candidate to be rejected, e.g. by setting $q = 1$. What is the smallest uniform quota q that ensures consistency, hence leads to a well-defined Arrowian aggregator? According to the anonymous Intersection Property, f_q is consistent if and only if $q \cdot \#\mathcal{G} \geq \#\mathcal{G} - 1$ for any critical family $\mathcal{G}$, hence if

$$q \geq 1 - \frac{1}{\kappa},$$

where κ is the size of the largest critical family.

Anonymous Aggregation of Acyclic Orderings

As an example in the context of preference aggregation consider the family of all acyclic (strict) orderings $\mathcal{A}cy(A)$ embedded in the $[\#A \cdot (\#A-1)]$-dimensional hypercube with $H_{a \succ b}$ corresponding to the property "$a \succ b$" for every ordered pair (a,b), and $H^c_{a \succ b}$ corresponding to "not $(a \succ b)$." The subset $\mathcal{A}cy(A)$ is comprehensive with the empty relation as default since removing binary comparisons from an acyclic ordering keeps the ordering acyclic. As above, the Intersection Property thus immediately implies the existence of anonymous monotone Arrowian aggregators.[6]

2.6 Conditional Entailment

When does a given property space admit non-dictatorial monotone Arrowian aggregators? Or anonymous monotone Arrowian aggregators? Or monotone Arrowian aggregators of some other kind? Note that while the Intersection Property characterizes the consistency of particular voting-by-issue schemes $f_{\mathcal{W}}$ as the canonical candidates for monotone Arrowian aggregation, it does not, of course, say anything directly about their existence. In a few cases such as in the case of comprehensive sets, the existence

[6] The aggregation of orderings into acyclic social orderings has been first studied by Mas-Colell and Sonnenschein (1972); see Moulin (1988) for an overview of the results.

question is straightforward. In general, however, it depends on the global combinatorial structure of the entire class of critical families, and on the specific requirements imposed on the aggregator.

It turns out that much of the relevant structure can be compactly summarized by a relation of "conditional entailment" between properties as follows. Say that H *conditionally entails* G, written as $H \geq^0 G$ if $H \neq G^c$ and there exists a critical family containing both H and G^c. Intuitively, $H \geq^0 G$ thus means that, given *some* combination of other properties, property H entails property G. More precisely, let $H \geq^0 G$, i.e. let $\{H, G^c, G_1, ..., G_l\}$ be a critical family; then with $A = \cap_{j=1}^{l} G_j$ one has both $A \cap H \neq \emptyset$ ("property H is compatible with the combination A of properties") and $A \cap G^c \neq \emptyset$ ("property G^c is compatible with A as well") but $A \cap \; H \cap G^c = \emptyset$ ("properties H and G^c are jointly incompatible with A"). Note that, by definition, $H \geq^0 G \Leftrightarrow G^c \geq^0 H^c$, that is, $\geq^0$ is "negation adapted." We write $\geq$ for the transitive closure of $\geq^0$, and $\equiv$ for the symmetric part of $\geq$.

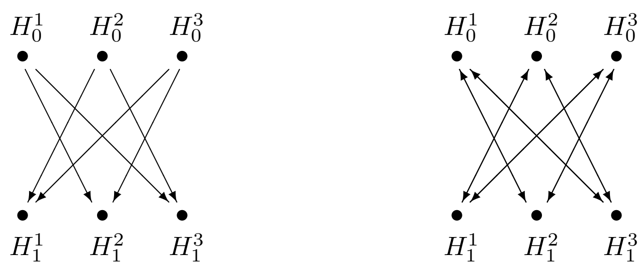
As a simple example, consider once again the family $\{H_{(a,b)}, H_{(b,c)}, H^c_{(a,c)}\}$ in case of aggregation on $(\mathcal{L}in(A), \mathcal{H}_{\mathcal{R}})$. Criticality of this family can be paraphrased as saying that, conditional on $a \succ b$, $b \succ c$ entails $a \succ c$.

The key role of the conditional entailment relation in our context derives from the following lemma.

Lemma 1 (Contagion Lemma) *If $\{\mathcal{W}_H\}_{H \in \mathcal{H}}$ satisfies the Intersection Property and $H \geq G$, then $\mathcal{W}_H \subseteq \mathcal{W}_G$.*

Proof By transitivity, it suffices to show that $H \geq^0 G \Rightarrow \mathcal{W}_H \subseteq \mathcal{W}_G$. Thus, suppose that $\{H, G^c\} \subseteq \mathcal{G}$ for some critical family $\mathcal{G}$. By the Intersection Property, $W \cap W' \neq \emptyset$ for any $W \in \mathcal{W}_H$ and any $W' \in \mathcal{W}_{G^c}$, hence by (2.2), $\mathcal{W}_H \subseteq \mathcal{W}_G$.

One can visualize the conditional entailment relation by means of a directed graph on the set of all properties such that an edge goes from H to G if and only if $H \geq^0 G$. Figure 2 shows two examples. The graph in Fig. 2(a) shows the entailment relation induced by the comprehensive subset of $\{0,1\}^3$ in which only the point $(0,0,0)$ is infeasible, corresponding to the critical family $\{H_0^1, H_0^2, H_0^3\}$. Note that, by definition, a property is never connected by an edge with its complementary property.



(a) (b)

Figure 2: Two Conditional Entailment Graphs

The graph in Fig. 2(b) shows the conditional entailment relation of the property space $(\mathcal{L}in(A), \mathcal{H}_{\mathcal{R}})$ shown in Fig. 1 which is isomorphic to the feasible set $\{0,1\}^3 \setminus \{(0,0,0),(1,1,1)\}$.

Observe that different property spaces can give rise to the same conditional entailment relation, thus there is a loss of information entailed by moving from the collection of all critical families to the conditional entailment relation. It is remarkable that the information contained in the latter is nevertheless sufficient for most of the characterization results to be presented now.

3 Characterization Results

3.1 Non-Dictatorial Aggregation

Given a monotone Arrowian aggregator f on $(X, \mathcal{H})$, an individual i is said to have a *veto on property* H if $x_i \in H^c \Rightarrow f(x_1, ..., x_n) \in H^c$, for all $(x_1, ..., x_n)$. Evidently, individual i has a veto on H if and only if $\{i\} \in \mathcal{W}_{H^c}$. The aggregator f is called *dictatorial* if there exists some individual i such that, for all $x_1, ..., x_n$, $f(x_1, ..., x_n) = x_i$. Evidently, f is dictatorial if and only if some individual i has a veto on *all* properties, i.e. if and only if $\{i\} \in \mathcal{W}_H$ for all $H \in \mathcal{H}$.

Definition (Total Blockedness) Say that $(X, \mathcal{H})$ is *totally blocked* if, for all $H, G \in \mathcal{H}$, $H \geq G$, i.e. if there exists a sequence of conditional entailments from every property to every other property.

In graph-theoretic terms, total blockedness simply says that between any two properties there is a directed path of conditional entailments. Evidently, spaces with the conditional entailment graph in 2(b) are totally blocked, while those in 2(a) are not.

Theorem 1 *A property space $(X, \mathcal{H})$ admits non-dictatorial and monotone Arrowian aggregators if and only if it is not totally blocked.*

Proof of necessity By the Contagion Lemma 1, all winning coalitions must be the same, $\mathcal{W}_H = \mathcal{W}_0$ for all $H \in \mathcal{H}$; in the later (standard) terminology, the Arrowian aggregator must be *neutral*. This does not bode well for the possibility of non-dictatorial aggregation, since very little flexibility is left. And, indeed, the following "Veto Lemma" forces a dictatorship, since total blockedness is easily seen to imply the existence of a critical family with at least three elements.

Lemma 2 (Veto Lemma) *Suppose that a structure of winning coalitions satisfies the Intersection Property and that $\{G_1, G_2, G_3\} \subseteq \mathcal{G}$ for some critical family $\mathcal{G}$. If $\mathcal{W}_{G_1^c} \subseteq \mathcal{W}_{G_2}$, then $\{i\} \in \mathcal{W}_{G_3^c}$ for some $i \in N$.*

Proof of Lemma 2 Let $\tilde{W}_1$ be a minimal element of $\mathcal{W}_{G_1}$, and let $i \in \tilde{W}_1$. By (2.2) and minimality of $\tilde{W}_1$, one has $(\tilde{W}_1^c \cup \{i\}) \in \mathcal{W}_{G_1^c}$. By assumption, $\mathcal{W}_{G_1^c} \subseteq \mathcal{W}_{G_2}$, hence $(\tilde{W}_1^c \cup \{i\}) \in \mathcal{W}_{G_2}$. Now consider any $W_3 \in \mathcal{W}_{G_3}$. By the Intersection Property, $\cap_{j=1}^3 W_j \neq \emptyset$ for any selection $W_j \in \mathcal{W}_{G_j}$. In particular, $\tilde{W}_1 \cap (\tilde{W}_1^c \cup \{i\}) \cap W_3 \neq \emptyset$. Since $\tilde{W}_1 \cap (\tilde{W}_1^c \cup \{i\}) = \{i\}$, this means $i \in W_3$ for all $W_3 \in \mathcal{W}_{G_3}$. By (2.2), this implies $\{i\} \in \mathcal{W}_{G_3^c}$.

Note the simplicity of the argument (with the Intersection Property in the background, of course): The Contagion Lemma forces the monotone Arrowian aggregator to be neutral. Neutrality in the presence of critical families of cardinality greater than two implies a veto, hence a dictatorship.

Since, moreover, by the sufficiency part of Theorem 1 (proved in the appendix), total blockedness characterizes dictatorial problems, Theorem 1 shows in fact that, if an Arrowian impossibility can be demonstrated for a particular space at all, the proof can always take this simple form once total blockedness of the space has been established.

To establish total blockedness of a given space is typically fairly straightforward, as it involves coming up with sufficiently many instances of conditional entailment; in particular, it is not necessary to determine the set of critical families exhaustively. By contrast, in order to show that a domain is non-dictatorial, in principle one needs to determine the transitive hull of the entire conditional entailment relation; this may be difficult. However, an easily verifiable and frequently applicable sufficient condition is that there be at least one property not contained in any non-trivial critical family. Indeed, if H is only contained in the trivial critical family $\{H, H^c\}$, one has $H \not\geq^0 G$ for all G, and therefore $H \not\geq H^c$, which implies that the underlying property space is not totally blocked.

3.2 Locally Non-Dictatorial and Anonymous Aggregation

As a possibility result, Theorem 1 is not completely satisfactory since non-dictatorial aggregation rules can still be rather degenerate, e.g. by giving almost all decision power to one agent, or by specifying different "local" dictators for different issues. In this subsection, we therefore characterize the problems for which locally non-dictatorial monotone Arrowian aggregators exist. It turns out that this is also exactly the class of problems for which anonymous monotone Arrowian aggregators exist.

An aggregator f is called *locally dictatorial* if there exists an individual i and an issue (H, H^c) such that i has a veto on H and on H^c, i.e. for all $x_1, ..., x_n$, $f(x_1, ..., x_n) \in H \Leftrightarrow x_i \in H$. Note that there may exist several local dictators (over different issues); also observe that an anonymous aggregator is necessarily locally non-dictatorial.

Definition (Blockedness, Unblockedness, Quasi-Unblockedness) Say that a property $H \in \mathcal{H}$ is *blocked* if $H \equiv H^c$, i.e. if there exists a sequence of conditional entailments from H to its complement H^c, and vice versa. Call a property space $(X, \mathcal{H})$ *blocked* if some $H \in \mathcal{H}$ is blocked; otherwise, if no H is blocked, $(X, \mathcal{H})$ is called *unblocked*. Finally, for each $G \in \mathcal{H}$, let $\mathcal{H}_{\equiv G} := \{H \in \mathcal{H} : H \equiv G\}$, and say that a property space is *quasi-unblocked* if for any $G \in \mathcal{H}$ and any critical family $\mathcal{G}$, $\#(\mathcal{H}_{\equiv G} \cap \mathcal{G}) \leq 2$, whenever G is blocked.

Evidently, quasi-unblockedness is intermediate in strength between not total blockedness and unblockedness. Examples of unblocked spaces are the property spaces corresponding to the entailment graph shown in Fig. 2(a). More generally, all comprehensive sets of feasible committees (cf. first example in Section 2.5 above) give rise to unblocked spaces.

Theorem 2 *Let $(X, \mathcal{H})$ be a property space. The following are equivalent.*
(i) *$(X, \mathcal{H})$ admits locally non-dictatorial and monotone Arrowian aggregators.*
(ii) *$(X, \mathcal{H})$ admits anonymous and monotone Arrowian aggregators.*
(iii) *$(X, \mathcal{H})$ is quasi-unblocked.*

A noteworthy corollary of Theorem 2 is the fact that any property space that admits locally non-dictatorial monotone Arrowian aggregators even admits anonymous such aggregators.

In Appendix A, we show that there are spaces that are quasi-unblocked yet blocked. However, these appear quite contrived, and it seems unlikely that they are relevant in applications. Moreover, on such spaces anonymous aggregation rules exist only for an odd number of agents; hence, the possibility obtained in these cases is not robust.[7]

A cleaner and more satisfying characterization is obtained for property spaces admitting anonymous rules for an arbitrary number of agents, as follows.

Definition (Median Point) Let $(X, \mathcal{H})$ be a property space. An element $\hat{x} \in X$ is called a *median point* if, for any critical family $\mathcal{G}$, $\#\{G \in \mathcal{G} : \hat{x} \in G\} \leq 1$.

Thus, a state is a median point if every critical family contains at most one of its constituent properties. Note that the default in a comprehensive subset of the hypercube is always a median point. The set of all median points is denoted by $M(X)$. The notion of a median point and its characterization in Proposition 3.1 below are due to Nehring (2010); see also Nehring and Puppe (2002, Section 6) for the original statement.

In median spaces, and only in these, *every* element is a median point. Indeed, if all critical families have exactly two elements then evidently each point is a median point. Moreover, if $\mathcal{G}$ is a critical family with at least three elements and $G \in \mathcal{G}$, then any $x \in \cap(\mathcal{G} \setminus \{G\} \neq \emptyset$ is not a median point. Median spaces are important for the theory of abstract Arrowian aggregation by securing the possibility of various neutrality properties, see Section 3.3 below.

The characterization of the spaces that admit anonymous monotone Arrowian aggregators relies on the following characterization of unblocked spaces.

Proposition 3.1 *A property space $(X, \mathcal{H})$ is unblocked if and only if $M(X) \neq \emptyset$, i.e. if and only if $(X, \mathcal{H})$ admits at least one median point.*

Median points play a central role in our present context because they are canonically associated with unanimity rules. An Arrowian aggregator f is called a *unanimity rule* if there exists $\hat{x} \in X$ such that for all $H \in \mathcal{H}_{\hat{x}}$,

$$f(x_1, ..., x_n) \in H \Leftrightarrow x_i \in H \text{ for some } i \in N. \tag{3.1}$$

That is, each of the properties H in $\mathcal{H}_{\hat{x}}$ (ie. the constituent properties of $\hat{x}$) holds in the aggregate unless there is unanimous agreement on H^c.

Clearly, a state $\hat{x}$ such that (3.1) is satisfied for all $H \in \mathcal{H}_{\hat{x}}$ is uniquely determined and is referred to as the *status quo*. Henceforth, we denote the unanimity rule with status quo $\hat{x}$ by $f_{\hat{x}}$. Note that the properties determined in (3.1) may not be jointly consistent, so that an Arrowian unanimity rule of the form $f_{\hat{x}}$ may or may not exist.

Proposition 3.2 *A property space $(X, \mathcal{H})$ admits an Arrowian unanimity rule of the form $f_{\hat{x}}$ if and only if $\hat{x} \in M(X)$.*

The following result summarizes the characterizations entailed by Propositions 3.1 and 3.2, and shows that the existence of median points is also necessary for the existence of anonymous monotone Arrowian aggregators for any number of individuals.

[7]Arguably, as pointed out by an editor, the non-robustness is itself not that robust, in that "nearly anonymous" monotone Arrowian aggregators exist on quasi-unblocked yet blocked spaces also for an even number of agents: simply make "odd" out of "even" by giving one agent two votes, and all others one.

Theorem 3 *Let $(X, \mathcal{H})$ be a property space. The following are equivalent.*
(i) *$(X, \mathcal{H})$ admits anonymous and monotone Arrowian aggregators for some even n.*
(ii) *$(X, \mathcal{H})$ admits anonymous and monotone Arrowian aggregators for all $n \geq 2$.*
(iii) *$(X, \mathcal{H})$ admits some Arrowian unanimity rule.*
(iv) *$(X, \mathcal{H})$ is unblocked.*
(v) *$(X, \mathcal{H})$ admits a median point.*

Examples

1. The Discursive Dilemma A special class of aggregation problems arises by considering a set of binary propositions that can be split into a set of "premises" and a set of "conclusions" which depend on the evaluation of the premises. A simple example arises by taking a conclusion d that is logically equivalent to the conjunction of its premises c_1 and c_2. The so-called "discursive dilemma" (see List and Pettit (2002), following Pettit (2001) and Kornhauser and Sager (1986)) consists in the observation that in this case natural aggregation methods, such as proposition-wise majority voting, may yield inconsistent collective judgements. If one embeds the problem in the 3-dimensional hypercube with the conclusion corresponding to the third coordinate, one obtains the feasible set $\{(0,0,0),(1,0,0),(0,1,0),(1,1,1)\}$, where "1" stands for affirmation and "0" for negation of the corresponding proposition. As is easily seen, negating all propositions (i.e. the point $(0,0,0)$) constitutes a median point. Hence the Arrowian unanimity rule according to which all propositions are collectively negated unless there is unanimous agreement to affirm them is consistent. It can easily be shown that this is in fact the only anonymous monotone Arrowian aggregator in this case. For extensions of this basic finding, see Nehring and Puppe (2008, 2010), Dokow and Holzman (2009), and Dietrich and Mongin (2010).

2. Electing Candidates (cont.) Consider again the K-dimensional hypercube and the subset $X_{(K;k,k')} \subseteq \{0,1\}^K$ of all binary sequences with at least k and at most k' coordinates having the entry 1, where $0 \leq k \leq k' \leq K$. Thus, feasible committees must have at least k and at most k' members.

If $k = 0$ and $k' = K$, we obtain the full hypercube in which evidently every point is a median point. Next, assume that $k > 0$. If $k' = K$, the underlying subset is comprehensive and the non-trivial critical families are exactly the subsets of $\{H_0^1, H_0^2, ..., H_0^K\}$ with $K - k + 1$ elements ("if already $K - k$ candidates have been rejected, then all of the remaining candidates must be elected"). As noted above, the default of a comprehensive subset is always a median point, hence the Arrowian unanimity rule according to which all candidates are elected unless there is unanimous agreement to reject them is consistent.

Let now $0 < k \leq k' < K$. Then, in addition to all subsets of $\{H_0^1, H_0^2, ..., H_0^K\}$ with $K - k + 1$ elements also the subsets of $\{H_1^1, H_1^2, ..., H_1^K\}$ with $k' + 1$ elements form critical families. The corresponding spaces are totally blocked whenever $K \geq 3$ since one has $H_0^k \geq^0 H_1^j$ and $H_1^k \geq^0 H_0^j$ for all distinct k, j (see Fig. 2(b) for the conditional entailment graph if $K = 3$). By Theorem 1, any monotone Arrowian aggregator is dictatorial.

As a variation of this example, consider a non-empty subset $J \subseteq \{1, ..., K\}$ representing a subgroup of candidates, and suppose that at least one candidate from the set of all candidates has to be elected, but at most m out of the subgroup J, where $1 \leq m \leq \#J$. Denote the corresponding subspace by $X_{(K;m,J)}$. If $\#J < K$, none of the spaces $X_{(K;m,J)}$ is totally blocked. Indeed, for all $k \notin J$, the property "candidate

k is elected" is not an element of any non-trivial critical family. Thus, by the remark after Theorem 1 above, the space is not totally blocked. On the other hand, if $\#J > 2$, then the subspace corresponding to the coordinates in J is totally blocked. It can be shown that, therefore, all monotone Arrowian aggregators on $X_{(K;m,J)}$ are locally dictatorial whenever $2 < \#J < K$. On the other hand, if $\#J = 2$ the corresponding spaces admit at least one median point. Hence by Theorem 3, there exist anonymous monotone Arrowian aggregators in this case.

As another variation, suppose that l of the K candidates are women and that a regulation requires that at least as many women be hired as men. Evidently, the state in which all women and no men are elected is a median point. There may be other median points, but in general the space is not a median space; for instance, the space that results from taking $l = 2$ and $K = 3$ is isomorphic to the space $X_{(3;1,3)}$ above, since there is only one infeasible state, the state in which the man is elected while both women are rejected.

3.3 Neutral Aggregation

The unanimity rules considered in the previous subsection treat properties in an extremely asymmetric way. It is therefore natural to ask when a property space admits monotone Arrowian aggregators that treat properties symmetrically. This question is answered in this subsection.

Definition (Neutrality, Unbiasedness, Uniformity) Let x_i, y_i, $i = 1, ..., n$, and two properties H and H' be given such that, for all i, $x_i \in H \Leftrightarrow y_i \in H'$. An aggregator f is called *neutral with respect to* H *and* H' if in this situation $f(x_1, ..., x_n) \in H \Leftrightarrow f(y_1, ..., y_n) \in H'$. An aggregator f is called neutral within issues or *unbiased* if, for all H, f is neutral with respect to H and H^c; moreover, f is called neutral across issues or *uniform* if, for all H and H', f is neutral with respect to H and H' or with respect to H^c and H'; finally, f is called (fully) *neutral* if it is neutral with respect to all H and H'.

Examples of aggregators that are uniform but not unbiased are the unanimity rules, or more generally, supermajority rules with a uniform quota $> 1/2$ for each issue. An example of an aggregator that is unbiased but not uniform is *weighted* issue-by-issue majority voting where the weights differ across issues. Specifically, let $\mathcal{H} = \{H_1, H_1^c, ..., H_K, H_K^c\}$; for all k and i, denote by $w_i^k \geq 0$ the weight of voter i in issue k, and assume that $\sum_i w_i^k = 1$ for all $k = 1, ..., K$. Weighted issue-by-issue majority voting is defined by

$$f(x_1, ..., x_n) \in H_k :\Leftrightarrow \sum_{i: x_i \in H_k} w_i^k > 1/2.$$

The difference in weights across issues may be the natural result of voters having different stakes and/or different expertise in different dimensions.

Theorem 4 *Let* $(X, \mathcal{H})$ *be a property space.*

a) $(X, \mathcal{H})$ *admits non-dictatorial and uniform monotone Arrowian aggregators if and only if* $(X, \mathcal{H})$ *admits a median point.*

b) $(X, \mathcal{H})$ *admits locally non-dictatorial and unbiased monotone Arrowian aggregators if and only if* $(X, \mathcal{H})$ *is a median space.*

c) $(X, \mathcal{H})$ *admits non-dictatorial and neutral monotone Arrowian aggregators if and only if* $(X, \mathcal{H})$ *is a median space.*

The important role of median spaces in the context of neutral Arrowian aggregation has already been emphasized in Nehring and Puppe (2007), where part c) of Theorem 4 is proved.

Note that, in accordance with part b), non-dictatorial and unbiased aggregators may exist also outside the class of median spaces. In particular, if a property space can be decomposed into components that do not interact with each other one can specify different local dictators on different components. On the other hand, if a space is "indecomposable" one obtains the stronger result that already the existence of non-dictatorial and unbiased monotone Arrowian aggregators requires a median space. Here, $(X, \mathcal{H})$ is called *decomposable* if $\mathcal{H}$ can be partitioned into two non-empty subfamilies $\mathcal{H}_1$ and $\mathcal{H}_2$ such that each critical family is either entirely contained in $\mathcal{H}_1$ or entirely contained in $\mathcal{H}_2$; otherwise, $(X, \mathcal{H})$ is called *indecomposable.* One can easily show that a property space is decomposable if and only if it can be represented as the Cartesian product of (at least) two property spaces. In the context of Arrowian (i.e. independent) aggregation, indecomposability can be assumed without loss of generality, since the admissible Arrowian aggregators on a decomposable space are derived from combining the admissible Arrowian aggregators on its components; this is a straightforward implication of the Intersection Property.

Proposition 3.3 *Suppose that* $(X, \mathcal{H})$ *is indecomposable. Then, any monotone Arrowian aggregator that is unbiased is also uniform, hence neutral.*

Example: Neutral Aggregation of Weak Orderings Consider again the property space $(\mathcal{W}eak(A), \mathcal{H}_{\mathcal{L}})$ defined in Section 2.3 above. In this space, all non-trivial critical families have the form $\{H_L, H_{L'}\}$ for $L, L' \subseteq A$ with $L \not\subseteq L'$ and $L' \not\subseteq L$. In particular, all critical families have exactly two elements, i.e. the space is a median space. By Theorem 4, $(\mathcal{W}eak(A), \mathcal{H}_{\mathcal{L}})$ admits neutral monotone Arrowian aggregators; for instance, issue-by-issue majority voting with an odd number of individuals is consistent on $(\mathcal{W}eak(A), \mathcal{H}_{\mathcal{L}})$. Moreover, the space $(\mathcal{W}eak(A), \mathcal{H}_{\mathcal{L}})$ is indecomposable, hence any unbiased monotone Arrowian aggregator is fully neutral by Proposition 3.3.

3.4 Independence without Monotonicity

The proofs of our main characterization results provided in the appendix rely on the characterization of all monotone Arrowian aggregators in terms of the Intersection Property. We have already noted that the monotonicity condition is crucial in this characterization. While the monotonicity condition seems conceptually uncontroversial, mathematically, it may be interesting to explore the aggregation possibilities without it. Some results have already been established in that direction. Dokow and Holzman (2010a) identify a condition ("non-affineness") that together with total blockedness characterizes the aggregation problems on which all Arrowian aggregators (with or without monotonicity) are dictatorial. In a similar vein, Dietrich and List (2007, 2010) show how Theorem 4b) and 4c) can be adapted to the non-monotone case. Finally, Dietrich and List (2009) show that Theorem 3 remains valid also without the monotonicity requirement.

4 Application: Preference Aggregation on Restricted Domains

4.1 Strategy-Proof Social Choice on Generalized Single-Peaked Domains

The results of the present paper allow one to derive corresponding results on the existence of strategy-proof social choice functions on a large class of domains, the "generalized single-peaked domains" introduced in Nehring and Puppe (2002, 2007), as follows.

Definition ($\mathcal{H}$-Betweenness) Let $(X, \mathcal{H})$ be a property space, and let $x, y, z \in X$. Say that y is *$\mathcal{H}$-between* x and z if $\mathcal{H}_y \supseteq (\mathcal{H}_z \cap \mathcal{H}_x)$, i.e. if y shares all properties common to x and z.

Definition (Generalized Single-Peakedness) Consider $\succ \in \mathcal{L}in(X)$ with top element ("peak") $x^* \in X$. The preference ordering $\succ$ is called *generalized single-peaked* on $(X, \mathcal{H})$ if, for all distinct y, z,

$$y \succ z \text{ whenever } y \text{ is } \mathcal{H}\text{-between } x^* \text{ and } z.$$

The class of all generalized single-peaked preferences on $(X, \mathcal{H})$ is denoted by $\mathcal{S}_{(X,\mathcal{H})}$.

Thus, a preference is generalized single-peaked on $(X, \mathcal{H})$ whenever points that share more properties with the ideal point are strictly preferred. The underlying geometric intuition is especially transparent in those cases in which the betweenness of $\mathcal{H}$ is derived from its graph. This includes many naturally occurring domains of generalized single-peaked preferences. The graph $\Gamma_{\mathcal{H}}$ of a property space $(X, \mathcal{H})$ is the set of pairs of points $\{x, z\}$ without any other points between them: y $\mathcal{H}$-between x and z implies $y \in \{x, z\}$. For any graph Γ on X, y is *graphically between*, or *Γ-between*, x and z if it is on a shortest path connecting x and z. The property space $(X, \mathcal{H})$ is said to be *graphic* if its property and graphic betweenness coincide.

The three property spaces shown in Fig. 3 are graphic, defining important and frequently studied classes of generalized single-peaked domains.

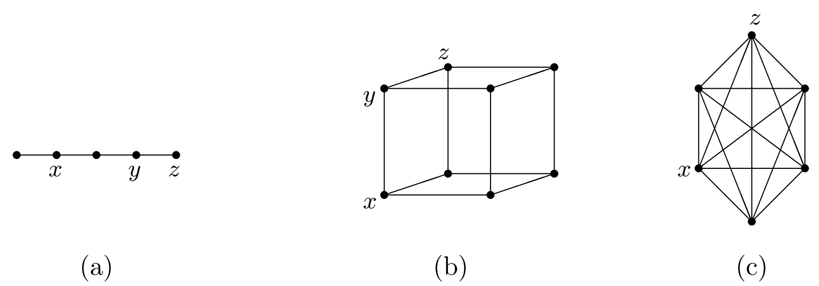


Figure 3: Graphic Property Spaces

The linear graph in Fig. 3(a) corresponds to the collection of all properties of the form "lying to the left of" (resp. "lying to the right of") some given point. Evidently, y is between x and z in Fig. 3(a), and a preference ordering is generalized single-peaked if and only if it is single-peaked in the usual sense. The domain $\mathcal{S}_{(X,\mathcal{H})}$ is thus the classical domain of single-peaked preferences introduced by Black (1958) and further studied in Moulin (1980).

The graph in Fig. 3(b) corresponds to the hypercube $X = \{0,1\}^K$ (depicted here for $K = 3$). The displayed triple of points again provides an instance of betweenness with y between x and z. As is easily seen, a preference ordering $\succ$ is generalized single-peaked if and only if it is *separable* in the sense that, for all $x, y \in \{0,1\}^K$ and all $k = 1, ..., K$, $x \succ (x^{-k}, y^k) \Leftrightarrow (y^{-k}, x^k) \succ y$. Thus, in this case the domain $\mathcal{S}_{(X,\mathcal{H})}$ coincides with the domain of all separable preferences studied in Barberà, Sonnenschein and Zhou (1991).

Finally, the complete graph shown in Fig. 3(c) corresponds to the property space $(X, \mathcal{H})$ with $\mathcal{H} = \{\{x\}, X \setminus \{x\} : x \in X\}$. Evidently, no point is ever between two other points, hence the domain $\mathcal{S}_{(X,\mathcal{H})}$ of all generalized single-peaked preferences coincides with the unrestricted preference domain $\mathcal{L}in(X)$.

Consider now a social choice function (scf) of the form $F : \mathcal{D}^n \to X$ for some $\mathcal{D} \subseteq \mathcal{S}_{(X,\mathcal{H})}$. The scf F is said to satisfy "peaks only" if it only depends on the top elements of the preference orderings. In this case, the scf F induces an aggregator $f : X^n \to X$ by letting $f(x_1, ..., x_n) := F(\succ_1, ..., \succ_n)$ where the x_i are the top elements of the $\succ_i$.

In Nehring and Puppe (2002, 2007) it is shown that an scf F defined on a sufficiently rich domain of generalized single-peaked preferences on $(X, \mathcal{H})$ is strategy-proof and onto if and only if (i) it satisfies peaks only, and (ii) the induced aggregator f satisfies independence, monotonicity and unanimity, i.e. is a monotone Arrowian aggregator on $(X, \mathcal{H})$.[8] The question of the existence of strategy-proof scfs on generalized single-peaked domains of a certain type thus boils down to the question of the existence of appropriate monotone Arrowian aggregators. In particular, Theorem 1 above provides the necessary and sufficient condition of when a generalized single-peaked domain on a property space only admits dictatorial strategy-proof and onto scfs. We thus obtain the following result.

Theorem 5 *A property space $(X, \mathcal{H})$ admits only dictatorial strategy-proof and onto scfs $F : \mathcal{S}^n_{(X,\mathcal{H})} \to X$ if and only if $(X, \mathcal{H})$ is totally blocked.*

Since the unrestricted preference domain is a special generalized single-peaked domain, Theorem 5 constitutes a maximal generalization of the Gibbard-Satterthwaite theorem to domains of generalized single-peaked preferences.[9] Similarly, Theorems 2 - 4 entail corresponding characterizations of the generalized single-peaked domains that admit locally non-dictatorial, anonymous and neutral strategy-proof and onto scfs, respectively.[10]

4.2 Arrowian Aggregation on Domains of Single-Peaked Preferences

We have noted that the characterization of dictatorial monotone Arrowian aggregation provided by Theorem 1 applies to standard preference domains such as the set of all

[8]The proof of the "peaks only" property is based on prior work by Barberà, Masso and Neme (1997), and others.

[9]A weaker generalization of the Gibbard-Satterthwaite theorem to domains of generalized single-peaked preferences has been obtained in Nehring and Puppe (2007). Aswal, Chatterji and Sen (2003) provide a generalization of the Gibbard-Satterthwaite theorem in a different direction; their result is neither implied, nor implies our Theorem 1 above.

[10]This is described in detail with further examples in the unpublished manuscript Nehring and Puppe (2005).

linear orders $\mathcal{L}in(A)$ on a set A of alternatives. An issue that has attracted fairly wide attention is the possibility of achieving positive results by restricting the domain of preferences. We will demonstrate the versatility of the tools and results developed here by studying the possibility of Arrowian preference aggregation on the class of preferences that are single-peaked relative to a graph.

Let A be a set of alternatives, and let $\Gamma \subseteq A \times A$ be a (non-directed) connected graph on A. Recall that, for any graph Γ on A, y is graphically between x and z if it is on a shortest path connecting x and z.

Definition (Graphic Single-Peakedness) A strict linear ordering $\succ$ on A with top element x^* will be called *single-peaked with respect to* Γ if, for all distinct y, z,

$$y \succ z \text{ whenever } y \text{ lies graphically between } x^* \text{ and } z.$$

The class of all linear orderings on A that are single-peaked with respect to Γ is denoted by $\mathcal{S}_{(A,\Gamma)}$.

Quite a few graphs do not come from property spaces; conversely, not every property space is graphic. Hence, the preference domains considered here and in the previous subsection are mutually incomparable. However, the domain of overlap appears to contain most of the natural instances of generalized single-peakedness (relative to either notion of betweenness).[11]

One could apply the results of the present paper directly to obtain results on monotone Arrowian aggregators that map profiles of single-peaked preferences to single-peaked preferences, i.e. to aggregators of the form $f : \mathcal{S}^n_{(A,\Gamma)} \to \mathcal{S}_{(A,\Gamma)}$.[12] This may not be viewed as fully satisfactory, especially if an impossibility is derived, since one main rationale for imposing restrictions on group preferences is the enabling of a well-defined optimum for all choice sets. This rationale motivates only transitivity but not single-peakedness constraints. Thus it is more natural to study aggregators of the form $f : \mathcal{S}^n_{(A,\Gamma)} \to \mathcal{L}in\,(A)$.

At first sight, this seems to render the methods and results here inapplicable but this inference would be too rash. To describe the appropriate adaptation of our methodology to this case, consider monotone Arrowian aggregators of the form $f : Y^n \to X$ where $(X, \mathcal{H})$ is a property space and $Y \subseteq X$. It is still true that any monotone Arrowian aggregator takes the form of consistent voting by issues. The Intersection Property is still (trivially) sufficient for consistency, but no longer necessary since some inconsistencies may be precluded from materializing due to the domain restriction to Y. To overcome this difficulty we provide a modified necessary condition that, although not sufficient for consistency, is powerful enough to derive dictatorship in many cases, as follows.

Definition (Effective Critical Family) Say that $\mathcal{G}$ is an *effective critical family* if

[11] A necessary and sufficient condition for when a graph (A, Γ) can be endowed with the structure of a property space $(A, \mathcal{H})$ such that the notions of Γ-betweenness and $\mathcal{H}$-betweenness coincide, i.e. such that $\mathcal{S}_{(A,\Gamma)} = \mathcal{S}_{(A,\mathcal{H})}$, is given in Nehring and Puppe (2007, Fact 2.2).

[12] Note that while the domains of individual preferences are single-peaked linear orderings on a property space $(X, \mathcal{H})$ both here and in the previous subsection, the aggregation exercise is very different due to the "peaks only" condition in the previous subsection which implies that the strategy-proof aggregation of preferences reduces to a monotone aggregation of peaks. For instance, in case of the K-dimensional hypercube the number of properties is $2K$ in the context of strategy-proof social choice (with "peaks only") while here every *pair* of distinct elements of the hypercube defines a property, thus there are $2^K \cdot (2^K - 1)/2$ properties in our present context.

$\cap\mathcal{G} = \emptyset$ and, for all $G \in \mathcal{G}$, $(\cap\mathcal{G} \setminus \{G\}) \cap Y \neq \emptyset$.

Definition (Restricted Intersection Property) A structure of winning coalitions $\mathcal{W}$ satisfies the *restricted Intersection Property* if, for any effective critical family $\{G_1, ..., G_l\} \subseteq \mathcal{H}$ and any selection $W_j \in \mathcal{W}_{G_j}$,

$$\bigcap_{j=1}^{l} W_j \neq \emptyset.$$

Thus, a critical family is *effective* if its criticality, i.e. the consistency of any proper subfamily, can be verified by means of elements in the restricted domain Y. This modified concept will prove useful via the following result, the proof of which duplicates the argument for the necessity of the original Intersection Property for consistency given in Proposition 2.1 above.

Proposition 4.1 *Let $(X, \mathcal{H})$ be a property space and $Y \subseteq X$. Any monotone Arrowian aggregator $f : Y^n \to X$ takes the form of voting by issues with a structure of winning coalitions satisfying the restricted Intersection Property.*

Say that a property H *restricted conditionally entails* a property G, written as $H \geq_Y^0 G$ if $H \neq G^c$ and there exists an effective critical family containing both H and G^c. Moreover, denote by $\geq_Y$ the transitive closure of the restricted conditional entailment relation, and say that Y is *totally blocked in* $(X, \mathcal{H})$ if, for all $H, G \in \mathcal{H}$, $H \geq_Y G$.

The restricted Intersection Property may be far from characterizing monotone Arrowian aggregators $f : Y^n \to X$ in general, but it will be powerful if the restricted conditional entailment relation is rich. For example, from Proposition 4.1 and the proof of Theorem 1 above it is immediate that if Y is totally blocked in $(X, \mathcal{H})$ then the only monotone Arrowian aggregators $f : Y^n \to X$ are dictatorships.

We apply this now to the aggregation of preferences that are single-peaked with respect to a graph, viewed as a subset of the set $(\mathcal{L}in(A), \mathcal{H}_{\mathcal{R}})$ of all linear orderings with the standard "relational" property space structure.

Definition (Bi-Connectedness) Say that a graph (A, Γ) is *bi-connected* if it is connected even after removal of a single vertex and all of its edges.

For instance, the line in Fig. 3(a) is clearly not bi-connected, while the hypercube graph on $\{0,1\}^K$ (see Fig. 3(b) for the case $K = 3$) is bi-connected for all $K \geq 2$, and the complete graph (cf. Fig. 3(c)) is bi-connected whenever $\#A \geq 3$.

Theorem 6 *Let (A, Γ) be a connected graph, and $\mathcal{S}_{(A,\Gamma)}$ the set of all single-peaked preferences on A with respect to Γ. Suppose that $\mathcal{L}in(A)$ is endowed with the standard (relational) property structure $\mathcal{H}_{\mathcal{R}}$.*

a) *If (A, Γ) is bi-connected, then any monotone Arrowian aggregator $f : \mathcal{S}^n_{(A,\Gamma)} \to \mathcal{L}in(A)$ is dictatorial.*

b) *If (A, Γ) has a vertex with only one neighbor, then there exist non-dictatorial monotone Arrowian aggregators $f : \mathcal{S}^n_{(A,\Gamma)} \to \mathcal{L}in(A)$.*

c) *There exist locally non-dictatorial monotone Arrowian aggregators $f : \mathcal{S}^n_{(A,\Gamma)} \to \mathcal{L}in(A)$ if and only if (A, Γ) is a line. In this case, issue-by-issue majority voting is consistent.*

The paradigmatic cases to which part b) applies are acyclic graphs, i.e. trees. The consistency of issue-by-issue majority voting on the line asserted in part c) has already been observed in the literature (see Moulin (1988)).[13]

The application to "concrete Arrowian" aggregation presented in this subsection takes a step beyond the rest of the paper and most of the existing literature on judgement aggregation by considering situations in which the domain of individual evaluations (single-peaked orderings) may be strictly smaller than that of social evaluations (allowed to be any linear ordering). The Intersection Property continues to be sufficient but ceases to be necessary for monotone Arrowian aggregation. By contrast, the introduced "restricted" Intersection Property is necessary and not sufficient, but it is strong enough to obtain the almost-characterization in Theorem 6. Especially for the fans of impossibility theorems, the restricted Intersection Property could be useful in the further study of abstract Arrowian aggregation with restricted domains.

[13] Theorem 6 falls short of a full characterization, since a gap is left between the sufficient conditions for dictatorship given in part a) and those for non-dictatorship in b). An example of a graph that is not covered by the theorem is the union of two disjoint triangles linked by a single edge. We conjecture that all domains with missing characterization are dictatorial; that is, for a connected graph Γ all monotone Arrowian aggregators $f : \mathcal{S}^n_{(A,\Gamma)} \to \mathcal{L}in(A)$ are dictatorial if and only if all all vertices have at least two neighbors. Since the verification of this conjecture appears to involve additional, somewhat ad-hoc arguments, we did not pursue this further.

Appendix A: Anonymity without Median Points

Consider the subspace $X \subseteq \{0,1\}^5$ shown in Figure 4 below. The two cubes to the right correspond to a "1" in coordinate 4 (i.e. to the property H_1^4), similarly, the two top cubes correspond to a "1" in coordinate 5 (i.e. to H_1^5). Missing points of the 5-hypercube are indicated by blank circles. For the purpose of better illustration, the edges connecting different points across the four subcubes have been omitted in the figure.

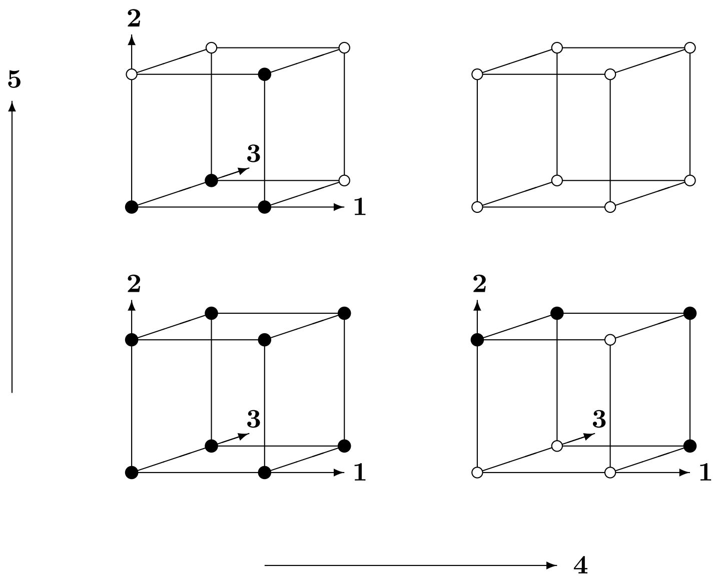


Figure 4: A quasi-unblocked space without median points

This space is characterized by the following critical families: $\mathcal{G}_1 = \{H_1^1, H_0^3, H_1^4\}$, $\mathcal{G}_2 = \{H_1^1, H_1^3, H_1^5\}$, $\mathcal{G}_3 = \{H_0^1, H_0^2, H_1^4\}$, $\mathcal{G}_4 = \{H_0^1, H_1^2, H_1^5\}$, $\mathcal{G}_5 = \{H_0^2, H_0^3, H_1^4\}$, $\mathcal{G}_6 = \{H_1^2, H_1^3, H_1^5\}$ and $\mathcal{G}_7 = \{H_1^4, H_1^5\}$. For instance, the criticality of $\{H_1^4, H_1^5\} = \mathcal{G}_7$ reflects the fact that the top-right cube contains no element of X, and is a maximal subcube with this property. As is easily verified, one has $H_0^k \equiv H_1^k$ for $k = 1, 2, 3$, i.e. the first three coordinates are blocked. For instance, one has $H_0^1 \geq^0 H_1^2$ and $H_0^2 \geq^0 H_1^1$ due to criticality of $\mathcal{G}_3$, etc. By Proposition 3.1, the underlying space admits no median points. Nevertheless, denoting by q_1^k the quota corresponding to H_1^k, the following anonymous choice rule is easily seen to be consistent if the number of voters is odd: The final outcome lies in the top left subcube if and only if all voters endorse property H_1^5 (i.e. $q_1^5 = 1$); similarly, the choice is in the bottom right subcube if and only if all voters endorse H_1^4 (i.e. $q_1^4 = 1$). In all other cases, the outcome lies in the bottom left subcube ($q_0^5 = q_0^4 = 0$). In addition, the location of the outcome within any of the three admissible subcubes is decided by majority vote in each of the first three coordinates ($q_1^1 = q_1^2 = q_1^3 = \frac{1}{2}$). By Proposition 2.2 this rule is in fact the *only* anonymous and monotone Arrowian aggregator in the present example. Note in particular that, in accordance with Theorem 3, there is no anonymous rule for an even number of voters.

Appendix B: Remaining Proofs

We start with the following lemma that turns out to be very useful in the subsequent proofs. Let $(X, \mathcal{H})$ be a property space. Partition $\mathcal{H}$ as follows.

$$\begin{aligned} \mathcal{H}_0 &:= \{H \in \mathcal{H} : H \equiv H^c\}, \\ \mathcal{H}_1^+ &:= \{H \in \mathcal{H} : H > H^c\}, \\ \mathcal{H}_1^- &:= \{H \in \mathcal{H} : H^c > H\}, \\ \mathcal{H}_2 &:= \{H \in \mathcal{H} : \text{ neither } H \geq H^c \text{ nor } H^c \geq H\}. \end{aligned}$$

Thus, $\mathcal{H}_0$ is the family of all blocked properties, $\mathcal{H}_1^+$ denotes the family of all those properties the complement of which can be reached by a sequence of conditional entailments, but not vice versa. The family of the complements of the properties in $\mathcal{H}_1^+$ is denoted by $\mathcal{H}_1^-$, and the remaining properties are collected in $\mathcal{H}_2$.

Lemma 3 a) *For any critical family $\mathcal{G}$, if $G \in \mathcal{G} \cap \mathcal{H}_1^-$, then $\mathcal{G} \setminus \{G\} \subseteq \mathcal{H}_1^+$.*
b) *For any critical family $\mathcal{G}$, if $\mathcal{G} \cap \mathcal{H}_0 \neq \emptyset$, then $\mathcal{G} \subseteq \mathcal{H}_0 \cup \mathcal{H}_1^+$.*
c) *Take any $\tilde{H} \in \mathcal{H}_2$. Then there exists a partition of $\mathcal{H}_2$ into $\mathcal{H}_2^-$ and $\mathcal{H}_2^+$ with $\tilde{H} \in \mathcal{H}_2^-$ such that $G \in \mathcal{H}_2^- \Leftrightarrow G^c \in \mathcal{H}_2^+$, and for no $G \in \mathcal{H}_2^-$ and $H \in \mathcal{H}_2^+$, $G \geq H$.*

Proof of Lemma 3 a) Suppose $G \in \mathcal{G} \cap \mathcal{H}_1^-$, i.e. $G^c > G$. Consider any other $H \in \mathcal{G}$. We have $H \geq G^c > G \geq H^c$, hence $H > H^c$, i.e. $H \in \mathcal{H}_1^+$.
b) Suppose $G \in \mathcal{G} \cap \mathcal{H}_0$ and let $H \in \mathcal{G}$ be different from G. We have $H \geq G^c \equiv G \geq H^c$, hence $H \geq H^c$. But this means $H \in \mathcal{H}_0 \cup \mathcal{H}_1^+$.
c) The desired partition into $\mathcal{H}_2^- = \{G_1, ..., G_l\}$ and $\mathcal{H}_2^+ = \{G_1^c, ..., G_l^c\}$ will be constructed inductively. Set $G_1 = \tilde{H}$, and suppose that $\{G_1, ..., G_r\}$, with $r < l$, is determined such that $G_j \not\geq G_k^c$ for all $j, k \in \{1, ..., r\}$. Take any $H \in \mathcal{H}_2 \backslash \{G_1, G_1^c, ..., G_r, G_r^c\}$ and set

$$G_{r+1} := \begin{cases} H & \text{if for no } j \in \{1, ..., r\} : \quad G_j \geq H^c \\ H^c & \text{if for some } j \in \{1, ..., r\} : \quad G_j \geq H^c \end{cases}$$

First note that $G_{r+1} \not\geq G_{r+1}^c$ since $H \in \mathcal{H}_2$. Thus, the proof is completed by showing that for no $k \in \{1, ..., r\}$, $G_k \geq G_{r+1}^c$ (and hence also not $G_{r+1} \geq G_k^c$). To verify this, suppose first that $G_{r+1} = H$; then, the claim is true by construction. Thus, suppose $G_{r+1} = H^c$; by construction, there exists $j \leq r$ with $G_j \geq H^c$, hence also $H \geq G_j^c$. Assume, by way of contradiction, that $G_k \geq G_{r+1}^c$, i.e. $G_k \geq H$. This would imply $G_k \geq H \geq G_j^c$, in contradiction to the induction hypothesis.

We first use Lemma 3 to prove Proposition 3.1, and then move on to the proof of Theorem 1. Proposition 3.1 is due to Nehring (2010); we include its proof here to make the exposition self-contained.

Proof of Proposition 3.1 Suppose that $(X, \mathcal{H})$ is unblocked, i.e. suppose that $\mathcal{H}_0$ is empty. Partition $\mathcal{H}$ into $\mathcal{H}_1^-$, $\mathcal{H}_1^+$, $\mathcal{H}_2^-$ and $\mathcal{H}_2^+$ according to Lemma 3. Then, any critical family $\mathcal{G}$ can meet $\mathcal{H}_1^- \cup \mathcal{H}_2^-$ at most once. Indeed, by Lemma 3a), $H \in \mathcal{G} \cap \mathcal{H}_1^-$ implies $\mathcal{G} \setminus \{H\} \subseteq \mathcal{H}_1^+$. Furthermore, if $\{H, H'\} \subseteq \mathcal{G} \cap \mathcal{H}_2^-$, one would obtain $H' \geq H^c$ which contradicts the construction of $\mathcal{H}_2^-$. But this implies that $\cap(\mathcal{H}_1^- \cup \mathcal{H}_2^-)$ is nonempty (otherwise it would contain a critical family), and by H3, it consists of a single element, say $\hat{x}$. By definition, $\hat{x} \in M(X)$.

Conversely, let $\hat{x} \in M(X)$, and consider any $H \in \mathcal{H}_{\hat{x}}$. By definition, $H \geq^0 G$ means that $\{H, G^c\} \subseteq \mathcal{G}$ for some critical family $\mathcal{G}$. Since $\hat{x} \in M(X)$, $\mathcal{G}$ contains at most one element of $\mathcal{H}_{\hat{x}}$, hence $G^c \notin \mathcal{H}_{\hat{x}}$, which implies $G \in \mathcal{H}_{\hat{x}}$. This observation immediately implies $H \not\equiv H^c$. Hence, $(X, \mathcal{H})$ is unblocked.

Proof of Theorem 1 (Sufficiency of Non-Total Blockedness) Let $(X, \mathcal{H})$ be not totally blocked and partition $\mathcal{H}$ as above. If $\mathcal{H}_1^+ \cup \mathcal{H}_1^-$ is non-empty, set $\mathcal{W}_H = 2^N \setminus \{\emptyset\}$ for all $H \in \mathcal{H}_1^-$ and $\mathcal{W}_H = \{N\}$ for all $H \in \mathcal{H}_1^+$; moreover, choose a voter $i \in N$ and set $\mathcal{W}_G = \{W \subseteq N : i \in W\}$ for all other $G \in \mathcal{H}$. Clearly, the corresponding voting by issues is non-dictatorial. It also satisfies the Intersection Property. Indeed, the only problematic case is when a critical family $\mathcal{G}$ contains elements of $\mathcal{H}_1^-$. However, by Lemma 3a), if $G \in \mathcal{G} \cap \mathcal{H}_1^-$, we have $\mathcal{G} \setminus \{G\} \subseteq \mathcal{H}_1^+$, in which case the Intersection Property is clearly satisfied.

Next, suppose that $\mathcal{H}_1^+ \cup \mathcal{H}_1^-$ is empty, and consider first the case in which both $\mathcal{H}_0$ and $\mathcal{H}_2$ are non-empty. By Lemma 3b), no critical family $\mathcal{G}$ can meet both $\mathcal{H}_0$ and $\mathcal{H}_2$. Hence, we can specify two different dictators on $\mathcal{H}_0$ and $\mathcal{H}_2$, respectively, by setting $\mathcal{W}_H = \{W : i \in W\}$ for all $H \in \mathcal{H}_0$ and $\mathcal{W}_G = \{W : j \in W\}$ for all $G \in \mathcal{H}_2$ with $i \neq j$. Clearly, the Intersection Property is satisfied in this case.

Now suppose that $\mathcal{H}_2$ is also empty, i.e. $\mathcal{H} = \mathcal{H}_0$. Since $(X, \mathcal{H})$ is not totally blocked, $\mathcal{H}$ is partitioned in at least two equivalence classes with respect to the equivalence relation $\equiv$. Since, obviously, no critical family can meet two different equivalence classes, we can specify different dictators on different equivalence classes while satisfying the Intersection Property.

Finally, if $\mathcal{H}_0$ is empty, $(X, \mathcal{H})$ admits a median point $\hat{x}$ by Proposition 3.1. Set $\mathcal{W}_H = 2^N \setminus \{\emptyset\}$ for all $H \in \mathcal{H}_{\hat{x}}$ and $\mathcal{W}_H = \{N\}$ for all $H \notin \mathcal{H}_{\hat{x}}$. By the Intersection Property the corresponding voting by issues is consistent. Evidently, it coincides with the Arrowian unanimity rule $f_{\hat{x}}$ which is non-dictatorial. This completes the proof of Theorem 1.

Proof of Theorem 2 Obviously, (ii) implies (i). Thus, it suffices to show that (i) implies (iii), and that (iii) implies (ii).

"(i) $\Rightarrow$ (iii)" We prove the claim by contraposition. Assume that $(X, \mathcal{H})$ is not quasi-unblocked. This means that there exists $G \in \mathcal{H}$ with $G \equiv G^c$ and some critical family $\mathcal{G}$ such that $(\mathcal{H}_{\equiv G} \cap \mathcal{G}) \supseteq \{H, H', H''\}$ for three distinct H, H', H''. Consider a structure of winning coalitions satisfying the Intersection Property. By Lemma 1, $\mathcal{W}_H = \mathcal{W}_G$ for all $H \in \mathcal{H}_{\equiv G}$. By Lemma 2, applied to the critical family $\mathcal{G} \supseteq \{H, H', H''\}$, there exists i, such that $\{i\} \in \mathcal{W}_H$ for all $H \in \mathcal{H}_{\equiv G}$. Hence, i is a dictator on $\mathcal{H}_{\equiv G}$, which proves the claim.

"(iii) $\Rightarrow$ (ii)" We will construct an anonymous Arrowian aggregator by specifying an appropriate structure of winning coalitions, provided that $(X, \mathcal{H})$ is quasi-unblocked. Partition $\mathcal{H}$ as in Lemma 3 above. Let n be odd, and set

$$\begin{array}{lll} \mathcal{W}_H = \{W : \#W > n/2\} & \text{if} & H \in \mathcal{H}_0, \\ \mathcal{W}_H = 2^N \setminus \{\emptyset\} & \text{if} & H \in \mathcal{H}_1^- \cup \mathcal{H}_2^-, \\ \mathcal{W}_H = \{N\} & \text{if} & H \in \mathcal{H}_1^+ \cup \mathcal{H}_2^+. \end{array}$$

Clearly, this structure of winning coalitions is anonymous; we will show that it satisfies the Intersection Property. Let $\mathcal{G}$ be a critical family; we distinguish three cases.

Case 1: $\mathcal{G} \cap (\mathcal{H}_1^- \cup \mathcal{H}_2^-) \neq \emptyset$. If $G \in \mathcal{G} \cap \mathcal{H}_1^-$, then by Lemma 3a), $\mathcal{G} \setminus \{G\} \subseteq \mathcal{H}_1^+$, and the Intersection Property is clearly satisfied. Thus, suppose that there exists $H \in \mathcal{G} \cap \mathcal{H}_2^-$. By Lemma 3b), we must have $\mathcal{G} \cap \mathcal{H}_0 = \emptyset$, and by Lemma 3a), $\mathcal{G} \cap \mathcal{H}_1^- = \emptyset$.

Hence, if there exists $H' \in \mathcal{G} \setminus \{H\}$ with $\mathcal{W}_{H'} \neq \{N\}$, we must have $H' \in \mathcal{H}_2^-$. But then $H \geq (H')^c$ contradicts the construction of $\mathcal{H}_2^-$ and $\mathcal{H}_2^+$ in Lemma 3c). Thus, if $H \in \mathcal{G} \cap \mathcal{H}_2^-$, one has $\mathcal{W}_{H'} = \{N\}$ for any other element $H' \in \mathcal{G}$, in which case the Intersection Property is satisfied.

Case 2: $\mathcal{G} \cap \mathcal{H}_0 \neq \emptyset$. First, observe that $G_1 \equiv G_2$ whenever $\{G_1, G_2\} \subseteq \mathcal{G} \cap \mathcal{H}_0$. Indeed, $G_1 \equiv G_2$ follows at once from $G_1 \geq G_2^c$, $G_2 \geq G_1^c$, $G_1 \equiv G_1^c$ and $G_2 \equiv G_2^c$. Thus, by quasi-unblockedness, $\mathcal{G}$ can contain at most two elements of $\mathcal{H}_0$. By Lemma 3b), for any $H \in \mathcal{G} \setminus \mathcal{H}_0$ one has $\mathcal{W}_H = \{N\}$. Hence, the Intersection Property is also satisfied in Case 2.

Case 3: If $\mathcal{G}$ does not meet $\mathcal{H}_0$, $\mathcal{H}_1^-$ and $\mathcal{H}_2^-$, then $\mathcal{G} \subseteq (\mathcal{H}_1^+ \cup \mathcal{H}_2^+)$, in which case the Intersection Property is trivially satisfied. This completes the proof of Theorem 2.

Proof of Proposition 3.2 Let $f_{\hat{x}}$ be an Arrowian unanimity rule and consider the set $\mathcal{H}_{\hat{x}}$ of all properties possessed by $\hat{x}$. As is easily verified, $f_{\hat{x}}$ corresponds to voting by issues with $\mathcal{W}_H = 2^N \setminus \{\emptyset\}$ for all $H \in \mathcal{H}_{\hat{x}}$ and $\mathcal{W}_H = \{N\}$ for all $H \notin \mathcal{H}_{\hat{x}}$. Suppose that there exists a critical family $\mathcal{G}$ and two distinct H, H' with $H, H' \in \mathcal{H}_{\hat{x}} \cap \mathcal{G}$; then one can choose $W \in \mathcal{W}_H$ and $W' \in \mathcal{W}_{H'}$ with $W \cap W' = \emptyset$, violating the Intersection Property. Thus, $\#(\mathcal{H}_{\hat{x}} \cap \mathcal{G}) \leq 1$ for every critical family $\mathcal{G}$, i.e. $\hat{x} \in M(X)$.

Conversely, it is immediate from the Intersection Property (and already observed at the end of the proof of Theorem 1 above) that for any median point $\hat{x} \in M(X)$ voting by issues with $\mathcal{W}_H = 2^N \setminus \{\emptyset\}$ for all $H \in \mathcal{H}_{\hat{x}}$ and $\mathcal{W}_H = \{N\}$ for all $H \notin \mathcal{H}_{\hat{x}}$ is consistent. Evidently, it corresponds to the Arrowian unanimity rule $f_{\hat{x}}$.

Proof of Theorem 3 The equivalences "(iv) $\Leftrightarrow$ (v)" and "(iii) $\Leftrightarrow$ (v)" follow at once from Propositions 3.1 and 3.2, respectively. The implications "(iii) $\Rightarrow$ (ii)" and "(ii) $\Rightarrow$ (i)" are evident. Thus, the proof is completed by verifying the implication "(i) $\Rightarrow$ (iv)." This is done by contraposition. Thus, assume that H is blocked, i.e. $H \equiv H^c$. By Lemma 1 this implies $\mathcal{W}_H = \mathcal{W}_{H^c}$ for any structure of winning coalitions satisfying the Intersection Property. Under anonymity, this implies, using (2.1), $\mathcal{W}_H = \mathcal{W}_{H^c} = \{W \subseteq N : \#W > n/2\}$, which is compatible with (2.2) only if the number of voters is odd. This completes the proof of Theorem 3.

Proof of Theorem 4 a) By Proposition 3.2, if there exists a median point there also exists an Arrowian unanimity rule, and any such rule is uniform and non-dictatorial.

Conversely, let $f : X^n \to X$ be voting by issues satisfying the Intersection Property. We show by contraposition that if f is non-dictatorial and uniform, then $(X, \mathcal{H})$ must admit a median point. By contradiction, assume there is none. By Proposition 3.1, some property H is blocked, i.e. $H \equiv H^c$. By Lemma 1, this implies $\mathcal{W}_H = \mathcal{W}_{H^c}$, hence f is fully neutral, i.e. $\mathcal{W}_H = \mathcal{W}_0$ for all H and some fixed $\mathcal{W}_0$. Since $(X, \mathcal{H})$ is not a median space, there exists a critical family $\mathcal{G}$ with at least three elements, say $\mathcal{G} \supseteq \{G_1, G_2, G_3\}$. By Lemma 2, $\{i\} \in \mathcal{W}_{G_3^c} = \mathcal{W}_0$, i.e. voter i is a dictator.
b) Median spaces are characterized by the property that all critical families have cardinality two. By the Intersection Property this implies that issue-by-issue majority voting with an odd number of agents is consistent on any median space, and evidently, issue-by-issue majority voting with an odd number of agents is neutral, in particular unbiased.

Conversely, let $f : X^n \to X$ be voting by issues satisfying the Intersection Property. We show by contraposition that if f is locally non-dictatorial and unbiased, then $(X, \mathcal{H})$ must be a median space. Thus, suppose that $(X, \mathcal{H})$ is not a median space. Then

there exists a critical family $\mathcal{G}$ with at least three elements, say $\mathcal{G} \supseteq \{G_1, G_2, G_3\}$, in particular, $G_j \geq G_k^c$ for distinct $j, k \in \{1, 2, 3\}$. By Lemma 1, $\mathcal{W}_{G_j} \subseteq \mathcal{W}_{G_k^c}$ for distinct $j, k \in \{1, 2, 3\}$. Under unbiasedness this implies at once that $\mathcal{W}$ assigns identical families of winning coalitions to G_1, G_2, G_3 and their respective complements. By Lemma 2 above, $\{i\} \in \mathcal{W}_{G_3^c}$, i.e. voter i is a local dictator.
c) As in part b), an underlying median space guarantees the existence of a fully neutral aggregator. The converse follows from part b) together with the observation that, under full neutrality, a local dictator must even be a global dictator. This completes the proof of Theorem 4.

Proof of Proposition 3.3 Suppose that $(X, \mathcal{H})$ is indecomposable, and consider any two properties H and G. Since $(X, \mathcal{H})$ is indecomposable, there exists a sequence $H_1, H_2, ..., H_l$ such that $H = H_1$, $H_l = G$ and such that for each $j = 1, ..., l-1$ there is a critical family $\mathcal{G}_j$ that contains either H_j and H_{j+1}, or H_j^c and H_{j+1}, or H_j and H_{j+1}^c, or H_j^c and H_{j+1}^c. The claim thus follows immediately from the Intersection Property and Lemma 1.

Proof of Proposition 4.1 The proof is obtained by a straightforward adaption of the proof of necessity of the Intersection Property for consistency given in Proposition 2.1 above.

Proof of Theorem 6 a) Let (A, Γ) be a bi-connected graph, and consider a monotone Arrowian aggregator $f : \mathcal{S}^n_{(A,\Gamma)} \to \mathcal{L}in(A)$. A subset $C \subseteq A$ is identified with its induced subgraph. Thus C is a cycle if the corresponding induced subgraph is a cycle. A cycle $C \subseteq A$ is *geodesic* if the graph distance on C equals the graph distance on A, where the "graph distance" between two points is simply the number of edges of a shortest path that connects them. For each pair $(a, b) \in A \times A$, denote by $H_{a \succ b} := \{\succ \in \mathcal{L}in(A) : a \succ b\}$. The proof proceeds in several steps. First, we establish the following.

Claim 1 Let C be an geodesic cycle. For any triple of distinct points $\{a, b, c\} \subseteq C$, $\mathcal{G} = \{H_{a \succ b}, H_{b \succ c}, H_{c \succ a}\}$ is an effective critical family.

Proof of Claim 1 Evidently, by transitivity, $\cap \mathcal{G} = \emptyset$. We thus have to show that $\mathcal{S}_{(A,\Gamma)}$ has an element in common with the intersection of each pair of $\mathcal{G}$. If neither of the three points a, b, c lies on a shortest path connecting the two others points (on C and therefore by assumption also on (A, Γ)) this is obvious since single-peakedness imposes no restriction. Thus suppose that one point lies on a shortest path connecting the other two points, w.l.o.g. suppose that b lies on a shortest path connecting a and c.

The set $H_{a \succ b} \cap H_{b \succ c} \cap \mathcal{S}_{(A,\Gamma)}$ is non-empty since it contains any single-peaked preference $\succ \in \mathcal{S}_{(A,\Gamma)}$ with peak a.

The set $H_{b \succ c} \cap H_{c \succ a} \cap \mathcal{S}_{(A,\Gamma)}$ is non-empty since it contains any single-peaked preference $\succ \in \mathcal{S}_{(A,\Gamma)}$ with peak at b and such that $c \succ a$. Such preferences exist by Szpilrajn's theorem since $\succ_b \cup \{(c, a)\}$ is acyclic, where $\succ_b$ denotes the partial order induced by single-peakedness and the fact that b is the top alternative, i.e. $e \succ_b f$ if and only if e is on a shortest path between b and f.

To see that $H_{c \succ a} \cap H_{a \succ b} \cap \mathcal{S}_{(A,\Gamma)}$ is non-empty, let e be a point on C that is maximally distant from b. Then $\succ_e \cup \{(c, a), (a, b)\}$ is acyclic. Hence, appealing to Szpilrajn's theorem again, any linear extension of this relation is contained in $H_{c \succ a} \cap H_{a \succ b} \cap \mathcal{S}_{(A,\Gamma)}$.

The next two claims show that the family $\mathcal{H}' = \{H_{a \succ b} : \{a, b\} \in \Gamma\}$ is "totally blocked" with respect to $\geq_{\mathcal{S}_{(A,\Gamma)}}$.

Claim 2 Let $\{a, b\}$ and $\{c, d\}$ be any pair of edges contained in a common cycle in (A, Γ). Then $H_{a \succ b} \geq_{\mathcal{S}_{(A,\Gamma)}} H_{c \succ d}$.

Proof of Claim 2 The proof is by induction on the length of the cycle. It holds vacuously for cycles of length $l \leq 2$ since such cycles do not exist. Consider a cycle $C = \{a_1, ..., a_l\}$ with $a_1 \Gamma a_2 \Gamma ... \Gamma a_l \Gamma a_1$ of length l, and assume that the claim holds for all cycles of length strictly less than l.

If C is geodesic, then the assertion follows straightforwardly from Claim 1. If C is not geodesic, there exist $a_k, a_{k'} \in C$ with $k < k'$ such that $d_\Gamma(a_k, a_{k'}) < (k' - k)$, where d_Γ denotes the graph distance. For illustration, see Figure 5 which shows a non-geodesic cycle $(a_1 \Gamma a_2 \Gamma a_3 \Gamma a_4 \Gamma a_1)$ in a graph with five edges on four vertices; in the figure one has $k = 2$ and $k' = 4$ and $d_\Gamma(a_2, a_4) = 1$.

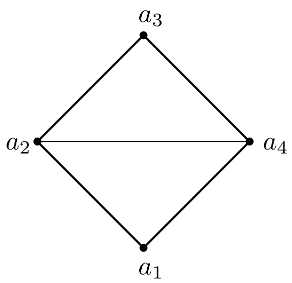


Figure 5: A Non-geodesic Cycle

Let $\{a_k, b_1, ..., b_m, a_{k'}\}$ denote a shortest path from a_k to $a_{k'}$. Then the sets $C' := \{a_k, b_1, ..., b_m, a_{k'}, a_{k'+1}, ...a_l, a_1, ...a_{k-1}\}$ and $C'' := \{a_k, b_1, ..., b_m, a_{k'}, a_{k'-1}, ...a_{k+1}\}$ are paths of length *strictly less than* l and such that $C \subset C' \cup C''$. If $\{a, b\}$ and $\{c, d\}$ are edges both of C', or both of C'', we are done by induction assumption. Suppose thus that $\{a, b\}$ is an edge of C' while $\{c, d\}$ is an edge of C''. Then by the induction assumption $H_{a \succ b} \geq_{\mathcal{S}_{(A,\Gamma)}} H_{a_k \succ b_1}$ and $H_{a_k \succ b_1} \geq_{\mathcal{S}_{(A,\Gamma)}} H_{c \succ d}$, hence $H_{a \succ b} \geq_{\mathcal{S}_{(A,\Gamma)}} H_{c \succ d}$ by transitivity.

Claim 3 Any two edges $\{a, b\}$ and $\{c, d\}$ of (A, Γ) are contained in a common cycle.

Proof of Claim 3 This is a straightforward consequence of Menger's Theorem (see, e.g. Diestel (2005, p.62)) according to which in a bi-connected graph, any two vertices can be connected by two vertex-disjoint paths, as follows.

Suppose without loss of generality that $d_\Gamma(b, c) \leq \min\{d_\Gamma(a, c), d_\Gamma(a, d), d_\Gamma(b, d)\}$. Let $\pi = \{b, e_1, ..., e_m, c\}$ be a shortest path from b to c. Clearly π contains neither a nor d. Hence $\pi' = \{a, b, e_1, ..., e_m, c, d\}$ is a path from a to d. By Menger's Theorem, there are two paths π'' and π''' connecting a and d such that $\pi'' \cap \pi''' = \{a, d\}$. If $\pi'' \cap \{b, c\} = \emptyset$ or $\pi''' \cap \{b, c\} = \emptyset$, the asserted cycle is given by $\pi' \cup \pi''$ or $\pi' \cup \pi'''$. If $\pi'' \cap \{b, c\} = \{b, c\}$ or $\pi''' \cap \{b, c\} = \{b, c\}$, the asserted cycle is given by $\pi'' \cup \pi'''$. Otherwise, without loss of generality, $\pi'' \cap \{b, c\} = \{b\}$ and $\pi''' \cap \{b, c\} = \{c\}$. Let $\widetilde{\pi}''$ denote the path from a to d replacing the subpath in π'' from a to b by the edge $\{a, b\}$. Likewise, let $\widetilde{\pi}'''$ denote the path from a to d replacing the subpath in π'' from c to d by the edge $\{c, d\}$. Then $\widetilde{\pi}'' \cap \widetilde{\pi}''' = \{a, d\}$. The asserted cycle is thus given by $\widetilde{\pi}'' \cup \widetilde{\pi}'''$.

Combining Claims 2 and 3, we obtain that $\mathcal{H}'$ is totally blocked with respect to $\geq_{\mathcal{S}_{(A,\Gamma)}}$. Hence by Theorem 1, f is dictatorial on $\mathcal{H}'$ with dictator i^*.

Take any pair $\{a, b\}$ that is not an edge of Γ. To complete the proof that f is dictatorial on all of $\mathcal{H}$, we need to show that $\{i^*\} \in \mathcal{W}_{a \succ b}$ for any such $\{a, b\}$. To verify this, take any profile $(\succ_1, \succ_2, ..., \succ_n)$ of single-peaked preferences such that i^* has peak a and all $i \neq i^*$ have peak b. Since i^* is a dictator on $\mathcal{H}'$ for all edges (c, d) such that c is Γ-between a and d, we have

$$f(\succ_1, ..., \succ_n) \subseteq H_{c \succ d}. \tag{B.1}$$

Consider any shortest path $\pi = \{e_0, e_1, ..., e_n\}$ from a to b, setting $a = e_0$ and $b = e_n$. By (B.1), $f(\succ_1, ..., \succ_n) \subseteq H_{e_m \succ e_{m+1}}$ for $m = 0, ..., n-1$. By transitivity of f as a linear order, therefore $f(\succ_1, ..., \succ_n) \subseteq H_{a \succ b}$. Since $\{i^*\} = \{i : \succ_i \in H_{a \succ b}\}$ it follows that $\{i^*\} \in \mathcal{W}_{a \succ b}$. This completes the proof of Part a).

b) Let $a \in A$ be such that it has only one neighbor in the graph, say b. Then, b is Γ-between a and all $c \in A \setminus \{a\}$. A consistent and non-dictatorial voting by issues rule is obtained by setting $\mathcal{W}_{H_{a \succ b}} = \{N\}$, $\mathcal{W}_{H_{b \succ a}} = 2^N \setminus \{\emptyset\}$, and dictatorship of some i elsewhere. Clearly, if i's peak is not on a, i's entire preference ordering is adopted; the same applies if all voters (including i) have their peak at a. If i's peak is at a but some other voter's peak is not at a, then b is the peak of the social ordering while all other comparisons follow i's preference (in particular a is socially the second best alternative).

c) Suppose f is locally non-dictatorial. By Claims 1 and 2 of the proof of part a), Γ cannot contain a cycle. Thus, Γ must be acyclic, i.e. a tree. Unless it is a line, there are three alternatives $\{a, b, c\}$ neither of which is between the other two. By consequence, the triples of properties $\{H_{a \succ b}, H_{b \succ c}, H_{c \succ a}\}$ and $\{H_{b \succ a}, H_{c \succ b}, H_{a \succ c}\}$ are both critical families implying local dictatorship in view of Theorem 2.

Conversely, if (A, Γ) is a line, then *all* critical families have the form $\{H_{a \succ b}, H_{c \succ b}\}$, where b is Γ-between a and c. Therefore, $\mathcal{S}_{(A,\Gamma)}$ is a median space, and issue-by-issue majority defines a consistent aggregation method by Proposition 2.3.

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