Nash implementable domains for the Borda count*

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Summary. We characterize the preference domains on which the Borda count satisfies Maskin monotonicity. The basic concept is the notion of a "cyclic permutation domain" which arises by fixing one particular ordering of alternatives and including all its cyclic permutations. The cyclic permutation domains are exactly the maximal domains on which the Borda count is strategy-proof (when combined with every tie breaking rule). It turns out that the Borda count is monotonic on a larger class of domains. We show that the maximal domains on which the Borda count satisfies Maskin monotonicity are the "cyclically nested permutation domains." These are the preference domains which can be obtained from the cyclic permutation domains in an appropriate recursive way.

Keywords: Maskin monotonicity, Borda count, restricted preference domains

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1 Introduction

A social choice correspondence satisfies Maskin monotonicity if and only if a chosen alternative remains a possible choice whenever in no individual's ranking its relative position to all other alternative decreases. It is well-known that Maskin monotonicity, which we shall henceforth refer to simply as "monotonicity," is a necessary condition for Nash implementability; moreover, combined with a no veto power condition it is also sufficient (Maskin (1999/1977)). In this paper, we characterize the preference domains on which the Borda count satisfies monotonicity. Since the Borda count satisfies the no veto power condition whenever there are sufficiently many voters, the result thereby also yields the preference domains on which the Borda count is Nash implementable.

The celebrated Muller-Satterthwaite theorem (Muller and Satterthwaite (1977)) establishes that, for social choice *functions* (i.e. single-valued social choice correspondences), monotonicity is equivalent to strategy-proofness, provided that all preference profiles are admissible. By contrast, while strategy-proofness always implies monotonicity, the converse need not be true on restricted domains. In fact, the main result of the present paper provides an illustration of this, showing that there exist preference domains on which the Borda count is monotonic but not strategy-proof when combined with a tie breaking rule.

The preference domains on which the Borda count (with tie breaking) is strategy-proof have been characterized in the companion paper Barbie, Puppe and Tasnádi (2006). There, we have shown that, if all individuals face the same domain restriction, the maximal strategy-proof domains for the Borda count are obtained by fixing one particular ordering of the alternatives and including all its cyclic permutations. We refer to such domains as *cyclic permutation domains*. Here, we show that monotonicity of the Borda count imposes weaker restrictions and allows one to construct domains on which possibility results emerge in a recursive way from the cyclic permutation domains. The corresponding domains are referred to as *cyclically nested permutation domains*. Specifically, we prove that, under a mild richness condition, the cyclically nested permutation domains are exactly the domains on which the Borda count is monotonic, maintaining the assumption that all individuals face the same domain restriction.

Cyclically nested permutation domains have a more complicated structure than the cyclic permutation domains from which they are recursively constructed. This is the price to be paid when moving from the stronger condition of strategy-proofness to the less demanding condition of monotonicity. Note, however, that in the context of the Borda count, monotonicity seems to be the more natural condition. Indeed, the Borda count is naturally defined as a social choice *correspondence* while the definition of strategy-proofness requires a social choice function. Thus, in order to analyze strategy-proofness, the Borda count has first to be transformed into a social choice function using a tie breaking rule.[1]

There is a large literature on domain restrictions in social choice (see Gaertner (2001) for a recent state-of-the-art summary). Most contributions in this area, however, have studied majority voting and its generalizations, taking Black's (1948) seminal contribution on the notion of single-peaked preferences as the starting point. Some

[1]In Barbie, Puppe and Tasnádi (2006), we show that in fact some results do depend on the way ties are broken. Strictly speaking, the above mentioned characterization result asserts that the Borda count combined with *every* tie breaking rule is strategy-proof if and only if the underlying domain is a cyclic permutation domain.

papers, such as Kalai and Muller (1977) and Kalai and Ritz (1980), have analyzed abstract Arrovian aggregation on restricted domains and obtained characterizations of those domains that admit possibility results.

The closest relative in the literature to the present paper is the work of Bochet and Storcken (2005). To the best of our knowledge, this is the only other paper studying Maskin monotonicity on restricted preference domains in the framework of the abstract social choice model.[2] These authors analyze both maximal strategy-proof and maximal monotonic domains for general social choice functions. However, unlike the present paper in which every individual faces the same preference restriction, Bochet and Storcken (2005) consider restrictions of the preference domain of exactly *one* individual. By consequence, the social choice functions found to satisfy the desired properties of strategy-proofness and monotonicity have a very special hierarchical structure and are in fact "almost" dictatorial.

2 Basic Notation and Definitions

Let X be a finite universe of social states or social alternatives and q be its cardinality. By $\mathcal{P}_X$, we denote the set of all linear orderings (irreflexive, transitive and total binary relations) on X, and by $\mathcal{P} \subseteq \mathcal{P}_X$ a generic subdomain of the unrestricted domain $\mathcal{P}_X$.

Definition (Social choice rule) A mapping $f : \bigcup_{n=1}^{\infty} \mathcal{P}^n \rightarrow 2^X \setminus \{\emptyset\}$ that assigns a set of (most preferred) alternatives $f(\succ_1, ..., \succ_n) \in 2^X \setminus \{\emptyset\}$ to each n-tuple of linear orderings and all n is called a *social choice rule (SCR).*

Let $rk[x, \succ]$ denote the *rank* of alternative x in the ordering $\succ$ (i.e. $rk[x, \succ] = 1$ if x is the top alternative in the ranking $\succ$, $rk[x, \succ] = 2$ if x is second-best, and so on).

Definition (Borda count) The SCR f^B associated with the *Borda count* is given as follows: for all n and all $\succ_1, \ldots, \succ_n \in \mathcal{P}_X$ we have

$$x \in f^B(\succ_1, \ldots, \succ_n) \Leftrightarrow \sum_{i=1}^{n} rk[x, \succ_i]) \leq \sum_{i=1}^{n} rk[y, \succ_i] \quad \text{for all } y \in X.$$

We shall denote by $L(x, \succ) = \{y \in X \mid x \succ y\}$ the *lower contour set* and by $U(x, \succ) = \{y \in X \mid y \succ x\}$ the *upper contour set* of a voter having preference $\succ$ at the alternative $x \in X$. A SCR f is called *monotonic* on $\mathcal{P}$ if for all $x \in X$, all n and all $\succ_1, \ldots, \succ_n$, $\succ_1' \ldots, \succ_n' \in \mathcal{P}$ we have

$$[x \in f(\succ_1, ..., \succ_n),\ L(x, \succ_i) \subseteq L(x, \succ_i') \text{ for all } i = 1, ..., n] \ \Rightarrow\ x \in f(\succ_1', ..., \succ_n').$$

We call a domain $\mathcal{P}$ *Borda monotonic* if f^B is monotonic on $\mathcal{P}$. Given a profile of preferences $(\succ_1, \ldots, \succ_n) \in \mathcal{P}^n$, we say that alternatives $A \subseteq X$ are *indifferent on the top* if $A = f^B(\succ_1, ..., \succ_n)$.

We will only be interested in preference domains that are minimally rich since on

[2] There is a distantly related literature on monotonic *extensions* of social choice rules. For instance, the work of Erdem and Sanver (2005) is also motivated by the observation that the Borda count, and in fact any scoring method, violates the monotonicity condition on an unrestricted domain. However, the monotonic extensions are again defined on the unrestricted preference domain; therefore, the analysis does not contribute to the question on which preference domains the *original* (non-extended) social rule would satisfy monotonicity.

"small" preference domains properties such as monotonicity or strategy-proofness can be satisfied in a trivial way.[3] Specifically, we will impose the following condition.

Definition (Rich domain) A domain $\mathcal{P}$ is called *rich* if, for any $x \in X$, there exists (i) $\succ \in \mathcal{P}$ such that $rk[x, \succ] = 1$, and (ii) $\succ' \in \mathcal{P}$ such that $rk[x, \succ'] = q$.

Thus, our richness condition requires that each alternative must be (i) most preferred by at least one preference ordering, and (ii) least preferred by some (other) preference ordering. This is slightly stronger than the richness condition used in Barbie, Puppe and Tasnádi (2006) which consisted of part (i) only. Part (ii) of the present condition is needed in Lemma 3.3 and in Substep 2B of the proof of our main result below.

Cyclically nested permutation domains

An ordering $\succ'$ is called a *cyclic permutation* of $\succ$ if $\succ'$ can be obtained from $\succ$ by sequentially shifting the bottom element to the top while leaving the order between all other alternatives unchanged. Thus, for instance, the cyclic permutations of the ordering *abcd* are *dabc*, *cdab* and *bcda*. The set of all cyclic permutations of a fixed ordering $\succ$ is denoted by $\mathcal{Z}(\succ)$, which we also call a *cyclic permutation domain.* In Barbie, Puppe and Tasnádi (2006), we have shown that the cyclic permutation domains are exactly the domains on which the Borda count is strategy-proof when combined with any conceivable deterministic tie-breaking rule.[4] The cyclic permutation domains will be the building blocks of the Borda monotonic domains.

We will define the cyclically nested permutation (henceforth, CNP) domains recursively. First, cyclic permutation domains are CNP domains of depth 1. Second, we define CNP domains of depth 2. Assume that $q = q_1 q_2$. We introduce the set of *second order pseudo alternatives* $X^{(2)} = \{X_1^{(2)}, \ldots, X_{q_2}^{(2)}\}$. Pick a preference $\succ^{(2)} \in \mathcal{P}_{X^{(2)}}$ on the set of second order pseudo alternatives and let us start with the pseudo domain $\mathcal{Z}(\succ^{(2)})$. Next, we will replace each second order pseudo alternative $X_i^{(2)}$ with a cyclical permutation domain defined on the set of alternatives X_i with cardinality q_1, where $X_1, \ldots, X_{q_2}$ is a partition of X. For instance, if $q_2 = 3$ and $q_1 = 2$, then first we obtain the domain at the left hand side of Table 1 and thereafter the domain at the right hand side of this table.

Table 1: Constructing CNP domains

$\succ_1^{(2)}$	$\succ_2^{(2)}$	$\succ_3^{(2)}$
a_1	a_2	a_3
a_2	a_3	a_1
a_3	a_1	a_2

$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$	$\succ_5$	$\succ_6$
b_1	b_2	c_2	c_1	d_2	d_1
b_2	b_1	c_1	c_2	d_1	d_2
c_1	c_2	d_2	d_1	b_1	b_2
c_2	c_1	d_1	d_2	b_2	b_1
d_1	d_2	b_1	b_2	c_1	c_2
d_2	d_1	b_2	b_1	c_2	c_1

[3]Obviously, every social choice function (i.e. single-valued social choice rule) is strategy-proof and monotonic on any domain consisting of only one preference ordering.

[4]Combined with particular, appropriately chosen tie-breaking rules the Borda count can be strategy-proof on a larger class of domains, see Barbie, Puppe and Tasnádi (2006).

However, we must restrict the admissible replacements of pseudo alternatives. To see this consider Table 2. Pick a profile Π consisting of one voter of each type. Then $f^B(\Pi) = \{b_1, b_2, c_1, c_2, d_1, d_2\}$ and monotonicity is violated at alternative b_2 if, for instance, a voter of type $\succ_3$ switches to type $\succ_6$.

Table 2: A non-monotonic domain

$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$	$\succ_5$	$\succ_6$
b_1	b_2	c_2	c_1	d_2	d_1
b_2	b_1	c_1	c_2	d_1	d_2
c_1	c_2	d_2	d_1	b_2	b_1
c_2	c_1	d_1	d_2	b_1	b_2
d_1	d_2	b_1	b_2	c_1	c_2
d_2	d_1	b_2	b_1	c_2	c_1

We restrict the admissible replacements of second order pseudo alternatives by cyclical permutation domains as follows. We have to specify those pairs of alternatives, derived from different second order pseudo alternatives, that must maintain their rank differences whenever they are ordered in the same way by two distinct preferences. We can see, for example, in Table 1 that the rank differences between b_1 and c_1 are the same in those preferences, which rank b_1 higher than c_1. We can observe similar relationships between the following pairs of alternatives: (b_1, d_1), (b_2, c_2), (b_2, d_2), (c_1, d_1), (c_1, b_2), (c_2, d_2), (c_2, b_1), (d_1, b_2), (d_1, c_2), (d_2, b_1) and (d_2, c_1). More generally, to define a CNP domain we must also specify for all $i, j \in \{1, \ldots, q_2\}$, $i \neq j$ bijections $\varphi_{i,j} : X_i \to X_j$ such that x and $\varphi_{i,j}(x)$ maintain their rank differences for all $x \in X_i$ whenever x is ranked above $\varphi_{i,j}(x)$.

Assume that we have already defined all CNP domains of depth $n-1$ and that $q = \prod_{i=1}^{n} q_i$. Now we introduce the set of n^{th} *order pseudo alternatives* $X^{(n)} = \{X_1^{(n)}, \ldots, X_{q_n}^{(n)}\}$ to define CNP domains of depth n. Pick a preference $\succ^{(n)} \in \mathcal{P}_{X^{(n)}}$ on the set of nth order pseudo alternatives and we start with the pseudo domain $\mathcal{Z}(\succ^{(n)})$. Then we replace for all $i = 1, \ldots, q_n$ each instance of an nth order pseudo alternative $X_i^{(n)}$ with the same CNP domain of depth $n-1$, size q/q_n and an associated factorization $q/q_n = \prod_{i=1}^{n-1} q_i$. We shall denote by $X_i \subseteq X$ the set of alternatives derived from the nth order pseudo alternative $X_i^{(n)}$ for any $i = 1, \ldots, q_n$. Note that $X_1, \ldots, X_{q_n}$ partitions X. Again, we restrict the admissible replacements of nth order pseudo alternatives by CNP domains of depth $n-1$. We specify those pairs of alternatives derived from different nth order pseudo alternatives that must maintain their rank differences whenever they are ordered in the same way by two distinct preferences. More formally, for all $i, j \in \{1, \ldots, q_n\}$, $i \neq j$ we need to define bijections $\varphi_{i,j} : X_i \to X_j$ such that x and $\varphi_{i,j}(x)$ maintain their rank differences for all $x \in X_i$ whenever x is ranked above $\varphi_{i,j}(x)$.

We provide an example of a CNP of depth 3 with $q_1 = 2$, $q_2 = 3$ and $q_3 = 2$ to illustrate the definition of CNP domains. The first pseudo domain is a cyclical permutation domain defined on two alternatives as shown in Table 3. Next we have to replace both pseudo alternatives with CNP domains of depth 2 with associated factorizations $2 \cdot 3$. We derive these two CNP domains from the second order pseudo alternatives and

Table 3: Third order pseudo domain

$\succ_1^{(3)}$	$\succ_2^{(3)}$
a_1	a_2
a_2	a_1

preferences shown in Table 4. The second order pseudo alternatives $b_1, \ldots, b_6$ must be

Table 4: Two second order CNP pseudo domains of depth 2

$\succ_1^{(2)}$	$\succ_2^{(2)}$	$\succ_3^{(2)}$	$\succ_4^{(2)}$	$\succ_5^{(2)}$	$\succ_6^{(2)}$
b_1	b_2	b_3	b_4	b_5	b_6
b_2	b_3	b_1	b_5	b_6	b_4
b_3	b_1	b_2	b_6	b_4	b_5

replaced by cyclical permutation domains each defined on two alternatives. We replace b_i with $\mathcal{Z}\left(x_{2i-1} \succ x_{2i}\right)$ for all $i = 1, \ldots, 6$. Considering pseudo alternatives b_1, b_2, b_3 and picking bijections $\varphi_{1,2}(x_1) = x_3$, $\varphi_{1,2}(x_2) = x_4$, $\varphi_{1,3}(x_1) = x_5$, $\varphi_{1,3}(x_2) = x_6$, $\varphi_{2,3}(x_3) = x_5$, $\varphi_{2,3}(x_4) = x_6$, $\varphi_{2,1} = \varphi_{1,2}^{-1}$, $\varphi_{3,1} = \varphi_{1,3}^{-1}$, $\varphi_{3,2} = \varphi_{2,3}^{-1}$, we obtain the CNP domain shown at the left hand side of Table 5. In an analogous way one obtains the CNP domain shown at the right hand side of Table 5. Finally, we have to insert

Table 5: Two CNP domains of depth 2 with $q_1 = 2$ and $q_2 = 3$

$\succ_1^{(1)}$	$\succ_2^{(1)}$	$\succ_3^{(1)}$	$\succ_4^{(1)}$	$\succ_5^{(1)}$	$\succ_6^{(1)}$	$\succ_7^{(1)}$	$\succ_8^{(1)}$	$\succ_9^{(1)}$	$\succ_{10}^{(1)}$	$\succ_{11}^{(1)}$	$\succ_{12}^{(1)}$
x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}
x_2	x_1	x_4	x_3	x_6	x_5	x_8	x_7	x_{10}	x_9	x_{12}	x_{11}
x_3	x_4	x_5	x_6	x_1	x_2	x_9	x_{10}	x_{11}	x_{12}	x_7	x_8
x_4	x_3	x_6	x_5	x_2	x_1	x_{10}	x_9	x_{12}	x_{11}	x_8	x_7
x_5	x_6	x_1	x_2	x_3	x_4	x_{11}	x_{12}	x_7	x_8	x_9	x_{10}
x_6	x_5	x_2	x_1	x_4	x_3	x_{12}	x_{11}	x_8	x_7	x_{10}	x_9

these two CNP domains of Table 5 into Table 3. To obtain Table 6 we pick bijections $\psi_{1,2}(x_i) = x_{i+6}$ for all $i = 1, \ldots, 6$ and $\psi_{2,1} = \psi_{1,2}^{-1}$.

3 Monotonic domains

The following is our main result.

Proposition 1 *A rich domain is Borda monotonic if and only if it is a CNP domain.*

For the proof of Proposition 1, we need a series of lemmas some of which are interesting on their own right. If there are k given preferences $\succ_1, \ldots, \succ_k \in \mathcal{P}$ and k

Table 6: A CNP domain of depth 3

$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$	$\succ_5$	$\succ_6$	$\succ_7$	$\succ_8$	$\succ_9$	$\succ_{10}$	$\succ_{11}$	$\succ_{12}$
x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}
x_2	x_1	x_4	x_3	x_6	x_5	x_8	x_7	x_{10}	x_9	x_{12}	x_{11}
x_3	x_4	x_5	x_6	x_1	x_2	x_9	x_{10}	x_{11}	x_{12}	x_7	x_8
x_4	x_3	x_6	x_5	x_2	x_1	x_{10}	x_9	x_{12}	x_{11}	x_8	x_7
x_5	x_6	x_1	x_2	x_3	x_4	x_{11}	x_{12}	x_7	x_8	x_9	x_{10}
x_6	x_5	x_2	x_1	x_4	x_3	x_{12}	x_{11}	x_8	x_7	x_{10}	x_9
x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_1	x_2	x_3	x_4	x_5	x_6
x_8	x_7	x_{10}	x_9	x_{12}	x_{11}	x_2	x_1	x_4	x_3	x_6	x_5
x_9	x_{10}	x_{11}	x_{12}	x_7	x_8	x_3	x_4	x_5	x_6	x_1	x_2
x_{10}	x_9	x_{12}	x_{11}	x_8	x_7	x_4	x_3	x_6	x_5	x_2	x_1
x_{11}	x_{12}	x_7	x_8	x_9	x_{10}	x_5	x_6	x_1	x_2	x_3	x_4
x_{12}	x_{11}	x_8	x_7	x_{10}	x_9	x_6	x_5	x_2	x_1	x_4	x_3

given positive integers $n_1, \ldots, n_k$, then we shall denote by $\Pi = (n_1 \cdot \succ_1, \ldots, n_k \cdot \succ_k)$ a preference profile in which $n_1, \ldots, n_k$ voters have preferences $\succ_1, \ldots, \succ_k$, respectively.

Lemma 3.1 *Suppose that $\mathcal{P}$ is a rich domain. If there exist two distinct preferences $\succ, \succ' \in \mathcal{P}$ and two alternatives $x, y \in X$ satisfying $rk\,[x, \succ] = 1$, $rk\,[y, \succ] = 2$ and $d := rk\,[y, \succ'] - rk\,[x, \succ'] \geq 2$, then Borda count violates monotonicity on $\mathcal{P}$.*

Proof of Lemma 3.1 Let $\succ'' \in \mathcal{P}$ be a preference with top alternative y, $d' = rk\,[x, \succ''] - rk\,[y, \succ'']$ and $k = \left\lceil \frac{2q+1}{d'+1} \right\rceil$.[5] We consider the following two profiles of $k(d'+1) - 1$ individuals: $\Pi = ((kd'-1) \cdot \succ, k \cdot \succ'')$ and $\Pi' = ((kd'-3) \cdot \succ, 2 \cdot \succ', k \cdot \succ'')$. Observe that the Borda score of y is greater than that of x by 1 in profile Π and since y dominates the remaining alternatives we have $f^B(\Pi) = \{y\}$. For profile Π' the choice of k assures[6] that x and y receive higher Borda scores than any other alternative. Thus, $f^B(\Pi') = \{x\}$ by the assumptions imposed on $\succ$ and $\succ'$. Finally, the precondition of monotonicity for the alternative x is satisfied as we switch from Π' to Π, but y becomes the Borda winning alternative in Π. This completes the proof.

Lemma 3.2 *If $\mathcal{P}$ is a Borda monotonic rich domain, then for any two preferences in $\mathcal{P}$ having the same top alternative the second ranked alternatives have to be identical.*

Proof of Lemma 3.2 Suppose that there are preferences $\succ, \succ' \in \mathcal{P}$ such that $rk\,[x, \succ] = 1$, $rk\,[y, \succ] = 2$, $rk\,[x, \succ'] = 1$, $rk\,[z, \succ'] = 2$ and $y \neq z$. Then $rk\,[y, \succ'] > 2$ and Lemma 3.1 applies.

Lemma 3.3 *If $\mathcal{P}$ is a Borda monotonic rich domain, then*

$$\pi(x) = \{y \in X \mid \exists \succ \in \mathcal{P} \text{ such that } rk[x, \succ] = 1 \text{ and } rk[y, \succ] = 2\}$$

defines a one-to-one correspondence (permutation) on X.

[5] In what follows $\lfloor x \rfloor$ stands for the largest integer not greater than x and $\lceil x \rceil$ stands for the smallest integer not less than x.

[6] Any larger integer for k does the job.

Proof of Lemma 3.3 The statement is obviously true in case of $q \leq 3$. Therefore, we only have to consider the case of $q > 3$. Suppose that x is ranked first by $\succ \in \mathcal{P}$ and ranked second by $\succ', \succ'' \in \mathcal{P}$. We shall denote the top alternatives of $\succ' \in \mathcal{P}$ and $\succ'' \in \mathcal{P}$ by y and z, respectively. Any $\succ^* \in \mathcal{P} \setminus \{\succ, \succ', \succ''\}$ has to rank y or z lower than x; since otherwise, y and x or z and x violate Lemma 3.1. Hence, $\mathcal{P}$ cannot satisfy part (ii) of the richness condition, a contradiction.

Lemma 3.4 *Suppose that $\mathcal{P}$ is a Borda monotonic rich domain. Then we cannot find two distinct preferences $\succ, \succ' \in \mathcal{P}$ and an alternative $y \in X$ such that*

- $rk\,[y, \succ] > 2$
- $U(y, \succ) = U(y, \succ')$,
- $\forall x \in U(y, \succ) : rk\,[x, \succ] \neq rk\,[x, \succ']$.

Proof of Lemma 3.4 Suppose that there exist two distinct preferences $\succ, \succ' \in \mathcal{P}$ and an alternative $y \in X$ such that $d = rk\,[y, \succ] > 2$, $U(y, \succ) = U(y, \succ')$ and $rk\,[x, \succ] \neq rk\,[x, \succ']$ for all $x \in U(y, \succ)$. Let $\succ'' \in \mathcal{P}$ be a preference with top alternative y and $U(y, \succ) = \{x_1, \ldots, x_{d-1}\}$. Observe that y dominates all alternatives in $X \setminus U(y, \succ)$ in all profiles consisting only of preferences $\succ, \succ'$ and $\succ''$. We define values $d_m = 2d - rk\,[x_m, \succ] - rk\,[x_m, \succ']$ and $d'_m = rk\,[x_m, \succ''] - 1$ for all $m \in \{1, \ldots, d-1\}$. Now let $S = \arg\min_{s \in \{1, \ldots, d-1\}} \frac{d'_s}{d_s}$ and $A = \{x_s \in X \mid s \in S\}$. For any $s \in S$ it can be verified that a profile consisting of d'_s preferences of type $\succ$, d'_s preferences of type $\succ'$ and d_s preferences of type $\succ''$ makes alternatives $\{y\} \cup A$ indifferent on the top. Let $\Pi = (d'_s \cdot \succ, d'_s \cdot \succ', d_s \cdot \succ'')$. Hence, $f^B(\Pi) = \{y\} \cup A$.

First, if there exists an $s \in S$ such that x_s is ranked higher in $\succ'$ than in $\succ$, then pick an arbitrary alternative $x_m \in A$ achieving the highest rank increase by replacing one voter of type $\succ$ with one voter of type $\succ'$. In this case we construct Π' from Π by replacing one preference $\succ$ with one preference $\succ'$. It can be checked that $y \notin f^B(\Pi')$, while $x_m \in f^B(\Pi')$. Second, if for all $s \in S$ we have that x_s is ranked higher in $\succ$ than in $\succ'$, then pick an arbitrary alternative $x_m \in A$ achieving the highest rank decrease from $\succ$ to $\succ'$. In this second case we construct Π' from Π by replacing one preference $\succ'$ with one preference $\succ$. Again, we have $y \notin f^B(\Pi')$, while $x_m \in f^B(\Pi')$. We obtained in both cases a violation of monotonicity at y; a contradiction.

Lemma 3.5 *Any CNP domain $\mathcal{P}$ on X consists of q preferences and for all $x \in X$, all $i \in \{1, \ldots, q\}$ there exists a preference $\succ \in \mathcal{P}$ such that $rk\,[x, \succ] = i$.*

Proof of Lemma 3.5 Following the recursive construction of a CNP domain, we obtain a pseudo domain of cardinality q_n, a pseudo domain of cardinality $q_n q_{n-1}$, and so on til we obtain a CNP domain of cardinality q. This proves the first part of the statement.

The second part of the statement can be established by an induction on the depth of CNP domains. Cyclical permutation domains clearly satisfy our statement. Assume that our statement holds true for CNP domains of depth $n-1$. Take a CNP domain $\mathcal{P}$ of depth n, which has to be constructed from a cyclical permutation domain on nth order pseudo alternatives and from CNP domains of depth $n-1$ replacing the nth order pseudo alternatives. Employing the induction hypothesis for the CNP domains of depth $n-1$ and the structure of a cyclical permutation domain (on the nth order pseudo alternatives), we obtain our statement.

From Lemma 3.5 we obtain the following corollary.

Corollary 1 *$f^B(1\cdot \succ_1, \dots, 1\cdot \succ_q) = X$ if $\mathcal{P} = \{\succ_1, \dots, \succ_q\}$ is a CNP domain on X.*

Lemma 3.6 *Let $\mathcal{P}$ be a Borda monotonic rich domain, $X' \subseteq X$ and $\mathcal{P}' \subseteq \mathcal{P}$. Assume that $q' := \#X' = \#\mathcal{P}'$ and that the restriction of $\mathcal{P}'$ to its top q' alternatives gives a CNP domain on X'. Then for any preference $\succ \in \mathcal{P}$ there exists a preference $\succ' \in \mathcal{P}'$ such that the alternatives from X' must follow each other consecutively in the same order in $\succ$ as in $\succ'$.*

Proof of Lemma 3.6 The restriction of $\mathcal{P}'$ to its top q' alternatives equals $\mathcal{P}'_{|X'}$, which is a CNP domain on X', by the assumptions of Lemma 3.6. We employ an induction on the depth of the CNP domain on X'. Lemma 3.1 implies that Lemma 3.6 is satisfied whenever $\mathcal{P}'_{|X'}$ is a CNP domain of depth 1.

Assume that the statement holds true for any CNP domain $\mathcal{P}'_{|X'}$ of depth less than n. Now let $\mathcal{P}'_{|X'}$ be a CNP domain of depth n. The nth order pseudo alternatives generating $\mathcal{P}'_{|X'}$, partition X' into sets $X_1, \dots, X_k$ of cardinality q'/k. Observe that $\mathcal{P}'_{|X_i}$ are CNP domains of depth $n-1$ for all $i = 1, \dots, k$. Thus, for all preferences $\succ \in \mathcal{P}$ and all $i = 1, \dots, k$ there exists a preference $\succ' \in \mathcal{P}'$ such that the alternatives from X_i must follow each other consecutively in the same order in $\succ$ as in $\succ'$ by our induction hypothesis. Pick an arbitrary preference $\succ \in \mathcal{P} \setminus \mathcal{P}'$ and suppose that there does not exist a preference $\succ' \in \mathcal{P}'$ such that the alternatives from X' must follow each other consecutively in the same order in $\succ$ as in $\succ'$. Let x_1 be the highest ranked X' alternative by $\succ$. We can assume without loss of generality that $x_1 \in X_1$. We shall denote by $\succ' \in \mathcal{P}'$ the preference ranking x_1 on top. We assume for notational convenience that $\succ'$ ranks X_i above X_{i+1} for all $i = 1, \dots, k-1$. Let $j \in \{1, \dots, k\}$ be the largest index such that the alternatives $\cup_{i=1}^{j-1} X_i$ follow each other consecutively in the same order in $\succ$ as in $\succ'$. We shall denote by x_j the highest ranked X_j alternative in $\succ'$ and by $\succ'' \in \mathcal{P}'$ the preference with top alternative x_j. There exists positive integers a and b such that profile $\Pi = (a\cdot \succ', b\cdot \succ'')$ has x_j and $U \subseteq \cup_{i=1}^{j-1} X_i$ indifferent on the top. We shall denote by u the lowest ranked alternative from U by $\succ'$. Let $d' = rk\,[x_j, \succ'] - rk\,[u, \succ']$ and $d = rk\,[x_j, \succ] - rk\,[u, \succ]$. We must have $d' < d$ by the definition of j and our induction hypothesis. Let $c = \left\lceil \frac{d'}{d-d'} \right\rceil$. We can assume that $a > c$, since otherwise, we can take an appropriate multiple of a and b to have $f^B(\Pi) = \{x_j\} \cup U$ and $a > c$. Let $\Pi' = ((a-1)\cdot \succ', b\cdot \succ'')$ and $\Pi'' = (c\cdot \succ, (a-1-c)\cdot \succ', b\cdot \succ'')$. If a and b were chosen large enough so that no other alternative can interfere, then $f^B(\Pi') = \{x_j\}$ and $u \in f^B(\Pi'')$, and therefore, monotonicity is violated at u by switching from Π'' to Π'.

4 Proof of the Main Result

Proof of Proposition 1 *Sufficiency* can be shown by employing an induction on the depth of CNP domains. If $\mathcal{P}$ is a CNP domain of depth 1, then $\mathcal{P}$ is a simple cyclic permutation domain. Pick an arbitrary profile Π and any alternative $x \in f(\Pi)$.

Note that for any cyclic permutation domain $L(x,\succ) \subseteq L(x,\succ')$ implies for any other alternative $y \in X \setminus \{x\}$ either equal rank differences in $\succ$ and $\succ'$ between x and y or $y \succ x$ and $x \succ' y$. Thus, x cannot be overtaken by other alternatives if we replace preferences in Π with other preferences in a way that the precondition of monotoniciy is satisfied. Thus, a cyclic permutation domain has to be monotonic.

Assume that CNP domains of depth $n-1$ are monotonic. Now take an arbitrary CNP domain $\mathcal{P}$ of depth n. We shall denote by X_i the alternatives derived from the nth order pseudo alternative $X_i^{(n)}$ for all $i = 1, \dots, q_n$. By our construction of CNP domains there are for all $i, j \in \{1, \dots, q_n\}$ and $i \neq j$ bijections $\varphi_{i,j} : X_i \to X_j$ such that $x \in X_i$ and $\varphi_{i,j}(x)$ maintain their rank differences whenever x is ranked above $\varphi_{i,j}(x)$. Since each instance of an nth order pseudo alternative $X_i^{(n)}$ is replaced with the same CNP domain of depth $n-1$, which are monotonic, we can only have a violation of monotonicity by considering two alternatives derived from two different nth order pseudo alternatives. Thus, pick two distinct nth order pseudo alternatives $X_i^{(n)}$ and $X_j^{(n)}$. Take an arbitrary profile Π such that $x \in f(\Pi)$, where $x \in X_i$. Alternative x can be overtaken by alternative $y \in X_j$ $(i \neq j)$ by replacing preferences in Π without violating the precondition of monotonicity only if we can find voters of type $\succ$ in Π and a preference $\succ' \in \mathcal{P}$ such that $L(x,\succ) \subseteq L(x,\succ')$ and either

(a) $x \succ y$, $x \succ' y$ and $rk\,[y,\succ] - rk\,[x,\succ] > rk\,[y,\succ'] - rk\,[x,\succ']$ or

(b) $y \succ x$, $y \succ' x$ and $rk\,[x,\succ] - rk\,[y,\succ] < rk\,[x,\succ'] - rk\,[y,\succ']$.

We only consider case (a) since case (b) can be established in an analogous way. Let $x' = \varphi_{i,j}^{-1}(y)$, $d = rk\,[y,\succ] - rk\,[x',\succ] = rk\,[y,\succ'] - rk\,[x',\succ']$, $d_1 = rk\,[y,\succ] - rk\,[x,\succ]$ and $d_2 = rk\,[y,\succ'] - rk\,[x,\succ']$. We cannot have $x \succ x'$ and $x' \succ' x$, since this would violate $L(x,\succ) \subseteq L(x,\succ')$. Moreover, $x' \succ x$ and $x \succ' x'$ cannot be the case, since this would imply $d_1 < d < d_2$, which is in contradiction with $d_1 > d_2$. The remaining two subcases $x \succ x'$ and $x \succ' x'$, and $x' \succ x$ and $x' \succ' x$ would imply the non-monotonicity of the CNP subdomain of depth $n-1$ on X_i by Corollary 1 since each subdomain of depth $n-1$ does also appear on 'top' of a subdomain of $\mathcal{P}$; a contradiction.

Now we turn to the *necessity* of CNP domains. We need the following notations to prove the necessity of our condition. For any $1 \leq i \leq j \leq q = \#X$ let $\succ_{|[i,j]}$ be the restriction of $\succ$ ranging from the ith position to the jth position of $\succ$, i.e., $\succ_{|[i,j]} = \succ_{|\{x_i, x_{i+1}, \dots, x_j\}}$ where $x_1 \succ \cdots \succ x_i \succ \cdots \succ x_j \succ \cdots \succ x_q$. In addition, for any $1 \leq i \leq j \leq q$, we define $\mathcal{P}_{[i,j]} = \left\{ \succ_{|[i,j]} | \succ \in \mathcal{P} \right\}$. Furthermore, for any linear ordering $\succ$ on $X' \subseteq X$ we define $T_i(\succ) = \{x \in X' \mid rk\,[x,\succ] \leq i\})$ and $M_{i,j}(\succ) = \{x \in X' \mid i \leq rk\,[x,\succ] \leq j\}$. We divide our proof into several steps.

Step 1: Lemma 3.3 implies that the top two alternatives determine a permutation π of X. The cycles of permutation π partition X into sets $X_1, \dots, X_p$. We shall denote by X' an arbitrary set X_i $(i = 1, \dots, p)$, by $x_1, \dots, x_m$ its alternatives and by $\succ_k \in \mathcal{P}$ an arbitrary preference with top alternative x_k $(k = 1, \dots, m)$.[7] Clearly, $m \geq 2$. Let $\mathcal{P}' = \{\succ_1, \dots, \succ_m\}$. In what follows we can assume without loss of generality that $rk\,[x_{k \oplus_m 1}, \succ_k] = 2$.[8]

[7]It will turn out that the preference having x_k on top is unique.

[8]For two integers $k, l \in \{1, \dots, m\}$, if $k + l \neq m$ and $k + l \neq 2m$, we define $k \oplus_m l := (k + l) \bmod m$, while if $k + l = m$ or $k + l = 2m$, we define $k \oplus_m l := m$.

We determine the top m alternatives of $\mathcal{P}'$. We must have $rk\,[x_{k\oplus_m 2}, \succ_k] = 3$ for all $k = 1, \ldots, m$ by Lemma 3.1. Moreover, it follows from Lemma 3.1 by induction that $rk\,[x_{k\oplus_m l}, \succ_k] = l + 1$ for all $l = 1, \ldots, m-1$ and all $k = 1, \ldots, m$. But this implies that the top m alternatives of the preferences in $\mathcal{P}'$ follow the pattern shown in Table 7. Moreover, the restriction to its top m alternatives of any preference in $\mathcal{P}$ with a

Table 7: A full cycle on the top

$\succ_1$	$\succ_2$	$\ldots$	$\succ_{m-1}$	$\succ_m$
x_1	x_2	$\ldots$	x_{m-1}	x_m
x_2	x_3	$\ldots$	x_m	x_1
$\vdots$	$\vdots$	$\iddots$	$\vdots$	$\vdots$
x_{m-1}	x_m	$\ldots$	x_{m-3}	x_{m-2}
x_m	x_1	$\ldots$	x_{m-2}	x_{m-1}
$\vdots$	$\vdots$		$\vdots$	$\vdots$

top alternative from X' equals the restriction to its top m alternatives of a preference in $\mathcal{P}'$. In addition, $\mathcal{P}'$ prescribes the possible orderings of the alternatives from X' by any preference in $\mathcal{P}$ by Lemma 3.6.

Clearly, we are finished if $p = 1$. Hence, in what follows we will assume that $p > 1$.

Step 2: Let $X_1, \ldots, X_p$ be a partition of X, $m_i = \#X_i$ and $\mathcal{P}^i = \{\succ \in \mathcal{P} \mid \exists x \in X_i$ such that $rk\,[x, \succ] = 1\}$ for all $i = 1, \ldots, p$. Assume that we have already established that $T_{m_i}(\succ) = X_i$ for all $\succ \in \mathcal{P}^i$ and that $\mathcal{P}_{|X_i}$ are CNP domains on X_i for all $i = 1, \ldots, p$.

We will demonstrate in Step 2 that Borda monotonicity implies the existence of a set of indices $I \subseteq \{1, \ldots, p\}$ such that $\#I \geq 2$ and $\mathcal{P}'_{|Y}$ is a CNP domain on Y, where $Y = \cup_{i \in I} X_i$, $\mathcal{P}' = \cup_{i \in I} \mathcal{P}^i$ and $T_{\#Y}(\succ) = Y$ for all $\succ \in \mathcal{P}'$.[9]

We can assume without loss of generality that $m_1 \leq m_i$ for all $i = 1, \ldots, p$ and we simply write m for m_1. Our proof of Step 2 will require three substeps.

Substep A: We claim that there exists an $i \in \{2, \ldots, p\}$ such that $\mathcal{P}^1_{|X_1}$ and $\mathcal{P}^i_{|X_i}$ have identical associated factorizations, and furthermore, $M_{m+1,2m}(\succ) = X_i$ for all $\succ \in \mathcal{P}^1$. In addition, there exists a bijection $\varphi_{1,i} : X_1 \to X_i$ such that $x \in X_1$ and $\varphi_{1,i}(x)$ maintain their rank differences in $\mathcal{P}^1$. The claim of Substep A implies by Lemmas 3.4 and 3.6 that $\mathcal{P}^1_{|X_i} = \mathcal{P}^i_{|X_i}$ and $m_1 = \#\mathcal{P}^1_{|X_1} = \#\mathcal{P}^1_{|X_1 \cup X_i}$. We prove our claim by induction.

Initial step of Substep A: We consider a subdomain $\mathcal{P}'$ of $\mathcal{P}^1$ with a cyclic permutation domain on top. Note that $\mathcal{P}' = \mathcal{P}^1$ if $\mathcal{P}^1_{|X_1}$ is a CNP domain of depth 1. It follows from Lemma 3.4 that there cannot be an alternative $x \in X$ that is ranked by two distinct preferences $\succ$ and $\succ'$ in $\mathcal{P}'$ at the $m+1$th position. We shall denote the n distinct alternatives ranked mth by the preferences in $\mathcal{P}'$ by $y_1, \ldots, y_n \in X$, the corresponding preferences by $\succ_1, \ldots, \succ_n$ and the corresponding top alternatives by

[9]This implies that $m_i = m_j$ for all $i, j \in I$, that the CNP domains $\mathcal{P}^i_{|X_i}$ possess the same factorizations $m_i = \prod_{j=1}^{k} q_j$ for all $i \in I$ and that the factorization associated with $\mathcal{P}'_{|Y}$ is $\prod_{j=1}^{k} q_j \cdot \#I$.

$z_1, \ldots, z_n$, respectively. Let $Y = \{y_1, \ldots, y_n\}$ and $Z = \{z_1, \ldots, z_n\}$. We can assume without loss of generality that $\mathcal{P}'_{|Z} = \mathcal{Z}(\succ^*)$, where $z_1 \succ^* z_2 \succ^* \ldots \succ^* z_n$.

We show that $rk\,[y_{k \oplus_n 1}, \succ_k] = m+2$ for all $k = 1, \ldots n$. This assures by Lemma 3.6 that the preferences in $\mathcal{P}'$ look like in Table 8. For notational convenience we will only

Table 8: Substep A

$\succ_1$	$\succ_2$		$\cdots$	$\succ_{n-1}$	$\succ_n$
z_1	z_2		$\cdots$	z_{n-1}	z_n
z_2	z_3		$\cdots$	z_n	z_1
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
z_{n-1}	z_n		$\cdots$	z_{n-3}	z_{n-2}
z_n	z_1		$\cdots$	z_{n-2}	z_{n-1}
$\vdots$	$\vdots$			$\vdots$	$\vdots$
y_1	y_2		$\cdots$	y_{n-1}	y_n
y_2	y_3		$\cdots$	y_n	y_1
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
y_{n-1}	y_n		$\cdots$	y_{n-3}	y_{n-2}
y_n	y_1		$\cdots$	y_{n-2}	y_{n-1}
$\vdots$	$\vdots$			$\vdots$	$\vdots$

show that $rk\,[y_2, \succ_1] = m+2$. We shall denote by $\succ'$ a preference with top alternative y_2. Arguing in an even simpler way than in Lemma 3.4, we can find positive integers a and b such that y_2 together with at least another alternative from set X_1 receive the highest Borda scores in profile $\Pi = (a\cdot \succ_2, b\cdot \succ')$ and the lead of y_2 over the alternatives from $X_1 \setminus f^B(\Pi)$ is at least m. Let $U \subseteq X_1$ be the set of those alternatives that are ranked higher by $\succ_1$ than by $\succ_2$.[10]

Suppose that $f^B(\Pi) \cap U \neq \emptyset$. Pick arbitrary alternative $u \in f^B(\Pi) \cap U \neq \emptyset$. Then there exists a $k \in \{1, \ldots, m/n\}$ such that $rk\,[u, \succ_2] = kn$. We shall denote by $v \in X_1$ the $(k-1)n+1$th ranked alternative by $\succ_2$.[11] Let $d = rk\,[y_2, \succ_2] - rk\,[u, \succ_2]$ and $d' = rk\,[u, \succ'] - rk\,[y_2, \succ']$. Since $u, y_2 \in f^B(\Pi)$, we must have $ad = bd'$, $u \succ' v$ and by Lemma 3.6 $rk\,[v, \succ'] = d'+2$. Let us compare the Borda score of v with that of u in Π. On the one hand v receives $a(n-1)$ points more than u and on the other hand u receives b points more than v. Therefore, we must have

$$a(n-1) \leq b \Leftrightarrow a(n-1) \leq a\frac{d}{d'} \Leftrightarrow d'(n-1) \leq d;$$

a contradiction, since $d < m \leq d'$ and $n \geq 2$ by the Assumptions of Step 2. Thus, $f^B(\Pi) \cap U = \emptyset$.

[10]From the structure of $\mathcal{P}^1_{|X_1}$ it follows that $rk\,[u, \succ_2] - rk\,[u, \succ_1] = n-1$ for any $u \in U$ and $rk\,[x, \succ_1] - rk\,[x, \succ_2] = 1$ for any $x \in X_1 \setminus U$.

[11]Observe that $rk\,[u, \succ_1] = (k-1)n+1$ and $rk\,[v, \succ_1] = (k-1)n+2$.

Let z be the highest ranked alternative from $f^B(\Pi) \setminus \{y_2\}$ by $\succ'$, $\delta = rk\,[y_2, \succ_1] - rk\,[y_1, \succ_1]$, $d = rk\,[y_2, \succ_2] - rk\,[z, \succ_2]$ and $d' = rk\,[z, \succ'] - rk\,[y_2, \succ']$. Observe that z has to be the lowest ranked alternative in $\succ_2$ from set $f^B(\Pi) \setminus \{y_2\}$. Suppose that $\delta \geq 2$, which would mean that y_2 does not follow immediately y_1 in $\succ_1$. We have to incorporate at least one voter of type $\succ_1$ appropriately in order to obtain a contradiction with $\delta \geq 2$. First, we omit a voter of type $\succ_2$, which makes y_2 the single Borda winner with a lead of d over z. Second, we compensate this lead by replacing $c = \left\lceil \frac{d}{\delta-1} \right\rceil$ voters of type $\succ_2$ with voters of type $\succ_1$. If $a \leq c$, then by taking an appropriate multiple of Π, we can ensure that we have more voters of type $\succ_2$ than c. Hence, we can assume $a > c$ without loss of generality. Third, we have to take care about not making an alternative $u \in U$ the Borda winning alternative. If z does not lead by cn over alternatives $u \in U$ in Π, then this can be guaranteed by starting already with an appropriate multiple of Π.[12] Again, we can assume without loss of generality that a and b satisfy this latter requirement. Finally, let $\Pi' = (c\cdot \succ_1, (a-c-1)\cdot \succ_2, b\cdot \succ')$ and $\Pi'' = ((a-1)\cdot \succ_2, b\cdot \succ')$. It can be verified that monotonicity is violated at z by switching from Π' to Π'', since $z \in f^B(\Pi')$ and $\{y_2\} = f^B(\Pi'')$. Thus, we must have $\delta = 1$.

Induction hypotheses of Substep A: Assume that we have already obtained a partition $\mathcal{P}^{1,1}, \ldots, \mathcal{P}^{1,t}$ of $\mathcal{P}^1$, disjoint subdomains $\mathcal{P}^{2,1}, \ldots, \mathcal{P}^{2,t} \subseteq \mathcal{P} \setminus \mathcal{P}^1$ with respective top $n = \frac{m}{t}$ alternatives $X^{j,i}$ ($j = 1, 2$ and $i = 1, \ldots, t$) such that $t \geq 2$, $n = \#\mathcal{P}^{1,i}_{[1,n]} = \#\mathcal{P}^{2,i}_{[1,n]}$, $\mathcal{P}^{1,i}_{[m+1,m+n]} = \mathcal{P}^{2,i}_{[1,n]}$ are CNP domains with associated factorizations $n = q_1 \cdot \ldots \cdot q_l$ for all $i = 1, \ldots, t$ and there exist bijections $\varphi : X^{1,i} \to X^{2,i}$ satisfying that x and $\varphi_i(x)$ maintain their rank differences in $\mathcal{P}^{1,i}$ for all $x \in X^{1,i}$ and all $i = 1, \ldots, t$.[13] This implies that the factorization associated with $\mathcal{P}^1$ equals $q_1 \cdot \ldots \cdot q_l \cdot \ldots \cdot q_{l'}$ for some l' and $q_{l+1}, \ldots, q_{l'}$.

Induction step of Substep A: Let $r = q_{l+1}$, $h = t/r$ and $\mathcal{P}^{j,i} = \{\succ^{j,i}_1, \ldots, \succ^{j,i}_t\}$ for all $j = 1, 2$ and all $i = 1, \ldots, t$. We shall denote by $X^{j,i}$ the set of top alternatives of $\mathcal{P}^{j,i}$. Hence, if we denote the lth order pseudo alternatives of $\mathcal{P}^1_{|X_1}$ by $X^{1,1}, \ldots, X^{1,t}$, then the first r pseudo preferences of the pseudo domain associated with $\mathcal{P}^1_{|X_1}$ look like in Table 9 supposed that we have labeled the sets $X^{1,i}$ appropriately. In what follows we shall focus, for notational convenience, on $\mathcal{P}^{1,1}$ and $\mathcal{P}^{1,2}$. In addition, we can assume without loss of generality that the alternatives and preferences are labeled in a way that $rk\left[x^{1,i}_n, \succ^{1,2}_k\right] = (i-2)n + 1 + (n-k)$ for all $k = 1, \ldots, n$ and all $i = 2, \ldots, t$ for which $i-1$ is not divisible by r, and otherwise, $rk\left[x^{1,i}_n, \succ^{1,2}_k\right] = (i+r-2)n+1+(n-k)$ for all $k = 1, \ldots, n$ and all $i = 1, \ldots, t$.

We shall denote by $y_1, \ldots, y_n$ the n distinct alternatives ranked $m+1$th by the preferences $\succ^{1,2}_1, \ldots, \succ^{1,2}_n$, respectively. Let $Y = \{y_1, \ldots, y_n\}$. Moreover, for all $i = 1, \ldots, n$ we simply write $\succ_i$, $\succ'_i$ and $\succ''_i$ for $\succ^{1,1}_i$, $\succ^{1,2}_i$ and $\succ^{2,2}_i$, respectively.

We can find positive integers a and b such that Y and at least a set of alternatives $X' \subseteq X_1$ receives the highest Borda score in profile $\Pi = (a\cdot \succ'_1, \ldots, a\cdot \succ'_n, b\cdot \succ''_1, \ldots, b\cdot \succ''_n)$. Let $U \subseteq X_1$ be the set of those alternatives that are ranked higher by $\succ_1$ than by $\succ'_1$. Observe that $U = \cup_{i=0}^{h-1} X^{1,ir+1}$.

[12]More precisely, we should have first defined $c = \left\lceil \frac{d}{\delta-1} \right\rceil$ and a, b afterwards. However, we have followed a different order for expositional reasons.

[13]Our initial step assured the existence of a partition with $l = 1$.

Table 9: Pseudo domain on top

$\mathcal{P}^{1,1}$	$\mathcal{P}^{1,2}$	$\dots$	$\mathcal{P}^{1,r}$
$X^{1,1}$	$X^{1,2}$	$\dots$	$X^{1,r}$
$X^{1,2}$	$X^{1,3}$	$\dots$	$X^{1,1}$
$\vdots$	$\vdots$		$\vdots$
$X^{1,r}$	$X^{1,1}$	$\dots$	$X^{1,r-1}$
$\vdots$	$\vdots$		$\vdots$
$X^{1,t-r+1}$	$X^{1,t-r+2}$	$\dots$	$X^{1,t}$
$X^{1,t-r+2}$	$X^{1,t-r+3}$	$\dots$	$X^{1,t-r+1}$
$\vdots$	$\vdots$		$\vdots$
$X^{1,t}$	$X^{1,t-r+1}$	$\dots$	$X^{1,t-1}$
$\vdots$	$\vdots$		$\vdots$

Suppose that there exists an $i = 1, \dots, h$ such that $u \in f^B(\Pi) \cap X^{1,(i-1)r+1}$. Pick an arbitrary alternative $y \in Y$. Since $\{u, y\} \in f^B(\Pi)$, we must have $a[(h-i)r+1]n^2 = bd'$, where $d' = \sum_{i=1}^{n} rk\,[u, \succ_i''] - rk\,[y, \succ_i'']$. Take an alternative v from $X^{1,(i-1)r+2}$ such that $d'' = \sum_{i=1}^{n} rk\,[v, \succ_i''] - rk\,[y, \succ_i'']$ is as small as possible. Since the Borda score of v cannot be greater than that of u in Π, we must have $an(r-1)n \leq b\,(d'' - d')$. Therefore,

$$\frac{bd'}{[(h-i)r+1]n^2} n^2(r-1) \leq b\,(d'' - d') \Leftrightarrow d'(r-1) \leq (d'' - d')\,[(h-i)r+1]. \quad (4.1)$$

By the induction hypothesis of Step 2 and by Lemmas 3.4 and 3.6 we must have $d' \geq mn = tn^2$. The value $d'' - d'$ would be the largest if u is ranked higher than v by any preference $\succ_i'' \in \mathcal{P}^{2,2}$. Then the alternatives from $X^{1,(i-1)r+2}$ must follow immediately the alternatives from $X^{1,(i-1)r+1}$ in any $\succ_i'' \in \mathcal{P}^{2,2}$ by Lemma 3.6. Moreover, we have $d'' - d' \leq n^2$ by Lemma 3.6 and by the choice of v, which together with equation (4.1) implies

$$tn^2 \leq d'(r-1) \leq (d'' - d')\,[(h-i)r+1] \leq n^2[(h-i)r+1].$$

It follows from these inequalities that $rh = t \leq [(h-i)r+1]$, which implies $ir \leq 1$. Therefore, since $r, n \geq 2$, $h \geq 1$ and $i \geq 1$ we obtained a contradiction and we conclude that $f^B(\Pi) \cap U = \emptyset$.

Define $v = \max\{i = 1, \dots, t \mid X^{1,i} \cap f^B(\Pi) \neq \emptyset\}$ and pick an alternative z from $X^{1,v} \cap f^B(\Pi)$. Let $\succ' \in \mathcal{P}^{1,2}$ the preference that ranks z highest. For notational convenience we can assume that $\succ' = \succ_n'$ and $\succ'$ ranks y_n as the highest alternative from Y.[14] Hence, z is the highest ranked alternative from $X^{1,v}$ by $\succ_n'$. Observe that from the way how we labeled the alternatives of X_1 and our assumption on $\varphi : X^{1,2} \to X^{2,2} = Y$ it follows that $rk\,[y_n, \succ_k'] - rk\,\left[x_n^{1,2}, \succ_k'\right] = m$ for all $k = 1, \dots, n$. In addition, we can assume for notational convenience that $\succ_{k|X^{1,v}} = \succ'_{k|X^{1,v}}$ for all $k = 1, \dots, n$. We will show that $rk\,[y_n, \succ_k] = m + n + 1 + (n - k)$ for all $k = 1, \dots, n$. Observe that $rk\,[y_n, \succ_k] > m + n$, since the shortest sequence of alternatives that must follow an

[14] Otherwise, we would relabel the alternatives of X_1 and Y as well as the preferences of $\mathcal{P}^{1,2}$.

already prescribed order is of length n and by Lemma 3.4 none of the alternatives of Y can be ranked $m+1$th by a preference of $\mathcal{P}^{1,1}$. Now take an arbitrary index $k=1,\ldots,n$ and let $\delta_k = rk\,[y_n,\succ_k] - (m+1+n-k)$.

Suppose that $\delta_k > n$. By replacing a preference $\succ'_1$ with $\succ'_n$, we can achieve that $f^B(\Pi)$ contains only y_n from Y and only z from $X^{1,v}$. In what follows we shall denote this modified profile by Π with a slight abuse of notation. Let $d = rk\,[y_n,\succ'_n] - rk\,[z,\succ'_n]$ and $d' = \sum_{i=1}^{n} rk\,[z,\succ''_i] - rk\,[y_n,\succ''_i]$. Note that $d' \geq mn$ by the assumptions of Step 2 and by Lemmas 3.4 and 3.6. Now we have to incorporate at least one voter of type $\succ_k$ into Π in order to obtain a contradiction with $\delta_k > n$. First, we omit a voter of type $\succ'_1$, which makes y_n the single Borda winner with a lead of d over z. Second, we compensate this lead by replacing $c = \left\lceil \frac{d}{\delta_k - n} \right\rceil$ voters of type $\succ'_k$ with voters of type $\succ_k$. If $a \leq c$, then by starting with an appropriate multiple of Π, we can ensure that we have more than c voters of type $\succ'_k$. Hence, we can assume $a > c$ without loss of generality. Third, we have to take care about not making an alternative $u \in U$ the Borda winning alternative. If z does not lead by cm over alternatives $u \in U$ in Π, then this can be guaranteed by starting already with an appropriate multiple of Π.[15] Thus, we can assume without loss of generality that a and b satisfy this latter requirement. Let $\Pi' = (c\cdot \succ_k, (a-2)\cdot \succ'_1, a\cdot \succ'_2, \ldots, a\cdot \succ'_{k-1}, (a-c)\cdot \succ'_k, a\cdot \succ'_{k+1}, \ldots, a\cdot \succ'_{n-1}, (a+1)\cdot \succ'_n, b\cdot \succ''_1, \ldots, b\cdot \succ''_n)$ and $\Pi'' = ((a-2)\cdot \succ'_1, a\cdot \succ'_2, \ldots, a\cdot \succ'_{n-1}, (a+1)\cdot \succ'_n, b\cdot \succ''_1, \ldots, b\cdot \succ''_n)$. It can be verified that monotonicity is violated at z by switching from Π' to Π'', since $\{z\} = f^B(\Pi')$ and $\{y_n\} = f^B(\Pi'')$. Thus, we cannot have $\delta_k > n$.

Suppose that $\delta_k < n$ for some k. Then alternative y_n has to be ranked by at least two different preferences of $\mathcal{P}^{1,1}$ at the same position, since $\delta_k \leq n$ for all $i=1,\ldots,n$. However, this is in contradiction with Lemmas 3.4 and 3.6. Hence, we must have $\delta_k = n$ for all $k=1,\ldots,n$.

Therefore, since the shortest sequence of alternatives that must follow an already prescribed order is of length n and by Lemma 3.6 none of the alternatives of Y can be ranked $m+1$th by a preference of $\mathcal{P}^{1,1}$, we obtained that $n = \#\mathcal{P}^{1,2}_{|X_1\cup Y}$ and $\mathcal{P}^{1,1}_{[m+n+1,m+2n]} = \mathcal{P}^{2,2}_{[1,n]}$. Thus, the pseudo domain of Table 9 extends to a pseudo domain as illustrated in Table 10. Now Lemma 3.6 implies that the alternatives $\cup_{i=1}^{r} Y_i$ must form a CNP domain of depth $l+1$ with an associated factorization $\prod_{i=1}^{l+1} q_i$. Therefore, our induction works and the induction hypothesis is true for depth $l+1$. Arriving to l', we see that the claim of Substep A is true, since it follows from Lemma 3.6 that there exists an $i=2,\ldots,p$ for which $\mathcal{P}^{2,1},\ldots,\mathcal{P}^{2,t}$ partitions $\mathcal{P}^i$.

Substep B: Substep A implies that there exists an $I \subseteq \{1,\ldots,p\}$ such that for all $i \in I$ the subdomains $\mathcal{P}^i_{|X_i}$ have all identical factorizations, there exists a $j \in I \setminus \{i\}$ for which $M_{m+1,2m}(\succ) = X_j$ for all $\succ \in \mathcal{P}^i$ and there exists a bijection $\varphi_{i,j} : X_i \to X_j$ such that $x \in X_i$ and $\varphi_{i,j}(x)$ maintain their rank differences in $\mathcal{P}^i$. We shall assume for notational convenience that $I = \{1,\ldots,r\}$. Hence, there exists a $\sigma : \{1,\ldots,r\} \to \{1,\ldots,r\}$ telling us, which set $X_{\sigma(i)}$ of alternatives must follow immediately the top set X_i of alternatives for all $i=1,\ldots,r$. In Substep B we demonstrate that σ is a bijection. This is clearly the case if $r=2$.

Thus, we can assume that $r > 2$. Our proof will be similar to that of Lemmas 3.1-3.3, but we have to replace the alternatives appearing in those proofs with "nested

[15] More precisely, we should have first defined $c = \left\lceil \frac{d}{\delta_k - n} \right\rceil$ and a, b afterwards. Again, we have followed a different order for expositional reasons.

Table 10: Extended pseudo domain

$\mathcal{P}^{1,1}$	$\mathcal{P}^{1,2}$	$\dots$	$\mathcal{P}^{1,r}$
$X^{1,1}$	$X^{1,2}$	$\dots$	$X^{1,r}$
$X^{1,2}$	$X^{1,3}$	$\dots$	$X^{1,1}$
$\vdots$	$\vdots$		$\vdots$
$X^{1,r}$	$X^{1,1}$	$\dots$	$X^{1,r-1}$
$\vdots$	$\vdots$		$\vdots$
$X^{1,t-r+1}$	$X^{1,t-r+2}$	$\dots$	$X^{1,t}$
$X^{1,t-r+2}$	$X^{1,t-r+3}$	$\dots$	$X^{1,t-r+1}$
$\vdots$	$\vdots$		$\vdots$
$X^{1,t}$	$X^{1,t-r+1}$	$\dots$	$X^{1,t-1}$
Y_1	Y_2	$\dots$	Y_r
Y_2	Y_3	$\dots$	Y_1
$\vdots$	$\vdots$		$\vdots$

cycles", which will complicate the argument.

Suppose that σ does not define a bijection. Then there exists $i, i', i'' \in \{1, \dots r\}$ such that $i \neq i'$, $i \neq i''$, $i' \neq i''$, $\sigma(i') = i$ and $\sigma(i'') = i$. Moreover, $m = m_i = m_{i'} = m_{i''}$ by Substep A. For notational convenience let $X' = \{x_1, \dots, x_m\} = X_i$, $Y = \{y_1, \dots, y_m\} = X_{i'}$, $Z = \{z_1, \dots, z_m\} = X_{i''}$ such that $\varphi_{i',i}(y_l) = x_l$ and $\varphi_{i'',i}(z_l) = x_l$ for all $l = 1, \dots, m$. Pick preferences $\succ_1, \dots, \succ_m, \succ'_1, \dots, \succ'_m, \succ''_1, \dots, \succ''_m \in \mathcal{P}$ with respective top alternatives $x_1, \dots, x_m, y_1, \dots, y_m, z_1, \dots, z_m$. If $r = 3$, then the alternatives from X' cannot be lowest ranked alternatives; a contradiction. If $r > 3$, then we can assume the existence of a preference $\succ \in \mathcal{P}$ in which the alternatives from Y are ranked above the alternatives from Z and the alternatives from Z are ranked above the alternatives from X' by part (ii) of the richness condition. For notational convenience we can assume that

$$y_l \succ'_l \dots \succ'_l y_{l'} \succ'_l x_1 \succ'_l \dots \succ'_l x_m \succ'_l \dots$$
$$z_k \succ''_k \dots \succ''_k z_{k'} \succ''_k x_1 \succ''_k \dots \succ''_k x_m \succ''_k \dots$$

and

$$\dots \succ y_s \succ \dots \succ y_{s'} \succ \dots \succ z_t \succ \dots \succ z_{t'} \succ \dots \succ x_1 \succ \dots \succ x_m$$

for some $l, l', k, k', s, s', t, t' \in \{1, \dots, m\}$.

Let $J = \arg\min_{j \in \{1,\dots,m\}} \sum_{u=1}^{m} rk\,[y_j, \succ_u]$ and $Y' = \{y_j \in Y \mid j \in J\}$. Then there exist positive integers a and b such that $b > m^2 + 1$ and that profile

$$\Pi = (a \cdot \succ_1, \dots, a \cdot \succ_m, b \cdot \succ'_1, \dots, b \cdot \succ'_m),$$

has alternatives $Y' \cup X'$ indifferent on top with a lead of at least $(m^2 + 1)q$ over the alternatives from $X \setminus (X' \cup Y)$. We consider profile

$$\Pi' = (a \cdot \succ_1, \dots, a \cdot \succ_m, (b-1) \cdot \succ'_1, \dots, (b-1) \cdot \succ'_m),$$

in which the top alternatives X' have a lead of m^2 over alternatives Y'. To obtain Π'' from Π' we replace $m^2 + 1$ preferences of type $\succ'_s$ with $m^2 + 1$ preferences of type $\succ$.

It can be verified that $f^B(\Pi') = X'$ and $f^B(\Pi'') = Y'$. Thus, we have a violation of monotonicity at any alternative $y \in Y'$ if we switch from Π'' to Π'.

Substep C: Substep B established that the cycles of permutation σ partition $\{1, \dots, r\}$ into sets $I_1, \dots, I_s$. In what follows we consider, for notational convenience, the case of $I = I_1 = \{1, \dots, k\}$ and $\sigma(1) = 2, \dots, \sigma(k-1) = k, \sigma(k) = 1$. First, in an analogous way to Step 1 we show in Substep C that the cycles formed by alternatives $X_1, \dots, X_k$ follow each other in a cyclic pattern in $\mathcal{P}_1 \cup \ldots \cup \mathcal{P}_k$; that is, for all $i = 1, \dots, k$ we have in $\mathcal{P}^i$ that the alternatives from $X_{i \oplus_k 1}$ follow those from X_i, the alternatives from $X_{i \oplus_k 2}$ follow those from $X_{i \oplus_k 1}$, and so on. Second, we claim that there exist bijections $\tau_{i,j} : X_i \to X_j$ $(i, j = 1, \dots, k)$ such that

$$[x \in X_i, v = \tau_{i,j}(x), x \succ v, x \succ' v] \Rightarrow rk\,[v, \succ] - rk\,[x, \succ] = rk\,[v, \succ'] - rk\,[x, \succ'] \quad (4.2)$$

for all $\succ, \succ' \in \mathcal{P}_1 \cup \ldots \cup \mathcal{P}_k$ and all $i \neq j$, $i, j = 1, \dots, k$.

Clearly, both claims are true for the case of $k \leq 2$. Hence, we can assume that $k \geq 3$. We know by Substep B that the first claim of Substep C is true for the top $2m$ alternatives of any preferences in $\mathcal{P}_i$ and that we can define bijections $\tau_{i,i \oplus_k 1} : X_i \to X_{i \oplus_k 1}$ in a way that equation (4.2) holds true for all $i = 1, \dots, k$ if we restrict ourselves to the top $2m$ alternatives of $\mathcal{P}_1 \cup \ldots \cup \mathcal{P}_k$.

Our induction hypotheses is that the claim holds true for the top lm alternatives, where $l \in \{2, \dots, k-1\}$, of any preference in $\mathcal{P}_1 \cup \ldots \cup \mathcal{P}_k$.[16] For purely notational convenience let $Y = \{y_1, \dots, y_m\} = X_l$, $Z = \{z_1, \dots, z_m\} = X_{l+1}$, and $\tau_{l,l+1}(y_i) = z_i$ for all $i = 1, \dots, m$. We will just consider the case of z_1. The other elements of Z can be handled in the same way. We shall denote by $\succ \in \mathcal{P}_1$ the preference that ranks y_1 as the highest ranked Y alternative; i.e., $rk\,[y_1, \succ] = m(l-1) + 1$. Moreover, let $\succ'$ be a preference ranking y_1 on the top and let $\succ''$ be a preference ranking z_1 on the top. There exists positive integers a and b such that profile $\Pi = (a \cdot \succ', b \cdot \succ'')$ has alternatives z_1 and $Y' \subseteq Y$ on the top. Let $y_s = \max\{t \in \{1, \dots, j\} \mid y_s \in Y'\}$.

Suppose that alternative z_1 does not immediately follow y_m in $\succ$; i.e, $rk\,[z_1, \succ] > ml + 1$. Let $\delta = rk\,[z_1, \succ] - (ml + 1)$, $d = rk\,[z_1, \succ'] - rk\,[y_s, \succ']$ and $c = \lceil \frac{d}{\delta} \rceil$. Then considering profiles $\Pi' = ((a-1) \cdot \succ', b \cdot \succ'')$ and $\Pi' = (c \cdot \succ, (a - c - 1) \cdot \succ', b \cdot \succ'')$, we can verify that $y_s \in f^B(\Pi'')$ and $f^B(\Pi') = \{z_1\}$ if a and b were selected large enough so that no other alternative can interfere and $a > c + 1$. Monotonicity is now violated at y_s if we switch from Π'' to Π'.

It follows from the above defined $\tau_{l,l+1} : X_l \to X_{l+1}$, $\tau_{1,l} : X_1 \to X_l$ and transitivity that we obtained a bijection $\tau_{1,l+1} : X_1 \to X_{l+1}$ in a way that equation (4.2) holds true for all $l = 1, \dots, k-1$ if we restrict ourselves to the top $(l+1)m$ alternatives of $\mathcal{P}_1 \cup \ldots \cup \mathcal{P}_k$. One can obtain the remaining bijections in an analogous way.

We conclude that we have constructed the required sets $Y = \cup_{i=1}^k X_i$ and $\mathcal{P}' = \cup_{i=1}^k \mathcal{P}^i$ by induction.

Step 3: The partition $X_1, \dots, X_p$ of X in Step 1 satisfies the requirements of Step 2. Finally, it follows by induction from Step 2 that $\mathcal{P}$ has to be a CNP domain.

[16]This includes that we have defined bijections $\tau_{i,i \oplus_k l-1} : X_i \to X_{i \oplus_k l-1}$ in a way that equation (4.2) holds true for all $i = 1, \dots, k$ if we restrict ourselves to the top lm alternatives of $\mathcal{P}_1 \cup \ldots \cup \mathcal{P}_k$.

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