The Structure of Strategy-Proof Social Choice

Part I: General Characterization and Possibility Results on Median Spaces *

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Abstract *We define a general notion of single-peaked preferences based on abstract betweenness relations. Special cases are the classical example of single-peaked preferences on a line, the separable preferences on the hypercube, the "multi-dimensionally single-peaked" preferences on the product of lines, but also the unrestricted preference domain. Generalizing and unifying the existing literature, we show that a social choice function is strategy-proof on a sufficiently rich domain of generalized single-peaked preferences if and only if it takes the form of voting by issues ("voting by committees") satisfying a simple condition called the "Intersection Property."*

Based on the Intersection Property, we show that the class of preference domains associated with "median spaces" gives rise to the strongest possibility results; in particular, we show that the existence of strategy-proof social choice rules that are non-dictatorial and neutral requires an underlying median space. A space is a median space if, for every triple of elements, there is a fourth element that is between each pair of the triple; numerous examples are given (some well-known, some novel), and the structure of median spaces and the associated preference domains is analyzed.

Keywords: Strategy-proofness, generalized single-peakedness, median spaces, intersection property

JEL Classification: D71, D72

1 Introduction

By the Gibbard-Satterthwaite impossibility theorem, non-degenerate social choice functions can be strategy-proof only on restricted domains. In response to this fundamental result, a large literature has taken up the challenge of determining domains on which possibility results emerge. In economic environments in which it is assumed that individuals care only about certain aspects of social alternatives, the well-known class of Groves mechanisms offers a rich array of strategy-proof social choice functions under the additional assumption of quasi-linear utility. By contrast, in contexts of "pure" social choice ("voting") individuals care about all aspects of the social state. Here, a path-breaking paper by Moulin (1980) demonstrated the existence of a large class of strategy-proof social choice functions in the Hotelling-Downs model in which social states can be ordered from left to right as in a line, and in which preferences are single-peaked with respect to that ordering. Moulin showed that all strategy-proof social choice functions can be understood as generalizations of the classical median voter rule. His result inspired a sizeable literature that obtained related characterizations for other particular domains or proved impossibility results (see, among others, Border and Jordan (1983) and Barberà, Sonnenschein and Zhou (1991)). Remarkably, it turned out that when a positive result could be obtained, the class of strategy-proof social choice functions had a structure similar to that uncovered by Moulin which we shall refer to as "voting by issues" ("voting by committees" in the terminology of Barberà, Sonnenschein and Zhou (1991)).

In this paper, we introduce a large class of preference domains, referred to as "generalized single-peaked" domains, and show that strategy-proof social choice can be characterized in terms of voting by issues on these domains. This allows us then to determine exactly which domains admit strategy-proof social choice functions exhibiting fundamental additional properties such as non-dictatorship, anonymity, neutrality, and efficiency. While part of this work is left to companion papers (see Nehring and Puppe (2005a) and (2005b)), we shall identify here the class of domains on which the strongest possibility results obtain; these are characterized geometrically as "median spaces" and described in more detail below.

Generalized Single-peaked Domains

The basic idea underlying our approach is to describe the space of alternatives geometrically in terms of a three-place *betweenness* relation, and to consider associated domains of preferences that are *single-peaked* in the sense that individuals always prefer social states that are between a given state and their most preferred state, the "peak".

Following Nehring (1999), we shall conceptualize betweenness more specifically in terms of the differential possession of *relevant properties*: a social state y is between the social states x and z if y shares all relevant properties common to x and z. (Generalized) single-peakedness means that a state y is preferred to a state z whenever y is between z and the peak x^*, i.e. whenever y shares all properties with the peak x^* that z shares with it (and possibly others as well). Throughout, it will be assumed that a property is relevant if and only if its negation is relevant, so that each property together with its negation defines an *issue* to be decided upon. As further illustrated below, a great variety of preference domains that arise naturally in applications can be described as single-peaked domains with respect to betweenness relations of the kind just described. In fact, our assumptions encompass almost all domains that have been shown to enable non-degenerate strategy-proof social choice in a voting context. For instance, the standard betweenness relation in case of a line is derived from properties of the form "to the right (resp. left) of a given state." But generalized single-peaked domains can also easily give rise to impossibility results. For instance, the unrestricted domain envisaged by the Gibbard-Satterthwaite theorem can be described as the set of all single-peaked preferences with respect to a vacuous betweenness relation that declares no social state between any two other states; the corresponding relevant properties are, for every social state x, "being equal to x," and "being different from x."

The Structure of Strategy-Proof Social Choice

Building on previous work culminating in Barberà, Massó and Neme (1997), we show that strategy-proof social choice on generalized single-peaked domains can be described in a unified manner as "voting by issues" (Theorem 2). This structure has two aspects. First, the social choice depends on individuals' preferences through their most preferred alternative only, i.e. it satisfies "peaks only." Second, the social choice is determined by a separate "vote" on each issue: an individual is construed as voting for a property over its negation if and only if her top-ranked alternative has the property. For example, in the special case in which voting by issues is anonymous and neutral it takes the form of majority voting on issues; that is, a chosen state has a particular property if and only if the majority of agents' peaks have that property. In general, the chosen state has a property if and only if the individuals voting for that property form a winning coalition.

Crucially, the voting by issues structure describes only an implication of strategy-proofness, not a characterization, since it does not by itself allow one to generate well-defined social choice functions. Indeed, without restrictions on the family of properties deemed relevant and/or the structure of winning coalitions, the chosen properties may well be mutually incompatible. Consider, for example, majority voting on issues on a domain of three states, and take as relevant the six properties of being equal to or different from any particular of these states, corresponding to the unrestricted domain of preferences. If there are three agents with distinct peaks, a majority of agents votes for each property of the form "is different from state x." Since no social state is different from *all* social states (including itself), the social choice is therefore empty. A structure of winning coalitions is called *consistent* if the chosen properties are always jointly realizable (irrespective of voters' preferences). We show that a structure of winning coalitions is consistent if and only if it satisfies a simple condition, called the "Intersection Property." This leads to a unifying characterization of the class of all strategy-proof social choice functions on all generalized single-peaked domains, namely as voting by issues satisfying the Intersection Property (Theorem 3 below). For each particular generalized single-peaked domain, it allows one to describe the subclass of *anonymous* strategy-proof social choice functions in terms of a system of linear inequalities representing bounds on the admissible quotas.

Median Spaces as Distinguished Domains

The restrictions imposed by the Intersection Property on the admissible sets of winning coalitions reflect the structure of the underlying space. The Intersection Property thereby provides the crucial tool for determining on which generalized single-peaked domains there exist *well-behaved* (e.g. non-dictatorial, anonymous, neutral, etc.) strategy-proof social choice functions, but it does not answer this question by itself. This is the central concern of the two companion papers Nehring and Puppe (2005a) and (2005b). Here, we seek to determine those domains admitting a maximally rich class of strategy-proof social choice functions. It turns out that these domains are exactly the domains on which strategy-proof social choice functions exist that are both anonymous and neutral, amounting to majority voting on issues. We show that majority voting on issues is consistent if and only if the betweenness relation has the property that, for all three distinct states, there exists a state between each pair of them (Corollary 5 below). Such a state is called a *median* of the triple, and the resulting space a *median space*. In the case of three agents, for example, majority voting on issues boils down to choosing the median of the agents' peaks. The median can be viewed as a natural compromise between the voters' preferences, since, by the single-peakedness, every voter ranks the median above the other two agents' peaks; thus, the median wins a majority vote against every voter's peak in pairwise comparison. We then prove that, under strategy-proofness and non-dictatorship, neutrality *alone* requires the underlying space to be a median space (Theorem 4). Moreover, median spaces are characterized by the property that, for each alternative x, there exists a "minority veto rule" with x as status quo (Theorem 5).

Median spaces represent the natural generalization and unification of the known cases in which well-behaved strategy-proof social choice functions have been shown to exist, the line and its multi-dimensional extensions on the one hand, and trees on the other; see Border and Jordan (1983) as well as Barberà, Gul and Stacchetti (1993) for the former, and Demange (1982) for the latter. In Nehring and Puppe (2005b) we show that efficiency presupposes an underlying median space structure, unless the social choice is dictatorial; thus, from this point of view as well median spaces are central.

Since median spaces turn out to play such a distinguished role, one would like to understand their associated preference domains directly, not merely indirectly via the associated betweenness geometry. To this behalf, we show that single-peaked preferences on a median space can be described in terms of two economically fundamental types of preference restrictions, convexity and separability. In the special case of the line (or, more generally, in trees), the single-peaked preferences are simply the convex ones; in the case of the hypercube considered in Barberà, Sonnenschein and Zhou (1991), the single-peaked preferences are those that are separable. These are the two pure cases; in general, single-peaked preferences on a median space are characterized by a combination of convexity and separability restrictions.

Relation to the Literature

This paper was inspired by the remarkable paper Barberà, Massó and Neme (1997) which demonstrated that strategy-proof social choice functions can be characterized in terms of voting by issues ("generalized median voter schemes" in their terminology) much more generally than thought previously. These authors looked at the domain of all single-peaked preferences defined on a fixed product of lines, and considered subdomains of preferences by restricting the peaks to lie in arbitrary prespecified subsets interpreted as "feasible sets." By contrast, in this paper we assume an arbitrary fixed set of social states and consider a wide range of different preference domains over that set. This fixed set is understood to reflect all feasibility constraints that may be relevant. Our central assumption is that the "betweenness geometry" implicit in the domain can be described in terms of an abstract "property space." Sometimes these properties can be understood as characteristics in the manner of Lancaster (1966), but at other times they are merely useful mathematical constructs.

Since states in a property space can be viewed as appropriately positioned points in a sufficiently high-dimensional hypercube, there is a close mathematical relationship between the setup of Barberà, Massó and Neme (1997) and ours. Indeed, for the subclass of preference domains consisting of *all* single-peaked preferences compatible with a given betweenness relation, their first main result yields via an extension argument the "peaks-only" property that is a central starting point for our analysis.[1] In our adaptation (Theorem 2 below), we state more general sufficient conditions on subdomains of single-peaked preferences that still deliver the "peaks-only" property and that are frequently required in applications. In terms of approach, our main step beyond Barberà, Massó and Neme (1997) is however our demonstration that many naturally occurring preference domains on a fixed set of social states *without given structure* can be analyzed fruitfully as single-peaked preferences in a property space (subset of a hypercube); see Section 2.4 below for details, especially Theorem 1 and Examples 6 and 7.

Barberà, Massó and Neme (1997) also provided a characterization of consistency in terms of a condition they called "intersection property" as well. Their condition is less transparent and workable than the one obtained here; for instance, in the anonymous case of "voting by quota," our condition directly translates into a system of linear inequalities, representing appropriate bounds on the quotas (see Section 3.3 below). In contrast to their condition, our condition makes direct reference to the combinatorial structure of the property space via the notion of a "critical family" which has no counterpart in their analysis. This feature is crucial in enabling us in related work to characterize exactly the property spaces that admit strategy-proof social choice functions with various desirable properties such as non-dictatorship, anonymity, efficiency, etc. (Nehring and Puppe (2005a,b)). In the present paper, the focus is on median spaces. Due to the canonical simplicity of their combinatorial structure identified by the Intersection Property they are associated with a maximally rich set of strategy-proof social choice functions, as pointed out above.

While median spaces are a well-known and well-studied object in abstract convexity theory (see e.g. van de Vel (1993)), they do not appear to have been considered anywhere in the strategy-proofness literature. Implicitly, however, the properties of median spaces play a central role in Barberà, Sonnenschein and Zhou (1991) and Barberà, Gul and Stacchetti (1993).

The remainder of the paper is organized as follows. Section 2 describes the preference domains to which our characterization results apply. In particular, it introduces the central concepts of single-peaked preference orderings with respect to general betweenness relations, and of betweenness relations derived from property spaces. We also provide a simple characterization of when a given set of linear orderings can be represented as a sufficiently rich domain of single-peaked preferences with respect to an appropriate betweenness relation (Theorem 1).

In Section 3, we use these concepts to provide a generalization and unification of the existing literature. Specifically, we show that every strategy-proof social choice function

[1] The appropriate extension argument is provided in an earlier version of this paper, see Nehring and Puppe (2002).

on a sufficiently rich domain of generalized single-peaked preferences satisfying a weak condition of "voter sovereignty" must be voting by issues (Theorem 2). We then derive a simple necessary and sufficient condition for the consistency of the structure of winning coalitions, the "Intersection Property." We thus obtain a unifying characterization of strategy-proof social choice on generalized single-peaked domains, namely as voting by issues satisfying the Intersection Property (Theorem 3).

Section 4 introduces the notion of a median space. We show that a domain enables neutral and non-dictatorial strategy-proof social choice if and only if the underlying domain of social states is a median space (Theorem 4), and that median spaces are maximally rich in the range of consistent "minority veto rules" (Theorem 5). We also show that on a median space a preference ordering is single-peaked if and only if it is convex and separable, and we introduce and analyze a stronger, cardinal notion of convexity. Section 5 concludes, and all proofs are collected in Appendix 2.

2 Generalized Single-Peaked Domains

In this section, we describe the preference domains to which our later characterization of strategy-proof social choice functions applies. Throughout, we assume that the relevant preference restrictions are independent and identical across voters, so that the domains are n-fold Cartesian products of one common set of individually admissible preferences where n is the number of voters. The individual domains, in turn, can be described as sufficiently rich sets of orderings that are "single-peaked" with respect to an appropriately defined betweenness relation. For expository convenience, we consider only the case of *linear* orderings here; the more general case of weak orderings and even partial orders is treated in an earlier working paper version Nehring and Puppe (2002).

2.1 Single-Peakedness with Respect to General Beetweenness Relations

The classical example of a preference domain admitting non-dictatorial and strategy-proof social choice is the domain of all single-peaked preferences on a line. Suppose that the social alternatives are ordered from left to right as in Figure 1a below. A preference ordering $\succ$ with top element x^* is single-peaked if $y \succ z$ whenever y is *between* z and the peak x^*. Here, the relevant notion of "betweenness" is of course the standard one corresponding to the left-to-right scale of the line. The aim of this paper is to study the structure of strategy-proof social choice on domains of preferences that are "single-peaked" with respect to more general betweenness relations. Formally, we will consider a ternary relation T on a finite universe X of social states or social alternatives with $\#X \geq 3$. The interpretation of the ternary relation T is that $(x, y, z) \in T$ if the social state y is **between** the social states x and z. By convention, let $(x, x, z) \in T$ and $(x, z, z) \in T$ for all x, z, i.e. every state is (weakly) between itself and every other state. The "betweenness" terminology will be justified in the sequel by the requirement of further axiomatic properties on the ternary relation.

Definition (Generalized Single-Peakedness) A preference ordering $\succ$ on X is *single-peaked with respect to* T if there exists $x^* \in X$ such that for all $y \neq z$,

$$(x^*, y, z) \in T \Rightarrow y \succ z. \tag{2.1}$$

We say that a preference ordering is *generalized single-peaked* if it is single-peaked with respect to some betweenness relation T on X.

Thus, in analogy to the standard definition, a preference is single-peaked with respect to T if every state y that is "T-between" the peak x^* and another state z is preferred to that state. The set of all linear orderings on X that are single-peaked with respect to T will be denoted by $\hat{\mathcal{S}}_{X,T}$.

As a first illustration, consider the three graphs in Figure 1 below with the nodes representing social states. To each graph one can associate the corresponding *graphic betweenness* according to which a social state y is between the two states x and z if y lies on some shortest path connecting x and z.[2] For instance, both y and y' are between x and z in Figures 1a and 1b, while w is not between x and z in Figures 1b and 1c. The graphic betweenness associated with the line in Figure 1a is of course the standard betweenness and the corresponding notion of single-peakedness is the usual one. The graph in Figure 1b can be viewed as the (3-dimensional) "hypercube" corresponding to the set $\{0, 1\}^3$ of binary sequences of length 3. A preference is single-peaked with respect to the graphic betweenness on a hypercube if and only if it is *separable* in the sense of Barberà, Sonnenschein and Zhou (1991).[3] Finally, the graph in Figure 1c is the complete graph in which each state is connected to every other state by an edge. By consequence, the corresponding graphic betweenness is *vacuous* in the sense that no state is between any two other states.[4] Clearly, *every* linear preference ordering is single-peaked with respect to this vacuous betweenness relation; therefore, the set of all generalized single-peaked preferences is the unrestricted preference domain in this case.

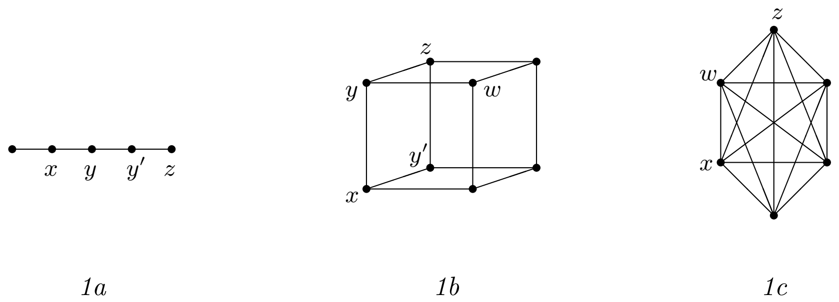


Figure 1: Three graphic betweenness relations.

In all what follows, we will consider subdomains $\mathcal{D} \subseteq \hat{\mathcal{S}}_{X,T}$ of single-peaked preferences, imposing the following "richness" conditions with respect to T. For all x, y,

[2]Formally, consider a graph ϕ on X, i.e. a symmetric binary relation on X. A *shortest path* connecting two distinct elements x_1 and x_n is a minimal set $\{x_1, ..., x_n\}$ such that $(x_i, x_{i+1}) \in \phi$ for all $i = 1, ..., n-1$. Note that shortest paths connecting two elements need not be unique. The *graphic betweenness* T_ϕ associated with ϕ is defined by $(x, y, z) \in T_\phi$ if $[y = x = z$ or y is an element of some shortest path connecting x and $z]$.

[3]This assertion follows at once from Fact 2.1 below.

[4]Note that the vacuous betweenness is not the empty set since it contains all triples of the form (x, x, z) and (x, z, z); it is "vacuous" in the sense that it contains no other instances of betweeness, i.e. no non-trivial instances of betweenness.

denote by

$$[x, y] := \{w \in X : (x, w, y) \in T\}$$

the *segment* between x and y, and say that two distinct elements x and y are *neighbours* if $[x, y] = \{x, y\}$, i.e. x and y are neighbours if no other point is between them.

R1 For all neighbours x, y there is $\succ \in \mathcal{D}$ such that for all $w \in X \setminus \{x, y\}$, $x \succ y \succ w$.
R2 For all x, y, z with $y \notin [x, z]$, there is $\succ \in \mathcal{D}$ with peak x such that $z \succ y$.

Condition R1 requires that, for every pair of neighbours, there is a preference ordering that has one of them as peak and the other as the second best element. Condition R2 states that, for each triple x, y, z such that y is not between x and z, there is a preference with peak x that ranks z above y. Henceforth, we will say that a domain $\mathcal{D} \subseteq \hat{\mathcal{S}}_{X,T}$ of single-peaked preferences is **rich with respect to** T if it satisfies conditions R1 and R2. The properties imposed on the betweenness relation in the following will guarantee that a rich domain includes for each x at least one preference ordering with peak x (see conditions T1-T5 in Section 2.4 below).

2.2 Betweenness Relations Derived from Property Spaces

For the purpose of characterizing the class of all strategy-proof social choice functions on generalized single-peaked domains, one needs additional structure on the underlying betweenness relation. Throughout, we will rely on the assumption that the betweenness relation can be derived from a "property space," as follows.

Suppose that the elements of X are distinguished by different *basic properties*. Formally, let these properties be described by a non-empty family $\mathcal{H} \subseteq 2^X$ of subsets of X where each $H \in \mathcal{H}$ corresponds to a property possessed by all alternatives in $H \subseteq X$ but by no alternative in the complement $H^c := X \setminus H$. The basic properties are thus identified *extensionally*: for instance, the basic property "the tax rate on labour income is 10% or less" is identified with the *set* of all social states in which the tax rate satisfies the required condition. We assume that the list $\mathcal{H}$ of basic properties satisfies the following three conditions.

H1 (Non-Triviality) $H \in \mathcal{H} \Rightarrow H \neq \emptyset$.
H2 (Closedness under Negation) $H \in \mathcal{H} \Rightarrow H^c \in \mathcal{H}$.
H3 (Separation) for all $x \neq y$ there exists $H \in \mathcal{H}$ such that $x \in H$ and $y \notin H$.

Condition H1 says that every basic property is possessed by some element in X. Condition H2 asserts that for each basic property corresponding to the set H there is also the complementary property possessed by all alternatives not in H. We will refer to a pair (H, H^c) as an *issue*. Finally, condition H3 says that every two distinct elements are distinguished by at least one basic property. A pair $(X, \mathcal{H})$ satisfying H1-H3 will be called a **property space**.

Following Nehring (1999), a property space $(X, \mathcal{H})$ gives rise to a natural betweenness relation $T_{\mathcal{H}}$ as follows. For all x, y, z,

$$(x, y, z) \in T_{\mathcal{H}} :\Leftrightarrow [\text{ for all } H \in \mathcal{H} : \{x, z\} \subseteq H \Rightarrow y \in H]. \tag{2.2}$$

Thus, y is between x and z in the sense of $T_{\mathcal{H}}$ if y possesses all basic properties that are common to x and z (and possible some more).

The following result characterizes single-peakedness in terms of the basic properties from which the betweenness is derived.

Fact 2.1 *Let $(X, \mathcal{H})$ be a property space. A preference ordering $\succ$ is single-peaked with respect to $T_{\mathcal{H}}$ if and only if there exists a partition $\mathcal{H} = \mathcal{H}_g \cup \mathcal{H}_b$ with $\mathcal{H}_g \cap \mathcal{H}_b = \emptyset$ such that*

(i) $H \in \mathcal{H}_g \Leftrightarrow H^c \in \mathcal{H}_b$,
(ii) *$y \succ z$ whenever $y \neq z$ and for all $H \in \mathcal{H}_g$, $z \in H \Rightarrow y \in H$, and*
(iii) *there exists x^* such that $x^* \in H$ for all $H \in \mathcal{H}_g$.*

In view of conditions (i) and (ii), single-peakedness with respect to $T_{\mathcal{H}}$ requires that it must be possible to partition all basic properties into a set of "good" properties (those in $\mathcal{H}_g$) and a set of "bad" properties (those in $\mathcal{H}_b$) in a *separable* way: a property is good or bad no matter with which other properties it is combined. Indeed, by condition (ii), possessing an additional "good" property is always preferred. In addition to separability, single-peakedness also requires, by condition (iii), that all good properties are jointly compatible, that is: possessed by some ideal point x^* which corresponds to the preference peak.

The ordinally separable representation in Fact 2.1 suggests a cardinal strengthening, in which preferences have an additive utility representation of the form

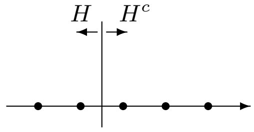
$$u(x) = \sum_{H \in \mathcal{H}_g, H \ni x} \lambda_H,$$

where $\lambda_H > 0$ for all $H \in \mathcal{H}_g$. As is easily verified, the domain of all additive preferences in this sense is rich with respect to $T_{\mathcal{H}}$.

2.3 Basic Examples

To illustrate the above concepts, consider the following examples of generalized single-peaked domains. The first three correspond to the graphic betweenness relations in Figure 1 above.

Example 1 (Single-Peakedness on Line) Let X be linearly ordered by $\geq$, and consider the betweenness relation T given by $(x, y, z) \in T :\Leftrightarrow [x \geq y \geq z$ or $z \geq y \geq x]$ (cf. Figure 1a). This betweenness can be derived via (2.2) from the family $\mathcal{H}$ of all sets of the form $H_{\geq w} := \{y \geq w :$ for some $w \in X\}$ or $H_{\leq w} := \{y \leq w :$ for some $w \in X\}$. Each basic property is thus of the form "lying to the right of w" or "lying to the left of w" (see Figure 2a).



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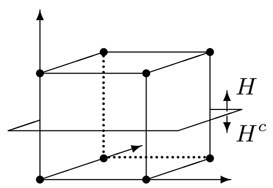

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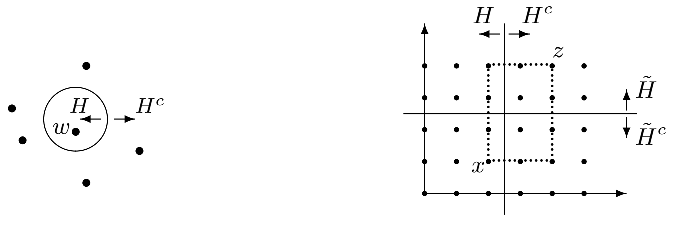

2c 2d

Figure 2: Basic properties underlying Examples 1-4.

Example 2 (Separability on the Hypercube) Let $X = \{0,1\}^K$, which we refer to as the K-dimensional hypercube (cf. Figure 1b). An element $x \in \{0,1\}^K$ is thus described as a sequence $x = (x^1, ..., x^K)$ with $x^k \in \{0,1\}$, and the natural betweenness is given by $(x, y, z) \in T :\Leftrightarrow$ [for all k : $x^k = z^k \Rightarrow y^k = x^k = z^k$]. As is easily verified, this betweenness coincides with the graphic betweenness in Figure 1b above. Geometrically, y is between x and z if and only if y is contained in the "subcube" spanned by x and z (see Figure 1b above and note, for instance, that the whole 3-hypercube is between w and y'). This betweenness can be derived from the basic properties of the form $H_1^k := \{x : x^k = 1\}$ and $H_0^k := \{x : x^k = 0\}$ for all k (see Figure 2b which depicts the two basic properties corresponding to the vertical coordinate). In view of Fact 2.1, a preference $\succ$ is single-peaked with respect to T if and only if it is separable in the sense that, for all x, y and all k,

$$x \succ (x^{-k}, y^k) \Leftrightarrow (y^{-k}, x^k) \succ y.$$

Example 3 (The Unrestricted Domain) The vacuous betweenness on X, defined by $(x, y, z) \in T :\Leftrightarrow y \in \{x, z\}$, can be derived via (2.2) from the family $\mathcal{H}$ of all properties of the form $\{x\}$ ("being equal to x") and $X \setminus \{x\}$ ("being different from x") for all $x \in X$ (see Figure 2c which depicts the property $H = \{w\}$). As noted above, every linear preference ordering is single-peaked with respect to the vacuous betweenness relation, i.e. the set of all single-peaked preferences is the unrestricted domain.

Example 4 (Products) The hypercube betweenness of Example 2 above is an instance of a product betweenness. More generally, let $X = X^1 \times ... \times X^K$, where the alternatives in each factor X^k are described by a list $\mathcal{H}^k$ of basic properties referring to coordinate k. Let $\mathcal{H} := \{H^k \times \prod_{j \neq k} X^j :$ for some k and $H^k \in \mathcal{H}^k\}$, and denote by $T_{\mathcal{H}^k}$ the betweenness relation on X^k induced by $\mathcal{H}^k$. The *product betweenness* T induced by $\mathcal{H}$ according to (2.2) is given by,

$$(x, y, z) \in T \Leftrightarrow [\text{ for all } k : (x^k, y^k, z^k) \in T_{\mathcal{H}^k}].$$

Figure 2d depicts the product of two lines; the alternatives between x and z are precisely the alternatives contained in the dotted rectangle spanned by x and z.

2.4 Rich Domains of Single-Peaked Preferences

Obviously, every given preference ordering $\succ$ is single-peaked with respect to *some* appropriate betweenness relation (for instance, with respect to the betweenness relation according to which a state is between two other states if and only if it is intermediate in terms of the preference ordering $\succ$). The essence of the domain restrictions considered in this paper is of course that all voters' preferences be single-peaked with respect to *the same* betweenness relation. Say that a preference domain $\mathcal{D}$ is a *(generalized) single-peaked domain* on X if there exists a betweenness relation T on X such that every preference ordering in $\mathcal{D}$ is single-peaked with respect to T. Furthermore, say that $\mathcal{D}$ is a *rich single-peaked domain* if there exists a betweenness relation T on X such that every preference ordering in $\mathcal{D}$ is single-peaked with respect to T and if $\mathcal{D}$ is rich with respect to T. In this subsection, we address the question which preference domains can be described as rich single-peaked domains with respect to appropriate betweenness relations. To provide an answer, we first need to find the conditions under which a betweenness relation can be derived from a property space.

2.4.1 When is a Betweenness Relation Induced by a Property Space?

The ternary betweenness relation $T_{\mathcal{H}}$ induced by a property space $(X, \mathcal{H})$ via (2.2) satisfies the following four conditions. For all x, y, z, x', z',

T1 (Reflexivity) $y \in \{x, z\} \Rightarrow (x, y, z) \in T$.
T2 (Symmetry) $(x, y, z) \in T \Leftrightarrow (z, y, x) \in T$.
T3 (Transitivity) $[(x, x', z) \in T$ and $(x, z', z) \in T$ and $(x', y, z') \in T] \Rightarrow (x, y, z) \in T$.
T4 (Antisymmetry) $[(x, y, z) \in T$ and $(x, z, y) \in T] \Rightarrow y = z$.

The reflexivity condition T1 and the symmetry condition T2 follow at once from the definition of $T_{\mathcal{H}}$. Note that it is the symmetry condition that justifies a geometric interpretation of T as "betweenness" relation. The transitivity condition T3 is also easily verified; it states that if both x' and z' are between x and z, and moreover y is between x' and z', then y must also be between x and z. Finally, the antisymmetry condition T4 easily follows, using H2, from the separation property H3.

For the next condition, we need some additional terminology. Say that a set $A \subseteq X$ is **convex** if for all x, y, z,

$$[\{x, z\} \subseteq A \text{ and } (x, y, z) \in T] \Rightarrow y \in A. \tag{2.3}$$

Hence, in accordance with the usual notion of convexity in a Euclidean space, a set is convex if it contains with every two elements all elements that are between them. Furthermore, say that a subset $H \subseteq X$ is a **half-space** if both H and its complement H^c are non-empty and convex.

T5 (Separation) If $(x, y, z) \notin T$, then there exists a half-space H such that

$$H \supseteq \{x, z\} \text{ and } y \notin H.$$

Fact 2.2 *Let T be a ternary relation on X. There exists a collection $\mathcal{H}_T$ of basic properties satisfying H1-H3 such that $T = T_{(\mathcal{H}_T)}$, i.e. such that T is derived from $\mathcal{H}_T$ via (2.2), if and only if T satisfies T1-T5.*

Necessity of the conditions T1-T5 are straightforward (cf. Nehring (1999)); their sufficiency follows from defining the underlying property space $\mathcal{H}_T$ as the collection of all half-spaces induced by T.[5]

As is easily verified, all graphic betweenness relations satisfy T1, T2 and T4. They need not satisfy T3 and T5, as illustrated by the following graph.[6]

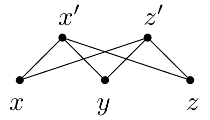

Figure 3: A graphic betweenness violating T3 and T5.

However, if a graphic betweenness satisfies T5, it also satisfies T3;[7] thus, by Fact 2.2, a graphic betweenness can be derived from a property space if and only if it satisfies the separation condition T5. The following example further illustrates this.

Example 5 (Cycles) Let $X = \{x_1, ..., x_l\}$, and consider the l-cycle on X, i.e. the graph with the edges (x_i, x_{i+1}), where indices are understood modulo l so that $x_{l+1} = x_1$ (see Figure 4 for the case $l = 6$). A subset is convex with respect to the graphic betweenness if it contains with every two points also a shortest path connecting them. In particular, if l is even, all half-spaces are of the form $\{x_j, x_{j+1}, ..., x_{j-1+\frac{l}{2}}\}$, and if l is odd, the family of half-spaces consists of all sets of the form $\{x_j, x_{j+1}, ..., x_{j-1+\frac{l+1}{2}}\}$ or $\{x_j, x_{j+1}, ..., x_{j-1+\frac{l-1}{2}}\}$. As is easily verified, the graphic betweenness on the l-cycle satisfies T5, and can thus be derived from a property space by taking all half-spaces as the basic properties; for even l, these are the connected "half-cycles" (see Figure 4).

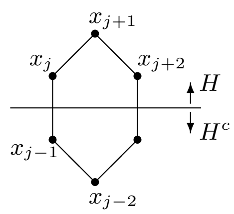

Figure 4: The 6*-cycle.*

2.4.2 When is a Domain a Rich Single-Peaked Domain?

Consider now an arbitrary domain $\mathcal{D}$ of linear preference orderings on X, and define a ternary relation $T_{\mathcal{D}}$ as follows. For all x, y, z with $y \neq z$,

$$(x, y, z) \in T_{\mathcal{D}} :\Leftrightarrow [\ y \succ z \text{ for all } \succ \in \mathcal{D} \text{ with peak } x]. \tag{2.4}$$

[5]Note, however, that the underlying property space is not uniquely determined by T, and frequently it is not necessary to consider the collection of *all* half-spaces. For instance, every non-empty set $A \subseteq X$ is convex with respect to the vacuous betweenness, hence every non-trivial set is a half-space in this case. However, as shown in Example 3 above, to generate the vacuous betweenness it suffices to take all sets of the form $\{x\}$ and their complements as the basic properties.

[6]Both x' and z' in Figure 3 are between x and z, moreover y is between x' and z'; however, y is not between x and z, in violation of T3. Similarly, every convex set containing x and z must also contain x' and z', and hence also y, in violation of T5.

[7]This follows from van de Vel (1993, Proposition 4.15, p.83).

Also, by convention, $(x, z, z) \in T_{\mathcal{D}}$ for all x, z. By construction, one has $\mathcal{D} \subseteq \hat{\mathcal{S}}_{X,T_{\mathcal{D}}}$, i.e. every preference in $\mathcal{D}$ is single-peaked with respect to $T_{\mathcal{D}}$. On the other hand, a given domain $\mathcal{D}$ will typically not include *all* single-peaked orderings with respect to $T_{\mathcal{D}}$. For instance, the betweenness relation associated with the unrestricted domain via (2.4) is the vacuous betweenness according to which no state is between any two other states. But this is also the betweenness associated with every domain such that, for every pair x, y, there is a preference with x as top element and y as second best element. Also note that every domain $\mathcal{D}$ satisfies the richness condition R2 with respect to $T_{\mathcal{D}}$ by construction. For example, this means that every domain such that, for every pair x, y, there is a preference with x as top element and y as second best element is a rich single-peaked domain with respect to the vacuous betweenness relation.

The following result characterizes the class of all domains that can be represented as rich single-peaked domains.

Theorem 1 *Let $\mathcal{D}$ be a set of linear orderings on X. There exists a list of basic properties $\mathcal{H}$ such that $\mathcal{D}$ is a rich single-peaked domain with respect to $T_{\mathcal{H}}$ if and only if (i) $T_{\mathcal{D}}$ satisfies symmetry (T2), transitivity (T3) and separation (T5), (ii) every $x \in X$ is the peak of some element in $\mathcal{D}$, and (iii) the following "closure" condition holds: for all x, y,*

$$[\forall w \notin \{x, y\}\ \exists \succ \in \mathcal{D}_x :\ x \succ y \succ w] \Rightarrow [\exists \succ \in \mathcal{D}_x\ \forall w \notin \{x, y\} :\ x \succ y \succ w], \quad (2.5)$$

where $\mathcal{D}_x$ denotes the subset of all preferences in $\mathcal{D}$ with peak x.

The closure condition (2.5) says that, if for every $w \notin \{x, y\}$, there is a preference with peak x that ranks y above w, then there must exist a preference with peak x that has y as second best element.

We conclude this section with two further examples illustrating the derivation of the underlying property space via $T_{\mathcal{D}}$ described in Theorem 1.

Example 6 ("Doing the Opposite") Suppose that, for each state x, there exists a state $\bar{x}$ ("the opposite of x") such that $\bar{\bar{x}} = x$. Consider then the domain $\mathcal{D}$ of all linear orderings $\succ$ such that $x \succ y \Leftrightarrow \bar{y} \succ \bar{x}$, i.e. if a state is deemed better than another, then its opposite must be worse than that state's opposite. The betweenness relation $T_{\mathcal{D}}$ associated with $\mathcal{D}$ according to (2.4) is given by

$$(x, y, z) \in T_{\mathcal{D}} \Leftrightarrow [z = \bar{x} \text{ or } y \in \{x, z\}],$$

i.e. every element is between opposite pairs, but no element is between two non-opposite states. As is easily verified $T_{\mathcal{D}}$ is the graphic betweenness corresponding to the graph in which each point is connected by an edge to all other points but to its opposite element. A subset $A \neq X$ is convex with respect to $T_{\mathcal{D}}$ if and only if $[x \in A \Rightarrow \bar{x} \notin A]$, and $H \neq X$ is a half-space if and only if $[x \in H \Leftrightarrow \bar{x} \notin H]$. In particular, $T_{\mathcal{D}}$ satisfies the separation condition T5; by Fact 2.2, it can thus be derived from a property space. In contrast to some of the examples above, however, the basic properties (i.e. the half-spaces) do not have a meaningful interpretation as "Lancasterian characteristics" here.

Observe that a preference is single-peaked with respect to $T_{\mathcal{D}}$ whenever the opposite of the peak is the least preferred alternative. By contrast, for a preference in $\mathcal{D}$, the ranking between *any* pair is uniquely determined by the ranking of the opposite pair. Thus, $\mathcal{D}$ is much smaller than the domain $\hat{\mathcal{S}}_{X,T_{\mathcal{D}}}$ of all single-peaked preferences. Nevertheless, $\mathcal{D}$ clearly satisfies the closure condition (2.5), hence it represents a rich single-peaked domain by Theorem 1. As we shall see, this implies that the induced betweenness $T_{\mathcal{D}}$ is all what matters for the analysis of strategy-proofness.

Example 7 (Additive Preferences over Public Goods) There are $K+1$ public goods, which can be supplied in non-negative discrete quantities. Denote by $x^k \in \mathbf{N}_0$ the quantity of good $k = 0, 1, ..., K$, and suppose that feasibility requires $\sum_k x^k \leq M$ for some fixed amount M, i.e. take the prices of each public good to be 1, for simplicity. Furthermore, suppose that preferences can be represented by additive utility functions of the form $\sum_k u^k(x^k)$, where each u^k is increasing and concave. By the resulting monotonicity of preferences, the choice will always lie on the budget line $\sum_k x^k = M$. We can therefore eliminate the coordinate corresponding to good 0, and consider the set $X = \{x \in \mathbf{N}_0^K : \sum_{k=1}^K x^k \leq M\}$ as the set of states. The utility functions on X can be written as follows,

$$u(x^1, ..., x^K) = \sum_{k=1}^{K} u^k(x^k) + u^0(M - \sum_{k=1}^{K} x^k). \tag{2.6}$$

As is easily verified, the geometric structure of the domain $\mathcal{D}$ of all additive preferences of the form (2.6) is described by the following betweenness relation:

$$(x, y, z) \in T_{\mathcal{D}} :\Leftrightarrow \left\{ y^k \in [x^k, z^k] \text{ for all } k, \text{ and } \sum_k y^k \in \left[\sum_k x^k, \sum_k z^k \right] \right\}.$$

For instance, for $K = 2$, $T_{\mathcal{D}}$ is the graphic betweenness corresponding to the following graph.

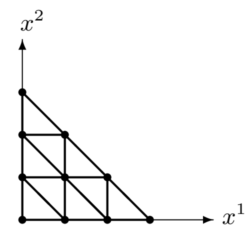

Figure 5: Additive preferences over three public goods.

Again, $T_{\mathcal{D}}$ satisfies T1-T5, and $\mathcal{D}$ satisfies (2.5). $\mathcal{D}$ is therefore yet another instance of a rich domain of single-peaked preferences. Interestingly, in this case there is an "endogenous" ideal point that depends on the budget set; indeed, due to the strict monotonicity of preferences, there is no ideal point on their natural domain $\mathbf{N}_0^K$. This is in contrast to the interpretation of Barberà, Massó and Neme (1997) who assume that the "true" ideal points are in fact feasible.[8]

[8]See Appendix 1 for further discussion of the relation to Barberà, Massó and Neme (1997).

3 The Structure of Strategy-Proof Social Choice on Generalized Single-Peaked Domains

3.1 Voting by Issues

Let $N = \{1, ..., n\}$ be a set of voters. Each voter i is characterized by a linear preference ordering $\succ_i$ in some domain $\mathcal{D}$; the best element of X with respect to $\succ_i$ is denoted by x_i^*. A *social choice function* is a mapping $F : \mathcal{D}^n \to X$ that assigns to each preference profile $(\succ_1, ..., \succ_n)$ in $\mathcal{D}^n$ a unique social alternative $F(\succ_1, ..., \succ_n) \in X$. In the following, we will assume that X is endowed with the structure of a property space, say $(X, \mathcal{H})$; in the next subsection, we will then consider social choice functions defined on rich domains of generalized single-peaked preferences.

An important class of social choice functions are those that only depend on the peaks of voters' preferences; these are referred to as "voting schemes." A social choice function F is a *voting scheme* if there exists a function $f : X^n \to X$ such that for all $(\succ_1, ..., \succ_n)$, $F(\succ_1, ..., \succ_n) = f(x_1^*, ..., x_n^*)$, where x_i^* is voter i's peak. In this case, we say that F satisfies *peaks only.* With slight abuse of terminology, we will also refer to any $f : X^n \to X$ as a voting scheme, since any such function f naturally induces a social choice function satisfying peaks only.

Given a description of alternatives in terms of their properties, a natural way to generate a social choice is to determine the final outcome via its properties. This is described now in detail.

Henceforth, *families of winning coalitions* of agents are denoted by $\mathcal{W}$, where $\mathcal{W}$ is a non-empty family of non-empty subsets of N (the "winning coalitions") satisfying $[W \in \mathcal{W}$ and $W' \supseteq W] \Rightarrow W' \in \mathcal{W}$.

For instance, *majority voting* corresponds to $\mathcal{W}_{\frac{1}{2}} := \{W \subseteq N : \#W > \frac{1}{2} \cdot n\}$. Majority voting is a special case of a *quota rule*: for each $q \in (0, 1)$, voting by quota q corresponds to $\mathcal{W}_q := \{W \subseteq N : \#W > q \cdot n\}$.

Definition (Structure of Winning Coalitions) A *structure of winning coalitions* on a property space $(X, \mathcal{H})$ is a mapping that assigns a family $\mathcal{W}_H$ of winning coalitions to each basic property $H \in \mathcal{H}$ satisfying the following condition:

$$W \in \mathcal{W}_H \Leftrightarrow W^c \notin \mathcal{W}_{H^c}. \tag{3.1}$$

As is easily verified, (3.1) implies that, for every basic property H, the families of winning coalitions corresponding to H and H^c are interrelated as follows.

$$\mathcal{W}_H = \{W \subseteq N : W \cap W' \neq \emptyset \text{ for all } W' \in \mathcal{W}_{H^c}\}. \tag{3.2}$$

Consider now the following voting procedure, adapted to the present framework from Barberà, Sonnenschein and Zhou's (1991) "voting by committees."

Definition (Voting by Issues) Given a property space $(X, \mathcal{H})$ and a structure of winning coalitions $\{\mathcal{W}_H : H \in \mathcal{H}\}$, *voting by issues* is the mapping $f_{\mathcal{W}} : X^n \to 2^X$ such that, for all $\xi \in X^n$,

$$x \in f_{\mathcal{W}}(\xi) :\Leftrightarrow \text{ for all } H \in \mathcal{H} \text{ with } x \in H : \{i : \xi_i \in H\} \in \mathcal{W}_H. \tag{3.3}$$

Voting by issues thus amounts to deciding, for each particular property, whether the final outcome is to possess that property or its negation. Note that $f_{\mathcal{W}}(\xi) \subseteq X$ is not assumed to be non-empty; in particular, $f_{\mathcal{W}}$ does not yet define a voting scheme in the sense of the above definition.

Definition (Consistency) A structure $\{\mathcal{W}_H : H \in \mathcal{H}\}$ of winning coalitions is called *consistent* if $f_{\mathcal{W}}(\xi) \neq \emptyset$ for all $\xi \in X^n$. If $\mathcal{W}$ is consistent, the corresponding voting procedure $f_{\mathcal{W}}$ will also be referred to as consistent.

Example (Inconsistency of issue-by-issue majority voting on an unrestricted domain) As a simple example of an inconsistent structure of winning coalitions, consider the vacuous betweenness (see Example 3 above) on $X = \{x, y, z\}$, and assume that voting by issues takes the form of *issue-by-issue majority voting*, i.e. $\mathcal{W}_H = \mathcal{W}_{\frac{1}{2}}$ for all H. Moreover, suppose that there are three voters with pairwise distinct preference peaks. In this situation, each of the following basic properties gets a majority of two votes: $\{y, z\}$ ("being different from x"), $\{x, z\}$ ("being different from y"), and $\{x, y\}$ ("being different from z"). But clearly, $\{y, z\} \cap \{x, z\} \cap \{x, y\} = \emptyset$, i.e. the basic properties determined according to (3.3) are jointly incompatible.

While the basic properties determined by voting by issues via (3.3) may be inconsistent as in the preceding example, the following fact shows that under consistency the outcome of voting by issues is *uniquely* determined.

Fact 3.1 *If $f_{\mathcal{W}}(\xi) \neq \emptyset$, then $f_{\mathcal{W}}(\xi)$ is single-valued. In particular, voting by issues defines a voting scheme whenever it is consistent.*

With slight abuse of notation, we may thus identify voting by issues with the corresponding function $f_{\mathcal{W}} : X^n \to X$ if the underlying structure of winning coalitions is consistent.

If $f_{\mathcal{W}}$ is consistent, one has for all H and ξ,

$$f_{\mathcal{W}}(\xi) \in H \Leftrightarrow \{i : \xi_i \in H\} \in \mathcal{W}_H \qquad (3.4)$$

by (3.3) and (3.1). Since $N \in \mathcal{W}_H$ for all H, this implies that $f_{\mathcal{W}}$ satisfies *unanimity*, i.e. for all $x \in X$, $f(x, x, ..., x) = x$. In particular, $f_{\mathcal{W}}$ is *onto* whenever it is consistent, i.e. each $x \in X$ is in the range of $f_{\mathcal{W}}$.

Voting by issues is characterized by the following monotonicity condition. Say that a voting scheme $f : X^n \to X$ is *monotone in properties* if, for all ξ, ξ', H,

$$[f(\xi) \in H \text{ and } \{i : \xi_i \in H\} \subseteq \{i : \xi'_i \in H\}] \Rightarrow f(\xi') \in H.$$

Monotonicity in properties states that if the final outcome has some property H and the voters' support for this property does not decrease, then the resulting final outcome must have this property as well.

Proposition 3.1 *Let $(X, \mathcal{H})$ be a property space. A voting scheme $f : X^n \to X$ is monotone in properties and onto if and only if it is voting by issues on $(X, \mathcal{H})$ with a consistent structure of winning coalitions.*

For a structure of winning coalitions $\{\mathcal{W}_H : H \in \mathcal{H}\}$, denote by $F_{\mathcal{W}} : \mathcal{D}^n \to 2^X$ the mapping defined by $F_{\mathcal{W}}(\succ_1, ..., \succ_n) = f_{\mathcal{W}}(x_1^*, ..., x_n^*)$, where for each i, x_i^* is the peak of $\succ_i$ on X. The mapping $F_{\mathcal{W}}$ will also be referred to as voting by issues. As above, we will identify voting by issues with the function $F_{\mathcal{W}} : \mathcal{D}^n \to X$ if the underlying structure of winning coalitions is consistent.

A social choice function F is called **anonymous** if it is invariant with respect to permutations of individual preferences, i.e. if $F(\succ_1, ..., \succ_n) = F(\succ_{\sigma(1)}, ..., \succ_{\sigma(n)})$ for every permutation $\sigma : N \to N$. The following fact shows that anonymous voting by issues takes the form of a *quota rule*; the first part is immediate from the definitions, the second part follows at once from (3.2).

Fact 3.2 *Voting by issues $F_{\mathcal{W}}$ is anonymous if and only if it is a quota rule, i.e. for all H there exists $q_H \in [0,1]$ such that $\mathcal{W}_H = \mathcal{W}_{q_H}$ if $q_H < 1$ and $\mathcal{W}_H = \mathcal{W}_1 := \{N\}$ if $q_H = 1$.*[9] *If $F_{\mathcal{W}}$ is consistent, the quotas can be chosen such that, for all $H \in \mathcal{H}$, $q_{H^c} = 1 - q_H$.*

The appropriate formulation of neutrality in our context turns out to be as follows. Say that a profile $(\succ_1, ..., \succ_n)$ is *simple* if $\#\{\succ_1, ..., \succ_n\} \leq 2$, i.e. if it contains at most two different preference orderings. A social choice function F is called **neutral** if, for all simple profiles $(\succ_1, ..., \succ_n)$, $(\succ'_1, ..., \succ'_n)$ and all permutations $\sigma : X \to X$ such that $x \succ_i y \Leftrightarrow \sigma(x) \succ'_i \sigma(y)$ for all x, y and i, $F(\succ'_1, ..., \succ'_n) = \sigma(F(\succ_1, ..., \succ_n))$.

Fact 3.3 *Let $(X, \mathcal{H})$ be a property space, and let $\mathcal{S}$ be a rich single-peaked domain with respect to $T_{\mathcal{H}}$. Voting by issues $F_{\mathcal{W}} : \mathcal{S}^n \to X$ is neutral if and only if $\mathcal{W}$ is constant, i.e. for all $H, H' \in \mathcal{H}$, $\mathcal{W}_H = \mathcal{W}_{H'}$.*

Note that by Facts 3.2 and 3.3 consistent voting by issues is anonymous and neutral if and only if it is issue-by-issue majority voting, i.e. if and only if, for all H, $\mathcal{W}_H = \mathcal{W}_{\frac{1}{2}}$. Note that, by (3.2), this requires on odd number of voters.

3.2 The Equivalence of Strategy-Proofness and Voting by Issues

Throughout, let $\mathcal{S}$ denote a single-peaked domain on a property space $(X, \mathcal{H})$ that is rich with respect to $T_{\mathcal{H}}$. A social choice function $F : \mathcal{S}^n \to X$ is *strategy-proof* (on $\mathcal{S}$) if for all i, $\succ_i, \succ'_i \in \mathcal{S}$ and $\succ_{-i} \in \mathcal{S}^{n-1}$,

$$F(\succ_i, \succ_{-i}) \succeq_i F(\succ'_i, \succ_{-i}).$$

Furthermore, say that F satisfies *voter sovereignty* if F is onto, i.e. if every $x \in X$ is in the range of F.

Proposition 3.2 *Let $(X, \mathcal{H})$ be a property space, let $\mathcal{S}$ be a rich single-peaked domain with respect to $T_{\mathcal{H}}$, and let $F : \mathcal{S}^n \to X$ be represented by the voting scheme $f : X^n \to X$. Then, F is strategy-proof if and only if f is monotone in properties.*

In combination with Proposition 3.1, this implies that a voting scheme is strategy-proof on a rich single-peaked domain and onto if and only if it is voting by issues with a consistent structure of winning coalitions. We now want to show that *any* strategy-proof social choice function $F : \mathcal{S}^n \to X$ satisfying voter sovereignty is voting by issues. For this, it remains to show that any such F is a voting scheme, i.e. that it satisfies peaks only.

[9] Note that the quotas q_H are not uniquely determined in the sense that different sets of quotas may define the same structure of winning coalitions.

Proposition 3.3 *Let $(X, \mathcal{H})$ be a property space, and let $\mathcal{S}$ be a rich single-peaked domain with respect to $T_{\mathcal{H}}$. Every strategy-proof social choice function $F : \mathcal{S}^n \to X$ that satisfies voter sovereignty is a voting scheme, i.e. satisfies peaks only.*

Combining Propositions 3.1 – 3.3 yields the following result.

Theorem 2 *Let $(X, \mathcal{H})$ be a property space, and let $\mathcal{S}$ be a rich single-peaked domain with respect to $T_{\mathcal{H}}$. A social choice function $F : \mathcal{S}^n \to X$ is strategy-proof and satisfies voter sovereignty if and only if it is voting by issues on $(X, \mathcal{H})$ with a consistent structure of winning coalitions.*

Theorem 2 is a counterpart and adaption of a fundamental result in Barberà, Massó and Neme (1997) that got this research started. We could not invoke their result ready-made[10] due to the added generality associated with considering rich single-peaked domains rather than maximal ones of the form $\mathcal{D} = \hat{\mathcal{S}}_{X,T}$. As illustrated by Examples 6 and 7, the generality added by this move is substantial. While the proof of Proposition 3.3 parallels that of Proposition 2 in Barberà, Massó and Neme (1997), ours adds two additional, significant steps not present in their analysis. The second of these introduces the notion of a "gated set" and yields substantial additional insights into the structure of strategy-proof social choice functions that are absent from the analysis of Barberà, Massó and Neme (1997). Specifically, for all voters i, denote by

$$o_i^F(\succ_{-i}) := \{x \in X : \text{ there exists } \succ_i \in \mathcal{S} \text{ such that } F(\succ_i, \succ_{-i}) = x\}$$

the *set of options* of voter i given the preference profile $\succ_{-i}$ of all voters other than i.

Definition (Gated set) A subset $Y \subseteq X$ is called *gated* if, for all $x \in X$, there exists an element $\gamma(x) \in Y$ such that $\gamma(x) \in [x, y]$ for all $y \in Y$, i.e. such that $\gamma(x)$ is between x and every element of Y. The element $\gamma(x)$ is called the *gate of Y to x*.

Observe that the universal set X and all singletons are always trivially gated.

Lemma 3.1 *Let $(X, \mathcal{H})$ be a property space, and let $\mathcal{S}$ be a rich single-peaked domain with respect to $T_{\mathcal{H}}$. Suppose that $F : \mathcal{S}^n \to X$ is strategy-proof and satisfies peaks only. Then, for all i and $\succ_{-i}$, the set of options $o_i^F(\succ_{-i})$ is gated.*

The significance of this result lies in the fact that property spaces frequently admit only few gated sets. For instance, the only gated sets with respect to the vacuous betweenness are the universal set and all singletons. On the other hand, it is easily verified that the existence of non-degenerate strategy-proof social functions requires the existence of non-trivial option sets. We thus obtain the following corollary.[11]

Corollary 1 (Generalized Gibbard-Satterthwaite Theorem) *Let $(X, \mathcal{H})$ be a property space and let $\mathcal{S}$ be a rich single-peaked domain with respect to $T_{\mathcal{H}}$. Moreover, let $F : \mathcal{S}^n \to X$ be strategy-proof and satisfy voter sovereignty and peaks only. If $\#X \geq 3$, and if only the universal set and all singletons are gated, then F must be dictatorial.*

[10] That is, via an extension argument, as had been done in an earlier working paper version Nehring and Puppe (2002), see also Appendix 1 below.

[11] In Corollaries 1 and 2 below, we assume the peaks-only property to obtain the results as corollaries from Lemma 3.1 without invoking Proposition 3.3 or Theorem 2.

This generalizes the Gibbard-Satterthwaite Theorem by showing that not only on the unrestricted domain but also on many other rich single-peaked domains only dictatorial social choice functions can be strategy-proof. For instance, also in Examples 6 and 7 above the universal set and all singletons are the only gated sets.

The following second corollary shows how restrictive neutrality is in our present context.

Corollary 2 *Let $(X, \mathcal{H})$ be a property space and let $\mathcal{S}$ be a rich single-peaked domain with respect to $T_{\mathcal{H}}$. Moreover, let $F : \mathcal{S}^n \to X$ be strategy-proof and satisfy voter sovereignty and peaks only. If F is neutral and non-dictatorial, then all segments are gated.*

The requirement that all segments be gated turns out to be a very natural property as it characterizes the class of "median spaces" analyzed in detail in Section 4 below. There, we will show that this condition is not only necessary but also sufficient for the existence of neutral and non-dictatorial strategy-proof social choice.

3.3 Consistent Structures of Winning Coalitions: The Intersection Property

By Theorem 2, a social choice function is strategy-proof on a rich domain of generalized single-peaked preferences and onto if and only if it is *consistent* voting by issues. It is, however, not self-evident whether a given structure of winning coalitions is consistent. The needed characterization of consistency is provided in this subsection. Consistency of voting by issues requires that the structure of winning coalitions be compatible with the combinatorial structure of basic properties. This structure is summarized by the class of minimally inconsistent families of basic properties; such families will be called "critical."

Definition (Critical Family) Say that a family $\mathcal{G} \subseteq \mathcal{H}$ of basic properties is a *critical family* if $\cap\mathcal{G} = \emptyset$ and for all $G \in \mathcal{G}$, $\cap(\mathcal{G} \setminus \{G\}) \neq \emptyset$.

Trivial instances of critical families are all pairs $\{H, H^c\}$ of complementary properties. Critical families admit a simple intuitive interpretation as they reflect the "entailment logic" of the underlying space. To illustrate, consider the line, labeled by the natural numbers $1, ..., m$. The basic properties are $H_{\geq j}$ ("being greater than or equal to j") and $H_{\leq k}$ ("being smaller than or equal to k") for appropriate j and k in $\{1, ..., m\}$. All critical families have the form $\{H_{\geq j}, H_{\leq k}\}$ for some $k < j$. The interpretation is that "$\geq j$" logically entails "not $\leq k$" whenever $k < j$. Thus, the critical family corresponds to the statement "for all x, $x \geq j$ implies (not $x \leq k$)." Similarly, consider the set $X = \{x_1, ..., x_m\}$ endowed with the vacuous betweenness. For each x_j, the set $H_j^c = X \setminus \{x_j\}$ corresponds to the basic property "being different from x_j." The critical family $\{H_1^c, ..., H_m^c\}$ thus describes the entailment: "if an alternative is different from $m-1$ distinct elements of X, it cannot be different from the remaining m-th element."

Intersection Property Say that voting by issues $F_{\mathcal{W}}$ satisfies the *Intersection Property* if for every critical family $\mathcal{G} = \{G_1, ..., G_l\}$, and every selection $W_j \in \mathcal{W}_{G_j}$,

$$\bigcap_{j=1}^{l} W_j \neq \emptyset.$$

Proposition 3.4 *Let* $(X, \mathcal{H})$ *be a property space. Voting by issues is consistent on* $(X, \mathcal{H})$ *if and only if it satisfies the Intersection Property.*

Combining this result with Theorem 2, we obtain the following characterization of all strategy-proof and onto social choice functions on every rich single-peaked domain.

Theorem 3 *Let* $(X, \mathcal{H})$ *be a property space, and let* $\mathcal{S}$ *be a rich single-peaked domain with respect to* $T_{\mathcal{H}}$*. A social choice function* $F : \mathcal{S}^n \to X$ *is strategy-proof and satisfies voter sovereignty if and only if it is voting by issues on* $(X, \mathcal{H})$ *satisfying the Intersection Property.*

Observe that by this result the class of all strategy-proof and onto social choice functions on a domain $\mathcal{S}$ only depends on the "betweenness geometry" of the underlying property space in the sense that any rich single-peaked domain defined on the same property space induces the same class of strategy-proof social choice functions. Also note that Theorem 3 may take on the character of an impossibility or a possibility result, depending on the restrictions that the Intersection Property imposes via the pattern of critical families of the underlying property space. For instance, in case of the vacuous betweenness, i.e. for the unrestricted domain, the Intersection Property directly entails the non-existence of anonymous social choice functions that are strategy-proof and onto, as shown presently. On the other spaces, the Intersection Property allows one to derive possibility results, as shown later.

Under anonymity, satisfaction of the Intersection Property translates into a system of linear inequalities on integers, as follows. Any anonymous voting by issues is characterized by a family $\{m_H : H \in \mathcal{H}\}$ of absolute quotas, where $m_H := \min\{\#W : W \in \mathcal{W}_H\}$. Note that, by definition, $m_H + m_{H^c} = n + 1$. It is easily verified that the Intersection Property on the structure of winning coalitions is equivalent to the following system of linear (in)equalities on the absolute quotas m_H,

$$\text{for all } H \in \mathcal{H} \quad : \quad m_H + m_{H^c} = n + 1 \tag{3.5}$$

$$\text{for all critical families } \mathcal{G} \quad : \quad \sum_{H \in \mathcal{G}} (n - m_H) < n. \tag{3.6}$$

The existence of an anonymous strategy-proof social choice rule is thus described by an integer programming problem. This can be restated and simplified into a linear programming problem by considering relative quotas, as described by the following result; by condition (ii), the integer problem does not disappear completely but becomes essentially trivial.

Fact 3.4 *Let* $(X, \mathcal{H})$ *be a property space and let* $\{q_H : H \in \mathcal{H}\}$ *be a system of relative quotas such that, for all* $H \in \mathcal{H}$, $q_H + q_{H^c} = 1$ *and* $q_H \cdot n$ *is not an integer other than* 0 *or* n*. If, for every critical family* $\mathcal{G}$,

$$\sum_{H \in \mathcal{G}} (1 - q_H) \leq 1, \tag{3.7}$$

then voting by issues $F_{\mathcal{W}}$ *with the structure* $\{\mathcal{W}_{q_H} : H \in \mathcal{H}\}$ *of winning coalitions is anonymous and consistent. Conversely, if voting by issues is anonymous and consistent, then there exist quotas* $\{q_H : H \in \mathcal{H}\}$ *such that (i) for all* $H \in \mathcal{H}$, $q_H + q_{H^c} = 1$, *(ii) for all* $H \in \mathcal{H}$, $q_H \cdot n$ *is not an integer other than* 0 *or* n*, and (iii) for every critical family* $\mathcal{G}$*, (3.7) is satisfied.*

The role of the integer condition (ii) is to ensure that the families $\mathcal{W}_{1-q_H}$ and $\mathcal{W}_{q_H}$ are adjoint in the sense of condition (3.1). This becomes important in situations in which all anonymous social choice functions require some quota q_H to be equal to $\frac{1}{2}$; clause (ii) implies in this case that n must be odd, which makes intuitive sense since majority voting is well-defined only for an odd number of individuals. The proof of the second part of Fact 3.4 in the appendix relies on the observation that if the absolute quotas m_H satisfy conditions (3.5) and (3.6), the relative quotas q_H defined by $q_H := \frac{m_H - 1}{n-1}$ satisfy conditions (i) and (3.7).

To illustrate the intuition behind the Intersection Property, we verify the necessity of (3.7) in the special case of the vacuous betweenness on $X = \{x_1, ..., x_m\}$; from this it is straightforward to infer the non-existence of anonymous, strategy-proof and onto social choice functions on an unrestricted domain if $m \geq 3$. Recall that the vacuous betweenness corresponds to the basic properties $H_j = \{x_j\}$ ("being equal to x_j") and $H_j^c = X \setminus \{x_j\}$ ("being different from x_j"), for $j = 1, ..., m$. The non-trivial critical families are $\{H_1^c, ..., H_m^c\}$ and, for all $j \neq k$, $\{H_j, H_k\}$. Consider the critical family $\{H_1^c, ..., H_m^c\}$, and suppose that (3.7) is violated, i.e. $\sum_j q_j^c < m - 1$, where q_j^c denotes the quota corresponding to H_j^c. If $q_j = 1 - q_j^c$ is the quota corresponding to H_j, one obtains $\sum_j q_j > 1$, say $\sum_j q_j = 1 + m \cdot \delta$ for some $\delta > 0$. Now assign to a fraction of $q_j - \delta$ voters the peak x_j. Since none of the properties $H_j = \{x_j\}$ reaches the quota, all complements are enforced; but since their intersection is empty, consistency is violated.

Theorem 3 generalizes Corollary 3 in Barberà, Massó and Neme (1997) which applies to the domains $\hat{\mathcal{S}}_{X,T}$ where X is some subset of a product of lines. In that context, these authors derive a condition also called "intersection property" that can be viewed as relating families of "inconsistent properties" to admissible winning coalitions. However, the condition obtained here is much simpler and more powerful due to the restriction to *minimal* such families. This gives rise to the characterization of anonymous and onto strategy-proof social choice functions in terms of a set of linear inequalities just described, and makes it possible to determine which property spaces admit strategy-proof social choice functions with various desirable properties such as non-dictatorship, anonymity and efficiency from their combinatorial structure (see Nehring and Puppe (2005a,b)).

The Intersection Property has a special, and particularly simple, structure when all critical families have cardinality two, for evidently $\{G, H^c\}$ is a critical family if and only if $G \subseteq H$, in other words, if and only if G entails H on its own. We shall call such property spaces **simple**. At an abstract level, the distinguished status of simple property spaces derives from the fact that in such spaces and in no others, a structure of winning coalitions is consistent if and only if it is *order-preserving* ($G \subseteq H \Rightarrow \mathcal{W}_G \subseteq \mathcal{W}_H$). Almost directly, this characterization ensures the existence of a rich set of strategy-proof social choice functions in simple property spaces, as borne out by the analysis of the following section. A basic but important example of this follows directly from the Intersection Property, as it is immediate from Fact 3.4 that issue-by-issue majority voting among an odd number of voters is consistent if and only if the underlying property space is simple. Indeed, by (3.7) voting by issues with a uniform quota $q_H = 1/2$ for all H can be consistent only if all critical families have two elements. In the next section, we characterize simple property spaces geometrically as "median spaces" and analyze their remarkable further structure in more detail.

4 Strong Possibility Results in Median Spaces

By Theorem 3 above, strategy-proof social choice on single-peaked domains takes the form of voting by issues satisfying the Intersection Property. For each *given* domain this yields a simple characterization of the class of all onto and strategy-proof social choice functions. On the other hand, it does not answer the question for which property spaces there exist *well-behaved* strategy-proof social choice functions on the associated domain of single-peaked preferences. In this section, we consider a distinguished class of property spaces, the class of median spaces, and show that they enable strategy-proof social choice functions that are well-behaved in a particularly strong sense.

4.1 Simple Property Spaces as Median Spaces

As an immediate consequence of the Intersection Property, we have seen that issue-by-issue majority voting is consistent if and only if the underlying property space is simple, i.e. every critical family has only two elements. What does that mean geometrically? To provide the intuition, consider three voters with peaks ξ_1, ξ_2, ξ_3 and denote by m the chosen state under issue-by-issue majority voting. Every basic property H possessed by both ξ_1 and ξ_2 gets a majority of at least two votes over H^c, hence we must have $m \in H$ (see Figure 6 below). By (2.2), this means that m is between ξ_1 and ξ_2. But the same argument applies to every basic property jointly possessed by ξ_1 and ξ_3, and to every basic property jointly possessed by ξ_2 and ξ_3. In other words, a necessary condition for issue-by-issue majority voting to be consistent is that every triple ξ_1, ξ_2, ξ_3 of social states admits a state $m = m(\xi_1, \xi_2, \xi_3)$ that is between any pair of them. Such a state will be called a "median" of the triple.

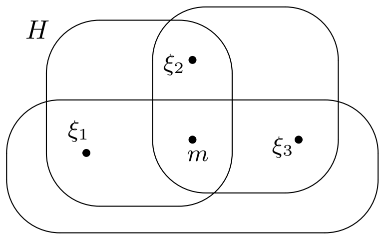

Figure 6: The median property.

Definition (Median Space) A property space $(X, \mathcal{H})$ is called a *median space* if the induced betweenness relation $T_{\mathcal{H}}$ satisfies the following condition. For all $x, y, z \in X$ there exists an element $m = m(x, y, z) \in X$, called the *median* of x, y, z, such that m is between any pair of $\{x, y, z\}$, i.e. such that $\{(x, m, y), (x, m, z), (y, m, z)\} \subseteq T_{\mathcal{H}}$.

Median spaces are a classic topic in abstract convexity theory (see, e.g., Bandelt and Hedlíková (1983) and the references in van de Vel (1993)). It is easily verified that due to the separation property H3, the median of a triple is uniquely determined. Moreover, one has the following result.

Proposition 4.1 *Let $(X, \mathcal{H})$ be a property space. The following are equivalent.*

(i) *All critical families have cardinality two, i.e. $(X, \mathcal{H})$ is simple.*

(ii) *$(X, \mathcal{H})$ is a median space.*

(iii) *All segments are gated.*

The betweenness relation of a median space is always a graphic betweenness (see van de Vel (1993, Chapter I.6)), and the median of a triple minimizes the sum of the graph distances (i.e. the number of edges in the underlying graph) to the triple. More generally, it is easily verified that in a median space the outcome of issue-by-issue majority voting with an odd number of agents minimizes the sum of the graph distances to the voters' peaks.

The simplest examples of median spaces are lines (Example 1 in Section 2 above) with the middle point of a triple as their median. More generally, the graphic betweenness associated with any *tree* (i.e. connected and acyclic graph) gives rise to a median space. To see this, consider for each triple of points in a tree the (unique) shortest paths connecting every pair. By the acyclicity, these three shortest paths have exactly one point in common, the median of the triple. Furthermore, all hypercubes (Example 2) are median spaces; a typical configuration is the triple x, z, w with the median y in Figure 1b above. More generally, products are median spaces if and only if every factor is a median space; indeed, by definition of the product betweenness (see Example 4), the median on a product is simply given by taking the median in each coordinate. Thus, our analysis shows that the common source of the possibilities of strategy-proof social choice derived in Moulin (1980), Barberà, Sonnenschein and Zhou (1991) and Barberà, Gul and Stacchetti (1993) is that in each case the underlying space is a median space.[12] Formally, we have the following corollaries to Theorem 3.

Corollary 3 (Barberà, Sonnenschein and Zhou (1991)) *Let $\mathcal{S}$ be the domain of all separable preference orderings on the hypercube. A social choice function $F : \mathcal{S}^n \to X$ is strategy-proof and onto if and only if it is voting by issues.*

Corollary 4 (Moulin (1980), Barberà, Gul and Stacchetti (1993)) *Let $\mathcal{S}$ be the domain of all single-peaked preference orderings on a product of lines. A social choice function $F : \mathcal{S}^n \to X$ is strategy-proof and onto if and only if it is voting by issues satisfying, for all G, H, $G \subseteq H \Rightarrow \mathcal{W}_G \subseteq \mathcal{W}_H$.*

4.2 Median Spaces Characterized by Possibility Results

One fundamental requirement of social choice is that all alternatives are treated on par, i.e. that the social choice function be neutral. We have seen that under voting by issues, there is a unique anonymous and neutral rule, issue-by-issue majority voting with an odd number of agents. On the other hand, there is a rich class of neutral rules that are not anonymous. Indeed, take any family of winning coalitions $\mathcal{W}_0$ satisfying $W \in \mathcal{W}_0 \Leftrightarrow W^c \notin \mathcal{W}_0$; then, defining $\mathcal{W}_H = \mathcal{W}_0$ for all H yields a neutral rule.

[12] By contrast, the property spaces underlying the other examples considered in Section 2 above are not median spaces. For instance, the triple x, z, w in Figure 1c above does not have a median. More generally, in Example 3 (the vacuous betweenness) *no* triple of pairwise distinct alternatives admits a median. The fact that cycles of length $\neq 4$ (Example 5) are not median spaces is exemplified by the triple x_{j-2}, x_j, x_{j+2} in Figure 4 above. As is easily verified, the spaces underlying Examples 6 and 7 are also not median spaces.

Combining Proposition 4.1 with Proposition 3.3 and Corollary 2 above, we can now show that under strategy-proofness neutral and non-dictatorial rules exist only on median spaces.

Theorem 4 *Let $(X, \mathcal{H})$ be a property space, and let $\mathcal{S}$ be a rich single-peaked domain with respect to $T_{\mathcal{H}}$. There exists a strategy-proof social choice function $F : \mathcal{S}^n \to X$ that is onto, neutral and non-dictatorial if and only if $(X, \mathcal{H})$ is a median space and $n \geq 3$.*

Moreover, on a median space $(X, \mathcal{H})$, a social choice function $F : \mathcal{S}^n \to X$ is onto, strategy-proof and neutral if and only if it is voting by issues such that $\mathcal{W}_H = \mathcal{W}_0$ for all $H \in \mathcal{H}$ and some family $\mathcal{W}_0$ of winning coalitions satisfying $W \in \mathcal{W}_0 \Leftrightarrow W^c \notin \mathcal{W}_0$.

The first part of Theorem 4 has also been proved in Nehring and Puppe (2005a), however, using a different argument.

In the anonymous case, we obtain the following corollary which entails that issue-by-issue majority voting is consistent if and only if the underlying space is a median space (cf. the remark at the end of Section 3 above).

Corollary 5 *Let $(X, \mathcal{H})$ be a property space, and let $\mathcal{S}$ be a rich single-peaked domain with respect to $T_{\mathcal{H}}$. There exists a strategy-proof social choice function $F : \mathcal{S}^n \to X$ that is onto, neutral and anonymous if and only if n is odd and $(X, \mathcal{H})$ is a median space.*

Neutrality is of additional interest in the context of strategy-proofness since a slightly weakened form of neutrality is necessary for efficiency, as we have shown in Nehring and Puppe (2005b). Note also that by the Intersection Property and Proposition 4.1 the necessary departures from neutrality outside median spaces are substantial. For instance, it follows at once from (3.7) that in the anonymous case at least one property must be chosen with supermajority of at least 2/3 if the underlying space is not a median space.[13]

In some contexts, however, one may want to treat properties and alternatives asymmetrically. For instance, in the context of constitutional change supermajority rules are frequently employed. Median spaces are also maximally rich in the range of strategy-proof choice rules that privilege particular properties and alternatives, as follows. For every family $\mathcal{W}$ of winning coalitions, let the *adjoint* $\mathcal{W}^a$ be defined by (3.1), or equivalently, by (3.2), i.e.

$$\mathcal{W}^a := \{W \subseteq N : W \cap W' \neq \emptyset \text{ for all } W' \in \mathcal{W}\}.$$

A family $\mathcal{W}$ of winning coalitions is called a *minority family* if it strictly contains its adjoint, i.e. if $\mathcal{W}^a \subseteq \mathcal{W}$ and $\mathcal{W}^a \neq \mathcal{W}$. Using the fact that $(\mathcal{W}^a)^a = \mathcal{W}$, it is easily verified that $\mathcal{W}$ is a minority family if and only if (i) there exist $W, W' \in \mathcal{W}$ with $W' \cap W = \emptyset$, and (ii) for all $W, W' \in \mathcal{W}^a$, $W \cap W' \neq \emptyset$. For instance, in the anonymous case a family of winning coalitions is a minority family if and only if it corresponds to a quota $< 1/2$.

Definition (Minority veto rule) A social choice function $F : \mathcal{S}^n \to X$ is called a *minority veto rule with status quo x* if there exists a minority family $\mathcal{W}$ such that $F(\succ_1, ..., \succ_n) = x$ whenever $\{i : \xi_i = x\} \in \mathcal{W}$, where ξ_i is the peak of $\succ_i$.

[13] To see this, note that, if $(X, \mathcal{H})$ is not a median space, there exists a critical family with at least three elements, by Proposition 4.1.

Theorem 5 *Let $(X, \mathcal{H})$ be a property space, and let $\mathcal{S}$ be a rich single-peaked domain with respect to $T_{\mathcal{H}}$. There exists, for each $x \in X$, a minority veto rule $F : \mathcal{S}^n \to X$ with status quo x if and only if $(X, \mathcal{H})$ is a median space.*

Moreover, if $(X, \mathcal{H})$ is a median space, there exists, for all x and every minority family, an associated minority veto rule $F : \mathcal{S}^n \to X$ with status quo x.

To illustrate the above results in the anonymous case, consider the graphic betweenness relation associated with the following graph.

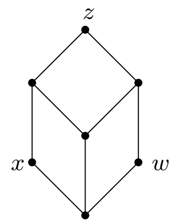

Figure 7: A seven point graph.

The basic properties of the underlying property space are the three 4-cycles through x, z and w, respectively, and their complements.[14] Note that the space is not a median space since the triple x, z, w does not admit a median. First, suppose that voters' peaks are evenly distributed among x, z and w. Evidently, consistency forces the chosen outcome to lie in at least one of the three 4-cycles. Hence, at least one of the three 4-cycles requires a quota of no more than 1/3. In particular, consistency is not compatible with neutrality. Now assume without loss of generality that it is the 4-cycle through w which requires a quota of at most 1/3, and consider a profile with 49% of the voters' peaks at x and 51% of the voters' peaks at w. Clearly, x is not chosen, which shows that there is no (anonymous) minority veto rule with status quo x. On the other hand, for every point in Figure 7, there exists a "*majority* rule" with the given point as status quo. For instance, suppose that a departure from the two 4-cycles through z and w, respectively, requires a simply majority, whereas a departure from the 4-cycle through x requires unanimous consent; by the Intersection Property this rule is consistent, and x is chosen whenever 51% of the voters have their peak at x.

4.3 Single-Peaked Preferences in Median Spaces

So far, we have described the preference domains on which possibility results emerge indirectly through their underlying geometry derived from the median space structure.

[14]A natural interpretation is in terms of three candidates, with the 4-cycle through x (resp. z, w) representing the admission of candidate A_x (resp. A_z, A_w). For instance, the point between x and z represents the state in which candidates A_x and A_z but not A_w are admitted; similarly, the center point represents the state in which all three candidates are admitted. Note that in the present example *at least one* candidate must be admitted, since the intersection of the complements of the three 4-cycles is empty.

We now show that one can characterize the single-peaked domains on median spaces directly through appropriate convexity and separability conditions. In this sense, it is ensured that all preference domains associated with median spaces are economically meaningful.

Throughout this subsection, $(X, \mathcal{H})$ denotes a median space. As noted above, the induced betweenness relation of a median space is always graphic. Say that y is *linearly between* x and z if y is on a unique shortest path connecting x and z in the underlying graph. Say that a preference ordering $\succ$ is **convex** on $(X, \mathcal{H})$ if $x \succ y \Rightarrow y \succ z$ whenever y is linearly between x and z and $y \neq z$. Moreover, say that $\succ$ is **separable** on $(X, \mathcal{H})$ if $x \succ y \Leftrightarrow z \succ w$ whenever (x, y) and (z, w) are two pairs of (graph) neighbours separated by the same basic property, i.e. such that $\{H \in \mathcal{H} : x \in H \text{ and } y \notin H\} = \{H \in \mathcal{H} : z \in H \text{ and } w \notin H\}$; note that in a median space, neighbours are always graph neighbours, and are always separated by exactly one basic property (cf. Lemma A.3 in the appendix).

Proposition 4.2 *On a median space, a preference ordering is generalized single-peaked if and only if it is convex and separable.*

On trees betweenness coincides with linear betweenness (since shortest paths are always unique), hence by Proposition 4.2 a preference is single-peaked on a tree if and only if it is convex. By contrast, in a hypercube the linear betweenness relation is vacuous, hence by Proposition 4.2 a preference is single-peaked on a hypercube if and only if it is separable.[15]

Note that while by Fact 2.1 above, any single-peaked preference on any property space is separable, a conclusion similar to that of Proposition 4.2 fails often outside the class of median spaces. For instance, consider a single-peaked preference ordering $\succ$ on the 6-cycle (cf. Figure 4 above): if y is opposite to the peak of $\succ$ and linearly between x and z, one has $x \succ y$ but $y \not\succ z$ in violation of convexity.

Frequently, taking as domain the class of *all* single-peaked preferences is unnaturally permissive. Indeed, the separability entailed by single-peakedness is quite weak. Often one would like to have *additive* separability. Similarly, in some contexts convexity may be too weak as well. For instance, in the product of two lines, convexity merely imposes the restriction that the sections of the upper contour sets are connected, i.e. lines. A stronger restriction results from the following *cardinal* notion of convexity. Say that a preference ordering $\succ$ is **cardinally convex and separable** on $(X, \mathcal{H})$ if it has an additive utility representation $u(x) = \sum_{H \in \mathcal{H}_g, H \ni x} \lambda_H$ (cf. Section 2.2 above) with $\lambda_G \geq \lambda_H$ whenever $G \supseteq H$. The appeal of this definition emerges from the following characterization.

Fact 4.1 *A preference ordering on a median space is cardinally convex and separable if and only if it has a utility representation $u : X \to \mathbf{R}$ such that u is concave on every linear path, and $u(x) - u(y) = u(z) - u(w)$ whenever (x, y) and (z, w) are two pairs of neighbours separated by the same basic property.*

For instance, in the context of the product of lines, a preference ordering is cardinally convex and separable if and only if it has a utility representation of the form

[15] The linear betweenness on the hypercube is vacuous since shortest paths between two distinct points that are not neighbours are never unique.

$u(x) = \sum_k u^k(x^k)$, where each u^k is concave.[16] Clearly, since in general a single-peaked preference need not be additively separable, not all single-peaked preferences are cardinally convex and separable. The following result shows that, on a median space, the subclass of all cardinally convex and separable preferences nevertheless forms a rich single-peaked domain.

Proposition 4.3 *On a median space, the set of all cardinally convex and separable preferences is a rich single-peaked domain.*

Propositions 4.2 and 4.3 establish that every median space is associated with economically natural and well-behaved domains of single-peaked preferences.

4.4 The Structure of Median Spaces

The above results show how remarkably well-behaved median spaces are for the purposes of the analysis of strategy-proof social choice. It remains to understand this class better in itself. The following figure shows the structure of the class of median spaces. An arrow indicates subsethood, and each class of spaces is the intersection of its two immediate predecessors.

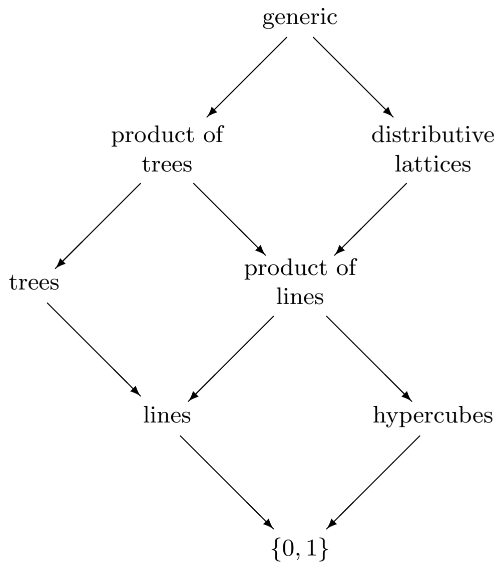

Figure 8: The class of median spaces.

The existing literature has studied the lower part of this diagram, Moulin (1980) lines, Demange (1982) trees, Barberà, Sonnenschein and Zhou (1991) hypercubes, and Barberà, Gul and Stacchetti (1993) the product of lines. As an example of a distributive lattice structure, consider the following problem of *constitutional choice*. Suppose that

[16]This follows from the observation that in a product of lines the linear paths are the paths parallel to the coordinate axes.

a set of countries, say the EU member states, have to decide on the procedures for their collective choices, i.e. they have to decide on their joint constitution. Specifically, consider the problem of determining on which of the issues $K = \{1, ..., k\}$ future decisions are to be made on the basis of majority voting. Individual preferences are thus taken to be over subsets of K ("constitutions") with the interpretation that $L \succ_i L'$ if country i prefers majority voting over exactly the issues in $L \subseteq K$ to majority voting over exactly the issues in $L' \subseteq K$. The assumption of single-peakedness does not seem implausible in that context; it requires that, for each single issue j, majority voting over issue j is preferred/not preferred independently of the corresponding preference over other issues. Observe, however, that this excludes a preference for the overall extent of majority voting (regardless on which issues), since in that case majority voting for one issue would be a substitute for majority voting over another issue.

In general, one cannot assume that the issues are independent from each other. In other words, one has to account for the "entailment logic" of the underlying problem. For instance, suppose that the issue j represents the joint defense policy of the countries, whereas j' represents their joint foreign policy. It is in general not possible to decide on defense policy by majority voting without also deciding at least on some foreign policy issues by majority voting. In particular, the set of all feasible constitutions will, in general, not be the entire power set 2^K. The entailment "majority voting over $j \Rightarrow$ majority voting over j'" thus corresponds to a critical family. If the space of feasible constitutions is constrained only by entailments of this form (with one antecedent and one conclusion), it forms a (necessarily distributive) lattice of sets. Another example of a distributive lattice that is not a product of lines arises in a quasi-linear version of the public goods Example 7 above (see Nehring and Puppe (2005a)).

The following figure shows an example of a "generic" median space that has neither a product structure nor that of a distributive lattice.

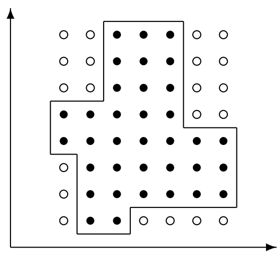

Figure 9: A median space embedded in the product of two lines.

More generally, any connected subset of the product of two lines such that every section is a line (in particular, every "discretization" of a convex subset of $\mathbf{R}^2$) is a median space. These spaces are of particular interest since they admit strategy-proof social choice functions that are efficient (see Nehring and Puppe (2005b), which also presents an economically natural location example with similar structure).

5 Conclusion

In this paper, we have defined a general notion of single-peakedness based on abstract betweenness relations. We have derived conditions under which a given subset of linear preferences can be represented as a rich single-peaked domain with respect to an appropriate betweenness relation (Theorem 1). We have then shown that a social choice function is onto and strategy-proof on such a rich single-peaked domain if and only if it takes the form of voting by issues satisfying the Intersection Property (Theorems 2 and 3). The concept of a median space, in which every triple of social states admits a fourth state that is between any pair of the triple, turned out to be fundamental for the existence of well-behaved strategy-proof social choice functions. Median spaces are distinguished from a number of different perspectives. Due to their simple and regular structure, median spaces give rise to a maximally rich class of strategy-proof social choice functions (Proposition 4.1). They are exactly the spaces that admit anonymous and neutral strategy-proof social choice rules, amounting to issue-by-issue majority voting (Corollary 5). More generally, neutral and non-dictatorial strategy-proof social choice functions only exist on median spaces (Theorem 4). Finally, median spaces are maximally rich in the range of admissible minority veto rules (Theorem 5), and their associated domain of single-peaked preferences can be exhaustively described by convexity and separability restrictions (Proposition 4.2).

The general characterization of strategy-proof social choice presented in this paper can be used to identify those preference domains that admit strategy-proof social choice functions satisfying various additional desirable properties. In Nehring and Puppe (2005a), we characterize the rich single-peaked domains that admit non-dictatorial strategy-proof social choice functions; we also provide characterizations of those domains that enable locally non-dictatorial and anonymous strategy-proof social choice rules, respectively. In Nehring and Puppe (2005b), we characterize the rich single-peaked domains that admit efficient and strategy-proof social choice functions.

Appendix 1: "Voting under Constraints" as a Special Case

The following clarifies the relation of our work to Barberà, Massó and Neme's "Voting under Constraints" (1997). Let $(X, \mathcal{H})$ be a property space with $\hat{\mathcal{S}}_{X,T_{\mathcal{H}}}$ as the associated domain of all single-peaked preferences. Suppose that only a subset Y of social alternatives is in fact feasible, *and* that voters' ideal points are known to be feasible (as pointed out by Barberà, Massó and Neme (1997), the latter assumption is clearly restrictive). Formally, let $\mathcal{D} := \{\succ|_Y : \succ \in \hat{\mathcal{S}}_{X,T_{\mathcal{H}}} \text{ with peak of } \succ \text{ in } Y\}$. One can show that $\mathcal{D}$ consists exactly of the preferences on Y that are single-peaked with respect to the restriction of $T_{\mathcal{H}}$ to Y, which is the betweenness associated with the relativization $(Y, \{H \cap Y : H \in \mathcal{H}\})$ of the underlying property space to Y. Every such domain $\mathcal{D}$ is therefore covered by our analysis. Barberà, Massó and Neme (1997) consider the special case in which the underlying property space $(X, \mathcal{H})$ is a product of lines. Conversely, *any* preference domain of the form $\hat{\mathcal{S}}_{X,T_{\mathcal{H}}}$, where $(X, \mathcal{H})$ is an arbitrary property space, is isomorphic to an appropriate relativized domain (in the above sense) embedded in a hypercube.

Appendix 2: Proofs

Proof of Fact 2.1 Let $\succ$ be single-peaked with respect to $T_{\mathcal{H}}$, and denote by x^* the peak of $\succ$; define $\mathcal{H}_g := \{H \in \mathcal{H} : x^* \in H\}$ and $\mathcal{H}_b := \{H \in \mathcal{H} : x^* \notin H\}$. Obviously, this partition of $\mathcal{H}$ satisfies all required properties.

Conversely, let the partition $\mathcal{H} = \mathcal{H}_g \cup \mathcal{H}_b$ satisfy (i), (ii) and (iii). It is straightforward to verify that $\succ$ is single-peaked with peak x^*.

Proof of Fact 2.2 in text.

Proof of Theorem 1 If $\mathcal{D}$ is a rich single-peaked domain with respect to $T_{\mathcal{H}}$, then $T_{\mathcal{D}} = T_{\mathcal{H}}$ by R2. Thus, any $T_{\mathcal{D}}$ associated with a single-peaked domain that is rich with respect to $T_{\mathcal{H}}$ satisfies T1-T5 since $T_{\mathcal{H}}$ does by Fact 2.2. Moreover, in a property space, every element has at least one neighbour. Thus, every $x \in X$ is the peak of at least one preference ordering in $\mathcal{D}$ due to the richness conditions R1 or R2. Finally, suppose that the antecedent in condition (2.5) is satisfied; this means that x and y are neighbours with respect to $T_{\mathcal{D}}$, and hence also neighbours with respect to $T_{\mathcal{H}}$. Thus, (2.5) follows from R1.

Conversely, suppose that $\mathcal{D}$ and $T_{\mathcal{D}}$ satisfy the stated conditions. The relation $T_{\mathcal{D}}$ satisfies T1 by construction, and T4 due to the antisymmetry of the preferences in $\mathcal{D}$ and the fact that each $x \in X$ is the peak of some element of $\mathcal{D}$. Hence, by Fact 2.2, $T_{\mathcal{D}}$ is the betweenness relation associated with some property space $(X, \mathcal{H})$. As noted in the text, the richness condition R2 is satisfied by construction. To verify R1, let x and y be neighbours with respect to $T_{\mathcal{D}}$; by definition, this means that, for all $w \notin \{x, y\}$, there exists a preference $\succ$ with peak x such that $x \succ y \succ w$, hence R1 follows from (2.5).

Proof of Fact 3.1 Suppose that $x \in f_{\mathcal{W}}(\xi)$ and consider any $y \neq x$. By condition H3, there exists $H \in \mathcal{H}$ such that $x \in H$ and $y \in H^c$. By definition of $f_{\mathcal{W}}$, $\{i : \xi_i \in H\} \in \mathcal{W}_H$. By (3.1), $\{i : \xi_i \in H\}^c = \{i : \xi_i \in H^c\} \notin \mathcal{W}_{H^c}$, hence by definition, $y \notin f_{\mathcal{W}}(\xi)$.

Proof of Proposition 3.1 Since families of winning coalitions are by definition closed under taking supersets, voting by issues is monotone in properties by (3.4). Furthermore, voting by issues is clearly onto since it satisfies unanimity.

Conversely, let $f : X^n \to X$ be onto and monotone in properties. For all $H \in \mathcal{H}$, define

$$\mathcal{W}_H := \{W \subseteq N : \exists \xi \text{ such that } \{i : \xi_i \in H\} = W \text{ and } f(\xi) \in H\}.$$

Note that by monotonicity of f, the definition of $\mathcal{W}_H$ does not depend on the choice of ξ. Since f is onto, $\mathcal{W}_H$ is non-empty. We verify that $\mathcal{W}_H$ is closed under taking supersets. Hence, suppose that $W \in \mathcal{W}_H$ and $W' \supseteq W$. Choose ξ such that $W = \{i : \xi_i \in H\}$ and $f(\xi) \in H$. Define ξ' as follows: $\xi_i = \xi_i$ whenever $i \in W$ or $i \in N \setminus W'$, and $\xi'_j \in H$ if $j \in W' \setminus W$. Then, $W' = \{i : \xi'_i \in H\}$ and, by monotonicity in properties, $f(\xi') \in H$. Hence, by definition, $W' \in \mathcal{W}_H$.

Next, we verify (3.1). It is easily seen that $W^c \notin \mathcal{W}_{H^c}$ implies $W \in \mathcal{W}_H$. To verify the converse implication, assume by way of contradiction that $W \in \mathcal{W}_H$ and $W^c \in \mathcal{W}_{H^c}$. Choose ξ with $\{i : \xi_i \in H\} = W$ and $f(\xi) \in H$, and ξ' with $\{i : \xi'_i \in H^c\} = W^c$ and $f(\xi') \in H^c$. Consider ξ'' defined by $\xi''_i = \xi_i$ for $i \in W$ and $\xi''_i = \xi'_i$ for $i \in W^c$. By monotonicity in properties, $f(\xi'') \in H$ and $f(\xi'') \in H^c$, a contradiction.

The proof is completed by noting that $f = f_{\mathcal{W}}$. Indeed, by definition of $\mathcal{W}$, one clearly has $f(\xi) \in f_{\mathcal{W}}(\xi)$, but $f_{\mathcal{W}}$ is single-valued by Fact 3.1.

Proof of Fact 3.2 in text.

Proof of Fact 3.3 Let $F_{\mathcal{W}} : \mathcal{S}^n \to X$ be neutral, and consider $H, H' \in \mathcal{H}$. We show that $\mathcal{W}_H \subseteq \mathcal{W}_{H'}$. Take any $W \in \mathcal{W}_H$ and choose $x \in H$ and $y \in H^c$ such that the segment $[x, y]$ is inclusion minimal. Using T3, it is easily seen that x and y are neighbours. Similarly, choose neighbours $x' \in H'$ and $y' \in (H')^c$. By the richness condition R1, there exist the following four single-peaked preferences: $\succ^x$ having x as its top element and y as the second best, $\succ^y$ with y as top and x as second best element, $\succ^{x'}$ with x' as top and y' as second best element, and $\succ^{y'}$ with y' as top and x' as second best element. Let $\sigma : X \to X$ be a permutation such that $w \succ^x z \Leftrightarrow \sigma(w) \succ^{x'} \sigma(z)$ and $w \succ^y z \Leftrightarrow \sigma(w) \succ^{y'} \sigma(z)$, for all w, z. In particular, $\sigma(x) = x'$ and $\sigma(y) = y'$. Denote by $(\succ^x; W, \succ^y; W^c)$ the simple profile in which all voters in W have the preference $\succ^x$ and all others have the preference $\succ^y$. Since $W \in \mathcal{W}_H$, we have $F_{\mathcal{W}}(\succ^x; W, \succ^y; W^c) \in H$ and in fact $F_{\mathcal{W}}(\succ^x; W, \succ^y; W^c) = x$, since clearly $F_{\mathcal{W}}(\succ^x; W, \succ^y; W^c) \in [x, y]$. By neutrality, $F_{\mathcal{W}}(\succ^{x'}; W, \succ^{y'}; W^c) = \sigma(x) = x'$, which implies $W \in \mathcal{W}_{H'}$ by definition of voting by issues. Thus, neutrality implies that $\mathcal{W}$ is constant.

The converse implication follows immediately from the from the following lemma.

Lemma A.1 *Let $x \neq y$, and suppose that $\mathcal{W}_H = \mathcal{W}_0$ for some $\mathcal{W}_0$ and all $H \in \mathcal{H}$. Then $F_{\mathcal{W}}(\succ^x; W, \succ^y; W^c) = x$ if and only if $W \in \mathcal{W}_0$.*

Proof of Lemma A.1 Clearly, if $F_{\mathcal{W}}(\succ^x; W, \succ^y; W^c) = x$, then W must be a winning coalition; indeed, otherwise W^c would be winning and could therefore enforce a basic property $H \ni y$ with $x \notin H$.

Conversely, suppose that $W \in \mathcal{W}_0$. Since $\mathcal{W}_H = \mathcal{W}_0$ for all $H \in \mathcal{H}$, W is winning for all basic properties. In particular, W enforces all basic property H that contain x. But their intersection contains the single point x by H3.

Proof of Proposition 3.2 Suppose $f : X^n \to X$ is monotone in properties. Consider an individual j with true peak ξ_j who reports $\hat{\xi}_j$. Let $H \in \mathcal{H}$ be any basic property such that $\xi_j \in H$ and $f(\hat{\xi}_j, \xi_{-j}) \in H$. Clearly, $\{i : (\hat{\xi}_j, \xi_{-j})_i \in H\} \subseteq \{i : \xi_i \in H\}$, hence by monotonicity in properties $f(\xi) \in H$. This shows that $f(\xi) \in [\xi_j, f(\hat{\xi}_j, \xi_{-j})]$, i.e. $f(\xi)$ is between the true peak ξ_j and the outcome $f(\hat{\xi}_j, \xi_{-j})$. By single-peakedness, this implies that $f(\xi) \succ_j f(\hat{\xi}_j, \xi_{-j})$ whenever $f(\xi) \neq f(\hat{\xi}_j, \xi_{-j})$.

Conversely, suppose that f is not monotone in properties; then there exist ξ, ξ' and H such that $W := \{i : \xi_i \in H\} \subseteq W' := \{i : \xi'_i \in H\}$, $f(\xi) \in H$ but $f(\xi') \in H^c$. Without loss of generality, we may assume that $W' = W \cup \{j\}$ for some individual $j \notin W$. Since $f(\xi')$ is not between ξ'_j and $f(\xi)$, there exists by the richness condition R2, a preference $\succ_j \in \mathcal{S}$ with top ξ'_j such that $f(\xi) \succ_j f(\xi')$. Clearly, if $\succ_j$ is the true preference of j, this voter will benefit from reporting ξ_j. Hence, F is not strategy-proof.

Proof of Proposition 3.3 The following proof is inspired by the proof of Barberà, Massó and Neme (1997, Proposition 2) which it augments by two significant intermediate steps; specifically, these are Facts A.1 and A.2, the latter of which is based on Lemma 3.1.

Let F be strategy-proof and onto. The proof of the "peaks-only" property proceeds by induction over the number of voters. Thus assume first $n = 2$. From the strategy-proofness of F it is immediate that

$$F(\succ_1, \succ_2) = \text{argmax}_{o_1(\succ_2)} \succ_1 = \text{argmax}_{o_2(\succ_1)} \succ_2, \tag{A.1}$$

i.e. $F(\succ_1, \succ_2)$ is the best element in the option set $o_i(\succ_j)$ with respect to $\succ_i$.

Denoting by $\tau(\succ) \in X$ the peak of $\succ$, one has

$$[\tau(\succ_1) = \tau(\succ_2) = x] \Rightarrow F(\succ_1, \succ_2) = x. \tag{A.2}$$

For verification, suppose that x is the common peak of $\succ_1$ and $\succ_2$. Since F is onto, there exist $\succ'_1$ and $\succ'_2$ such that $F(\succ'_1, \succ'_2) = x$, i.e. $x \in o_1(\succ'_2)$. By (A.1), $F(\succ_1, \succ'_2) = x$, i.e. $x \in o_2(\succ_1)$, hence again by (A.1), $F(\succ_1, \succ_2) = x$.

The following fact plays a key role in the proof of Lemma A.2 below.

Fact A.1 *Suppose that $y \in o_2(\succ_1)$ and $y' \in [y, \tau(\succ_1)]$, then $y' \in o_2(\succ_1)$.*

To verify this, we can assume that y' is a neighbour of y; from this the general claim then follows by induction using the transitivity condition T3. Thus, assume by way of contradiction that $y' \in [y, \tau(\succ_1)]$ is a neighbour of y with $y' \notin o_2(\succ_1)$. By the richness condition R1, there exists $\succ$ such that $y' \succ y \succ w$ for all $w \notin \{y, y'\}$. By (A.1), $F(\succ_1, \succ) = y$, and by (A.2), $F(\succ, \succ) = y'$. By the single-peakedness of $\succ_1$, voter 1 can therefore manipulate at $(\succ_1, \succ)$ via $\succ$, a contradiction.

For the next step, we need the following notation. For $F : \mathcal{S}^n \to X$ and every voter i, denote by

$$o^F_{-i}(\succ_i) := \{x \in X : \text{ there exists } \succ_{-i} \in \mathcal{S}^{n-1} \text{ such that } F(\succ_i, \succ_{-i}) = x\}.$$

In contrast to the set $o^F_i(\succ_{-i})$ defined in the main text, the set $o^F_{-i}(\succ_i)$ describes the social states that all voters *other* than i can induce, given a fixed preference for voter i. Note that for $n = 2$, one has $o^F_{-i}(\succ_j) = o^F_j(\succ_i)$, where $i \neq j$. When no confusion can arise, we will suppress the reference to the underlying social choice function and simply write $o_i(\succ_{-i})$ and $o{-}i(\succ_i)$.

Lemma A.2 *If $\tau(\succ_1) = \tau(\succ'_1)$, then $o_{-1}(\succ_1) = o_{-1}(\succ'_1)$.*

Proof of Lemma A.2 We first prove the result for $n = 2$. Suppose, by way of contradiction, that $x = \tau(\succ_1) = \tau(\succ_1')$ and $y \in o_2(\succ_1)$ but $y \notin o_2(\succ_1')$. By (A.2), one must have $y \neq x$. First, we show that y cannot be a neighbour of x. Otherwise, one could choose, by R1, a preference $\succ$ with $y \succ x \succ w$ for all $w \notin \{y, x\}$; by (A.1) one would obtain $F(\succ_1, \succ) = y$ and $F(\succ_1', \succ) = x$, but then voter 1 could manipulate at $(\succ_1, \succ)$ via $\succ_1'$.

Thus, y is not a neighbour of x. Choose a neighbour $y' \in [x, y]$ of y. By Fact A.1, $y' \in o_2(\succ_1)$. Suppose that also $y' \in o_2(\succ_1')$. By R1, there exists a preference $\succ'$ with $y \succ' y' \succ' w$ for all $w \notin \{y, y'\}$. By (A.1), $F(\succ_1, \succ') = y$ and $F(\succ_1', \succ') = y'$. But by the single-peakedness of $\succ_1$, we have $x \succ_1 y' \succ_1 y$; therefore, voter 1 can manipulate at $(\succ_1, \succ')$ via $\succ_1'$. Thus, we must have $y' \in o_2(\succ_1)$ and $y' \notin o_2(\succ_1')$. Now replace y by y' and repeat the argument until a neighbour of x is reached to derive a contradiction.

To prove the assertion for general n, define a social choice function $E : \mathcal{S}^2 \to X$ by $E(\succ_1, \succ_2) := F(\succ_1, \succ_2, ..., \succ_2)$. It is easily verified that E inherits the strategy-proofness and voter sovereignty from F. Hence, by the above arguments,

$$[\tau(\succ_1) = \tau(\succ_1')] \Rightarrow o_2^E(\succ_1) = o_2^E(\succ_1').$$

The proof is thus completed by showing that, for all $\succ_1$, $o_2^E(\succ_1) = o_{-1}^F(\succ_1)$. Clearly, one has $o_2^E(\succ_1) \subseteq o_{-1}^F(\succ_1)$. To show the converse inclusion, take any $x \in o_{-1}^F(\succ_1)$ and choose $\succ_2, ..., \succ_n$ such that $x = F(\succ_1, \succ_2, ..., \succ_n)$. Consider any preference $\succ$ with $\tau(\succ) = x$. By the strategy-proofness of F,

$$x = F(\succ_1, ..., \succ_n) = F(\succ_1, ..., \succ_{n-1}, \succ) = ... = F(\succ_1, \succ, ..., \succ) = E(\succ_1, \succ),$$

hence $x \in o_2^E(\succ_1)$. This concludes the proof of Lemma A.2.

For the case $n = 2$, we can now complete the proof of the "peaks-only" property. Indeed, that property follows at once from the fact that

$$[o_2(\succ_1) = o_2(\succ_1')] \Rightarrow F(\succ_1, \succ_2) = F(\succ_1', \succ_2). \tag{A.3}$$

To verify (A.3), assume by way of contradiction, that $x = F(\succ_1, \succ_2) \neq F(\succ_1', \succ_2) = x'$. By assumption there exist $\succ$ and $\succ'$ such that $F(\succ_1, \succ') = x'$ and $F(\succ_1', \succ) = x$. But then voter 2 can either manipulate at $(\succ_1, \succ_2)$ via $\succ'$ (if $x' \succ_2 x$), or manipulate at $(\succ_1', \succ_2)$ via $\succ$ (if $x \succ_2 x'$).

The proof for $n = 2$ is thus complete. For the induction argument, we need to first prove Lemma 3.1.

Proof of Lemma 3.1 Given any element $x \in X$ choose $\succ_i$ with $\tau(\succ_i) = x$, and set $\gamma(x) := F(\succ_1, \succ_2, ..., \succ_n)$. We claim that $\gamma(x)$ is the gate of $o_i(\succ_{-i})$. Assume, by way of contradiction, that $z \in o_i(\succ_{-i})$ is such that $\gamma(x) \notin [x, z]$. By the richness condition R2, there exists $\succ_i'$ with $\tau(\succ_i') = x$ and $z \succ_i' \gamma(x)$. By the strategy-proofness of F, $\gamma(x) \neq F(\succ_i', \succ_{-i})$; but this contradicts the "peaks-only" property of F.

Proof of Proposition 3.3 (cont.) For given $\succ_1$ define

$$G(\succ_2, ..., \succ_n) := F(\succ_1, ..., \succ_n),$$

and denote $Y := o_{-1}(\succ_1)$. Clearly, G is strategy-proof with range Y. Furthermore, Y is gated; indeed, as shown in the proof of Lemma A.2, we have $Y = o_{-1}^F(\succ_1) = o_2^E(\succ_1)$, and $o_2^E(\succ_1)$ is gated by Lemma 3.1. Let G_Y denote the restriction of G to the profiles of preference orderings in $\mathcal{S}$ that have their peak in Y. Now observe that for every $\succ \in \mathcal{S}$ with peak x, the restriction $\succ|_Y$ is single-peaked (with respect to the induced betweenness on Y) with peak $\gamma(x)$, where $\gamma(x)$ is the gate of Y to x. Moreover, the set of all restrictions is a rich domain on Y. Hence, by the induction hypothesis G_Y satisfies "peaks-only" and can therefore represented by a voting scheme $g : Y^{n-1} \to Y$.

Fact A.2 *$G(\succ_2, ..., \succ_n) = g(\gamma(\xi_2), ..., \gamma(\xi_n))$, where $\xi_i = \tau(\succ_i)$.*

This follows from $\gamma(\xi_i) = \text{argmax}_Y \succ_i$, using Lemma 3.1, and the observation that, by strategy-proofness, $G(\succ_2, ..., \succ_n) = G_Y(\succ_2|_Y, ..., \succ_n|_Y)$.

We now complete the proof by deriving a contradiction from the assumption that there exist $\succ_1$ and $\succ'_1$ with $\tau(\succ_1) = \tau(\succ'_1) =: x$ such that

$$y = F(\succ_1, \succ_2, ..., \succ_n) \neq F(\succ'_1, \succ_2, ..., \succ_n) = y'.$$

By Lemma A.2, $o_{-1}(\succ_1) = o_{-1}(\succ'_1) =: Y$. Define G, G_Y and g as above, and analogously, G', G'_Y and g'. By Fact A.2,

$$y = g(\gamma(\xi_2), ..., \gamma(\xi_n)) \neq g'(\gamma(\xi_2), ..., \gamma(\xi_n)) = y',$$

and by Propositions 3.1 and 3.2, g and g' are voting by issues on Y. Choose $H \in \mathcal{H}|_Y$ with $y \in H$, $y' \in H^c$ and, without loss of generality, $\gamma(x) \in H$. Let $W := \{i : \gamma(\xi_i) \in H\}$, $W' = \{i : \gamma(\xi_i) \in H^c\}$, and consider $\eta = (\eta_2, ..., \eta_n)$ where

$$\eta_i = \begin{cases} y' & \text{if} \quad i \in W' \\ \gamma(x) & \text{if} \quad i \notin W' \end{cases}.$$

Since g and g' are voting by issues, and since every basic property jointly possessed by y' and $\gamma(x)$ gets unanimous support, we have $\{g(\eta), g'(\eta)\} \subseteq [y', \gamma(x)]$. Moreover, $W = \{2, ..., n\} \setminus W'$ is winning for H in g, and W' is winning for H^c in g', hence $g(\eta) \in H$ and $g'(\eta) \in H^c$.

We show that $g'(\eta) \neq y'$. Otherwise, choose $\hat{\succ}_i$ with $\tau(\hat{\succ}_i) = \eta_i$ to obtain from $g(\eta) \in [y', \gamma(x)]$ and $g(\eta) \neq y'$,

$$g(\eta) = F(\succ_1, \hat{\succ}_2, ..., \hat{\succ}_n) \succ'_1 F(\succ'_1, \hat{\succ}_2, ..., \hat{\succ}_n) = y',$$

in contradiction to the strategy-proofness of F. Now repeat the argument replacing y' by $z' := g'(\eta) \in H^c$. The desired contradiction is then obtained by induction since the segment $[z', \gamma(x)]$ is strictly contained in $[y', \gamma(x)]$.

Proof of Corollary 1 We show, by contraposition, that if F is non-dictatorial, then under the stated assumptions there exist non-trivial gated sets. By Propositions 3.1 and 3.2, F is voting by issues. First, assume that all minimal elements of all $\mathcal{W}_H$ are singletons. Since F is non-dictatorial, there must exist two distinct voters i and j such that $\{i\} \in \mathcal{W}_H$ and $\{j\} \in \mathcal{W}_{H'}$. Since also the minimal elements of $\mathcal{W}_{H^c}$ and $\mathcal{W}_{(H')^c}$ are singletons by assumption, one must have $\{i\} \in \mathcal{W}_{H^c}$ and $\{j\} \in \mathcal{W}_{(H')^c}$ by (3.2); moreover, by the Intersection Property, all four intersections $H \cap H'$, $H^c \cap H'$, $H \cap (H')^c$ and $H^c \cap (H')^c$ are non-empty. In this case, $o_i(\succ_{-i})$ is clearly non-trivial, since if j's peak is in H', $o_i(\succ_{-i})$ is contained in H' and intersects both H and H^c. Hence by Lemma 3.1, there exist non-trivial gated sets.

Thus, suppose that $W \in \mathcal{W}_H$ is minimal winning and contains at least two different voters, say j and k. Since $\#X \geq 3$, we can choose $G \notin \{H, H^c\}$. Without loss of generality, we may assume that both $G^c \cap H$ and $G^c \cap H^c$ are non-empty; indeed, if one of these sets were empty, replace G by G^c. We distinguish two cases.

Case 1. Suppose that $\{j\} \in \mathcal{W}_G$. Then consider a profile $\succ_{-j}$ in which all voters other than j have their peak in $G^c \cap H^c$. By construction, $o_j(\succ_{-j})$ intersects both G and G^c; but since $\{j\}$ is not winning for H, we also have $o_j(\succ_{-j}) \subseteq H^c$. Thus, $o_j(\succ_{-j})$ is non-trivial, and by Lemma 3.1 it is gated.

Case 2. Now let $\{j\} \notin \mathcal{W}_G$. Consider a profile $\succ_{-j}$ in which all voters in $W \setminus \{j\}$ have their peak in $G^c \cap H$ and all voters outside W have their peak in $G^c \cap H^c$. By construction, $o_j(\succ_{-j})$ intersects both H and H^c; moreover, since $\{j\}$ is not winning for G, we also have $o_j(\succ_{-j}) \subseteq G^c$. Thus, the gated set $o_j(\succ_{-j})$ is again non-trivial.

Proof of Corollary 2 By Propositions 3.1 and 3.2, F is voting by issues, and by Fact 3.3 the corresponding structure of winning coalitions is constant, i.e. $\mathcal{W}_H = \mathcal{W}_0$. Let $W \in \mathcal{W}_0$ be minimal, and let $i \in W$. Consider any segment $[x, z]$ and a profile $\succ_{-i}$ in which all voters in $W \setminus \{i\} \neq \emptyset$ have their peak at x and all voters outside W have their peak at z. By construction, $[x, z] \subseteq o_i(\succ_{-i})$; indeed, by reporting $y \in [x, z]$ voter i enforces all basic properties possessed by y since all these are shared with x or with z. On the other hand, since F is non-dictatorial one also has $o_i(\succ_{-i}) \subseteq [x, z]$. Hence, $[x, z]$ coincides with $o_i(\succ_{-i})$, and is thus gated by Lemma 3.1.

Proof of Proposition 3.4 Suppose $f_{\mathcal{W}}$ is consistent, and let $\mathcal{G} = \{G_1, ..., G_l\}$ be a critical family. For $j = 1, ..., l$, consider any selection $W_j \in \mathcal{W}_{G_j}$. We will show $\cap_{j=1}^{l} W_j \neq \emptyset$ by a contradiction argument. Thus, assume that $\cap_{j=1}^{l} W_j = \emptyset$. Then, for all $i \in N$, there exists j_i such that $i \notin W_{j_i}$. For each i, pick an element $\xi_i \in G_{j_i}^c \cap (\cap_{j \neq j_i} G_j) = \cap_{j \neq j_i} G_j$ (observe that the latter set is non-empty by definition of a critical family). By construction, if $i \in W_j$, then $j \neq j_i$, hence $\xi_i \in G_j$. This shows that, for all j, $W_j \subseteq \{i : \xi_i \in G_j\}$. Therefore, $\{i : \xi_i \in G_j\} \in \mathcal{W}_{G_j}$, hence by (3.4), $f_{\mathcal{W}}(\xi_1, ..., \xi_n) \in G_j$ for all $j = 1, ..., l$. However, this contradicts the fact that $\{G_1, ..., G_l\}$ is a critical family.

Conversely, suppose $f_{\mathcal{W}}$ is not consistent, i.e. for some ξ, $f_{\mathcal{W}}(\xi) = \emptyset$. By (3.1) and (3.3), this implies that $\cap\{H \in \mathcal{H} : \{i : \xi_i \in H\} \in \mathcal{W}_H\} = \emptyset$. We show that $f_{\mathcal{W}}$ cannot satisfy the Intersection Property by contradiction. Thus assume $f_{\mathcal{W}}$ does satisfy the Intersection Property. Pick a critical family $\{G_1, ..., G_l\} \subseteq \{H \in \mathcal{H} : \{i : \xi_i \in H\} \in \mathcal{W}_H\}$. By the Intersection Property, $\cap_{j=1}^{l}\{i : \xi_i \in G_j\} \neq \emptyset$. Let $i_0 \in \{i : \xi_i \in G_j\}$ for all $j = 1, ..., l$. But then $\xi_{i_0} \in G_j$ for all j, contradicting the fact that $\{G_1, ..., G_l\}$ is a critical family.

Proof of Fact 3.4 To show the first statement, note that the structure of winning coalitions corresponding to the given set of quotas is well-defined due to the integer clause. It is then straightforward to derive the Intersection Property from (3.7) and the assumption that $q_H + q_{H^c} = 1$.

Conversely, suppose that voting by issues is anonymous and consistent. For each $H \in \mathcal{H}$, set $q_H := \frac{m_H - 1}{n-1}$ where m_H are the absolute quotas defined in the main text. If $m_H \neq n$, m_H is easily seen to be the smallest integer greater than $q_H \cdot n$. Thus, $\mathcal{W}_H = \mathcal{W}_{q_H} := \{W \subseteq N : \#W > q_H \cdot n\}$ if $m_H < n$. If, on the other hand, $m_H = n$, we clearly have $\mathcal{W}_H = \mathcal{W}_1 = \{N\}$. Thus, the given structure of winning coalitions can be described as a quota rule with the quotas q_H as specified. By (3.5), we have $m_H + m_{H^c} = n + 1$, hence $q_H + q_{H^c} = 1$. Finally, the Intersection Property implies

(3.6), and hence $\sum_{H \in \mathcal{G}}(1 - q_H) \leq 1$ as desired.

Proof of Proposition 4.1 We first show "(i) → (iii)." Thus, let $(X, \mathcal{H})$ be simple. For every pair v, w, denote by $\mathcal{H}_{\{v,w\}} := \{H \in \mathcal{H} : \{v, w\} \subseteq H\}$, and observe that $\cap\mathcal{H}_{\{v,w\}} = [v, w]$. Consider any segment $[x, y]$. We have to show that $[x, y]$ is gated. Clearly, $\gamma(z) = z$ if $z \in [x, y]$; thus, suppose $z \notin [x, y]$. We claim that the set $A := (\cap\mathcal{H}_{\{x,y\}}) \cap (\cap\mathcal{H}_{\{x,z\}}) \cap (\cap\mathcal{H}_{\{y,z\}})$ is non-empty. Indeed, if A were empty, the collection $\mathcal{H}_{\{x,y\}} \cup \mathcal{H}_{\{x,z\}} \cup \mathcal{H}_{\{y,z\}}$ would contain a critical family. But evidently, all pairs of this collection have a non-empty intersection; hence, since by assumption all critical families have cardinality two, A must be non-empty. Now let $\gamma(z) \in A$, then $\gamma(z)$ is the gate of $[x, y]$ to z. Indeed, pick an arbitrary $w \in [x, y]$, and consider any basic property $H \supseteq \{w, z\}$. Since w is between x and y, H must contain x or y, thus $H \in \mathcal{H}_{\{x,z\}}$ or $H \in \mathcal{H}_{\{y,z\}}$. In either case, $A \subseteq H$, hence $\gamma(z) \in H$, i.e. $\gamma(z)$ is between z and w.

The implication "(iii) → (ii)" is straightforward. Indeed, it is easily seen that the gate $\gamma(z)$ of the segment $[x, y]$ to z is the median of the triple x, y, z.

Thus, it remains to prove the implication "(ii) ⇒ (i)." By contraposition, suppose there exists a critical family with at least three elements, say $\mathcal{G} = \{H_1, H_2, H_3, ..., H_l\}$, and let $A := H_3 \cap ... \cap H_l$. By the criticality of $\mathcal{G}$, we can choose $x \in H_1 \cap H_2$, $y \in H_2 \cap A$, and $z \in H_1 \cap A$. By construction, the triple x, y, z cannot have a median m, since $m \in [x, z] \Rightarrow m \in H_1$, $m \in [x, y] \Rightarrow m \in H_2$, and $m \in [y, z] \Rightarrow m \in A$, but by assumption $H_1 \cap H_2 \cap A = \emptyset$.

Proof of Corollary 3 From Fact 2.1, we know that a preference ordering is separable on the hypercube if and only if it is single-peaked with respect to the corresponding betweenness. The result thus follows from the observation that, since all critical families on the (full) hypercube have the form $\{H, H^c\}$, every structure of winning coalitions satisfies the Intersection Property due to (3.1) or (3.2).

Proof of Corollary 4 The result follows at once from the observation that on a product of lines all critical families have the form $\{G, H^c\}$ with $G \subseteq H$. Note also that the "multi-dimensionally" single-peaked preference orderings on a product of lines defined in Barberà, Gul and Stacchetti (1993) are exactly the single-peaked ones as defined here, and that the conditions defining a "generalized median voter scheme" are exactly the restrictions imposed on a structure of winning coalitions by the Intersection Property. The result of Moulin (1980) corresponds to the one-dimensional case.

Proof of Theorem 4 By the Intersection Property, issue-by-issue majority voting with an odd number of voters is consistent on a median space. If $n > 3$ is even, by the same argument, one may take issue-by-issue majority voting among a subset of $n - 1$ voters. Conversely, suppose that F is strategy-proof, onto, neutral and non-dictatorial. By Theorem 2, F must be voting by issues; by Corollary 2, all segments must be gated, hence by Proposition 4.1, the underlying space must be a median space.

The second part follows from Fact 3.3 and the observation that, if $W \in \mathcal{W}_0 \Leftrightarrow W^c \notin \mathcal{W}_0$, the constant structure of winning coalitions $\mathcal{W}_H = \mathcal{W}_0$ for all H always satisfies the Intersection Property on a median space by Proposition 4.1.

Proof of Corollary 5 The proof is immediate from Theorem 4 and the observation that voting by issues is anonymous and neutral if and only if it is issue-by-issue majority voting with an odd number of voters.

Proof of Theorem 5 First, suppose that $(X, \mathcal{H})$ is a median space. For all $x \in X$, denote by $\mathcal{H}_{\{x\}} := \{H \in \mathcal{H} : H \ni x\}$. Take any minority family $\mathcal{W}_0$, and define a structure of winning coalitions by $\mathcal{W}_H = \mathcal{W}_0$ for all $H \in \mathcal{H}_{\{x\}}$, and by $\mathcal{W}_H = (\mathcal{W}_0)^a$ for $H \notin \mathcal{H}_{\{x\}}$. Note that (3.2) is thus satisfied by construction. Since $(X, \mathcal{H})$ is a median space, any critical family $\mathcal{G}$ has two members. If $\mathcal{G}$ contains an element of $\mathcal{H}_{\{x\}}$ (necessarily, at most one), the Intersection Property is satisfied by (3.2). On the other hand, if $\mathcal{G}$ does not intersect $\mathcal{H}_{\{x\}}$, the Intersection Property follows from $(\mathcal{W}_0)^a \subset \mathcal{W}_0$, i.e. the assumption that $\mathcal{W}_0$ is a minority family, and (3.2). Thus, in any case the corresponding minority veto rule is consistent. Note also that every minority veto rule is onto and strategy-proof.

We now show, by contraposition, that consistency of a minority veto rule for all $x \in X$ requires a median space. Thus, let F be a minority veto rule with status quo x and minority family $\mathcal{W}_0$. By definition of F and (3.4), we have $\mathcal{W}_0 \subseteq \mathcal{W}_H$ for all $H \in \mathcal{H}_{\{x\}}$. Suppose that the triple x, y, z has no median, i.e. suppose that $[x, y] \cap [y, z] \cap [x, z] = \emptyset$. This implies, in the notation of Proposition 4.1 above, that the family $\mathcal{H}_{\{x,y\}} \cup \mathcal{H}_{\{x,z\}} \cup \mathcal{H}_{\{y,z\}}$ contains a critical family $\mathcal{G}$. Clearly, at least two basic properties in $\mathcal{G}$ must contain x. Since $\mathcal{W}_0$ contains two disjoint winning coalitions, the Intersection Property applied to the critical family $\mathcal{G}$ is violated, hence F is not consistent.

For the results of Section 4.3, the following lemma will be useful. For all x, y, denote by $\mathcal{H}_{x\neg y} := \{H \in \mathcal{H} : x \in H \text{ and } y \notin H\}$.

Lemma A.3 *Let $(X, \mathcal{H})$ be a median space. Then, any two neighbours are separated by exactly one basic property. Moreover, x and z are connected by a unique shortest path if and only if $\mathcal{H}_{x\neg z}$ is linearly ordered by set inclusion.*

Proof of Lemma A.3 Let x, z be two neighbours. By contradiction, suppose that $\mathcal{H}_{x\neg z}$ contains two distinct elements H and H'. Without loss of generality, there exists $y \in H$ such that $y \notin H'$. By construction, the median of the triple x, y, z is different from x and z, but between these two elements, which is obviously not possible.

Let now $x_0, x_1, ..., x_l, x_{l+1}$ be a unique shortest path with $x_0 = x$ and $x_{l+1} = z$, and denote by H_j the unique property with $x_j \in H_j$ and $x_{j+1} \notin H_j$. We show that, for all j, $H_{j-1} \subseteq H_j$. By contradiction, suppose that there exists $y \in H_{j-1}$ with $y \in H_j^c$. Then, the median of the triple x_{j-1}, x_{j+1}, y is different from x_j but nevertheless between x_{j-1} and x_{j+1}, i.e. on a shortest path connecting these two elements. Obviously, this contradicts the assumption that the original shortest path is unique.

Conversely, suppose that $\mathcal{H}_{x\neg z}$ is linearly ordered by set inclusion, say $\mathcal{H}_{x\neg z} = \{H_0, H_1, ..., H_l\}$ with $H_0 \subseteq H_1 \subseteq ... \subseteq H_l$. For every j, consider the set $A_j := (H_j \setminus H_{j-1}) \cap [x, z]$. Since $H_j \setminus H_{j-1}$ is non-empty, and since $(X, \mathcal{H})$ is a median space, A_j is non-empty. Moreover, we will show that, for all j, A_j contains a unique element x_j. This immediately implies that $x, x_1, ..., x_l, z$ is a unique shortest path connecting x and z. Thus, assume by way of contradiction that y and y' are two distinct elements of A_j. By H3, let H be a basic property with $y \in H$ and $y' \in H^c$. Since both y and y' are between x and z, H must contain either x or z, but not both. Without loss of generality, assume that $x \in H$ and $z \notin H$. Then $H \in \mathcal{H}_{x\neg z}$, in contradiction to the fact that, obviously, $H \neq H_j$ for all $j = 0, ..., l$.

Proof of Proposition 4.2 First, let $\succ$ be single-peaked. By Fact 2.1, $\succ$ is separable. To show that $\succ$ is convex, suppose that y is linearly between x and z, and that $x \succ y$. Using Lemma A.3, let G be the maximal basic property in $\mathcal{H}_{x\neg z}$ with $y \in G^c$. By the single-peakedness of $\succ$ and Fact 2.1, we have $G \in \mathcal{H}_g$ since it must contain the peak of $\succ$. Indeed, if G^c contained the peak of $\succ$, y would be preferred to its neighbour in $[x, y]$, contradicting the single-peakedness. By Lemma A.3, this implies that $y \in H$ for all $H \in \mathcal{H}_g$ such that $z \in H$. Hence, again by Fact 2.1, $y \succ z$.

To prove that, conversely, every convex and separable preference ordering is single-peaked, we use the following fact (cf. van de Vel (1993, ch. I.6)). Let x, y, z be a triple of distinct elements in a median space such that $y \in [x, z]$. Then, there exists a "direct path" through y that connects x and z. Formally, there exists a sequence $y_0, y_1, ..., y_l, y_{l+1}$ with the following properties: $y_0 = x$, $y_{l+1} = z$, $y \in \{y_1, ..., y_l\}$, and for all $j = 0, ..., l$, y_j and y_{j+1} are neighbours such that $y_{j+1} \in [y_j, z]$.

Now consider any convex and separable preference ordering $\succ$ with peak x. We will show that $y \succ z$ for all $y \neq z$ with $y \in [x, z]$. As above, let $y_0, ..., y_{l+1}$ be a direct path through y connecting x and z. For all j, denote by Θ_j the set of neighbours of y_j in $[y_j, z]$. We show by induction that, for all j,

$$y_j \succ w \text{ for all } w \in \Theta_j. \tag{A.4}$$

By transitivity of $\succ$, (A.4) implies $y_j \succ z$ for all j, since $z = y_{l+1} \in \Theta_l$. For $j = 0$, (A.4) holds trivially since $y_0 = x$ is the peak of $\succ$. Thus, assume that (A.4) holds for $j-1$, and consider y_j along with a neighbour $w \in \Theta_j$. Let $\mathcal{H}_{y_{j-1} \neg y_j} = \{H\}$ and $\mathcal{H}_{y_j \neg w} = \{G\}$. There are two possible cases. First, if $H \subseteq G$ then y_j is linearly between y_{j-1} and w. By the induction hypothesis, $y_{j-1} \succ y_j$, hence $y_j \succ w$ by convexity. Otherwise, if $H \not\subseteq G$, there exists, using the median property, a neighbour v of y_{j-1} in $H \cap G^c$. Since $v \in \Theta_{j-1}$, we have $y_{j-1} \succ v$ by the induction hypothesis. This implies $y_j \succ w$ by separability, since the two pairs of neighbours (y_{j-1}, v) and (y_j, w) are separated by the same basic property G.

Proof of Fact 4.1 Suppose that $\succ$ has a utility representation u as required by cardinal convexity and separability. Clearly, if x and y are neighbours with $x \succ y$, we have $u(x) - u(y) = \lambda_H$ where $H \in \mathcal{H}_g$ is the unique basic property in $\mathcal{H}_{x \neg y}$. This directly implies the stated separability condition, and using Lemma A.3 the concavity on linear paths.

Conversely, let u be a utility representation such that $u(x) - u(y) = u(z) - u(w)$ for every two pairs of neighbours separated by the same basic property, and such that u is concave on every linear path. Define $\lambda : \mathcal{H} \to \mathbf{R}$ by $\lambda_H := \max\{u(x) - u(y), 0\}$ for every two neighbours x, y that are separated by H. By assumption, λ is well-defined. Clearly, the function $\tilde{u}$ defined by $\tilde{u}(x) := \sum_{H \ni x} \lambda_H$ equals u up to an additive constant. Concavity along linear paths implies $\lambda_G \geq \lambda_H$ if $G \supseteq H$ by Lemma A.3, since λ_H is the utility increment of moving one step in direction of the preference peak if the peak is contained in H.

Proof of Proposition 4.3 Evidently, every cardinally convex and separable preference ordering is single-peaked. To verify the richness condition R1, let the neighbours x and y be separated by $G \in \mathcal{H}_{\{x\}}$. Set $\lambda_H = 0$ for all $H \notin \mathcal{H}_{\{x\}}$, and choose generic, in particular pairwise distinct, numbers $\lambda_H > 0$ for all $H \in \mathcal{H}_{\{x\}}$, while respecting inclusion monotonicity. If $\lambda_G = \min\{\lambda_H : H \in \mathcal{H}_{\{x\}}\}$, the preference ordering represented by $u(x) = \sum_{H \ni x} \lambda_H$ has x as peak and y as second best element; the genericity of the λ_H for $H \in \mathcal{H}_{\{x\}}$ guarantees that the preference ordering displays no indifferences.

To verify R2, let $y \notin [x, z]$. Choose a cardinally convex and separable preference ordering $\succ$ with peak x, and a basic property $G \in \mathcal{H}_{\{x,z\}}$ such that $y \notin G$. If λ_G is larger than $\sum_{H \in \mathcal{H}_{\{x,y\}}} \lambda_H$, one obtains $z \succ y$. Since such a preference ordering clearly exists, R2 is satisfied.

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