Modelling Cost Complementarities in Terms of Joint Production*

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Abstract. *Cost complementarities arise from synergies in the production of heterogeneous goods. It is shown that synergies can be accounted for in terms of shared public inputs (roughly) if and only if synergies decrease as the scope of production increases. This case of "substitutive" synergies is argued to be typical. The key technical tool is a novel interpretation of conjugate Moebius inversion in terms of higher-order differences.* **Journal of Economic Literature** Classification Numbers: D24, L23

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1 Introduction

Economies of scope arise from synergies in the production of heterogeneous goods. The classical Marshallian notion of joint production (Marshall [4]) explains instances of such synergies by the fact that some factors of production are pure public inputs. That is, once the inputs have been used for producing one good, they become costlessly available for use in the production of others. We refer to this as the *joint public input* interpretation of economies of scope. In this paper, we demarcate the domain of applicability of the joint public input explanation to economies of scope by studying the entailed structure of synergies, i.e. the pattern of marginal cost reductions due to the production of related goods. It is straightforward to see that the economies of scope due to joint production entail cost complementarities. Indeed, the marginal cost of producing a given good decreases with the set of goods already produced since the larger the set of goods produced the more public inputs are already used, and are thus available for free in further production. Cost complementarity (technically: submodularity of the cost function) is clearly a stronger requirement than the presence of economies of scope in the sense of subadditivity (see Panzar and Willig [12], Baumol, Panzar and Willig [1] and Panzar [11]), but an economically very natural one. The bottom line of this paper is that assuming the existence of joint public inputs is only slightly more restrictive than assuming cost complementarity.

To focus on the combinatorial questions posed by the multi-dimensionality of the good space, we work in a discrete setting in which both inputs and outputs are formalized as *sets* of goods rather than vectors of continuous quantities. One interpretation of the good space is as a set of heterogeneous goods that can either be produced in unit quantity, or not produced. Alternatively, our analysis can also be applied to a setting with a finite number of *types* of goods that can be produced in arbitrary quantities, provided that the scope effects across types can be completely separated from any scale effects within types; clearly, that assumption is restrictive (see Section 2 for details).

The key to our analysis is the notion of synergy describing the local extent of cost complementarities. Specifically, the *synergy* between x and y is the reduction in the marginal cost of producing good x by the additional production of good y. Note that the synergy between x and y typically depends on the set of goods otherwise produced; submodularity can be paraphrased as the requirement that the synergy between two goods is always non-negative, no matter what other goods are being produced.

Under a joint public input interpretation, the synergy between two goods is simply the total cost of all inputs that are jointly required by both goods. Thus, cost functions derived from joint public inputs are always submodular; conversely, we show that any cost function satisfying a stronger condition of "*total* submodularity" can be represented mathematically as arising from the use of joint public inputs. The economic content of total submodularity is that the synergy between any pair of goods is decreasing as the scope of production increases; such decreasing (self-dampening) synergies are referred to as "substitutive" synergies. We show that the case of substitutive synergies is more fundamental than and presumably far more prevalent than the formally symmetric case of increasing (self-reinforcing) synergies, which turns out to be remarkably restrictive. Thus, we arrive at the main conclusion of the paper that *the step from the general notion of cost complementarities to the more structured joint public input interpretation is small, being largely a matter of assuming sufficient regularity.*

The key technical tool is conjugate Moebius inversion. The use of (non-conjugate)

Moebius inversion is standard in the related literature on cost sharing since Shapley's seminal contribution [14]; occasionally, one also finds references to its conjugate form (see Moulin [5] and Young [15]). As our main mathematical contribution, we sharpen this tool by providing a novel characterization of the conjugate Moebius inverse in terms of the higher-order derivatives[1] of a cost function (Theorem 2 below). This is economically relevant since the first derivative describes marginal costs and the second derivative the local synergies; moreover, positivity of the third derivative corresponds to decreasing ("substitutive") synergies.

The public input interpretation is fruitful especially because it opens the possibility to model economies of scope in a flexible way through appropriate assumptions on the pattern of inputs. For an analysis of how such patterns of inputs are reflected in the functional form of the cost function, we refer to the methods developed in Nehring [6] and Nehring and Puppe [8, 9].[2]

2 Two Interpretations of Discrete Cost Functions

Let X be a finite set of goods that can be potentially produced by a firm. For any subset $S \subseteq X$ denote by $c(S)$ the cost of producing exactly the goods in S; these costs are summarized in a cost function $c : 2^X \to \mathbf{R}$. Throughout, c is assumed to be monotone in the sense that $c(W) \leq c(S)$ whenever $W \subseteq S$, and normalized so that $c(\emptyset) = 0$.[3] By $m_x(S) := c(S \cup \{x\}) - c(S)$ we denote the marginal cost of producing good x given that the set S of goods is already produced. Note that monotonicity is equivalent to non-negativity of $m_x(S)$ for all x and S.

The elements of X are sometimes interpreted as heterogeneous *individual* objects, e.g. cars, or as *types* of goods ("product lines") distinguished by specific know-how, e.g. different car models sharing a basic common design. Under the latter interpretation, there are two distinct sources of costs: "fixed costs" of acquiring the capability of producing goods of a certain type, and "variable costs" of producing goods of a particular type in various quantities. For instance, fixed costs may correspond to R&D costs of developing a range of product lines. In the following, we describe the conditions under which the fixed costs of being able to produce different types of goods can be completely separated from the variable costs within each type. In this case, our methodology applies to the (then well-defined) fixed cost component of the cost function.

Let $X = \cup_{k \in Y} X_k$ with the X_k pairwise disjoint. The distinction between fixed and variable costs is formally captured by the following additively separable functional form of the cost function. For all S,

$$\tilde{c}(S) = c(\{k \in Y : S \cap X_k \neq \emptyset\}) + \sum_{k \in Y} f_k(\#(S \cap X_k)),$$

where $c : 2^Y \to \mathbf{R}$ represents fixed costs, and the $f_k : \{0, ..., \#X_k\} \to \mathbf{R}$ represent the variable cost functions for goods of type k. A cost function that admits such a separable representation will be referred to as *decomposable.*

[1]strictly speaking: higher-order differences.

[2]Although the interpretation is different in these papers, the underlying mathematical structure is the same.

[3]Formally, there are thus no fixed costs. Whenever we nevertheless refer to "fixed costs" in the following, strictly speaking we mean "quasi-fixed costs."

Proposition 1 *A cost function* $\tilde{c} : 2^X \to \mathbf{R}$ *is decomposable if and only if it satisfies the following two conditions.*
(i) For all $k \in Y$*,* $x, y \in X_k$*, and* $z \notin X_k$*,* $\tilde{m}_z(S) = \tilde{m}_z(S \cup \{x\})$ *whenever* $S \ni y$*, and*
(ii) For all S, W*,* $[\#(S \cap X_k) = \#(W \cap X_k)$ *for all* $k] \Rightarrow \tilde{c}(S) = \tilde{c}(W)$*.*

Moreover, the functions f_k *and* c *are uniquely determined up to a normalization of* $f_k(1)$ *for all* $k \in Y$*.*

Condition (i) states that once a single unit of some type k is produced, increasing the output of this type entails no further reduction of the marginal cost of producing goods of any other type j. Condition (ii) states that costs only depend on the number of units produced of each type.

Proof For each $k \in Y$, fix $f_k(1)$ arbitrarily such that $0 \leq f_k(1) \leq \tilde{c}(\{x_k\})$, where $x_k \in X_k$. For any k choose $x_k \in X_k$, and define $c : 2^Y \to \mathbf{R}$ as follows. For all $Y' \subseteq Y$, $c(Y') := \tilde{c}(\{x_k : k \in Y'\}) - \sum_{k \in Y'} f_k(1)$. By condition (ii), this does not depend on the choice of the x_k. The functions f_k are inductively defined as follows. Choose any set S that contains exactly $i \geq 2$ elements of X_k, and let $x_k \in S \cap X_k$. Then, set $f_k(i) := f_k(i-1) + [\tilde{c}(S) - \tilde{c}(S \setminus \{x_k\})]$. By conditions (i) and (ii), this definition does not depend on the choice of S and x_k. It is easily verified that the functions c and f_k yield the desired decomposition. q.e.d.

The function c captures potential economies of scope, while the f_k capture potential economies of scale. This clear-cut dichotomy, which is due to condition (i), seems appropriate in certain contexts such as that of R&D expenditure; it may not be acceptable in others where the cost reductions within one type depend on the quantity produced of other types. Summarizing then, our analysis applies to standard multi-product cost functions provided that the scope effects can be separated from any scale effects in the sense of condition (i) above.

3 Cost Complementarities as Synergies

Cost complementarities describe the reduction of marginal costs due to the production of other goods. Such *synergies* between goods can be described by second-order derivatives of the cost function. We define the (first) derivative[4] of the set function c at S with respect to x by

$$\nabla_x c(S) := m_x(S) = c(S \cup \{x\}) - c(S).$$

Thus, $\nabla_x c(S)$ is the marginal cost of producing x given that S is already produced. Clearly, for each $x \in X$, $\nabla_x(\cdot)$ is again a real-valued function on 2^X. A crucial role in the following analysis will be played by its derivative with respect y which we also refer to as the *cross-partial derivative* of c with respect to $\{x, y\}$:

$$\begin{aligned} \nabla_{\{x,y\}} c(S) &:= \nabla_y(m_x(S)) = \nabla_y(\nabla_x c(S)) \\ &= [c(S \cup \{x, y\}) - c(S \cup \{y\})] - [c(S \cup \{x\}) - c(S)] \end{aligned}$$

A cost function is said to be characterized by cost complementarities, i.e. decreasing marginal costs, if the cross-partials are always non-positive. Equivalently, the *synergy*

[4] strictly speaking: first-order difference.

between any x and y,

$$\mathrm{syn}_{\{x,y\}}(S) := -\nabla_{\{x,y\}} c(S), \tag{1}$$

i.e. the reduction of the marginal cost of producing x due to the production of y, is always non-negative. It is easily verified that non-negativity of synergies is equivalent to *submodularity* of the cost function, i.e. to the condition that, for all $S, W \subseteq X$,

$$c(S \cup W) + c(S \cap W) \leq c(S) + c(W).$$

The object under study is thus the class of all submodular cost functions on 2^X.

In writing down equation (1) one notes at once that the synergy between x and y depends on the set S of goods already produced. This dependence is significant. Independence of S would require that

$$\nabla_{\{x,y\}} c(S) = \nabla_{\{x,y\}} c(S')$$

for all S, S' that do not contain x or y. Such independence holds if and only if all third derivatives are zero, in which case the cost function will be referred to as *quadratic*. Despite its attractiveness from a computational point of view, a quadratic model turns out to be inappropriate in most cases. Typically, synergies depend on S in a significant and economically meaningful way.

To illustrate the role of the third derivative, consider a simple example of a seller delivering to a finite number of stores in a linear city. Suppose that there are n equidistant stores, so that $X = \{1, ..., n\}$. A seller located at the edge of town (at 0) wants to serve these stores. For simplicity, assume that the cost incurred by supplying store $x \in X$ consists in the transportation cost of driving from the starting point 0 to store x plus some constant cost $a > 0$ per store for unloading. With transportation costs proportional to the distance, the cost of serving store x is simply $c(\{x\}) = a + x$.

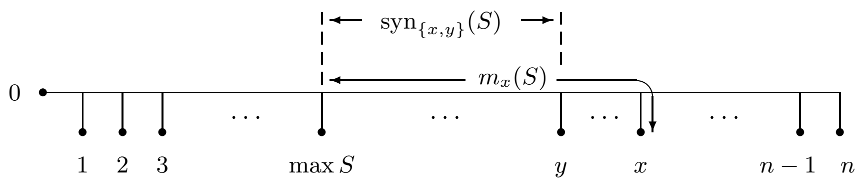

Figure 1: Serving stores in a linear city

Since in serving store x the seller has to drive beyond all stores $y < x$, the marginal cost of serving one of these is a once store x is served. The cost of supplying any set $S \subseteq X$ of stores is given by $c(S) = a \cdot (\#S) + \max S$, i.e. the costs of unloading plus the transportation cost of serving the farthest store in S. If $x \notin S$, marginal costs are thus $m_x(S) = a + [x - \max S]_+$, where $[z]_+$ is short for $\max\{z, 0\}$. Synergies are given by

$$\mathrm{syn}_{\{x,y\}}(S) = [\min\{x, y\} - \max S]_+,$$

(see Figure 1). In particular, the synergy between x and y is always non-negative, confirming the presence of cost complementarities. To illustrate, suppose that store

$z = \max S$ is already being served. The marginal cost $a + (x - z)$ of supplying any farther store $x > z$ is reduced by $y - z$ whenever an intermediate store y with $z < y < x$ is also served. The dependence of $\mathrm{syn}_{\{x,y\}}(S)$ on S is transparent: the reduction in the marginal cost $m_x(S)$ due to serving an intermediate store y with $\max S < y < x$ is smaller whenever $\max S$ is larger. In other words, $\mathrm{syn}_{\{x,y\}}(S)$ is decreasing in S: for all x, y and all S, S',

$$S \subseteq S' \Rightarrow \mathrm{syn}_{\{x,y\}}(S) \geq \mathrm{syn}_{\{x,y\}}(S'). \tag{2}$$

The case of decreasing synergies in the sense of (2) will be referred to as the case of *substitutive* synergies: synergies become weaker the more synergies are already being exploited. Since $\mathrm{syn}_{\{x,y\}}(S) = -\nabla_{\{x,y\}}c(S)$, (2) is equivalent to monotonicity of the cross-partials $\nabla_{\{x,y\}}c(\cdot)$, and hence to non-negativity of the third derivative of the cost function,[5]

$$\nabla_{\{x,y,z\}}c(\cdot) := \nabla_z[\nabla_{\{x,y\}}c(\cdot)].$$

This simple example demonstrates that the structure of synergies is in general complex. In particular, it shows that the qualitative behavior of synergies is closely related to the qualitative behavior of the third derivative of the cost function. A fundamental distinction concerns the *sign* of the third derivative. As outlined above, non-negativity of the third derivative everywhere corresponds to substitutive (decreasing) synergies. The polar case of *complementary* (increasing) synergies is also conceivable. However, we shall argue that these cases are not symmetric (see Section 6 below): substitutive synergies will turn out to be much more natural than complementary ones.

How can we understand the presence of substitutive synergies?

4 Substitutive Synergies due to Joint Public Inputs

The following example is meant to illustrate the notion of joint public inputs central to the further development of the theory. Consider the following very stylized description of the cost structure of producing BMWs (a well-known German car make).[6] First, to be able to produce any BMW x at all a certain amount F_{oh} of firm-wide overhead has to be incurred. Developing a specific product line, say the 5-series of BMW, requires large expenditures $F_{\mathrm{pl}(x)}$. Similarly, designing a particular model, such as the 525td, involves additional costs $F_{\mathrm{mo}(x)}$. Finally, the actual production of the individual car has unit costs $K_{\mathrm{mo}(x)}$. Thus, producing a single BMW x (think of $x \in X$ as one car of a particular model) has total cost $c(\{x\}) = F_{\mathrm{oh}} + F_{\mathrm{pl}(x)} + F_{\mathrm{mo}(x)} + K_{\mathrm{mo}(x)}$. For instance, suppose one 525td is being produced. Then, the marginal cost of producing a second 525td is $K_{525\mathrm{td}}$. By contrast, the marginal cost of, say a 528i, is $F_{528\mathrm{i}} + K_{528\mathrm{i}}$. More generally, the total cost of producing a set S of cars $c(S)$ is given by the sum of overhead costs, the (quasi-)fixed costs of any product category in S (line or model) plus the marginal costs of each individual car.

In this example, the presence of economies of scope is due to *joint public inputs*, i.e. inputs required by several goods (cars) that become freely available once used for one single good (car). For instance, the cost of producing a 525td and a 528i jointly

[5]Observe that the value of $\nabla_{\{x,y,z\}}c(S)$ is independent of the order of taking derivatives.

[6]While some readers may be less fond of BMWs than we are, they may nevertheless appreciate the pedagogical value of the example, especially of the crispness with which BMW's nomenclature conveys the qualitative structure of its product space.

is smaller than the sum of the cost of producing each of the two cars separately since both cars share common inputs, namely those required for developing the 5-series.

In general, the cost structure of production with joint public inputs can be described as follows. Let Ω be a set of public inputs with given prices p_ω, $\omega \in \Omega$. For any $\omega \in \Omega$ let $h(\omega) \subseteq X$ denote the set of those goods which require input ω. The total cost of producing the subset S of goods is thus given by

$$c(S) = \sum_{\omega : h(\omega) \cap S \neq \emptyset} p_\omega.$$

Note that each public input occurs only once in the sum on the right-hand side, since it becomes freely available for all outputs once it has been used by one. In the following, it will be convenient to identify inputs ω with their "extensions" $h(\omega) \subseteq X$, i.e. with the corresponding sets of goods that require these inputs. In particular, we will refer to a set A of goods as a "public input" whenever there is an $\omega \in \Omega$ such that $A = h(\omega)$, i.e. whenever there is some ω-input that is required exactly by all goods in A. Henceforth, a public input is thus simply a certain subset of goods, and the set of all public inputs is a collection of such subsets. The price of the "input" $A \subseteq X$ is then given by the aggregate cost of all ω-inputs required exactly by the goods in A:

$$\lambda_A := \sum_{\omega : h(\omega) = A} p_\omega,$$

with $\sum_\emptyset := 0$ by convention. Expressing costs in terms of the λ_A one thus obtains for all S,

$$c(S) = \sum_{A \subseteq X : A \cap S \neq \emptyset} \lambda_A. \tag{3}$$

We can view λ as a measure on 2^X and write $\lambda(\mathcal{A}) := \sum_{A \in \mathcal{A}} \lambda_A$ for any family $\mathcal{A} \subseteq 2^X$. Note that, in general, there will be many subsets A for which $\lambda_A = 0$, and observe that the family of all public inputs is given by the *support* $\Lambda := \{A \subseteq X : \lambda_A \neq 0\}$ of λ.

By (3), one obtains

$$m_x(S) = c(S \cup \{x\}) - c(S) = \lambda(\{A : x \in A \subseteq S^c\}) \tag{4}$$

and

$$\mathrm{syn}_{\{x,y\}}(S) = \lambda(\{A : \{x, y\} \subseteq A \subseteq S^c\}), \tag{5}$$

where S^c denotes the complement of S in X. The marginal cost of x at S is thus given by the aggregate cost of all inputs that are required by x but not already used by some good in S. Similarly, the synergy between x and y at S equals aggregate cost of all inputs common to x and y that are not required by any element of S. In particular, it is clear from (4) and (5) that any cost function of the form (3) is monotone and submodular, due to the non-negativity of λ. Moreover, it is also evident from the right-hand side of (5) that the extent of cost reductions is decreasing in S, i.e. that the synergies are substitutive. More generally, the higher-order derivatives of any cost function of the form (3) have alternating sign, beginning with a positive sign for the first derivative (see Theorem 2 below). Such cost functions will be called monotone and *totally submodular*. Equivalently, the class of totally submodular cost functions can be characterized by the property that the absolute value of any higher-order derivative is decreasing. Formally, for any $W = \{x_1, ..., x_m\} \subseteq X$, define the derivative of c with respect to W at S recursively by $\nabla_W c(S) := \nabla_{x_m}(\nabla_{W\setminus\{x_m\}} c(S))$. A monotone cost function is totally submodular if and only if, for all non-empty W, and all S, S',

$$S \subseteq S' \Rightarrow |\nabla_W c(S)| \geq |\nabla_W c(S')|. \tag{6}$$

Observe that submodularity corresponds to the case $\#W = 1$, and substitutivity of synergies to the case $\#W = 2$. We will refer to monotone and totally submodular cost functions as characterized by *regular* substitutive synergies.

5 Implicit Joint Inputs Obtained from Conjugate Moebius Inversion

Consider now a monotone and totally submodular cost function. The above procedure for aggregating product-group specific fixed costs can be inverted, as shown by the following result.

Theorem 1 (Conjugate Moebius Inversion) *For any set function $c : 2^X \to \mathbf{R}$ there exists a unique measure λ on 2^X, the so-called* conjugate Moebius inverse*, such that (3) holds, i.e. such that for all S,*

$$c(S) = \lambda(\{A \subseteq X : A \cap S \neq \emptyset\}) = \sum_{A\subseteq X: A\cap S\neq\emptyset} \lambda_A,$$

where $\lambda_A := \lambda(\{A\})$. The measure λ has the following representation. For all A,

$$\lambda_A = \sum_{S\subseteq A} c(S^c) \cdot (-1)^{\#(A\setminus S)+1}.$$

Moreover, λ is non-negative if and only if c is totally submodular.

The first part is a standard result in combinatorics and the theory of non-additive probabilities (see Rota [13], Chateauneuf and Jaffray [2]). The second part, also well-known, follows from Theorem 2 below.

The non-negativity of the conjugate Moebius inverse (henceforth: c.m.i.) allows us to interpret it as a *cost decomposition*. In the BMW example above the cost decomposition was exogeneously given, as total costs were computed based on presupposed fixed costs specific to certain groups of products. However, even if the public inputs are not part of the physical description of the technology, the decomposition (3) can still admit an economic interpretation in terms of *imputed* fixed costs, with the values λ_A representing the fixed costs of the public inputs "imputed" to exactly the goods in A. Theorem 1 thus entails that, up to conditions on derivatives of order ≥ 4 ("regularity"), *any* cost function characterized by substitutive synergies admits such a cost decomposition in terms of imputed fixed costs.

The use of (non-conjugate) Moebius inversion is standard in the literature on cost sharing since Shapley [14]. Occasionally, one also finds reference to its conjugate form as defined here. For instance, Young [15] refers to cost functions with the representation (3) as "decomposable." Both Young [15, p. 93] and Moulin [5, p. 140] note that in the totally submodular case, the Shapley value admits a particularly simple and intuitive representation. It amounts to assigning, for any implicit input, an equal cost share to each good that uses it: $Sh(x) = \sum_{A \ni x} \frac{\lambda_A}{\#A}$.

As a formal result, Theorem 1 can of course only secure the logical possibility of a joint public input interpretation. The mathematically identified inputs need not necessarily be economically meaningful.[7] Nonetheless, Theorem 1 establishes the remarkable generality of the joint public input *language* for talking about economies of scope *as if* originating from joint public inputs.

As a first illustration, consider again the example of serving stores in the linear city. Serving a store x requires the input "driving from $x-1$ to x." This input is in fact shared by all stores that are farther out than x. Hence, for each x, the set $\{x, ..., n\}$ is a public input with $\lambda_{\{x,...,n\}}$ as the transportation cost of driving from $x-1$ to x. In addition, each store x requires an idiosyncratic input $\{x\}$ with $\lambda_{\{x\}} = a$ representing the cost of unloading. The c.m.i. of the cost function $c(S) = a \cdot (\#S) + \max S$ is thus obtained by setting

$$\lambda_A = \begin{cases} 1 & \text{if } A = \{x, ..., n\} \text{ for } x < n, \\ a & \text{if } A = \{x\} \text{ for } x < n, \\ 1 + a & \text{if } A = \{n\}, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, the cost decompositon of the function $c(S) = a \cdot (\#S) + \max S$ in terms of joint public inputs is given by imputing one cost-unit to each set that consists of all stores that are farther away from the starting point than some given store x, and an unloading cost of a to each single store (see Figure 2). Note that the fixed cost $\lambda_{\{n\}}$ imputed to the farthest store subsumes both the transportation cost from $n-1$ to n and the unloading cost at n.

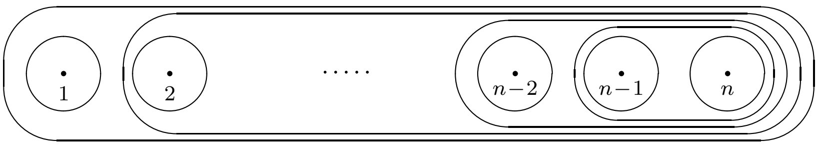

Figure 2: Extensionally identified public inputs in the linear city

The following result shows that the c.m.i. simultaneously describes the cost function and all its derivatives. Since derivatives play such a central role in the analysis of cost functions, it shows at a purely mathematical level the importance of the c.m.i.

Theorem 2 *Let $c : 2^X \to \mathbf{R}$ be a set function with c.m.i. λ. For all S and all non-empty $W \subseteq X$,*

$$\nabla_W c(S) = (-1)^{\#W+1} \cdot \lambda(\{A : W \subseteq A \subseteq S^c\}).$$

Proof The proof proceeds by induction over $\#W$. For $W = \{x\}$,

$$\begin{aligned} \nabla_x c(S) &= c(S \cup \{x\}) - c(S) \\ &= \lambda(\{A : A \ni x, A \cap S = \emptyset\}) \\ &= \lambda(\{A : x \in A \subseteq S^c\}). \end{aligned}$$

[7] See, however, Nehring and Puppe [10] where we provide a simple sufficient condition under which the economic meaningfulness of the imputed inputs is secured.

Next, let $\#W \geq 2$, and suppose the given formula applies to all derivatives of order $< \#W$. Then, for any $x \in W$,

$$\begin{aligned}
\nabla_W c(S) &= \nabla_x \left(\nabla_{W\setminus\{x\}} c(S)\right) \\
&= \nabla_{W\setminus\{x\}}[c(S \cup \{x\}) - c(S)] \\
&= (-1)^{\#W} [\lambda(\{A : (W \setminus \{x\}) \subseteq A \subseteq (S \cup \{x\})^c\}) \\
&\qquad\qquad -\lambda(\{A : (W \setminus \{x\}) \subseteq A \subseteq S^c\})] \\
&= (-1)^{\#W+1} \lambda(\{A : (W \setminus \{x\} \subseteq A \subseteq S^c, x \in A\}) \\
&= (-1)^{\#W+1} \lambda(\{A : W \subseteq A \subseteq S^c\}) .
\end{aligned}$$

q.e.d.

Taking $W = A$ and $S = A^c$ in Theorem 2 one obtains the following simple representation of the c.m.i.

Corollary *Let $c : 2^X \to \mathbf{R}$ be a set function with c.m.i. λ. For all A,*

$$\lambda_A = \nabla_A c(A^c) \cdot (-1)^{\#A+1}.$$

Observe that both results apply to arbitrary set functions. In particular, the equivalence of non-negativity of λ and total submodularity of c is an immediate consequence of Theorem 2.[8]

Theorem 2 can be used to demonstrate the restrictiveness of quadratic cost functions, thereby underlining the role of third and higher-order derivatives. Suppose that for all x, y and all S with $S \cap \{x, y\} = \emptyset$, the synergy $\mathrm{syn}_{\{x,y\}}(S)$ between x and y is strictly positive but does not depend on S. By Theorem 2, this implies that the support of the corresponding c.m.i. λ consists exactly of all one- and two-element subsets of X, i.e. $\Lambda = \{\{x, y\} : x, y \in X\}$. In particular, one obtains by (3), for all S,

$$c(S) \geq \frac{1}{2} \sum_{x \in S} c(\{x\}), \tag{7}$$

and more generally, $c(S \cup W) \geq \frac{1}{2} \sum_{x \in S} m_x(W)$. Note that (7) is easily violated when many goods in S share a common input, e.g. when there are significant overhead costs. Hence, modelling a cost function as quadratic entails a strong quantitative limitation on the extent of synergies.

Analogously, a vanishing $(k+1)$-th derivative means, by Theorem 2, that all public inputs are shared by at most k goods, which implies $c(S) \geq \frac{1}{k} \sum_{x \in S} c(\{x\})$ for all S.

6 The Privileged Status of Substitutive Synergies

As we have seen, the applicability of the joint public input interpretation, i.e. the interpretation of the c.m.i. as a cost decomposition, is limited to the case of substitutive (decreasing) synergies. This condition is not as restrictive as it may appear, since substitutivity of synergies is economically more natural than their complementarity. This

[8] Choquet [3, Sect. 14 and 26] introduced totally submodular set functions in terms of the alternating sign of the higher-order derivatives (thus calling them "alternating of infinite order"), and suggested that they occur more frequently and seem more useful than totally supermodular ones (belief functions). He did not state Theorem 2 nor its corollary; both seem to be new.

intuition is confirmed by the fact that complementary (i.e. increasing) synergies impose strong restrictions on the overall extent of synergies, as expressed by the following inequality. For all x,

$$\sum_{y\in X\setminus\{x\}} \mathrm{syn}_{\{x,y\}}(\emptyset) \leq c(\{x\}).$$[9]

In terms of total cost, complementary synergies entail the same strong restriction as the quadratic model (cf. (7)):

Theorem 3 *Let $c : 2^X \to \mathbf{R}$ be monotone and submodular. Furthermore, assume that, for all x, y, $syn_{\{x,y\}}(\cdot)$ is increasing, i.e. that the third derivative of c is non-positive everywhere. Then, for all S,*

$$c(S) \geq \frac{1}{2}\sum_{x\in S} c(\{x\}).$$

Proof Let $c : 2^X \to \mathbf{R}$ be monotone, submodular with non-positive third derivative. The restriction of c to any $S \subseteq X$ has these same properties; hence, for the proof it suffices to show that

$$c(X) \geq \frac{1}{2}\sum_{x\in X} c(\{x\}). \tag{8}$$

Define the average cost function $f : \{0, ..., \#X\} \to \mathbf{R}$ by $f(i) := \frac{1}{\#\mathcal{S}(i)}\sum_{S\in\mathcal{S}(i)} c(S)$, where $\mathcal{S}(i) := \{S \subseteq X : \#S = i\}$. Note that $f(0) = 0$, $n \cdot f(1) = \sum_{x\in X} c(\{x\})$ and $f(n) = c(X)$, where $n := \#X$. Consider the derivative ∇f defined by $\nabla f(i) := f(i + 1) - f(i)$. By assumption, $\nabla f : \{0, ..., n - 1\} \to \mathbf{R}$ is positive, decreasing and concave. By Jensen's inequality, one has for all $i = 1, ..., n - 1$,

$$f(i + 1) - f(i) = \nabla f(i) \geq \frac{n-1-i}{n-1} \cdot \nabla f(0).$$

Summing these inequalities, one obtains

$$f(n) = \sum_{i=0}^{n-1}[f(i + 1) - f(i)] \geq \left(\sum_{i=0}^{n-1} \frac{n-1-i}{n-1}\right) \cdot f(1) = \frac{n}{2} f(1),$$

i.e. (8). q.e.d.

Note that, by comparison, substitutive synergies entail no analogous restriction beyond monotonicity (i.e. $c(S) \geq \max_{x\in S} c(\{x\}) \geq \frac{1}{\#S}\sum_{x\in S} c(\{x\})$). The analogy to functions on the real line may be instructive. If $f : [0, \infty) \to \mathbf{R}$ is increasing and concave, then it is not possible that its third derivative is strictly negative everywhere.[10] By contrast, it is perfectly possible that its third derivative is strictly positive everywhere.

As we have already argued in the context of quadratic cost functions, the restrictions described above will be undesirable in many cases, for instance they are easily violated when overhead costs are significant. From this we conclude that the case of substitutive synergies is the by far more relevant case in applications.

[9] For verification, let $X = \{x, y_1, ..., y_m\}$, and observe that $\sum_{i=1}^m \mathrm{syn}_{\{x,y_i\}}(X \setminus \{x, y_1, ..., y_i\}) = m_x(\emptyset) - m_x(X \setminus \{x\}) \leq c(\{x\})$. The stated inequality thus follows from the assumption of increasing synergies. Note that, by contrast, in the substitutive case one can only deduce the inequality $\sum_{y\in X\setminus\{x\}} \mathrm{syn}_{\{x,y\}}(\emptyset) \leq (n - 1) \cdot c(\{x\})$.

[10] This is easily seen by considering the first derivative f'. By assumption, f' is non-negative and decreasing everywhere. Clearly, in this case f' must have a convex part.

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