Soc Choice Welfare (2002) 19: 415–430

Social Choice and Welfare

A simple procedure for finding equitable allocations of indivisible goods

Dorothea Herreiner, Clemens Puppe

Department of Economics, University of Bonn, Adenauerallee 24-42, 53113 Bonn, Germany (e-mail: clemens.puppe@econ3.uni-bonn.de)

Received: 3 March 2000/Accepted: 27 November 2000

Abstract. The paper investigates how far a particular procedure, called the "descending demand procedure," can take us in finding equitable allocations of indivisible goods. Both interpersonal and intrapersonal criteria of equitability are considered. It is shown that the procedure generally fares well on an interpersonal criterion of "balancedness"; specifically, the resulting allocations are Pareto-optimal and maximize the well-being of the worst-off individual. As a criterion of intrapersonal equitability, the property of envy-freeness is considered. To accommodate envy-freeness, a modification of the basic procedure is suggested. With two individuals, the modified procedure is shown to select the envy-free allocations that are balanced, i.e. the allocations that maximize the well-being of the worse-off individual among all envy-free allocations.

1 Introduction

The paper addresses the issue of how to find equitable allocations of indivisible goods among n individuals. In contrast to most of the literature on the subject (see e.g., Thomson 1997 for an overview), we do not consider the possibility of monetary compensation. Our motivation for studying a framework in which all goods are indivisible is twofold. First, we believe that there are important cases in which monetary compensation is either not possible, or usually not considered.[1] Secondly, one may view the allocation of the set of indivisible goods as a prior stage in a more comprehensive procedure of al-

Corresponding author: Clemens Puppe

We are grateful to Steven Brams and an anonymous referee for helpful comments.

[1] An example discussed in the literature is the assignment of administrative tasks, say to the members of an academic department. At least at German universities, extra monetary compensation for such tasks is not customary.

locating both indivisible and divisible goods. On such a view, the search for a "most equitable" allocation of the indivisible items can be motivated from the desire to minimize the amount of later compensation payments. In any case, a focus on the pure indivisible case allows one to work with minimal informational requirements on the preferences of the individuals involved. Specifically, the present inquiry uses no information beyond the ordinal preferences of individuals over the set of bundles of indivisible items to be allocated.

The procedures presented here are different variants of the following simple scheme which we refer to as the *descending demand procedure*. In the first round, individuals name in some prespecified ordering their most preferred bundle of items; in the second round, each individual names her second best bundle, and so on. The procedure stops when for the first time a feasible allocation can be combined from the bundles of items named up to that moment. In particular, each item goes to exactly one person, and each person receives one of the bundles she has already named. The purpose of this paper is to explore how far this method can take us in finding "equitable" allocations. In doing so, we focus on two different criteria, one criterion of *inter*personal equitability, and one criterion of *intra*personal equitability. Specifically, we show that the set of allocations that are obtained by the basic descending demand procedure consists of the Pareto-optimal allocations that maximize the "well-being" of the worst-off individual.[2] Thus, the procedure fares well in terms of an egalitarian criterion of interpersonal equitability. As a criterion of intrapersonal equitability, we consider the notion of *envy-freeness*. While the descending demand procedure may yield envy-free allocations under some circumstances, more often it fails to find envy-free allocations (even when such allocations exist).

Motivated by its failure to yield an envy-free allocation, we consider a modification of the descending demand procedure and show that its solution is always envy-free, *provided* that there exists a solution at all. In contrast to the two-person case, when there are more than two individuals, the modified procedure may, however, not provide a solution even when envy-free allocations exist. On the other hand, with two individuals the procedure is shown to fare well on *both* criteria. Specifically, we prove that the modified descending demand procedure finds the envy-free allocation(s) that are most equitable in the interpersonal sense, i.e. that maximize the well-being of the worse-off individual among all envy-free allocations. While such allocations may not be Pareto-optimal, we also discuss ways to find envy-free and Pareto-optimal allocations (whenever such allocations exist).

Variants of the descending demand procedure have been considered in Brams and Kilgour (1999), and Brams and Taylor (1996, 1999); however, there are also significant differences. Most importantly, the analysis of the so-called "fallback bargaining" procedure in Brams and Kilgour (1999) applies to social states rather than bundles of goods; in particular, their framework

[2] The terms "well-being" and "worst-off" are to be understood in an ordinal sense, as explained in Sect. 2.

does not allow to address the issue of envy-freeness. Moreover, in contrast to the descending demand procedure proposed here, fallback bargaining does not rely on a specific ordering of individuals. The "strict alternation" procedure of Brams and Taylor (1999) shares the sequential nature of the descending demand procedure; on the other hand, under strict alternation individuals are supposed to name *single* goods rather than bundles of goods. Most recently, the problem of fair division of indivisible items between two individuals has been addressed by Brams and Fishburn (2000). Their analysis is restricted to a certain class of preferences; specifically, these authors assume that individual preferences over sets of goods satisfy the axioms of qualitative probability, in particular an independence condition that rules out complementarities between goods. Moreover, they assume that the two individuals rank single items in the same way. None of these restrictions are imposed in the present paper. Edelman and Fishburn (2000) extend the analysis of Brams and Fishburn (1999) to more than two individuals; Brams et al. (2000a,b) suggest different criteria, and trade-offs among them, for evaluating the equitability of allocations.

Just as most procedures considered in the literature, the descending demand procedure described here is vulnerable to strategic manipulation. However, a successful manipulation in our context would require considerable knowledge of the other persons' preferences. The practical relevance of the theoretical possibilities for strategic manipulation thus seems limited. In any case, our analysis is not meant to contribute to the problem of *implementing* equitable allocations with sophisticated, strategically-thinking players, but rather to the combinatorial problem of how to *find* such allocations in the absence of strategic behaviour. We further comment on this issue in the conclusion.

The paper is organized as follows. Section 2 introduces the basic descending demand procedure. It is shown that its solutions are essentially the Pareto-optimal allocations that are "balanced" in the sense that they give the worst-off individual the best possible rank in her preference ordering. In Sect. 3, we consider an iterated procedure and show its solutions to correspond to an appropriate lexicographic refinement of the notion of balancedness. Section 4 is devoted to the criterion of envy-freeness. While balanced and Pareto-optimal allocations are not envy-free in general, we present a "modified" procedure that always yields envy-free solutions. However, when there are three or more individuals, the modified procedure may not yield a solution at all, even though envy-free allocations may exist. On the other hand, for two individuals the procedure yields a very appealing compromise between intra- and interpersonal equitability, as it always selects the balanced among all envy-free allocations. Section 5 concludes and discusses some directions for further research.

2 The descending demand procedure

Consider a set $N = \{1, \dots, n\}$ of individuals, indexed by $i \in N$, and a finite set $S = \{a, b, c, \dots\}$ of indivisible goods. Each individual i has a preference relation $\succeq_i$ over the family 2^S of all subsets of S. Throughout, we assume that each $\succeq_i$ is a *linear ordering*, i.e. a complete, transitive and antisymmetric relation. The preference ordering $\succeq_i$ is called *monotone* if it satisfies the following condition.

Monotonicity (MON). For all $A, B \in 2^S$,

$$A \supseteq B \Rightarrow A \succeq_i B.$$

Observe that, since indifferences between distinct subsets are ruled out, MON in particular implies $A \supset B \Rightarrow A \succ_i B$. In many contexts, monotonicity is a natural requirement as it reflects an assumption of all goods being desirable. Nevertheless, in some cases one may also wish to include "bads" such as undesirable tasks, for instance. With the exception of Proposition 4.1 below, none of our results hinge on monotonicity. However, for expository convenience monotonicity is assumed in the examples below.

An allocation $\mathbf{A} = (A_1, \dots, A_n)$ is a list that assigns a set $A_i \subseteq S$ of goods to each individual. Often, we will refer to a subset of S as a *bundle* of goods. An allocation $\mathbf{A} = (A_1, \dots, A_n)$ is *feasible* if

$$A_i \cap A_j = \emptyset \quad \text{for all } i \neq j \quad \text{and} \quad \bigcup_{i=1}^{n} A_i = S.$$[3]

The set of *Pareto-optimal* allocations is denoted by $\mathscr{P}$, i.e. $\mathbf{A} \in \mathscr{P}$ if $\mathbf{A}$ is feasible and there does not exist a feasible $\mathbf{A}'$ that is unanimously (weakly) preferred to $\mathbf{A}$ by all individuals with a strict preference for at least one individual. For $A \in 2^S$, we denote by $rk_i(A)$ the rank of A in the ordering $\succeq_i$; thus, $rk_i(A) = 1$ means that the set A is ranked top in the preference of individual i, $rk_i(A) = 2$ means that A is second-best, and so on. Given an allocation $\mathbf{A} = (A_1, \dots, A_n)$, denote by $r(\mathbf{A})$ the maximal rank over all individuals, i.e. $r(\mathbf{A}) := \max_i rk(A_i)$. An allocation $\mathbf{A}$ is called *balanced* if it is Pareto-optimal and there does not exist another Pareto-optimal allocation $\mathbf{A}'$ with $r(\mathbf{A}') < r(\mathbf{A})$. The set of balanced allocations is denoted by $\mathscr{BP}$. Intuitively, balancedness expresses an egalitarian criterion of interpersonal equitability, in the sense that balanced allocations minimize the maximal rank in the preference orderings across individuals. It is straightforward to verify that under our assumptions on individual preferences both $\mathscr{P}$ and $\mathscr{BP}$ are non-empty.

The basic descending demand procedure, henceforth DDP, works as follows.

The Descending Demand Procedure (DDP). A specific ordering of inviduals is determined; without loss of generality, let 1 be the first individual in the

[3] In some contexts, one may want to allow for the possibility of not distributing some of the goods, thus replacing the feasibility condition in the text by the weaker requirement that $A_i \cap A_j = \emptyset$ and $\bigcup_{i=1}^{n} A_i \subseteq S$. It is easily checked that our analysis remains valid with minor modifications under this weaker notion of feasibility. The only results that depend on the stronger notion of feasibility used in the text are Lemma 4.1 as well as Propositions 4.3 and 4.4 below.

ordering, 2 the second, and so on. In the first round, each individual i names her most preferred subset $A_i^1 \in 2^S$, where claims are made in the prespecified ordering. If $\mathbf{A}^1 = (A_1^1, \ldots, A_n^1)$ is feasible, the procedure stops and $\mathbf{A}^1$ is the solution.

If $\mathbf{A}^1$ is not feasible, the individuals go on naming their second best bundle A_i^2, again in the same ordering as before. The procedure stops at the moment when for the first time an individual j names, say in the k-th round, a set A_j^k such that there exists a feasible allocation $\mathbf{B} = (B_1, \ldots, B_n)$ with $B_j = A_j^k$, $B_i \in \{A_i^1, \ldots, A_i^k\}$ for $i < j$, and $B_i \in \{A_i^1, \ldots, A_i^{k-1}\}$ for $i > j$. Any allocation that is Pareto-optimal among the feasible allocations of this form is called a *solution* of the DDP.

Proposition 2.1. *Any solution of the DDP is an element of $\mathscr{BP}$.*

Proof. As above, let A_j^k be the subset of goods named by the j-th individual in round k of the DDP. Denote by $A_j^k\uparrow$ the set of all feasible allocations $\mathbf{B} = (B_1, \ldots, B_n)$ satisfying $B_j = A_j^k$, $B_i \in \{A_i^1, \ldots, A_i^k\}$ for $i < j$, and $B_i \in \{A_i^1, \ldots, A_i^{k-1}\}$ for $i > j$. Consider now a solution $\mathbf{A} = (A_1, \ldots, A_n)$ of the DDP for some ordering of individuals. Without loss of generality, we may rename individuals and take the ordering to be the standard ordering $1, 2, \ldots, n$. Let A_j be the subset at which the DDP stopped, and suppose that this happened in round k, so that $A_j = A_j^k$, i.e. $rk_j(A_j) = k$. First, we show that $A \in \mathscr{P}$. Suppose, by way of contradiction, there is a feasible allocation $\mathbf{B} = (B_1, \ldots, B_n)$ Pareto-superior to $\mathbf{A}$. By construction, $B_j = A_j$, since otherwise, i.e. if $B_j \succ_j A_j$, the DDP would have stopped earlier. But this implies $\mathbf{B} \in A_j^k\uparrow$; in particular, $\mathbf{A}$ is not Pareto-optimal in $A_j^k\uparrow$, a contradiction. The fact that $\mathbf{A}$ is balanced, i.e. an element of $\mathscr{BP}$, follows from noting that $r(\mathbf{A}) = k$. Hence, if there were a Pareto-optimal allocation $\mathbf{B}$ with $r(\mathbf{B}) < k$, the DDP would have stopped at the latest in round $k - 1$. q.e.d.

Given the result of Proposition 2.1, a natural question to ask is whether *any* element of $\mathscr{BP}$ can be obtained as the solution of the DDP for some ordering of the individuals. Perhaps somewhat surprisingly, the answer is positive for two individuals, but negative for three or more individuals. To see this, consider $N = \{1, 2\}$ and an allocation $\mathbf{A} = (A_1, A_2) \in \mathscr{BP}$. Suppose first that $rk_1(A_1) = rk_2(A_2)$; in this case, $\mathbf{A}$ is in fact the *unique* element of $\mathscr{BP}$, and it is easily verified that it is the solution of the DDP for both orderings of the two individuals. Hence, suppose that $rk_1(A_1)$ differs from $rk_2(A_2)$, say $rk_1(A_1) > rk_2(A_2)$. Then, $\mathbf{A}$ will be the solution of the DDP for the ordering in which individual 1 starts. Note that with two individuals, $\mathscr{BP}$ always consists of at most two elements. By the preceding observation, if there are two balanced allocations, each will be the solution of the DDP for one of the two orderings of the individuals.

With three or more individuals, there may exist elements of $\mathscr{BP}$ that are never solutions to the DDP. Essentially, the reason is that, for a given allocation $\mathbf{A} = (A_1, \ldots, A_n)$, there may be several individuals i such that $rk_i(A_i) = r(\mathbf{A})$. Specifically, consider the following example.

Example 1. Let $N = \{1,2,3\}$ and $S = \{a,b,c,d,e,f\}$. Suppose that all individuals prefer a greater number of goods to a smaller number of goods, so that each individual prefers each five-element subset of S to any four-element subset of S, and each four-element subset to all three-element subsets, and so on. In particular, this implies that individual preferences are monotone. The four top-ranked sets among the two-element subsets of S are given as follows.

1	2	3
$\vdots$	$\vdots$	$\vdots$
$\{a,b\}$	$\{a,c\}$	$\{a,c\}$
$\{a,e\}$	$\{a,d\}$	$\{a,f\}$
$\{c,d\}$	$\{a,b\}$	$\{b,f\}$
$\{a,d\}$	$\{c,d\}$	$\{e,f\}$
$\vdots$	$\vdots$	$\vdots$

It is easily verified that in this example, $\mathscr{BP}$ consists of the following three allocations: $\mathbf{A} = (\{a,b\},\{c,d\},\{e,f\})$, $\mathbf{B} = (\{a,e\},\{c,d\},\{b,f\})$ and $\mathbf{B}' = (\{c,d\},\{a,b\},\{e,f\})$. Denote by l the common maximal rank in these allocations, i.e. $l = r(\mathbf{A}) = r(\mathbf{B}) = r(\mathbf{B}')$. In allocation $\mathbf{A}$ both individuals 2 and 3 get the same (maximal) rank: $rk_2(A_2) = rk_3(A_3) = l$. By contrast, in $\mathbf{B}$ and $\mathbf{B}'$ the maximal rank is uniquely attained by individuals 2 and 3, respectively. This observation immediately implies that the DDP will never yield $\mathbf{A}$ as solution. Indeed, the solution will be $\mathbf{B}$ whenever individual 2 is ahead of individual 3 in the ordering, and the solution will be $\mathbf{B}'$ for all other orderings.

The example motivates the following refinement of $\mathscr{BP}$. For any feasible allocation $\mathbf{A} = (A_1,\ldots,A_n)$, denote by $J_{\mathbf{A}}$ the set of individuals j for which $rk_j(A_j) = r(\mathbf{A})$, i.e. the set of individuals who are assigned the maximal rank. Denote by $\mathscr{BP}^*$ the set of all balanced allocations $\mathbf{A}$ such that for no balanced allocation $\mathbf{A}'$, $J_{\mathbf{A}'} \subset J_{\mathbf{A}}$. Clearly, $\mathscr{BP}^* \subseteq \mathscr{BP}$, and $\mathscr{BP}^*$ is always non-empty; moreover, it has already been noted that $\mathscr{BP}^* = \mathscr{BP}$ whenever there are only two individuals.

Proposition 2.2. *Any element of $\mathscr{BP}^*$ can be obtained as a solution of the DDP for some ordering of the individuals.*

Proof. Let $\mathbf{A} \in \mathscr{BP}^*$ with $k = r(\mathbf{A})$; consider any ordering where the individuals in $J_{\mathbf{A}}$ are first to name their preferred bundles. Thus, suppose without loss of generality that $J_{\mathbf{A}} = \{1,\ldots,j\}$ with $j \leq n$. Clearly, the DDP cannot stop before round k, since by Proposition 2.1 this would imply the existence of a Pareto-optimal allocation $\mathbf{B}$ with $r(\mathbf{B}) < k$. In round k, the DDP cannot stop before individual j has named A_j^k, since otherwise there would have been a balanced allocation $\mathbf{B}$ with $J_{\mathbf{B}} \subset J_{\mathbf{A}}$. Hence, the DDP stops at A_j^k yielding $\mathbf{A}$ as one solution. q.e.d.

3 An iterated procedure

When there are many individuals and many goods, the DDP described so far has two inherent weaknesses: first, it does not provide an explicit method for determining whether there are feasible allocations in $A_j^k\uparrow$ once individual j has named A_j^k; secondly, given that there are such allocations, it does not provide an explicit method for finding Pareto-optimal allocations among these. We do not further address the first problem here.[4] However, we propose a solution for the second problem; specifically, in this section, we analyze the following *iterated* descending demand procedure.

The Iterated Descending Demand Procedure (IDDP). Let $n \geq 3$. A specific ordering of individuals is determined. The standard DDP is applied until for the first time an individual, say individual j, names in round k a set A_j^k such that there exist feasible allocations in $A_j^k\uparrow$. Individual j parts with the set A_j^k; the DDP is restarted with the remaining individuals $N\backslash\{j\}$ (in the same ordering as before) to determine an allocation of the remaining set of goods $S\backslash A_j^k$. In doing so, the individuals still name their preferred bundles from the *original* set S, i.e. they should ignore the fact that the items in A_j^k are no longer available. Feasibility of an allocation $(A_1, \ldots, A_{j-1}, A_{j+1}, \ldots, A_n)$ at this stage, however, requires $\bigcup_{i \neq j} A_i = S\backslash A_j^k$. The DDP is applied until for the first time an individual, say individual h, names a bundle that is part of a feasible allocation; individual h parts with that set, and the DDP is restarted with the remaining individuals. This is repeated until there are only two individuals left. Any solution of the DDP applied to the remaining two individuals, together with the bundles already taken by the individuals that parted, is called a solution of the IDDP.

It is easily verified that any solution of the IDDP is a balanced allocation, i.e. an element of $\mathscr{BP}$. What properties do the solutions of the IDDP have in addition, and which balanced allocations can be obtained as solutions of the IDDP? In addressing these questions, we focus for simplicity on the first round of iteration only; the extension to the general case of $n-2$ iterations then follows in a straightforward way.

For an allocation $\mathbf{A} = (A_1, \ldots, A_n)$, denote by $r^-(\mathbf{A})$ the second highest rank for some individual, i.e. $r^-(\mathbf{A}) := \max_i\{rk_i(A_i) : rk_i(A_i) < r(\mathbf{A})\}$.[5] Denote by $\mathscr{BP}^-$ the set of balanced allocations $\mathbf{A}$ such that either: (i) $\#J_{\mathbf{A}} \geq 2$, or (ii) $\#J_{\mathbf{A}} = 1$ and, for no balanced allocation $\mathbf{A}'$, $[J_{\mathbf{A}'} = J_{\mathbf{A}}$ and $r^-(\mathbf{A}') < r^-(\mathbf{A})]$. The set $\mathscr{BP}^-$ may thus be viewed as an appropriate *lexicographic* refinement of $\mathscr{BP}$. To illustrate, consider allocations represented by the rank that the bundles of goods have in the individual preference orderings; thus,

[4] Note that with only two persons, determining the set of feasible allocations in $A_j^k\uparrow$ is trivial, no matter how many goods there are. With more than two individuals, the complexity of the problem of determining the feasible allocations in $A_j^k\uparrow$ crucially depends on the number of goods. In particular, the problem becomes serious only when there are many individuals *and* a large number of goods.

[5] If all individuals enjoy the same rank in $\mathbf{A}$, we set $r^-(\mathbf{A})$ equal to zero.

e.g. the vector $(4, 4, 1)$ represents an allocation that gives both the first and second individual a bundle ranked at the fourth place in their preference ordering, while individual 3 gets the top ranked subset. By definition, if a balanced allocation $\mathbf{A}$ gives rise to the rankings $(4, 4, 1)$, the allocation is an element of $\mathscr{BP}^-$ since $\#J_{\mathbf{A}} = 2$. On the other hand, suppose that two balanced allocations $\mathbf{B}$ and $\mathbf{B}'$ give rise to the rankings $(4, 1, 3)$ and $(4, 2, 2)$, respectively; then, $\mathbf{B}'$ is an element of $\mathscr{BP}^-$, but $\mathbf{B}$ is not.

Proposition 3.1. *Suppose the DDP is iterated once, i.e. after the first individual has parted, the standard DDP is applied once to the remaining individuals. Then, any solution of the procedure is an element of* $\mathscr{BP}^-$.

Proof. Clearly, any solution is balanced. To verify the statement, we only have to show that, if $\mathbf{A}$ and $\mathbf{A}'$ are balanced allocations such that $J_{\mathbf{A}} = J_{\mathbf{A}'} = \{j\}$ for some individual j, and $r^-(\mathbf{A}') < r^-(\mathbf{A})$, then $\mathbf{A}$ cannot be a solution for any ordering of individuals. Thus, suppose in the first application of the DDP some individual parted with a bundle of goods. If this individual is different from j, the solution will be different from $\mathbf{A}$. Hence, suppose the parting individual was j. Then, the DDP applied to the remaining individuals will stop at the latest in round $r^-(\mathbf{A}')$, hence $\mathbf{A}$ is again not the solution. q.e.d.

To answer the converse question, of which balanced allocations can be obtained as solutions to the DDP iterated once, consider the following lexicographic refinement of $\mathscr{BP}^*$. For an allocation $\mathbf{A} = (A_1, \ldots, A_n)$, let $J_{\mathbf{A}}^-$ denote the (possibly empty) set of individuals that are assigned the second highest rank, i.e. $J_{\mathbf{A}}^- := \{i : rk_i(A_i) = r^-(\mathbf{A})\}$. Denote by $(\mathscr{BP}^*)^-$ the set of balanced allocations $\mathbf{A}$ such that for no balanced allocation $\mathbf{A}'$, $[J_{\mathbf{A}'} \subset J_{\mathbf{A}}$ or $(J_{\mathbf{A}'} = J_{\mathbf{A}}$ and $J_{\mathbf{A}'}^- \subset J_{\mathbf{A}}^-)]$.

Proposition 3.2. *Any allocation in* $(\mathscr{BP}^*)^-$ *can be obtained as a solution of the DDP iterated once for some ordering of individuals.*

Proof. Let $\mathbf{A} \in (\mathscr{BP}^*)^-$, and consider an ordering in which the individuals in $J_{\mathbf{A}}$ are first to name their preferred bundles, followed by the individuals in $J_{\mathbf{A}}^-$. Without loss of generality, let $J_{\mathbf{A}} = \{1, \ldots, j\}$ and $J_{\mathbf{A}}^- = \{j+1, \ldots, h\}$. First, consider the case $j > 1$, i.e. $\#J_{\mathbf{A}} \geq 2$. In this case, j is the individual to part with A_j^k after the first application of the DDP, where $k = r(\mathbf{A})$. By the argument in the proof of Proposition 2.2, the first iteration of the DDP (among the individuals $N \backslash \{j\}$) then stops again in round k when individual $j-1$ has named A_{j-1}^k. Hence, $\mathbf{A}$ is among the solutions.

Consider now the case $j = 1$. In this case, individual 1 parts in round k of the first application of DDP. Again by the argument of Proposition 2.2, the first iteration of the DDP among the remaining individuals stops when individual h has named $A_h^{k'}$ where $k' = r^-(\mathbf{A})$. Again, $\mathbf{A}$ must therefore be among the solutions of the DDP iterated once. q.e.d.

From the above analysis of the first round of iteration of the DDP it is clear that the general IDDP yields as solutions allocations in an appropriate further lexicographic refinement of $\mathscr{BP}$. Conversely, any allocation in an appropriate further lexicographic refinement of $\mathscr{BP}^*$ can be obtained as a solution to the IDDP. For the sake of notational simplicity, we do not describe the details here.

4 Envy-freeness

In this section, we turn to the question of how the DDP fares in terms of intrapersonal equitability. Specifically, we consider the criterion of envy-freeness. Formally, say that an allocation $\mathbf{A} = (A_1, \ldots, A_n)$ is *envy-free* if it is feasible and, for all i, $A_i \succeq_i A_j$ for all $j = 1, \ldots, n$. Denote by $\mathscr{E}$ the set of envy-free allocations. Furthermore, say that an allocation is *fair* if it is both Pareto-optimal and envy-free, and denote the set of fair allocations by $\mathscr{F}$. Note that both $\mathscr{E}$ and $\mathscr{F}$ may be empty.[6] For instance, when preferences are monotone, a necessary condition for $\mathscr{F} \neq \emptyset$ is that there are at least as many goods as there are individuals.

Consider now the descending demand procedure. First observe that by Pareto-optimality of the solution, a necessary condition for the DDP to yield an envy-free allocation is that $\mathscr{F}$ is non-empty. Are the allocations in $\mathscr{F}$ among the solutions of the DDP in that case? As we shall see, this is not generally the case. In fact, a positive answer is obtained only under very special circumstances. One simple example is when there are just as many goods as individuals, i.e. when $\#S = n$, and when each individual has monotone preferences. In this case, the only possibility for $\mathscr{F}$ to be non-empty is when each individual gets exactly one good which must be the good most preferred by her among all single goods. In particular, this implies that for each single good there has to be exactly one individual who prefers that good to all other single goods; clearly, this considerably restricts the set of admissible preference profiles. Nevertheless, assume that preferences are such that the allocation $\mathbf{A} = (\{a_1\}, \{a_2\}, \ldots, \{a_n\}) \in \mathscr{F}$, where $S = \{a_1, \ldots, a_n\}$. This allocation is then balanced; in fact, it is necessarily the *unique* balanced allocation. To see this, suppose that $\mathbf{B} = (B_1, \ldots, B_n)$ is some other Pareto-optimal allocation; in particular, this implies that $B_i \succ_i \{a_i\}$ for some individual i. Since $\{a_i\} \succ_i \{a_j\}$ for all $j \neq i$, the only possibility to make i better off is by giving her at least two goods, i.e. $\#B_i \geq 2$. But this implies that $B_j = \emptyset$ for some individual j, and therefore $r(\mathbf{B}) > r(\mathbf{A})$. Hence, since $\mathbf{A}$ is the unique balanced allocation, it is also the unique solution of the DDP by Proposition 2.1.

When there are two individuals, the argument just given can be somewhat generalized; specifically, one has the following result.

Proposition 4.1. *Suppose that there are two individuals, both having monotone preferences; furthermore, assume that $\#S \leq 3$. Then, the set of solutions of the DDP coincides with $\mathscr{F}$ whenever the latter set is non-empty.*

Proof. We have already argued above that a necessary condition for $\mathscr{F} \neq \emptyset$

[6] If one allows for the possibility of not distributing goods, the set $\mathscr{E}$ is always nonempty; indeed, the allocation in which nobody gets any good is clearly envy-free.

is $\#S \geq 2$. Furthermore, we have shown the statement to be valid if $\#S = 2$. Hence, it remains to consider the case $S = \{a_1, a_2, a_3\}$. Suppose that $\mathscr{F}$ is non-empty. By monotonicity, each individual must receive at least one good. Without loss of generality, by suitably renaming goods and individuals, suppose that $\mathbf{A} = (\{a_1, a_2\}, \{a_3\}) \in \mathscr{F}$. Using monotonicity of preferences and envy-freeness of $\mathbf{A}$, one easily verifies that $rk_1(\{a_1, a_2\}) \leq 4$ and $rk_2(\{a_3\}) = 4$. This implies that $\mathbf{A}$ is the unique balanced allocation. Indeed, any feasible allocation that would result in a lower rank for individual 2 would involve a rank ≥ 5 for individual 1. Given that $\mathbf{A}$ is the unique balanced allocation, it follows from Proposition 2.1 that it is also the unique solution of the DDP. q.e.d.

The conclusion of Proposition 4.1 fails when there are more than three goods, as shown by the following example.

Example 2. Let $S = \{a, b, c, d\}$ and suppose that both individuals prefer a greater number of goods to a smaller number of goods, so that their preferences are in particular monotone. The four top-ranked sets among all two-element subsets of S are given as follows.

1	2
⋮	⋮
$\{c, d\}$	$\{b, d\}$
$\{a, b\}$	$\{c, d\}$
$\{a, c\}$	$\{a, c\}$
$\{b, d\}$	$\{a, b\}$
⋮	⋮

In this example, the allocation $(\{a, c\}, \{b, d\})$ is the unique fair allocation. However, it is not balanced since the Pareto-optimal allocation $(\{a, b\}, \{c, d\})$ gives a lower rank to the worse-off individual. In fact, $(\{a, b\}, \{c, d\})$ is the unique balanced allocation, and thus the unique solution to the DDP. However, individual 1 envies individual 2 at that allocation.

The failure of the DDP to yield envy-free allocations when such allocations exist is not very surprising, since, by its very construction, the DDP is primarily concerned with interpersonal equitability while the concept of envy-freeness refers to intrapersonal equitability. Nevertheless, we now show that the DDP can be modified so that any solution is envy-free, *provided* the procedure gives a solution at all.

The Modified Descending Demand Procedure (MDDP). An ordering of individuals is determined; without loss of generality, let 1 be the first individual in the ordering, 2 the second, and so on. In the first round, each individual i names her most preferred bundle A_i^1, where claims are made in the prespecified ordering. If this results in a feasible allocation, that allocation is the solution, and the procedure stops. If not, some bundles named in the first round may have to be "marked." Specifically, at a moment when individual i claims a bundle A_i^1 that was already named by another individual $j < i$, both bundles, A_j^1 and A_i^1, are *marked*. The procedure goes to the second round in which individuals name their second best bundle.

In general, suppose that the procedure has come to round k and individual j has named A_j^k, where $rk_j(A_j^k) = k$. At that moment, each individual $i \leq j$ has named the list $(A_i^1, \ldots, A_i^k)$ of her most preferred bundles; similarly, each individual $i > j$ has named $(A_i^1, \ldots, A_i^{k-1})$. There are now two cases to consider.

Case 1. Suppose that A_j^k already occurred in the list of some other individual. In this case, all occurrences of this bundle are *marked* (including the occurrence in j's list). Denote by $(A_j^k\uparrow)^{ef}$ the set of feasible allocations $\mathbf{A} = (A_1, \ldots, A_n)$ in $A_j^k\uparrow$ such that, for all $i \neq j$, (i) A_i is unmarked, and (ii) $A_i \succ_i A_j^k$. If $(A_j^k\uparrow)^{ef}$ is non-empty, any allocation in $(A_j^k\uparrow)^{ef}$ that is Pareto-optimal *in that set* is a solution, and the procedure stops. If $(A_j^k\uparrow)^{ef}$ is empty, the procedure continues with the next individual naming her next preferred bundle.

Case 2. Suppose that no other individual has named A_j^k so far. If there exists a feasible allocation in $A_j^k\uparrow$ that consists of unmarked bundles only, any allocation that is Pareto-optimal among these is a solution, and the procedure stops. If not, the procedure continues with the next individual naming her next preferred bundle.

Proposition 4.2. *Any solution of the MDDP is envy-free.*

Proof. By construction, any solution of the MDDP is feasible. We show that any solution is envy-free. Suppose, by way of contradiction, that $\mathbf{A} = (A_1, \ldots, A_n)$ is a solution that is not envy-free, say $A_j \succ_i A_i$ for some individuals i and j. Then, the bundle A_j is contained in both individual i's and individual j's list at the moment when the MDDP stopped. However, in that case both occurrences of A_j are marked, and by the construction of the MDDP, $\mathbf{A}$ cannot be a solution. q.e.d.

Despite Proposition 4.2, the MDDP is not particularly attractive when there are more than two individuals. Indeed, in that case, the MDDP may not yield a solution at all even though $\mathscr{E}$ is non-empty. To illustrate this, consider the following example.

Example 3. Suppose $N = \{1, 2, 3\}$ and $S = \{a, b, c, d, e, f\}$. All individuals prefer a greater number of goods to a smaller number of goods. The three top-ranked two-element subsets are given as follows.

1	2	3
$\vdots$	$\vdots$	$\vdots$
$\{a, c\}$	$\{a, b\}$	$\{c, d\}$
$\{a, d\}$	$\{c, d\}$	$\{a, b\}$
$\{e, f\}$	$\{e, f\}$	$\{e, f\}$
$\vdots$	$\vdots$	$\vdots$

If the rankings between the other two-element subsets are completed in an appropriate way, and if, for instance, the rankings between all m-element subsets are identical for all $m \neq 2$, then $(\{e, f\}, \{a, b\}, \{c, d\})$ may be the unique envy-free allocation. However, the MDDP will not yield this allocation as a solution. The reason is that once individual 1 names $\{e, f\}$, the bundle $\{a, b\}$ in individual 2's list is alrady marked since it has also been named by individual 3 *before* the procedure reached $\{e, f\}$ in individual 1's list.

The example shows that the MDDP is of limited applicability when there are three or more individuals. However, for the case of two individuals the MDDP has very appealing features; specifically, it selects the balanced and Pareto-optimal allocations among the allocations in $\mathscr{E}$, whenever the latter set is non-empty. For the remainder of this section, we will focus on the case of two individuals.

Denote by $\mathscr{BE}$ the set of balanced envy-free allocations, i.e. $\mathbf{A} \in \mathscr{BE}$ if and only if $\mathbf{A}$ is envy-free and for no envy-free allocation $\mathbf{A}'$, $r(\mathbf{A}') < r(\mathbf{A})$. Observe that $\mathscr{BE}$ is non-empty whenever $\mathscr{E}$ is non-empty. With two individuals, any allocation in $\mathscr{BE}$ is automatically Pareto-optimal in $\mathscr{E}$; for future reference, we record this simple fact in the following lemma.

Lemma 4.1. *Suppose $N = \{1, 2\}$. Then, any allocation in $\mathscr{BE}$ is Pareto-optimal among all envy-free allocations.*

Proof. Let $\mathbf{A} \in \mathscr{BE}$. By feasibility, $\mathbf{A}$ is of the form $\mathbf{A} = (A, A^c)$ for some $A \in 2^S$, where $A^c := S \backslash A$ denotes the complement of A in S. We will show that $\mathbf{A}$ is Pareto-optimal in $\mathscr{E}$. Suppose, by way of contradiction, there exists an envy-free allocation $\mathbf{B} = (B, B^c)$ that is Pareto-superior to $\mathbf{A}$. Then necessarily $B \neq A$, hence by our assumptions on individual preferences, $B \succ_1 A$ and $B^c \succ_2 A^c$. In particular, this implies $r(\mathbf{B}) < r(\mathbf{A})$, contradicting the assumed balancedness of $\mathbf{A}$ in $\mathscr{E}$. q.e.d.

The following is the main result of this section.

Proposition 4.3. *Suppose $N = \{1, 2\}$. Then the MDDP yields a solution if and only if $\mathscr{E}$ is non-empty. In that case, any solution of the MDDP is in $\mathscr{BE}$. Conversely, any allocation in $\mathscr{BE}$ is a solution of the MDDP for one of the two orderings of individuals.*

Proof. By Proposition 4.2, any solution of the MDDP is envy-free. Suppose that $\mathscr{E} \neq \emptyset$, and let $\mathbf{A} = (A_1, A_2)$ be an element of $\mathscr{BE}$. We distinguish the following three cases.

Case 1. $rk_1(A_1) = rk_2(A_2) = r(\mathbf{A})$, say $r(\mathbf{A}) = k$. We will show that for both orderings of the individuals, the MDDP will yield $\mathbf{A}$ as the unique solution. First, observe that the MDDP cannot have stopped before both individuals 1 and 2 have named A_1 and A_2, respectively. Indeed, suppose, by way of contradiction, it had stopped before, yielding $\mathbf{A}'$ as solution. By Proposition 4.2, $\mathbf{A}'$ is envy-free; moreover, $\mathbf{A}'$ would be Pareto-superior to $\mathbf{A}$ which is not possible by Lemma 4.1. Thus it remains to show that for both orderings of individuals the MDDP stops at $\mathbf{A}$. Since $\mathbf{A}$ is envy-free, both bundles A_1 and A_2 are unmarked in round k; hence by Pareto-optimality of $\mathbf{A}$ in $\mathscr{E}$, $\mathbf{A}$ must be the solution of the MDDP.

Case 2. $rk_1(A_1) > rk_2(A_2)$, say $rk_1(A_1) = r(\mathbf{A}) = k$. Consider the ordering in which individual 1 is first to name her most preferred bundles. By the same argument as in the first case, the MDDP cannot have stopped before individual 1 named A_1 in round k. In round k, A_2 cannot be marked since, by envy-freeness, $A_1 \succ_1 A_2$; hence by Pareto-optimality of $\mathbf{A}$ among the elements of $\mathscr{E}$, $\mathbf{A}$ is the solution. Next, consider the ordering in which individual 2 is the first to name her most preferred bundles. Again, the MDDP cannot stop before round k, since this would imply the existence of an envy-free allocation $\mathbf{A}'$ with $r(\mathbf{A}') < r(\mathbf{A})$. Thus, suppose in round k individual 2 names A_2^k (note that $A_2^k \neq A_2$). If the procedure stops at this moment, the resulting solution is by construction of the MDDP an allocation in $\mathscr{BE}$. If the procedure does not stop after individual 2 has named A_2^k, it will stop when individual 1 names $A_1^k = A_1$ yielding $\mathbf{A}$ as solution, since A_2 is still unmarked then.

Case 3. $rk_1(A_1) < rk_2(A_2)$; this case is completely symmetric to Case 2.

We have thus shown that the MDDP yields an allocation in $\mathscr{BE}$, whenever $\mathscr{E}$ (and hence $\mathscr{BE}$) is non-empty. Since there are at most two such allocations, the argument also entails that any of them can be obtained as the solution for one of the orderings of the two individuals. q.e.d.

By Proposition 4.3, the MDDP yields a very appealing compromise between the intrapersonal criterion of envy-freeness and the interpersonal criterion of balancedness. Unfortunately, although Pareto-optimal among the allocations in $\mathscr{E}$, allocations in $\mathscr{BE}$ may still fail to be *globally* Pareto-optimal. By consequence, the MDDP as described so far cannot be expected to yield a fair allocation even when such allocations exist. To see this, consider the following example.

Example 4. Suppose that $S = \{a, b, c, d\}$; again suppose that both individuals always prefer having a greater number of goods. Among the two-element subsets, preferences are given as follows.

1	2
$\vdots$	$\vdots$
$\{b, d\}$	$\{a, d\}$
$\{a, d\}$	$\{a, b\}$
$\{b, c\}$	$\{c, d\}$
$\{c, d\}$	$\{b, c\}$
$\{a, c\}$	$\{a, c\}$
$\{a, b\}$	$\{b, d\}$
$\vdots$	$\vdots$

The unique fair allocation is $\mathbf{A} = (\{b, d\}, \{a, c\})$. However, the unique allocation in $\mathscr{BE}$ is $\mathbf{B} = (\{c, d\}, \{a, b\})$ which is Pareto-dominated by $(\{b, c\}, \{a, d\})$. By Proposition 4.3, the MDDP selects $\mathbf{B}$.

There is a simple sufficient condition under which the solutions of the MDDP applied to two individuals will be fair, whenever $\mathscr{F} \neq \emptyset$. Specifically, consider the following condition on individual preferences.

Limited Complementarity (LC). For all $A, B \in 2^S$,

$$A \succeq B \Leftrightarrow B^c \succeq A^c.$$

As our terminology suggests, condition LC limits the extent of complementarity between goods. To illustrate, consider the following three goods: an antique secretaire (a), the chair (b) belonging to it, and a chinese vase (c). The following preferences do not seem implausible: $\{a, b\} \succ \{a, c\} \succ \{b, c\} \succ \{c\} \succ \{a\} \succ \{b\}$. Given the set $S = \{a, b, c\}$, these preferences violate LC since $\{a, b\} \succ \{a, c\}$, but at the same time, $\{c\} \succ \{b\}$. Intuitively, LC thus rules out the complementarity between a and b. On the other hand, it is easily verified that LC does not necessarily rule out the stated preferences in a *larger* domain of goods, hence the name *limited* complementarity. The following result relies on Brams and Fishburn (2000, Theorem 4.4).

Proposition 4.4. *Suppose that $N = \{1, 2\}$ and that the preferences of both individuals satisfy LC. Then the MDDP yields a solution if and only if $\mathscr{E}$ is nonempty. Moreover, in that case any solution of the MDDP is fair.*

Proof. By Proposition 4.3, any solution of the MDDP is an element of $\mathscr{BE}$ (whenever $\mathscr{E} \neq \emptyset$). By Lemma 4.1, any solution is thus Pareto-optimal among the allocations in $\mathscr{E}$. By Theorem 4.4 in Brams and Fishburn (2000), condition LC implies that any allocation that is Pareto-optimal in $\mathscr{E}$ is in fact globally Pareto-optimal, hence fair. q.e.d.

While Proposition 4.4 provides a simple condition under which a solution of the modified descending demand procedure will be fair, there is also a more general strategy to find such allocations without additional assumptions on individual preferences. Indeed, one may simply apply the MDDP without stopping rule, until both individuals have arrived at their least preferred subset. In doing so, one separately records at any round the envy-free allocations (if such exist). In this manner, *any* envy-free allocation will be recorded. In particular, any fair allocation will be recorded.

5 Conclusion

In this paper we have investigated a particular method, the descending demand procedure, for finding equitable allocations of indivisible goods among n individuals. We have found that the procedure fares well on both interpersonal and intrapersonal criteria of equitability when there are two individuals. In terms of interpersonal equitability, the procedure is also appropriate for more than two people, but it is less attractive in terms of intrapersonal equitability in that case. The search for a better algorithm for determining equitable allocations with three or more individuals seems to be a worthwhile subject for future work. An important fact that one will have to consider in this context is that different criteria for "equitability" can easily be in conflict with each other. Indeed, even the two basic criteria of balanced Pareto-optimality and envy-freeness considered in this paper can be mutually incompatible. To illustrate, consider the following example.

Example 5. Suppose that $N = \{1, 2\}$ and $S = \{a, b, c, d\}$; assume that both individuals always prefer having a greater number of goods. Among the two-element subsets, preferences are given as follows.

1	2
$\vdots$	$\vdots$
$\{a, d\}$	$\{a, d\}$
$\{b, c\}$	$\{a, b\}$
$\{c, d\}$	$\{c, d\}$
$\{a, c\}$	$\{b, c\}$
$\{a, b\}$	$\{a, c\}$
$\{b, d\}$	$\{b, d\}$
$\vdots$	$\vdots$

The unique envy-free allocation is $\mathbf{A} = (\{c, d\}, \{a, b\})$. However, $\mathbf{A}$ is Pareto-dominated by $\mathbf{B} = (\{b, c\}, \{a, d\})$ which is the unique allocation in $\mathscr{BP}$. On the other hand, $\mathbf{B}$ is not envy-free, since individual 1 would rather have individual 2's bundle. By Proposition 2.1, the unmodified DDP selects $\mathbf{B}$, whereas by Proposition 4.3, the MDDP selects the envy-free but Pareto-inferior allocation $\mathbf{A}$.

In such examples, it is not obvious which criterion should have priority. With many individuals the conflict between different appealing criteria for equitability is likely to become even sharper; compromises of one or the other sort thus seem to be unavoidable.

Another open problem concerns strategic manipulability. Theoretically, the procedures presented here are vulnerable to strategic manipulation as are most of the procedures suggested in the literature (cf. Brams and Taylor 1996, 1999, or Brams and Fishburn 2000). In the purely indivisible case, very little work has been done to identify conditions under which strategy-proof mechanisms exist that implement equitable (in particular: envy-free) allocations. This, we believe, is an important area for future research.

References

Brams SJ, Edelman PH, Fishburn PC (2000a) Fair division of indivisible items. mimeo, New York University

Brams SJ, Edelman PH, Fishburn PC (2000b) Paradoxes of fair division. mimeo, New York University

Brams SJ, Fishburn PC (2000) Fair division of indivisible items between two people with identical preferences: Envy-freeness, pareto-optimality, and equity. Soc Choice Welfare 17: 247–267

Brams SJ, Kilgour M (1999) Fallback bargaining. Group Decision and Negotiation (forthcoming)

Brams SJ, Taylor AD (1996) Fair division. From cake-cutting to dispute resolution. Cambridge University Press, Cambridge UK

Brams SJ, Taylor AD (1999) The win-win solution: Guaranteeing fair shares to everybody. W.W. Norton, New York

Edelman PH, Fishburn PC (2000) Fair division of indivisible items among people with similar preferences. Math Soc Sci (forthcoming)

Thomson W (1997) Fair allocation rules. Manuscript, University of Rochester