The Structure of Strategy-Proof Social Choice

Part II: Non-Dictatorship, Anonymity and Neutrality*

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Abstract *Domains of generalized single-peaked preferences are classified in terms of the extent to which they enable well-behaved strategy-proof social choice. Generalizing the Gibbard-Satterthwaite Theorem, we characterize the domains that admit non-dictatorial strategy-proof social choice functions. We also provide characterizations of the domains that enable locally non-dictatorial, anonymous, and neutral strategy-proof social choice rules, respectively. Our findings imply that all domains that enable possibility results share a fundamentally similar geometry.*

JEL Classification D71, C72

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1 Introduction

In view of the celebrated Gibbard-Satterthwaite Impossibility Theorem, non-degenerate social choice functions can be strategy-proof only on restricted domains, that is: only when some a priori information on the possible preferences over social states is available. The classic example in a voting context is the Hotelling-Downs model in which social states are ordered as in a line, representing, for instance, policy choices that can be described in terms of a left-to-right scale. If preferences are single-peaked, then the selection of the Condorcet winner defines a strategy-proof social choice function with additional attractive properties such as anonymity and neutrality. The entire range of strategy-proof social choice functions on the domain of single-peaked preferences on the line was characterized in a path-breaking paper by Moulin (1980) which inspired a large literature obtaining possibility or impossibility results for particular domains. However, in spite of the considerable amount of attention that has been devoted to the topic over more than two decades, the demarcation between possibility and impossibility is still not well understood. The goal of this paper is to describe this boundary precisely for a large and flexible (though far from universal) class of preference domains that we shall refer to as "generalized single-peaked."[1] Within this class, we will classify domains according to the kinds of strategy-proof social choice functions they admit, using the fundamental properties of non-dicatorship, local non-dictatorship, anonymity and neutrality as classification criteria. It turns out, for example, that the scope of anonymous strategy-proof social choice extends far beyond the known examples in the literature. Moreover, it is shown that in our context anonymity is equivalent to the absence of local dictators. Likewise, under non-dictatorship, enabling neutrality is equivalent to enabling neutrality and anonymity together; thus, neutrality turns out to be substantially more demanding than anonymity.

Establishing the desired classification for arbitrary domains would clearly be an extremely difficult task for at least two reasons. First, domains as sets of preference orderings can be very heterogeneous, and their relevant structure may be hard to describe. Second, the structure of the social choice functions admitted may differ across domains, and it may not be possible to describe them in a unified way. For example, while in the great majority of cases, strategy-proof social choice functions depend only on the voters' most preferred alternatives, this does not hold for all domains. We thus concentrate our analysis on a large class of domains whose structure together with the structure of the possible strategy-proof social choice functions can be characterized in a unified, tractable way.

The basic idea underlying our approach is to describe the space of alternatives ("social states") geometrically in terms of a three-place *betweenness* relation, and to take the associated domain of preferences to consist of a sufficiently rich set of orderings that are single-peaked in the sense that individuals always prefer social states that are between a given state and their most preferred state, the "peak."

Following Nehring (1999), we shall conceptualize betweenness more specifically in terms of the differential possession of *relevant properties*: a social state y is between the social states x and z if y shares all relevant properties common to x and z. Single-peakedness means that a state y is preferred to a state z whenever y is between z and

[1] Throughout, we will assume that the a priori information about each individual is the same so that the domain of the social choice function is the n-fold copy of a fixed individual preference domain, where n is the number of voters.

the peak x^*, i.e. whenever y shares all properties with the peak x^* that z shares with it (and possibly others as well). Throughout, it will be assumed that a property is relevant if and only if its negation is relevant; a pair consisting of a property and its negation is referred to as an *issue*. As further illustrated below and discussed in detail in the companion paper Nehring and Puppe (2004), henceforth NP, a great variety of domains of preferences that arise naturally in applications can be described as single-peaked domains with respect to such betweenness relations. Note, for example, that the unrestricted domain envisaged by the Gibbard-Satterthwaite Theorem can be viewed as the set of all "single-peaked" preferences with respect to a vacuous betweenness relation that declares no social state between any two other states.

On generalized single-peaked domains, strategy-proof social choice can be described in a unified manner as "voting by committees" (following Barberá, Sonnenschein and Zhou (1991), Barberá, Massò and Neme (1997), and others, see NP). This structure has two aspects. First, the social choice depends on individuals' preferences through their most preferred alternative only. Second, the social choice is determined by a separate "vote" on each property: an individual is construed as voting for a property over its negation if and only if her top-ranked alternative has the property. For example, in the special case in which voting by committees is anonymous and neutral it takes the form of "issue-by-issue majority voting;" that is, a chosen state has a particular property if and only if the majority of agents' peaks have that property.

Crucially, in order to guarantee that the properties chosen by each committee are always jointly realizable for any profile of voters' preferences, the committees (i.e. voting rules) associated with each property must be consistent with each other. A main result in NP characterizes such consistency in terms of a simple condition called the "Intersection Property." Impossibility results obtain when consistency can be achieved only in degenerate ways, such as by giving the same agent full control over each property, leading to a dictatorial social choice function. The first main result of the present paper, Theorem 1, derives a combinatorial condition called "total blockedness" that is both necessary and sufficient for a generalized single-peaked domain to admit only dictatorial strategy-proof social choice functions. The unrestricted domain as well as many other single-peaked domains are totally blocked; examples are provided in Sections 2 and 4 below.

While this result ensures that if a space is not totally blocked non-dictatorial social choice functions exist, those choice functions may still be "almost dictatorial" by giving almost all decision power to a single agent. Thus, the negation of total blockedness cannot be viewed as securing genuine possibility results. The second main result of the paper, Theorem 2, therefore characterizes those domains that admit *anonymous* strategy-proof social choice functions ("voting by quota"), ensuring that all agents have equal influence on the chosen outcome. It turns out that within the class of generalized single-peaked domains, those that admit anonymous social choice rules are exactly those that admit *locally* non-dictatorial rules.

As illustrated by an example in Appendix B, the characterizing condition for the existence of some anonymous rule is necessarily complex. A simple sufficient and almost necessary condition is the existence of a *median point*. Following Nehring (2004), a median point is a point such that, given any other two points, there is an element between any pair of the three, their "median." Spaces that admit at least one median point are referred to as *quasi-median spaces*. Graphical examples will follow shortly. Anonymous rules exist in exceptional cases also outside quasi-median spaces, but they require an odd number of voters in such cases. Accordingly, the third main result, Theorem 3, shows that a domain admits anonymous strategy-proof social choice functions for *any* number of voters if and only if the underlying space is a quasi-median space.

While anonymous rules treat agents symmetrically, they typically treat social alternatives asymmetrically, for instance by applying different quotas to different issues. We therefore finally ask under what circumstances strategy-proofness is compatible with different notions of *neutrality*, i.e. symmetric treatment of social states. Our final main result, Theorem 4, shows first that a generalized single-peaked domain admits non-dictatorial strategy-proof social choice functions that are neutral *across* issues if and only if the underlying space is a quasi-median space. Strikingly, in terms of the kinds of strategy-proof social choice functions admitted, neutrality across issues thus requires the same underlying structure as anonymity (for any number of voters). Furthermore, Theorem 4 also shows that the existence of a rule that is neutral *within* issues and locally non-dictatorial is as demanding as the existence of a fully neutral and (globally) non-dictatorial rule, and that either condition requires *every* point to be a median point. Spaces in which all points are median points, i.e. in which every triple of points admits a fourth element in between any two of them, are called *median spaces* and are well-known in mathematics (see, e.g., van de Vel (1993)). The important role of median spaces in the context of strategy-proof social choice is analyzed in greater detail in NP. In particular, we show there that the structure of median spaces drives most of the possibility results in the literature.[2]

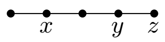
Possibility results in a strong sense thus require a median space; weaker possibility results still presuppose the existence of median points. Thus, while the range of domains with possibility results is expanded substantially beyond what is known, no radically different possibilities emerge. On the other hand, Theorem 1 provides the means of generating impossibility results for many new domains. On the whole, then, our results confirm the drift of the previous literature that possibility results require fairly parsimonious and highly structured preference domains.

The remainder of this paper is organized as follows. The following Section 2 offers a brief overview of the scope of our analysis. In Section 3, we provide the necessary background from NP. In particular, we introduce the notion of generalized single-peakedness on a property space and review the characterization of strategy-proof social choice on the associated preference domains. In Section 4, we generalize the Gibbard-Satterthwaite Theorem by characterizing the class of all single-peaked domains that only admit dictatorial strategy-proof social choice functions. Roughly, the characterizing condition ("total blockedness") says that there are too many families of mutually incompatible properties. We also investigate the existence of local dictators implied by an appropriate condition of local blockedness, and we discuss the relation of our analysis to the recent results of Aswal, Chatterji and Sen (2003). Section 5 and 6 provide the characterizations of the domains that admit strategy-proof social choice functions satisfying anonymity and neutrality, respectively. Section 7 concludes, and all proofs are collected in an appendix.

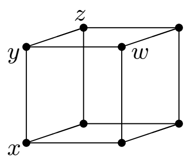
[2] See, among others, Moulin (1980), Demange (1982), Border and Jordan (1983), Barberá, Sonnenschein and Zhou (1991) and Barberá, Gul and Stacchetti (1993).

2 Overview and Related Literature

All of the following graphs describe economically meaningful preference domains that will reappear later in the paper.[3] Each graph corresponds to a different set of social states represented by its nodes. The relevant betweenness relation is the natural one: a state/node is between two other states/nodes if it lies on some shortest path connecting them.[4] Endowed with this notion of betweenness, the three graphs in the top row are all median spaces. Indeed, in Fig. 1a the betweenness relation is the standard one with the middle point as the median of any triple. In Fig. 1b and 1c, for instance, y is the median of x, z and w.



1a



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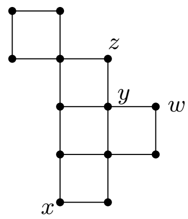
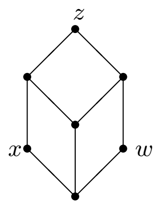
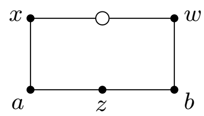


1c

Strong possibility on median spaces

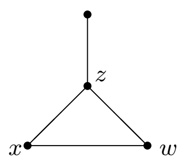


1d



1e

Partial possibility on quasi-median spaces

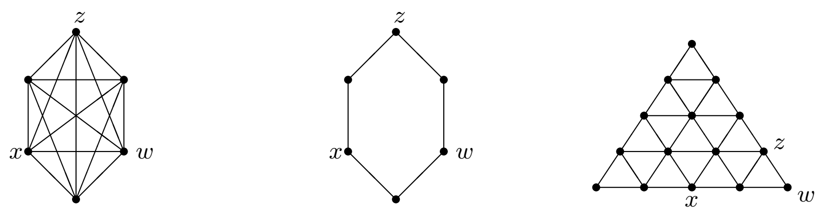


1f

Almost impossibility

[3]The graph in Fig. 1c describes an instance of a class of location problems analyzed in greater detail in Nehring and Puppe (2003).

[4]A shortest path is one with a minimal number of edges; note that such paths are, in general, not unique.



1g *1h* *1i*

Impossibility on totally blocked spaces

Figure 1: Examples of single-peaked domains based on graphs

Thus, by Theorem 4 below and the analysis in NP, the single-peaked domains associated with the three graphs in the top row of Fig. 1 give rise to possibility results in the strongest possible sense.

By constrast, none of the remaining graphs in Fig. 1 is a median space: in each case the indicated triple x, z, w does not admit a median.[5] In fact, as we will see, the three graphs in Fig. 1g, 1h and 1i give rise to strong impossibility results in the sense that the associated single-peaked domains only admit dictatorial strategy-proof social choice functions. For the single-peaked domain associated with the graph in Fig. 1g this follows from the Gibbard-Satterthwaite Theorem: since every point is connected with any other point by an edge, no point is between two other points; but in this case *any* preference is (vacuously) single-peaked, i.e. the associated domain of all single-peaked preferences is the unrestricted domain.

Examples of intermediate cases are given in Fig. 1d, 1e and 1f. For instance, as a non-median space the graph in Fig. 1d does not admit issue-by-issue majority voting; nonetheless, it does admit "qualified majority voting on properties." In this figure, the relevant properties are the three 4-cycles and their complements. For example, the rule according to which the social choice belongs to any of the 4-cycles if and only if at least one third of the voters' peaks are in that 4-cycle is consistent and strategy-proof. By constrast, while Fig. 1f does admit non-dictatorial strategy-proof social choice functions, none of these is anonymous. Fig. 1e, on the other hand, admits anonymous social choice functions; all of these are fairly degenerate, however, in that at least one property *must* be chosen unanimously. Accordingly, 1d and 1e admit median points, while 1f does not. Indeed, in Fig. 1d the median points are exactly the four non-labeled points (all points except x, z and w); similarly, the median points in Fig. 1e are the two points different from x, z and w. By contrast, in Fig. 1f there are no median points, since for any given alternative one can find two other alternatives such that the resulting triple has no median.

While our results show that the possibility of non-degenerate strategy-proof social choice is best understood geometrically in terms of the existence and structure of median points, a coarser look in terms of the nature of admitted cycles is also instructive, especially when the betweenness relation can be described by a graph. We show in Section 4 that graphs admitting locally non-dictatorial or, equivalently, anonymous stratgy-proof social choice functions cannot have odd cycles of any length, nor even

[5] The interpretation of the blank circle in Fig. 1e is that the shortest path connecting x and w comprises two edges; at the same time, no social state is (strictly) between x and w.

convex cycles of any length greater than four; a cycle is *convex* if no shortest path between any two points leaves the edges of the cycle.[6] Thus, all non-dictatorial domains share a fundamentally similar geometry.

The paper closest to ours is Aswal, Chatterji and Sen (2003). These authors make no structural assumption on the nature of the domain, and provide a sufficient condition for dictatorship. Adapted to our framework, their condition requires the existence of cycles of length three (and somewhat more, see Section 4.3 below). They also provide a sufficient condition for non-dictatorship, which however is very strong. In Fig. 1, their results allow to classify the domains 1a and 1f as non-dictatorial, and 1g and 1i as dictatorial. The only other paper in the literature we know of that considers domains with a variable geometry is Schummer and Vohra (2002), who embed the underlying spaces as closed sets in a finite-dimensional Euclidean space. Their domains are, however, not strictly comparable, since their underlying spaces are infinite and since their preference domains are defined somewhat differently. They find that the existence of *any* cycle precludes anonymity. This is consistent with our results, since all their cycles contain an infinite number of points.

3 Strategy-Proof Social Choice on Generalized Single-Peaked Domains

In this section, we briefly summarize the basic concepts and results from NP needed for the later analysis.

Property space A *property space* is a pair $(X, \mathcal{H})$, where X is a finite universe of social states or social alternatives, and $\mathcal{H}$ is a collection of subsets of X satisfying

H1 $H \in \mathcal{H} \Rightarrow H \neq \emptyset$,
H2 $H \in \mathcal{H} \Rightarrow H^c \in \mathcal{H}$,
H3 for all $x \neq y$ there exists $H \in \mathcal{H}$ such that $x \in H$ and $y \notin H$,

where, for any $S \subseteq X$, $S^c := X \setminus S$ denotes the complement of S in X. The elements $H \in \mathcal{H}$ are referred to as the *basic properties* (with the understanding that a property is extensionally identified with the subset of all social states possessing that property). A pair (H, H^c) is referred to as an *issue*.

Betweenness A property space $(X, \mathcal{H})$ induces a ternary *betweenness relation* $T \subseteq X^3$ according to

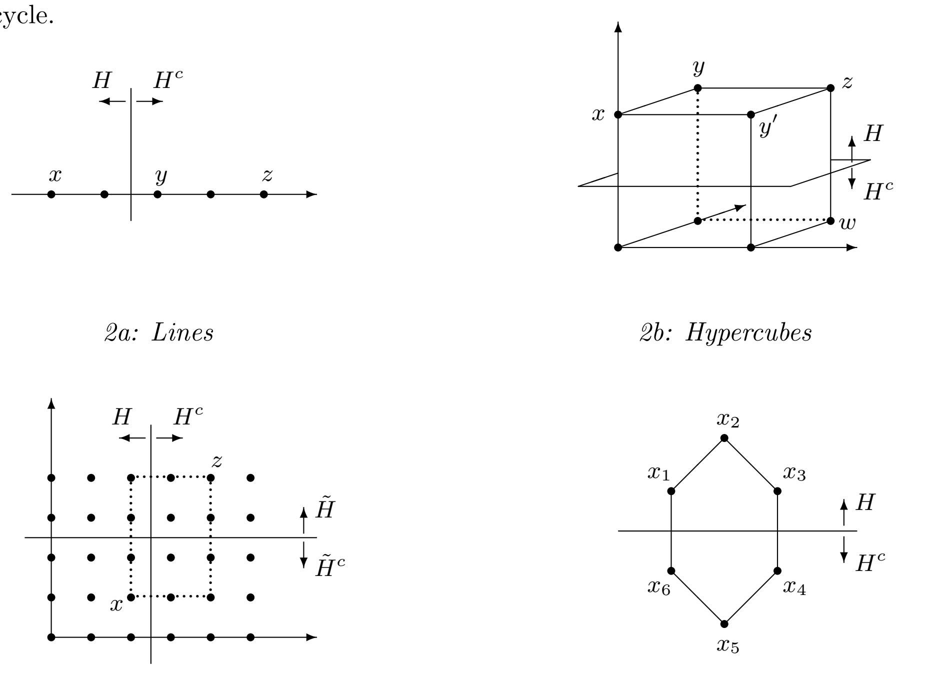
$$(x, y, z) \in T :\Leftrightarrow [\text{ for all } H \in \mathcal{H} : \{x, z\} \subseteq H \Rightarrow y \in H] \tag{3.1}$$

(cf. Nehring (1999)). Thus, $(x, y, z) \in T$ means that y shares all basic properties that are common to x and z, in which case we say that y is *between* x and z.

Figure 2 below shows some examples of property spaces. In the case of a line, the basic properties are of the form "lying to the left (resp. to the right)" of some given element (see Fig. 2a). The K-dimensional hypercube (cf. Fig.1b) is the set $\{0,1\}^K$ of all binary sequences of length K. The basic properties are, for all $k = 1, ..., K$, the sets H_0^k (resp. H_1^k) of all elements that have a zero (resp. a one) in coordinate k; the

[6] For instance, the 6-cycle in Fig. 1h is convex; in Fig. 1i, by contrast, all convex cycles have length three; finally, in Fig. 1g *any* three points form a convex cycle. In median spaces, all convex cycles must have length four.

3-hypercube is depicted in Fig. 2b. An element y is between x and z if it agrees with x and z in all coordinates in which these two elements agree; for instance, in Fig. 2b both y and y' are between x and z while the entire cube is between x and w. In a product $X = \prod X^k$, the basic properties are of the form $H^j \times \prod_{k \neq j} X^k$, where H^j is a basic property in coordinate j. Thus, e.g., in the product of two lines the basic properties are of the form $H^1 \times X^2$ and $X^1 \times H^2$, respectively. An element y is between x and z if it is contained in the rectangle spanned by x and z, i.e. if it is coordinatewise between x and z (see Fig. 2c). Finally, consider the graph in Fig. 2d, the 6-cycle already discussed above (cf. Fig. 1h). If one takes the family of basic properties to consist of all sets of three consecutive elements in the cycle (all "half-cycles"), the betweenness induced via (3.1) coincides with the *graphic betweenness* according to which a point is between two other points if and only if it lies on a shortest path connecting them. For instance, in Fig. 2d each state x_j is between the states x_{j-1} and x_{j+1}, and all states are between opposite pairs, such as x_1 and x_4. The graphic betweenness relation on a l-cycle for arbitrary $l \geq 3$ is obtained from a property space via (3.1) as follows. If l is even, the basic properties are all sets of $l/2$ consecutive elements of the cycle; if l is odd, the basic properties are all sets of $(l-1)/2$ and those of $(l+1)/2$ consecutive elements of the cycle.



2a: Lines *2b: Hypercubes*

2c: Products *2d: Cycles*

Figure 2: Examples of property spaces

Any property space $(X, \mathcal{H})$ canonically induces a graph as follows. Say that two distinct elements x and y are *neighbours* if no other element is between them, i.e. if $(x, w, y) \in T \Rightarrow [\, w = x \text{ or } w = y \,]$. The graph γ on X that connects each pair of neighbours by an edge will be referred to as the *underlying graph* of $(X, \mathcal{H})$. A property space $(X, \mathcal{H})$ is called **graphic** if the induced betweenness relation T according to (3.1) coincides with the graphic betweenness induced by γ, i.e. if

$$(x, y, z) \in T \Leftrightarrow \ y \text{ is on some shortest } \gamma\text{-path connecting } x \text{ and } z$$

Many property spaces that arise naturally in applications are graphic, and evidently, graphic spaces are particularly simple and useful for the purpose of illustration. But there is also a large class of non-graphic property spaces. The property spaces underlying the examples shown in Figures 1 and 2 are all graphic with the exception of the space corresponding to Fig. 1e.[7] In Appendix A, we describe the precise relation between property spaces and graphs in more detail. In particular, we provide a simple necessary and sufficient condition for a graphic betweenness to be derivable from a property space via (3.1), and discuss when the betweenness derived from a given property space can be representd by a graph.

Generalized Single-Peakedness A linear preference ordering $\succ$ on X is called *single-peaked* on $(X, \mathcal{H})$ if there exists $x^* \in X$ (the "peak") such that for all $y \neq z$,

$$(x^*, y, z) \in T \Rightarrow y \succ z.$$

A preference ordering is thus single-peaked whenever states between other states and the peak are preferred. Single-peaked preferences in this sense have been studied, among others, by Black (1958) and Moulin (1980) in the case of a line, by Barberá, Sonnenschein and Zhou (1991) in the hypercube (under the name of "separable preferences"), and by Barberá, Gul and Stacchetti (1993) in the product of lines (under the name of "multidimensionally single-peaked preferences").

The unrestricted preference domain is obtained as a generalized single-peaked domain by considering the collection of all basic properties of the form $H = \{x\}$ ("being equal to x") and their complements $H^c = X \setminus \{x\}$ ("being different from x"). The corresponding betweenness relation according to (3.1) is vacuous in the sense that no element x is between two other elements y and z (since y and z share the basic property "being different from x," a property not shared by x). By consequence, *any* linear preference ordering is single-peaked with respect to this betweenness relation.

Given a property space $(X, \mathcal{H})$, we denote by $\mathcal{S}_{(X,\mathcal{H})}$ the set of all single-peaked preferences, and by $\mathcal{S} \subseteq \mathcal{S}_{(X,\mathcal{H})}$ any subset of such preferences that is *rich* in the sense of the following two conditions. Note that both conditions are satisfied by the set $\mathcal{S}_{(X,\mathcal{H})}$ itself.

R1 For all neighbours x, y there exists a preference ordering in $\mathcal{S}$ that has x as peak and y as the second best element.

R2 For all x, y, z such that y is not between x and z there exists a preference ordering in $\mathcal{S}$ with peak x that ranks z above y.

Social Choice Function Let $N = \{1, ..., n\}$ be a set of voters. A *social choice function* on a single-peaked domain is a mapping $F : \mathcal{S}^n \to X$ that assigns to each preference profile $(\succ_1, ..., \succ_n) \in \mathcal{S}^n$ a unique social alternative $F(\succ_1, ..., \succ_n) \in X$.

The function F satisfies *voter sovereignty* if F is onto, i.e. if any $x \in X$ is in the range of F. Furthermore, a social choice function F is *strategy-proof* on $\mathcal{S}$ if for all i and $\succ_i, \succ_i' \in \mathcal{S}$,

$$F(\succ_1, ..., \succ_i, ..., \succ_n) \succeq_i F(\succ_1, ..., \succ_i', ..., \succ_n).$$

[7] The property space underlying Fig. 1e is given by the three properties $\{a, x\}$, $\{b, w\}$, $\{a, b, z\}$ and their respective complements. As is easily verified, the associated neighbourhood graph γ is the 5-cycle with the edges (x, a), (a, z), (z, b), (b, w), and (w, x); in particular, note that x and w are neighbours. However, the betweenness induced via (3.1) does not coincide with the graphic betweenness induced by γ since, for instance, both z and b are between a and w in the sense of T, but they are not on a shortest γ-path.

Voting by Committees as Voting by Properties A *committee* is a non-empty family $\mathcal{W}$ of subsets of N satisfying $[W \in \mathcal{W}$ and $W' \supseteq W] \Rightarrow W' \in \mathcal{W}$. The elements of $\mathcal{W}$ are called the *winning coalitions.* A *committee structure* on $(X, \mathcal{H})$ is a mapping $\mathcal{W} : H \mapsto \mathcal{W}_H$ that assigns a committee to each basic property $H \in \mathcal{H}$ satisfying $W \in \mathcal{W}_H \Leftrightarrow W^c \notin \mathcal{W}_{H^c}$; the latter condition is easily seen to imply

$$\mathcal{W}_H = \{W \subseteq N : W \cap W' \neq \emptyset \text{ for all } W' \in \mathcal{W}_{H^c}\}. \tag{3.2}$$

Voting by committees is the mapping $f_{\mathcal{W}} : X^n \to 2^X$ defined as follows. For all $\xi \in X^n$,

$$x \in f_{\mathcal{W}}(\xi) :\Leftrightarrow \text{ for all } H \in \mathcal{H} \text{ with } x \in H : \{i : \xi_i \in H\} \in \mathcal{W}_H.$$

Thus, x is the outcome of voting by committees if and only if, for any property H possessed by x, the coalition of those individuals whose peak have property H is winning for H. The induced mapping $F_{\mathcal{W}}(\succ_1, ..., \succ_n) := f_{\mathcal{W}}(x_1^*, ..., x_n^*)$, where x_i^* is the peak of $\succ_i$, is also referred to as *voting by committees.*

Consistency A committee structure $\mathcal{W}$ is called *consistent* if $f_{\mathcal{W}}(\xi) \neq \emptyset$ for all $\xi \in X^n$.

If voting by committees is consistent it is single-valued due to condition H3, and we will identify (with slight abuse of notation) $f_{\mathcal{W}}$ and $F_{\mathcal{W}}$ with the corresponding functions to X in that case.

The following two results are proved in NP. The first is an adaptation and generalization of a central result in Barberá, Massò and Neme (1997).

Theorem A *A social choice function $F : \mathcal{S}^n \to X$ satisfies voter sovereignty and is strategy-proof on a rich single-peaked domain $\mathcal{S}$ if and only if it is voting by committees with a consistent committee structure.*

Critical Family Say that a family $\mathcal{G} \subseteq \mathcal{H}$ of basic properties is a *critical family* in $(X, \mathcal{H})$ if $\cap \mathcal{G} = \emptyset$ and for all $G \in \mathcal{G}$, $\cap(\mathcal{G} \setminus \{G\}) \neq \emptyset$.

A critical family $\mathcal{G} = \{G_1, ..., G_l\}$ thus describes the exclusion of the combination of the corresponding basic properties in the sense that $G_1, ..., G_l$ cannot be jointly realized. "Criticality" (i.e. minimality) means that this exclusion is not implied by a more general exclusion in the sense that the basic properties in any proper subset of $\mathcal{G}$ are jointly realizable. Observe that all pairs $\{H, H^c\}$ of complementary basic properties are critical; they are referred to as the *trivial* critical families.

Intersection Property A committee structure satisfies the *Intersection Property* if for any critical family $\mathcal{G} = \{G_1, ..., G_l\}$, and any selection $W_j \in \mathcal{W}_{G_j}$, $\cap_{j=1}^l W_j \neq \emptyset$.

Theorem B *A social choice function $F : \mathcal{S}^n \to X$ satisfies voter sovereignty and is strategy-proof on a rich single-peaked domain $\mathcal{S}$ if and only if it is voting by committees satisfying the Intersection Property.*

In Theorems A and B, the notions of a rich single-peaked domain of preferences, of a consistent committee structure as well as the Intersection Property characterizing the latter are all understood relative to a given property space $(X, \mathcal{H})$. It is possible that a given preference domain $\mathcal{S}$ is "rich single-peaked" relative to more than one property space $(X, \mathcal{H})$. However, since Theorems A and B apply to any such property space, this multiplicity presents no special problems; the particular property space used for the analysis can be chosen for convenience.[8]

[8]The multiplicity of property spaces is tightly constrained, however: in NP, we show that all such property spaces must induce the same betweenness relation.

Taking a property space as a primitive is in line with much of the literature establishing possibility results which also describes preferences assuming a given (linear, multi-dimensional, etc.) structure of alternatives. Alternatively, one could take an unstructured preference domain as given, and ask whether its set of linear orders constitutes a rich single-peaked domain relative to an appropriate property space. A representation theorem to this purpose is provided in the companion paper NP, which also describes a constructive procedure of obtaining a suitable property space if it exists.

A social choice function F is *anonymous* if it is invariant with respect to permutations of voters' preferences. Voting by committees is anonymous if it takes the form of *voting by quota*: for all H, there exists $q_H \in [0,1]$ such that $\mathcal{W}_H = \{W : \#W > q_H \cdot n\}$ if $q_H < 1$ and $\mathcal{W}_H = \{N\}$ if $q_H = 1$. Note that the quotas q_H are not uniquely determined. Also observe that the quotas can be chosen such that $q_{H^c} = 1 - q_H$. In the anonymous case of voting by quota the Intersection Property simplifies to a system of linear inequalities, as follows. If, for any critical family $\mathcal{G}$,

$$\sum_{H \in \mathcal{G}} q_H \geq \#\mathcal{G} - 1, \tag{3.3}$$

then voting by quotas q_H for $H \in \mathcal{H}$ is consistent. Conversely, if anonymous voting by committees is consistent, then it can be represented by quotas satisfying (3.3).

4 Non-Dictatorship

In Subsection 4.1, we characterize the class of generalized single-peaked domains that only admit dictatorial rules. In Subsection 4.2, we investigate the existence of "local" dictators, and in Subsection 4.3 we provide conditions under which global dictatorship can be inferred from the existence of a local dictator.

4.1 Generalizing the Gibbard-Satterthwaite Theorem

By the Intersection Property, what strategy-proof social choice functions a particular property space admits is determined by its critical families. Clearly, the "more" critical families there are, the tighter the set of strategy-proof social choice functions is circumscribed. It turns out that, for the purpose of determining whether there exists at least one strategy-proof social choice function with particular well-behavedness properties such as non-dictatorship, anonymity or neutrality, *all* relevant information is summarized by the transitive closure of the following **conditional entailment** relation. For all basic properties $H, G \in \mathcal{H}$,

$$H \geq^0 G :\Leftrightarrow [H \neq G^c \text{ and there exists a critical family } \mathcal{G} \text{ with } \mathcal{G} \supseteq \{H, G^c\}]$$

Intuitively, $H \geq^0 G$ means that, given *some* combination of other basic properties, the basic property H "entails" the basic property G. More precisely, let $H \geq^0 G$, i.e. let $\{H, G^c, G_1, ..., G_l\}$ be a critical family; then with $A = \cap_{j=1}^{l} G_j$ one has both $A \cap H \neq \emptyset$ ("property H is compatible with the combination A of properties") and $A \cap G^c \neq \emptyset$ ("property G^c is compatible with A as well") but $A \cap\ H \cap G^c = \emptyset$ ("properties H and G^c are jointly incompatible with A").

The central role of conditional entailment derives from the following observation, where $\geq$ denotes the transitive closure of $\geq^0$.

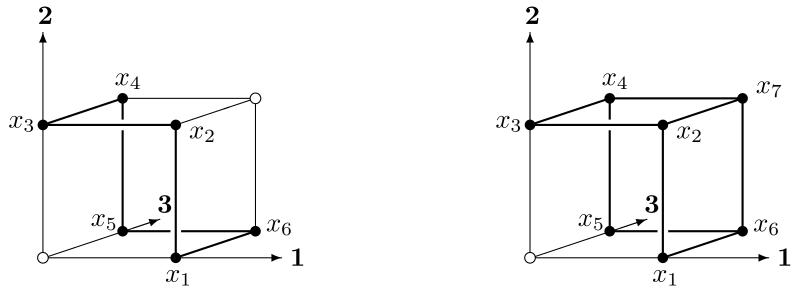
Fact 4.1 *Consider any committee structure $\mathcal{W}$ satisfying the Intersection Property. Then, for any pair of basic properties, $H \geq G \Rightarrow \mathcal{W}_H \subseteq \mathcal{W}_G$.*

To verify this, it suffices by transitivity to show that $H \geq^0 G \Rightarrow \mathcal{W}_H \subseteq \mathcal{W}_G$. Thus, suppose that $\{H, G^c\} \subseteq \mathcal{G}$ for some critical family $\mathcal{G}$. By the Intersection Property, $W \cap W' \neq \emptyset$ for any $W \in \mathcal{W}_H$ and any $W' \in \mathcal{W}_{G^c}$. By (3.2), this implies $\mathcal{W}_H \subseteq \mathcal{W}_G$.

By Fact 4.1, conditional entailment forces a strong relationship between the corresponding committees: if $H \geq G$, then any coalition that is winning for H (over its complement) must also be winning for G (over its complement).

As an illustration, consider again the 6-cycle and the seven-point graph of Fig. 1d above. For the present purpose, it is convenient to picture these graphs as embedded in a hypercube (see Figure 3 below). Denote by H_0^k the basic property corresponding to a zero in coordinate k, and by H_1^k the basic property corresponding to a one in coordinate k (in Fig. 3, the origin $(0,0,0)$ is the left-bottom-front point). Thus, for instance in Fig. 3a, the set H_1^1 (the right face of the cube) consists of the three points x_1, x_2 and x_6; similarly, for the set H_0^2 (the bottom face) one has $H_0^2 = \{x_1, x_5, x_6\}$. In Fig. 3b, on the other hand, one has $H_1^1 = \{x_1, x_2, x_6, x_7\}$ and again $H_0^2 = \{x_1, x_5, x_6\}$.

Viewed as a subspace of the three-dimensional hypercube, the seven-point subset in Fig. 3b is characterized by the following, single non-trivial critical family: $\mathcal{G}_0 := \{H_0^1, H_0^2, H_0^3\}$. Indeed, one has $\cap \mathcal{G}_0 = \emptyset$ corresponding to the fact that no element is simultaneously in the left, bottom and front faces of the cube. On the other hand, any two basic properties in $\mathcal{G}_0$ have a non-empty intersection, e.g. $H_0^1 \cap H_0^2 = \{x_5\}$. In terms of conditional entailment, criticality of $\mathcal{G}_0$ implies that $H_0^k \geq^0 H_1^{k'}$ for $k \neq k'$. Since there are no other non-trivial critical families, these are the only non-trivial instances of conditional entailment in Fig. 3b.



3a: The 6-cycle *3b: The seven-point graph*

Figure 3: Two graphs embedded in the three-dimensional hypercube

By contrast, consider the 6-cycle in Fig. 3a, which is characterized by the two critical families $\mathcal{G}_0 = \{H_0^1, H_0^2, H_0^3\}$ (no element is simultaneously in the left, bottom and front faces) and $\mathcal{G}_1 := \{H_1^1, H_1^2, H_1^3\}$ (no element is simultaneously in the right, top and back faces). Here, one has $H_0^k \geq^0 H_1^{k'}$ for all $k \neq k'$, and symmetrically, $H_1^k \geq^0 H_0^{k'}$ for all $k \neq k'$. This implies at once that for the 6-cycle, one has $H \geq G$ for *all* basic properties H and G. Thus, the relation $\geq$ is as large as it could possibly be; spaces with that property will be called "totally blocked." Specifically, denoting by $\equiv$ the symmetric part of $\geq$, i.e. $H \equiv G :\Leftrightarrow [H \geq G \text{ and } G \geq H]$, say that a property space $(X, \mathcal{H})$ is **totally blocked** if $H \equiv G$ for all $H, G \in \mathcal{H}$.

It follows at once from Fact 4.1 that consistent voting by committees on a totally blocked space must be *neutral* in the sense that all committees associated with the various basic properties are identical. While neutrality by itself is already quite restrictive as shown in Section 6 below, the stronger condition of total blockedness precludes all social choice functions but the dictatorial ones. Specifically, say that a social choice function is *dictatorial* if the chosen state always coincides with the peak of one fixed voter i, the dictator. Note that voting by committees is dictatorial with agent i as dictator if and only if $\{i\} \in \mathcal{W}_H$ for all H (i.e. i alone is winning for any basic property).

We will call a property space $(X, \mathcal{H})$ "dictatorial" if all strategy-proof and onto social choice functions $F : \mathcal{S} \to X$ on some rich domain (or, equivalently, all rich domains) of single-peaked preferences $\mathcal{S}$ are dictatorial. More generally, we will say that $(X, \mathcal{H})$ "has property P" if it forces all strategy-proof and onto social choice functions $F : \mathcal{S} \to X$ to have property P. In case P is a desirable property, we say that $(X, \mathcal{H})$ "is P" if it admits at least one strategy-proof and onto social choice function $F : \mathcal{S} \to X$ with property P.

Theorem 1 *A property space is dictatorial if and only if it is totally blocked.*

To use Theorem 1 to show that a given domain is dictatorial is typically fairly straightforward, as it involves coming up with sufficiently many instances of conditional entailment; in particular, it is not necessary to determine the set of critical families exhaustively. By contrast, in order to show that a domain is non-dictatorial, in principle one needs to determine the transitive hull of the entire conditional entailment relation; this may be difficult. However, an easily verifiable and frequently applicable sufficient condition is that there be at least one basic property not contained in any non-trivial critical family.[9]

Theorem 1 has the following corollary.

Corollary (The Gibbard-Satterthwaite Theorem) *If X contains three or more elements, then all onto strategy-proof social choice functions defined on an unrestricted domain of preferences are dictatorial.*

To see how the Gibbard-Satterthwaite Theorem follows from Theorem 1, consider the set $X = \{x_1, ..., x_m\}$ with the basic properties $H_j = \{x_j\}$ ("being equal to x_j") and $H_j^c = X \setminus \{x_j\}$ ("being different from x_j"), for all $j = 1, ..., m$. Recall that any preference is single-peaked with respect to the induced betweenness. The (non-trivial) critical families are $\{H_1^c, ..., H_m^c\}$ and, for any $j \neq k$, $\{H_j, H_k\}$. If $m \geq 3$, this implies at once that $(X, \mathcal{H})$ is totally blocked, hence the conclusion by Theorem 1.

We conclude this subsection by providing further examples of dictatorial domains.

Example (Ranking Sets of Applicants) Consider the K-dimensional hypercube and the subset $X_{(K;k,k')} \subseteq \{0,1\}^K$ of all binary sequences with at least k and at most k' coordinates having the entry 1, where $0 \leq k \leq k' \leq K$. A possible interpretation is that there are K applicants for a number vacant positions of which at least k *have* to be filled, and at most k' *can* be filled. A binary sequence in $X_{(K;k,k')}$ then simply specifies which applicants are admitted (those having entry 1). By considering preferences that are single-peaked on $X_{(K;k,k')}$, we are implicitly assuming that the ideal points are in $X_{(K;k,k')}$ as well, i.e. that all voters' most preferred state is one where at least k and at most k' positions are filled. This is clearly restrictive when $X_{(K;k,k')}$ is viewed as a set of feasible alternatives in the hypercube.

[9]Indeed, if H is only contained in the trivial critical family $\{H, H^c\}$, one has $H \not\geq^0 G$ for all G, and therefore $H \not\geq H^c$, which implies that the underlying property space is not totally blocked.

If $k = 0$ and $k' = K$, we obtain the full hypercube which is clearly not totally blocked. Thus, assume $k > 0$. If $k' = K$, the non-trivial critical families of the resulting space are exactly the subsets of $\{H_0^1, H_0^2, ..., H_0^K\}$ with $K - k + 1$ elements. The interpretation of such a critical family is that, if already $K - k$ applicants have been rejected, then all of the remaining applicants must be admitted. Also in this case one obtains a possibility result; for instance, by (3.3) above, the voting rule according to which an applicant is admitted as soon as at least a fraction of $1/(K-k+1)$ voters' vote for her is consistent. Note that the seven-point graph in Fig. 3b corresponds to the space $X_{(3;1,3)}$.

Let now $0 < k \leq k' < K$. Then, in addition to all subsets of $\{H_0^1, H_0^2, ..., H_0^K\}$ with $K - k + 1$ elements also any subset of $\{H_1^1, H_1^2, ..., H_1^K\}$ with $k' + 1$ elements forms a critical family. It is easily verified that the corresponding spaces are totally blocked whenever $K \geq 3$. By Theorem 1, any onto strategy-proof social choice function $F : \mathcal{S}^n \to X_{(K;k,k')}$ is dictatorial. Special cases are the 6-cycle corresponding to $X_{(3;1,2)}$, and the unrestricted domain on K alternatives which corresponds to $X_{(K;1,1)}$.

Another type of dictatorial domains are the l-cycles for $l \neq 4$, as shown by the following result.

Proposition 4.1 *An l-cycle is totally blocked if and only if $l \neq 4$.*

The fact that 4-cycles play a fundamentally different role can be explained by their isomorphism to the two-dimensional hypercube representing two independent issues.

4.2 Local Dictators

Non-dictatorial social choice functions on spaces that are not totally blocked can still be rather degenerate since they may possess "local" dictators, i.e. dictators on subdomains of preferences. Specifically, a voter i is called a **local dictator** if there exists a subdomain $\mathcal{D} \subseteq \mathcal{S}$ containing at least two preferences with different peaks such that for all $(\succ_1, ..., \succ_n) \in \mathcal{D}^n$, $F(\succ_1, ..., \succ_n) = x_i^*$, where x_i^* is the peak of $\succ_i$.

Fact 4.2 *Voting by committees possesses a local dictator if and only if $\{i\} \in \mathcal{W}_H$ and $\{i\} \in \mathcal{W}_{H^c}$ for some $H \in \mathcal{H}$ and some voter i.*

Theorem 1 has immediate implications for the existence of local dictators. To state these, we need some additional notation. Say that a subset $Y \subseteq X$ is **convex** if it corresponds to some combination of basic properties, i.e. if $Y = \cap\mathcal{H}_Y$ for an appropriate subfamily $\mathcal{H}_Y \subseteq \mathcal{H}$. For instance, the *segment* $[x, z] := \{y \in X : (x, y, z) \in T\}$ of all elements between x and z is a convex set, by (3.1) above. The use of the term "convex" is justified by the observation that any convex subset contains with any two elements x and z the entire segment $[x, z]$ between them; furthermore, the converse holds in any graphic space.[10] For any subset $S \subseteq X$, denote by coS the *convex hull* of S, i.e. the smallest convex subset containing S.

For any convex subset $Y \subseteq X$, denote by $(Y, \mathcal{H}|_Y)$ the induced property space on Y, where $\mathcal{H}|_Y := \{H \cap Y : H \in \mathcal{H},\ H \cap Y \neq \emptyset$ and $H^c \cap Y \neq \emptyset\}$. Say that $(X, \mathcal{H})$ is **locally blocked** if it contains a totally blocked subspace.

Proposition 4.2 *Any locally blocked property space is locally dictatorial.*

[10] See the discussion in Appendix A; in general, an additional regularity condition is needed to ensure that a set is convex whenever it contains with any two elements the entire segment between them.

In view of Proposition 4.1 above, this result yields a powerful local criterion for dictatorship when the underlying property space is graphic, namely the existence of a convex l-cycle with $l \neq 4$. In Fig. 1 above, for instance, convex 3-cycles occur in examples 1f, 1g and 1i. Note that while the 6-cycle in Fig. 1h is convex, also the graph in Fig. 1d contains a 6-cycle, the six "outer" points, but these do not form a convex set. In Section 5 below, we show that the graph in Fig. 1d admits a variety of anonymous rules, thus the existence of (non-convex) cycles of *even* length does not preclude genuine possibility results. However, the existence of *odd* cycles in a graphic space does, as shown by the following result.

Proposition 4.3 *If $(X, \mathcal{H})$ is graphic and locally non-dictatorial, then its graph contains no odd cycles.*

Graphs without odd cycles are called *bi-partite*, and are well-studied in graph theory. Note that the absence of convex cycles of length $\neq 4$ and the absence of odd cycles are necessary but not sufficient conditions for genuine possibility. For instance, both conditions are satisfied by the space $X_{(4;1,3)}$ ("4 applicants of which at least one must, but at most three can be admitted") which is totally blocked as already noted.

Example (Ranking Sets of Applicants cont.) Consider again the K-dimensional hypercube, and a non-empty subset $J \subseteq \{1, ..., K\}$ representing a subgroup of applicants. Suppose that at least one applicant has to be admitted, but at most m out of the subgroup J, where $1 \leq m \leq \#J$. Denote the correspoding subspace by $X_{(K;m,J)}$. If $\#J < K$, none of the spaces $X_{(K;m,J)}$ is totally blocked.[11] On the other hand, if $\#J > 2$, these spaces are locally blocked, since by the analysis of the preceding subsection, the convex subspace corresponding to the coordinates in J is totally blocked. If $\#J = 2$ the corresponding spaces are not locally blocked, and in fact admit anonymous strategy-proof social choice rules. As an example, consider Fig. 1e above interpreted as follows. There are three applicants, one of them has to be admitted; however, applicants 1 and 2 are relatives and therefore at most one of them can be admitted. Concretely, applicant 1 is admitted in exactly the states a and x, applicant 2 is admitted in states b and w, and applicant 3 is admitted in states a, b and z. While the decision on hiring each of the relatives may be made by majority voting, in any anonymous strategy-proof rule, the non-relative must be hired whenever at least one agent wants to hire her.[12]

4.3 From Local Dictatorship to Dictatorship

Local dictatorships are often easy to recognize. For instance, any triple of pairwise neighbours forces local dictatorship. Local dictators typically tend to "spread," often over the entire space. This observation is the basis of Aswal, Chatterji and Sen's

[11] Indeed, for all $k \notin J$, the property "applicant k is admitted" is not an element of any non-trivial critical family. Thus, by the remark after Theorem 1 above, the space is not totally blocked.

[12] To verify this, let $H_1^1 = \{a, x\}$, $H_0^1 = \{b, w, z\}$, $H_1^2 = \{b, w\}$, $H_0^2 = \{a, x, z\}$, $H_1^3 = \{a, b, z\}$ and $H_0^3 = \{x, w\}$. Then, the non-trivial critical families are $\{H_0^1, H_0^2, H_0^3\}$ ("at least one applicant has to be admitted") and $\{H_1^1, H_1^2\}$ ("at most one of applicants 1 and 2 can be admitted"). Using (3.3) it is easily seen that the class of all anonymous rules is given by the set of quotas satisfying $q^1 + q^2 \geq 1$ and $q^1 + q^2 + q^3 \leq 1$, where q^i is the quota needed for admission of applicant i. Note that consistency thus necessarily implies $q^3 = 0$, i.e. applicant 3 is rejected only if this is unanimously agreed upon, as claimed.

(2003) recent generalization of the Gibbard-Satterthwaite theorem. In our framework, the logic of the spreading is summarized by the following proposition.

Proposition 4.4 *If* $(Y, \mathcal{H}|_Y)$ *is totally blocked, and if* $x \notin Y$ *has at least two neighours in* Y*, then* $(Z, \mathcal{H}|_Z)$ *is totally blocked where* $Z = co(Y \cup \{x\})$*.*

Following and adapting Aswal, Chatterji and Sen (2003), say that $(X, \mathcal{H})$ is *linked* if the betweenness is graphic, if its graph contains a 3-cycle, and if for every convex proper subset $Y \subset X$ with $\#Y \geq 2$, there exists $x \notin Y$ with at least two neighbours in Y. The following is an immediate corollary of Proposition 4.4.

Corollary 4.1 (Aswal, Chatterji and Sen (2003)) *Any linked property space is totally blocked.*[13]

While frequently useful, the methodology underlying Proposition 4.4 has also clear limitations, since total blockedness is in general not a local phenomenon. For instance, all spaces $X_{(K;1,K-1)}$ are totally blocked, but none of them is linked; in fact, none of their convex subsets is totally blocked.

In the following class of examples, the above results allow one to classify all strategy-proof social choice functions on the associated single-peaked domain.

Example (Hotelling Model with Incomparabilities) Suppose that candidates for political office can be broadly ordered on a left-to-right spectrum; however, certain subgroups of candidates may not be unambiguously ordered in this way. For example, in a U.S. context, a Republican (*rep*), a Democrat (*dem*), a Socialist (*soc*), and a Green (*grn*) might run for president. While both the Socialist and the Green may be unambiguously to the left of the Democrat (i.e. everyone putting either of the two on top would prefer the Democrat over the Republican), there may be no unambiguous left-right ordering with respect to the Socialist and the Green, as partisans of both may prefer the Democrat over the other. This is illustrated in Figure 4a; in the symmetrically enlarged Figure 4b, there are two additional mutually non-comparable candidates on the right, say a Libertarian (*lib*) and a religious Fundamentalist (fun).

4a: locally dictatorial　　　　*4b: globally dictatorial*

Figure 4: Hotelling model with incomparabilities

Formally, denote by $\geq$ a partial order (transitive and antisymmetric binary relation) on X, and consider the property space induced by all basic properties of the form $H_{\geq x} := \{z \in X : z \geq x\}$ and $H_{\leq x} := \{z \in X : z \leq x\}$, and their respective complements. The corresponding betwenness according to (3.1) is given by $(x, y, z) \in T \Leftrightarrow [\, x \geq y \geq z \text{ or } z \geq y \geq x \,]$. In Figure 4, one has $x > y$ if and only if x is strictly to the right of y; moreover, in these examples the betweenness happens to be

[13] Aswal, Chatterji and Sen's (2003) original result is in fact more general than the stated corollary since it applies not only to generalized single-peaked preferences.

graphic (with the graphs as displayed). Note, however, that for general partial orders the derived betweenness need not be graphic.

Except for the case of a line, when $\geq$ is a linear order or when $\#X \leq 2$, all these spaces are locally dictatorial by Proposition 4.3 above, since any two mutually non-comparable elements are part of a 3-cycle. In fact, these spaces are globally dictatorial (as in Fig. 4b), unless there is a unique minimal and unique second-to-minimal element, or a unique maximal and second-to-maximal element (as in Fig. 4a).[14]

5 Genuine Possibility: Anonymity and the Absence of Local Dictators

By the results of the preceding section, the absence of a totally blocked convex subspace is necessary for local non-dictatorship. It seems natural to conjecture that local non-blockedness is also sufficient for local non-dictatorship; this, however, turns out to be false. To formulate the appropriate characterizing condition, we need some additional notation. Say that a basic property $H \in \mathcal{H}$ is *blocked* if $H \equiv H^c$; otherwise, if $H \not\equiv H^c$, H is called *unblocked.* For each $G \in \mathcal{H}$, let $\mathcal{H}_{\equiv G} := \{H \in \mathcal{H} : H \equiv G\}$, and say that a property space is **quasi-unblocked** if for any $G \in \mathcal{H}$ and any critical family $\mathcal{G}$, $\#(\mathcal{H}_{\equiv G} \cap \mathcal{G}) \leq 2$, whenever G is blocked. The following result entails that quasi-unblockedness implies the absence of a totally blocked subspace. In Appendix B, we show by means of an example that the converse does not hold and that quasi-unblockedness is indeed stronger than local non-blockedness.

Theorem 2 *Let $(X, \mathcal{H})$ be a property space. The following conditions are equivalent.*
(i) *$(X, \mathcal{H})$ is anonymous.*
(ii) *$(X, \mathcal{H})$ is locally non-dictatorial.*
(iii) *$(X, \mathcal{H})$ is quasi-unblocked.*

On generalized single-peaked domains, the existence of a rule without local dictators is thus in fact equivalent to the existence of an anonymous rule, and either condition

[14]To verify these claims, consider any two incomparable elements, such as *grn* and *soc* in Fig. 4a. If $\#X \geq 3$, each of these, say *grn*, either has an immediate predecessor, an immediate successor (*dem* in Fig. 4a), or there exists a third element that is not comparable to *grn*. In each case, one easily verifies the existence of 3-cycle containing *grn* and *soc*. Suppose there is a unique minimal and unique second-to-minimal element, or a unique maximal and second-to-maximal element, such as *rep* and *dem* in Fig. 4a. Then, the following type of rules is strategy-proof and non-dictatorial. Fix any voter i and a committee $\mathcal{W}_0$ such that $\{i\} \notin \mathcal{W}_0$, but $i \in W$ for any winning coalition $W \in \mathcal{W}_0$. Set the chosen state to be *rep* whenever the voters with peak *rep* form a winning coalition in $\mathcal{W}_0$; otherwise, the outcome is i's most preferred alternative among all elements in $X \setminus \{rep\}$. Evidently, i is a local but not a global dictator, since i alone cannot force the outcome *rep*.

Now suppose that there exists both a 3-cycle containing at least one minimal element, say *grn*, and a 3-cycle containing at least one maximal element, say fun as in Fig. 4b. By Proposition 4.3, there is a local dictator, say voter i, on any issue of the form $(H_y, (H_y)^c) := (\{y\}, X \setminus \{y\})$ where y is a minimal element of $(X, \geq)$, and a local dictator, say voter j, on any issue $(H_w, (H_w)^c) = (\{w\}, X \setminus \{w\})$ where w is a maximal element of $(X, \geq)$. However, since $\{H_{grn}, H_{fun}\}$ forms a critical family, the Intersection Property immediately implies $j = i$. Moreover, any basic property H either contains a minimal or a maximal element, hence $\{i\}$ is winning for any basic property, again by the Intersection Property, i.e. i is a dictator.

Finally, note that neither of the property spaces underlying the graphs in Fig. 4 is linked. While Aswal, Chatterji and Sen (2003) provide a sufficient condition that allows one to classify the domain in Fig. 4a as non-dictatorial, their results are silent on the domain corresponding to Fig. 4b.

is equivalent to quasi-unblockedness. However, the latter condition is complex and not easy to verify. Therefore, we now provide a simple geometric condition that is "almost" equivalent to quasi-unblockedness; it is based on the notion of a "median point," as follows.

An element $m = m(x, y, z)$ is called the **median** of x, y, z if m is between any pair of the triple, i.e. if $m \in [x, y] \cap [x, z] \cap [y, z]$. An element $\hat{x} \in X$ is called a **median point** if, for any y, z, the triple $\hat{x}, y, z$ admits a median; the set of median points is denoted by $M(X)$. A property space $(X, \mathcal{H})$ is called a **quasi-median space** if $M(X) \neq \emptyset$, and it is called a **median space** if every element is a median point, i.e. if $M(X) = X$. Median spaces are well-studied in the literature on abstract convexity theory (see, e.g., van de Vel (1993); the weaker concept of quasi-median space has been introduced in Nehring (2004).

Quasi-median spaces are always quasi-unblocked; conversely, there are spaces that are quasi-unblocked but still admit no median points. However, the latter phenomenon is not robust. The simplest example of a quasi-unblocked space without median points is 5-dimensional and is presented in Appendix B. Moreover, while spaces without median points may admit anonymous rules with an odd number of voters, they do not admit such rules for an *even* number of voters, as stated in Theorem 3 below.

By contrast, the existence of a median point guarantees the existence of anonymous strategy-proof social choice rules for *any* number of voters via "unanimity rules." A social choice function $F : \mathcal{S}^n \to X$ is a **unanimity rule** if there exists $\hat{x} \in X$ such that

$$F(\succ_1, ..., \succ_n) = \hat{x} \text{ whenever } \hat{x} \in \{x_1^*, ..., x_n^*\}, \tag{5.1}$$

where x_i^* denotes the peak of $\succ_i$. Clearly, a state $\hat{x}$ satisfying (5.1) is uniquely determined and is referred to as the *status quo.* Thus, a unanimity rule prescribes the choice of the status quo as soon as at least one voter endorses that outcome. In general, a unanimity rule is not fully determined by (5.1) since it does not specify a social choice if none of the peaks coincides with the status quo. However, among all unanimity rules with a given status quo $\hat{x}$ there is only one that has the structure of voting by committees. Denote by $F_{\hat{x}}$ voting by committees with $\mathcal{W}_H = 2^N \setminus \{\emptyset\}$ for all $H \ni \hat{x}$ and $\mathcal{W}_H = \{N\}$ for all $H \not\ni \hat{x}$.

Fact 5.1 *Voting by committees is a unanimity rule if and only if it is of the form $F_{\hat{x}}$ for some $\hat{x} \in X$.*

Proposition 5.1 *$F_{\hat{x}}$ is consistent if and only if $\hat{x} \in M(X)$. If $F_{\hat{x}}$ is consistent, $F_{\hat{x}}(\succ_1, ..., \succ_n)$ is the unique element in the convex hull $Co\{x_1^*, ..., x_n^*\}$ of the voters' peaks that is between $\hat{x}$ and any peak x_i^*.*

To see that consistency of $F_{\hat{x}}$ requires the status quo $\hat{x}$ to be a median point, consider two alternatives y, z and two voters with peaks at y and z, respectively. Since any property common to y and z gets unanimous support, the outcome under $F_{\hat{x}}$ must lie between the two peaks, i.e. $F_{\hat{x}}(\succ_1, \succ_2) \in [y, z]$. Moreover $F_{\hat{x}}(\succ_1, \succ_2) \in [\hat{x}, y]$ since any basic property jointly possessed by $\hat{x}$ and y gets the support of at least one voter, and by the same argument, $F_{\hat{x}}(\succ_1, \succ_2) \in [\hat{x}, z]$. In other words, the triple $\hat{x}, y, z$ must admit a median, namely $F_{\hat{x}}(\succ_1, \succ_2)$.

As an illustration, consider again the two subsets of the hypercube in Figure 3 above. As is easily verified, the 6-cycle in Fig. 3a has no median points. By comparison, the seven-point subset in Fig. 3b has the four median points x_2, x_4, x_6 and x_7. By Proposition 5.1, it therefore admits four different strategy-proof unanimity rules, each corresponding to one of the four median points as the status quo. Note that, while the space is a quasi-median space, it is not a median space since the triple x_1, x_3, x_5 does not admit a median.

Theorem 3 *Let $(X, \mathcal{H})$ be a property space. The following conditions are equivalent.*

(i) *$(X, \mathcal{H})$ admits an anonymous strategy-proof scf $F : \mathcal{S}^n \to X$ for some even n.*

(ii) *$(X, \mathcal{H})$ admits anonymous strategy-proof scfs $F : \mathcal{S}^n \to X$ for any $n \geq 2$.*

(iii) *$(X, \mathcal{H})$ admits locally non-dictatorial strategy-proof scfs $F : \mathcal{S}^n \to X$ for any $n \geq 2$.*

(iv) *$(X, \mathcal{H})$ admits some strategy-proof unanimity rule.*

(v) *All $H \in \mathcal{H}$ are unblocked.*

(vi) *$(X, \mathcal{H})$ is a quasi-median space.*

The equivalence of (v) and (vi) has been proved in Nehring (2004); for the sake of self-containedness, we reproduce the proof in the appendix below.

In the case of two voters, unanimity rules exhaust the class of all anonymous (or, equivalently, locally non-dictatorial) strategy-proof social choice functions $F : \mathcal{S}^2 \to X$. By the above results, all such rules can be described as follows: choose any median point $\hat{x} \in M(X)$ and set $F(\succ_1, \succ_2) = m(\hat{x}, x_1^*, x_2^*)$, where x_i^* is the peak of $\succ_i$. Thus, the final outcome is the median of $\hat{x}$ and the two voters' peaks; following Moulin (1980), the "status quo" $\hat{x}$ can also be interpreted as the peak of a "phantom voter."[15]

Example (Ranking Sets of Applicants cont.) Consider yet again the K-dimensional hypercube with each coordinate corresponding to an applicant, and suppose that m of these are women. Moreover, assume that a regulation requires that at least as many women be hired as men, so that not all points of the cube represent feasible states. Evidently, the state in which all women and no men are admitted is a median point, so that the underlying space is a quasi-median space. There may be other median points, but in general the space is not a median space; for instance, the seven-point graph in Fig. 3b results by taking $m = 2$ and $K = 3$. Using the Intersection Property, one easily verifies that the class of all anonymous rules that treat all women and all men symmetrically is a 1-dimensional family with the extreme points $(1, \frac{1}{m})$ and $(\frac{m}{m+1}, \frac{1}{m+1})$, where the first entry is the quota for hiring a man, and the second the quota for hiring a woman. Note the extent to which the regulation biases the hiring in favour of women under strategy-proofness.

6 Neutrality

Anonymous rules treat voters symmetrically. In this section, we are interested in social choice functions that treat alternatives symmetrically. Under voting by committees, a natural requirement is that all committees be identical, i.e. that $\mathcal{W}_H = \mathcal{W}_{H'}$ for all basic properties H, H'. Committee structures satisfying this condition will be called **neutral**. Neutrality can be decomposed into two conceptually distinct requirements: neutrality *within* issues and neutrality *across* issues. Formally, a committee structure

[15] A related result in the two voter case has been obtained by Bogomolnaia (1999).

is **neutral within issues** if, for all basic properties $H \in \mathcal{H}$, $\mathcal{W}_H = \mathcal{W}_{H^c}$, and it is **neutral across issues** if, for all $H, H' \in \mathcal{H}$, $\mathcal{W}_{H'} = \mathcal{W}_H$ or $\mathcal{W}_{H'} = \mathcal{W}_{H^c}$.[16]

An example of a social choice rule that is neutral within but not across issues is *weighted* issue-by-issue majority voting in the hypercube where the weights differ across issues. Specifically, for all k and i, let $w_i^k \geq 0$ be the weight of voter i in dimension k, and assume that $\sum_i w_i^k = 1$ for all k. Weighted issue-by-issue majority voting is defined by taking a coalition W as winning in dimension k if and only if $\sum_{i \in W} w_i^k > 1/2$. A difference in weights across issues may be the natural result of voters having different stakes and/or different expertise in different dimensions. By contrast, a natural class of examples of rules that are neutral across but not within issues are the unanimity rules, or more generally, supermajority rules with a uniform quota $> 1/2$ for each issue.

In NP, we have shown that, unless the social choice function is dictatorial, full neutrality presupposes an underlying median space, i.e. that *every* triple of elements admits a median. The following result shows that this conclusion remains true when neutrality is weakened to neutrality within issues while no-dictatorship is strengthened to the absence of local dictators; on the other hand, neutrality across issues can be realized under more general circumstances.

Theorem 4 a) *A property space $(X, \mathcal{H})$ admits a strategy-proof social choice function $F : \mathcal{S}^n \to X$ that is non-dictatorial and neutral across issues if and only if $(X, \mathcal{H})$ is a quasi-median space.*
b) *A property space $(X, \mathcal{H})$ admits a strategy-proof social choice function $F : \mathcal{S}^n \to X$ that is locally non-dictatorial and neutral within issues if and only if $(X, \mathcal{H})$ is a median space.*
c) *A property space $(X, \mathcal{H})$ admits a strategy-proof social choice function $F : \mathcal{S}^n \to X$ that is non-dictatorial and (fully) neutral if and only if $(X, \mathcal{H})$ is a median space.*

Note that, by part b), non-dictatorial rules that are neutral within issues may exist also outside the class of median spaces. However, if the underlying space is "indecomposable" then neutrality within issues is just as demanding as full neutrality, since consistency forces committee structures that are neutral within issues also to be neutral across issues. Specifically, say that $(X, \mathcal{H})$ is *decomposable* if $\mathcal{H}$ can be partitioned into (at least) two non-empty subfamilies $\mathcal{H}_1$ and $\mathcal{H}_2$ such that no critical family meets both $\mathcal{H}_1$ and $\mathcal{H}_2$; otherwise, $(X, \mathcal{H})$ is called *indecomposable.* One can easily show that a property space is decomposable if and only if it can be represented as the Cartesian product of (at least) two property spaces; for instance, among the property spaces illustrated in Fig. 1 above, only the hypercube (Fig. 1b) is decomposable.

Proposition 6.1 *Suppose that $(X, \mathcal{H})$ is indecomposable. If $F : \mathcal{S}^n \to X$ is strategy-proof and neutral within issues, then F is neutral across issues, hence fully neutral.*

By the above results, neutrality across issues is a substantially weaker requirement than neutrality within issues, since under no-dictatorship it only requires the existence of at least one median point while the latter essentially requires that all elements are median points. However, neutrality within issues seems to be the more natural and conceptually more fundamental condition. Strong possibility results can therefore only emerge on sufficiently homogeneous spaces, i.e. when the key geometric condition ("medianicity") is not only satisfied locally but throughout the entire space of alternatives.

[16] These neutrality conditions can be derived from corresponding conditions defined for general social functions $F : \mathcal{S}^n \to X$; for the derivation of full neutrality, see NP.

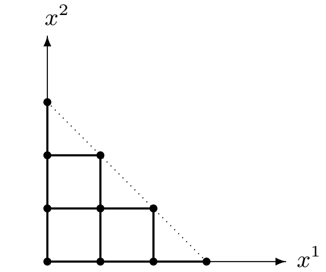
Basic examples of median spaces are lines, trees, the hypercube and products of these (see Figures 1a, 1b and 2a-c); further examples are appropriate subsets of median spaces (see, e.g., Fig. 1c). A more detailed analysis of median spaces is provided in NP. In Nehring and Puppe (2003), we show that efficiency requires a weak form of neutrality and, except under "almost-dictatorship," indeed an underyling median space.

As a final class of examples illustrating the spectrum from strong possibility on median spaces to impossibility on totally blocked spaces, consider the domain of additive preferences over public goods, as follows.

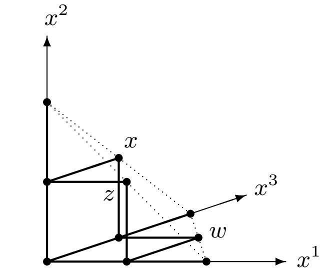
Example (Additive Preferences over Public Goods) There are $K+1$ public goods, which can be supplied in non-negative discrete quantities. Denote by $x^k \in \mathbf{N}_0$ the quantity of good $k = 0, 1, ..., K$, and suppose that feasibility requires $\sum_k x^k \leq M$ for some fixed amount M. Furthermore, suppose that preferences can be represented by additive utility functions of the form $\sum_k u^k(x^k)$, where each u^k is increasing and concave. By the resulting monotonicity of preferences, the choice will always lie on the budget line $\sum_k x^k = M$. We can therefore eliminate the coordinate corresponding to good 0, and consider the set $X = \{x \in \mathbf{N}_0^K : \sum_{k=1}^K x^k \leq M\}$ as the universe of alternatives. The utility functions on X can be written as follows,

$$u(x^1, ..., x^K) = \sum_{k=1}^{K} u^k(x^k) + u^0(M - \sum_{k=1}^{K} x^k). \tag{6.1}$$

Case 1 Suppose that preferences are *quasi-linear* with good 0 as the numeraire so that $u^0(x^0) = x^0$. Then, preferences can be represented by utility functions of the form $\tilde{u}(x) = \sum_k (u^k(x^k) - x^k)$. Since each summand $u^k(x^k) - x^k$ is concave, any such utility function represents a single-peaked preference with respect to the standard betweenness on X induced by the product $\mathbf{N}_0^K$ of K lines. For all K, the resulting property space $(X, \mathcal{H})$ is a quasi-median space with all points on the coordinate axes as median points. The following figure shows the cases $K = 2$ and $K = 3$. For $K = 2$ all elements are in fact median points, but not for $K \geq 3$ (see Fig. 5b in which the triple x, z, w has no median; all other states are median points).



5a: Median space for $K = 2$



5b: Quasi-median space for $K = 3$

Figure 5: Possibility under quasi-linearity

It is easily verified that, for all K, the set of all quasi-linear utility functions gives rise to a rich single-peaked domain on the quasi-median space $(X, \mathcal{H})$. Hence, by Theorem 3, there exist anonymous and strategy-proof social choice functions for any K. Examples are the unanimity rules corresponding to each of the median points on the coordinate axes. As is easily verified, another example is the rule that requires, for any fixed $x_0 \in X$ and all k, a quota of at least $(K-1)/K$ for any increase beyond x_0^k and majority voting below x_0^k.

Case 2 Consider now the general case without the quasi-linearity assumption. Then, the preferences represented by (6.1) are not necessarily single-peaked with respect to the product betweenness on $\mathbf{N}_0^K$. They are, however, single-peaked with respect to the following betweenness relation:

$$(x, y, z) \in \hat{T} :\Leftrightarrow \left\{ y^k \in [x^k, z^k] \text{ for all } k, \text{ and } \sum_k y^k \in \left[\sum_k x^k, \sum_k z^k \right] \right\}.$$

For instance, for $K = 2$ this is the graphic betweenness corresponding to the graph in Figure 6 below (cf. Fig.1i). It is easily verified that the underlying property space is totally blocked, and that the domain of all additive preferences of the form (6.1) is a rich single-peaked domain on that space. By Theorem 1, all strategy-proof and onto social choice functions are dictatorial on that domain.

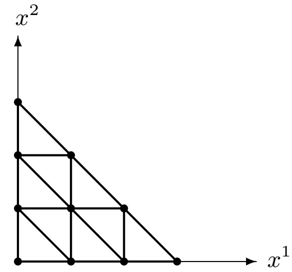


Figure 6: Impossibility without quasi-linearity

7 Conclusion

Based on the general characterization of strategy-proof social choice as voting by committees satisfying the Intersection Property, we have classified all generalized single-peaked domains in terms of the extent to which they enable well-behaved strategy-proof social choice rules. Specifically, we have characterized the domains that admit non-dictatorial, locally non-dictatorial, anonymous and neutral strategy-proof social choice functions, respectively. The class of domains that enable anonymous rules ("voting by quota") is only slightly larger than the class of domains admitting unanimity rules, according to which a departure from some "status quo" point requires unanimous consent. The spaces that admit such rules have a simple unifying geometric structure as quasi-median spaces. Specifically, a state can serve as status quo if and only if it is a "median point," i.e. if and only if it admits a median with any other pair of states. The requirement of symmetric treatment of alternatives turns out to be remarkably restrictive since, under no-dictatorship, strategy-proof social choice functions that are neutral require an underlying median spaces, i.e. that *every* state is a median point. For locally non-dictatorial rules, this holds even when neutrality is weakened to neutrality within issues.

From a technical point of view, due to the canonical representation of strategy-proof social choice functions in terms of voting by properties afforded by Theorems A and B, all the results of this paper are really results about voting by properties in general, and as such applicable to contexts quite different from strategy-proofness. For example, in Nehring (2003) a weak form of Arrow's celebrated Impossibility Theorem is derived as a consequence of our Theorem 1. More importantly, in Nehring and Puppe (2005) we show that the results of the present paper have broad applicability to the recent and rapidly growing literature on "judgement aggregation" and can be used to obtain interesting new results in that context.

Appendix A: Property Spaces Represented by Graphs

Any given property space $(X, \mathcal{H})$ can be embedded in a hypercube of dimension $K = \#\mathcal{H}/2$ such that each issue (H, H^c) corresponds to one coordinate. In particular, any betweenness T derived from a property space via (3.1) can be represented by a "graph with missing points," i.e. there exists a graph on a superset $Y \supseteq X$ such that the corresponding graphic betweenness relativized to X coincides with T (for an illustration, see Fig. 1e above in which the blank circle represents a "missing point.") A property space is graphic if and only if the representation is possible without missing points. For a well-known sufficient condition for this, the so-called "triangle property," see van de Vel (1993, p.97); a characterization is not known.

When is, conversely, a given graphic betweenness derivable from a property space via (3.1)? In NP, we derive the following necessary and sufficient condition. Let γ be a connected graph on X, and denote by T_γ the induced graphic betweenness according to which $(x, y, z) \in T_\gamma$ if and only if y in some shortest γ-path connecting x and z. Say that a set $A \subseteq X$ is *T_γ-convex* if for all x, y, z,

$$[\{x, z\} \subseteq A \text{ and } (x, y, z) \in T_\gamma] \Rightarrow y \in A.$$

Thus, a set is T_γ-convex if it contains with any two elements all elements that are on a shortest γ-path connecting them. Furthermore, say that a subset $H \subseteq X$ is a *half-space* if both H and its complement H^c are non-empty and T_γ-convex. The collection of all such half-spaces is denoted by $\mathcal{H}_{T_\gamma}$. The following condition states that points that are not T_γ-between two other points can be separated from them by a half-space.

T5 (Separation) If $(x, y, z) \notin T_\gamma$, then there exists a half-space H such that

$$H \supseteq \{x, z\} \text{ and } y \notin H.$$

Fact *Let γ be a connected graph. Then, there exists a property space $\mathcal{H}$ such that T_γ coincides with the betweenness derived from $\mathcal{H}$ via (3.1) if and only if T_γ satisfies T5.*

In this case, $\mathcal{H}_{T_\gamma}$ is in fact the largest such property space, and the convex sets coincide with the T_γ-convex sets. Thus, in graphic property spaces the convex sets can be assumed to coincide with the T_γ-convex sets, a property that otherwise holds only under an additional regularity condition.

Appendix B: Anonymity outside Quasi-Median Spaces

Consider the subspace $X \subseteq \{0,1\}^5$ shown in Figure 7 below. The two cubes to the right correspond to a "1" in coordinate 4 (i.e. to the basic property H_1^4), similarly, the two top cubes correspond to a "1" in coordinate 5 (i.e. to H_1^5). Missing points of the 5-hypercube are indicated by blank circles. For the purpose of better illustration, the edges connecting different points across the four subcubes have been omitted in the figure.

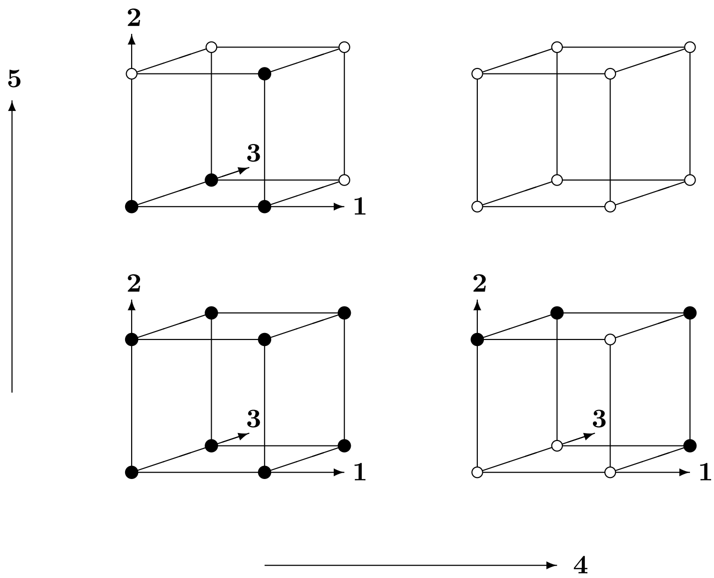


Figure 7: Anonymity and strategy-proofness without median points

This space is characterized by the following critical families: $\mathcal{G}_1 = \{H_1^1, H_0^3, H_1^4\}$, $\mathcal{G}_2 = \{H_1^1, H_1^3, H_1^5\}$, $\mathcal{G}_3 = \{H_0^1, H_0^2, H_1^4\}$, $\mathcal{G}_4 = \{H_0^1, H_1^2, H_1^5\}$, $\mathcal{G}_5 = \{H_0^2, H_0^3, H_1^4\}$, $\mathcal{G}_6 = \{H_1^2, H_1^3, H_1^5\}$ and $\mathcal{G}_7 = \{H_1^4, H_1^5\}$. For instance, the criticality of $\{H_1^4, H_1^5\} = \mathcal{G}_7$ reflects the fact that the top-right cube contains no element of X, and is a maximal subcube with this property. As is easily verified, one has $H_0^k \equiv H_1^k$ for $k = 1, 2, 3$, i.e. the first three coordinates are blocked; in particular, by Theorem 3, the underlying space admits no median points. Nevertheless, denoting by q_1^k the quota corresponding to H_1^k, the following anonymous choice rule is easily seen to be consistent if the number of voters is odd: The final outcome lies in the top left cube if and only if all voters have their peak in that cube ($q_1^5 = 1$); similarly, the choice is in the bottom right cube if and only if all voters have their peak there ($q_1^4 = 1$). In all other cases, the outcome lies in the bottom left cube ($q_0^5 = q_0^4 = 0$). In addition, the location of the outcome within any of the three admissible subcubes is decided by majority vote in each of the first three coordinates ($q_1^1 = q_1^2 = q_1^3 = \frac{1}{2}$). Using (3.3), it is easily verified that this rule is in fact the *only* anonymous strategy-proof social choice function in the present example. Note in particular that in accordance with Theorem 3, there is no anonymous rule for an even number of voters.

Clearly, the space shown in Fig. 7 is quasi-unblocked, hence by Theorem 2 also locally non-blocked. The following modification shows that quasi-unblockedness is indeed strictly stronger than local non-blockedness, hence that local non-blockedness does not guarantee the existence of an anonymous strategy-proof social choice function. Specifically, consider the subspace $(X, \mathcal{H})$ of the 6-dimensional hypercube characterized by the following critical families: $\mathcal{G}'_1 = \{H_1^1, H_0^3, H_1^4\}$, $\mathcal{G}'_2 = \{H_1^1, H_1^3, H_1^5\}$, $\mathcal{G}'_3 = \{H_0^1, H_0^2, H_1^4, H_1^6\}$, $\mathcal{G}'_4 = \{H_0^1, H_1^2, H_1^5, H_0^6\}$, $\mathcal{G}'_5 = \{H_0^2, H_0^3, H_1^4, H_1^6\}$, $\mathcal{G}'_6 = \{H_1^2, H_1^3, H_1^5, H_0^6\}$ and $\mathcal{G}'_7 = \{H_1^4, H_1^5\}$. As is easily verified, one now has $H_0^k \equiv H_1^k$ for $k = 1, 2, 3$ and $k = 6$, i.e. in addition to the first three coordinates also the sixth coordinate is blocked. Moreover, one has $H_1^1 \equiv H_1^2 \equiv H_1^3 \equiv H_1^6$, by consequence the underlying space is no longer quasi-unblocked. Nevertheless, it is locally unblocked. To see this, note that neither H_0^4 nor H_0^5 occur in any critical family. Since $H_1^4 \cap H_1^5 = \emptyset$, this implies that, for any convex subset $Y \subseteq X$, either $H_0^4 \cap Y \neq \emptyset$ and $H_1^4 \cap Y \neq \emptyset$, or $H_0^5 \cap Y$ and $H_1^5 \cap Y \neq \emptyset$; that is, either $H_0^4 \cap Y$ or $H_0^5 \cap Y$ is a basic property of $(Y, \mathcal{H}|_Y)$. Moreover, since the critical families of $(Y, \mathcal{H}|_Y)$ are obtained as relativizations of the critical families of $(X, \mathcal{H})$, either $H_0^4 \cap Y$ or $H_0^5 \cap Y$ is not contained in any critical family of $(Y, \mathcal{H}|_Y)$. As noted in the main text, this implies that $(Y, \mathcal{H}|_Y)$ is not totally blocked, hence $(X, \mathcal{H})$ is not locally blocked.

Appendix C: Proofs

In the following proofs we will sometimes refer to the fact that the conditional entailment relation $\geq$ is *complementation adapted* in the sense that $H \geq G \Leftrightarrow G^c \geq H^c$. Also note that $H \subseteq G \Rightarrow H \geq G$, since $H \subseteq G$ implies that $\{H, G^c\}$ is a critical family.

The following lemma plays a key role in the proofs of the theorems below.

Lemma 1 *Suppose that $\{G_1, G_2, G_3\} \subseteq \mathcal{G}$ for a critical family $\mathcal{G}$. If $\mathcal{W}_{G_1^c} \subseteq \mathcal{W}_{G_2}$, then $\{i\} \in \mathcal{W}_{G_3^c}$ for some $i \in N$.*

Proof of Lemma 1 Let $\tilde{W}_1$ be a minimal element of $\mathcal{W}_{G_1}$, and let $i \in \tilde{W}_1$. By definition of a committee structure and by minimality of $\tilde{W}_1$, one has $(\tilde{W}_1^c \cup \{i\}) \in \mathcal{W}_{G_1^c}$. By assumption, $\mathcal{W}_{G_1^c} \subseteq \mathcal{W}_{G_2}$, hence $(\tilde{W}_1^c \cup \{i\}) \in \mathcal{W}_{G_2}$. Now consider any $W_3 \in \mathcal{W}_{G_3}$. By the Intersection Property, $\cap_{j=1}^3 W_j \neq \emptyset$ for any selection $W_j \in \mathcal{W}_{G_j}$. In particular, $\tilde{W}_1 \cap (\tilde{W}_1^c \cup \{i\}) \cap W_3 \neq \emptyset$. Since $\tilde{W}_1 \cap (\tilde{W}_1^c \cup \{i\}) = \{i\}$, this means $i \in W_3$ for all $W_3 \in \mathcal{W}_{G_3}$. By (3.2), this implies $\{i\} \in \mathcal{W}_{G_3^c}$.

Proof of Theorem 1 Suppose that $(X, \mathcal{H})$ is totally blocked. By Fact 4.1, $\mathcal{W}_H = \mathcal{W}_0$ for some $\mathcal{W}_0$ and all H. Moreover, it is easily verified that any totally blocked space admits at least one critical family $\mathcal{G}$ with at least three elements, say $\mathcal{G} \supseteq \{G_1, G_2, G_3\}$. By Lemma 1, $\{i\} \in \mathcal{W}_{G_3^c} = \mathcal{W}_0$; but then voter i is a dictator.

Suppose then that $(X, \mathcal{H})$ is not totally blocked. To construct a non-dictatorial strategy-proof social choice function partition $\mathcal{H}$ as follows.

$$\begin{aligned}
\mathcal{H}_0 &:= \{H \in \mathcal{H} : H \equiv H^c\}, \\
\mathcal{H}_1^+ &:= \{H \in \mathcal{H} : H > H^c\}, \\
\mathcal{H}_1^- &:= \{H \in \mathcal{H} : H^c > H\}, \\
\mathcal{H}_2 &:= \{H \in \mathcal{H} : \text{ neither } H \geq H^c \text{ nor } H^c \geq H\}.
\end{aligned}$$

For future reference we note the following facts about this partition of $\mathcal{H}$. Part c) of the following lemma will only be used in the proof of Theorem 2 below.

Lemma 2 **a)** *For any critical family $\mathcal{G}$, if $G \in \mathcal{G} \cap \mathcal{H}_1^-$, then $\mathcal{G} \setminus \{G\} \subseteq \mathcal{H}_1^+$.*
b) *For any critical family $\mathcal{G}$, if $\mathcal{G} \cap \mathcal{H}_0 \neq \emptyset$, then $\mathcal{G} \subseteq \mathcal{H}_0 \cup \mathcal{H}_1^+$.*
c) *Take any $\tilde{H} \in \mathcal{H}_2$. Then there exists a partition of $\mathcal{H}_2$ into $\mathcal{H}_2^-$ and $\mathcal{H}_2^+$ with $\tilde{H} \in \mathcal{H}_2^-$ such that $G \in \mathcal{H}_2^- \Leftrightarrow G^c \in \mathcal{H}_2^+$, and for no $G \in \mathcal{H}_2^-$ and $H \in \mathcal{H}_2^+$, $G \geq H$.*

Proof of Lemma 2 a) Suppose $G \in \mathcal{G} \cap \mathcal{H}_1^-$, i.e. $G^c > G$. Consider any other $H \in \mathcal{G}$. We have $H \geq G^c > G \geq H^c$, hence $H > H^c$, i.e. $H \in \mathcal{H}_1^+$.
b) Suppose $G \in \mathcal{G} \cap \mathcal{H}_0$ and let $H \in \mathcal{G}$ be different from G. We have $H \geq G^c \equiv G \geq H^c$, hence $H \geq H^c$. But this means $H \in \mathcal{H}_0 \cup \mathcal{H}_1^+$.
c) The desired partition into $\mathcal{H}_2^- = \{G_1, ..., G_l\}$ and $\mathcal{H}_2^+ = \{G_1^c, ..., G_l^c\}$ will be constructed inductively. Set $G_1 = \tilde{H}$, and suppose that $\{G_1, ..., G_r\}$, with $r < l$, is determined such that $G_j \not\geq G_k^c$ for all $j, k \in \{1, ..., r\}$. Take any $H \in \mathcal{H}_2 \setminus \{G_1, G_1^c, ..., G_r, G_r^c\}$ and set

$$G_{r+1} := \begin{cases} H & \text{if for no } j \in \{1, ..., r\} : \quad G_j \geq H^c \\ H^c & \text{if for some } j \in \{1, ..., r\} : \quad G_j \geq H^c \end{cases}$$

First note that $G_{r+1} \not\geq G_{r+1}^c$ since $H \in \mathcal{H}_2$. Thus, the proof is completed by showing that for no $k \in \{1, ..., r\}$, $G_k \geq G_{r+1}^c$ (and hence, by complementation adaptedness, also not $G_{r+1} \geq G_k^c$). To verify this, suppose first that $G_{r+1} = H$; then, the claim is true by construction. Thus, suppose $G_{r+1} = H^c$; by construction, there exists $j \leq r$ with $G_j \geq H^c$, hence by complementation adaptedness also $H \geq G_j^c$. Assume, by way of contradiction, that $G_k \geq G_{r+1}^c$, i.e. $G_k \geq H$. This would imply $G_k \geq H \geq G_j^c$, in contradiction to the induction hypothesis.

Proof of Theorem 1 (cont.) If $\mathcal{H}_1^+ \cup \mathcal{H}_1^-$ is non-empty, set $\mathcal{W}_H = 2^N \setminus \{\emptyset\}$ for all $H \in \mathcal{H}_1^-$ and $\mathcal{W}_H = \{N\}$ for all $H \in \mathcal{H}_1^+$; moreover, choose a voter $i \in N$ and set $\mathcal{W}_G = \{W \subseteq N : i \in W\}$ for all other $G \in \mathcal{H}$. Clearly, the corresponding voting by committees is non-dictatorial. We show that it is consistent. By the Intersection Property, the only problematic case is when a critical family $\mathcal{G}$ contains elements of $\mathcal{H}_1^-$. However, by Lemma 2a), if $G \in \mathcal{G} \cap \mathcal{H}_1^-$, we have $\mathcal{G} \setminus \{G\} \subseteq \mathcal{H}_1^+$, in which case the Intersection Property is clearly satisfied.

Next, suppose that $\mathcal{H}_1^+ \cup \mathcal{H}_1^-$ is empty, and consider first the case in which both $\mathcal{H}_0$ and $\mathcal{H}_2$ are non-empty. By Lemma 2b), no critical family $\mathcal{G}$ can meet both $\mathcal{H}_0$ and $\mathcal{H}_2$. Hence, we can specifiy two different dictators on $\mathcal{H}_0$ and $\mathcal{H}_2$, respectively, by setting $\mathcal{W}_H = \{W : i \in W\}$ for all $H \in \mathcal{H}_0$ and $\mathcal{W}_G = \{W : j \in W\}$ for all $G \in \mathcal{H}_2$ with $i \neq j$. Clearly, the Intersection Property is satisfied in this case.

Now suppose that $\mathcal{H}_2$ is also empty, i.e. $\mathcal{H} = \mathcal{H}_0$. Since $(X, \mathcal{H})$ is not totally blocked, $\mathcal{H}$ is partitioned in at least two equivalence classes with respect to the equivalence relation $\equiv$. Since, obviously, no critical family can meet two different equivalence classes, we can specify different dictators on different equivalence classes while satisfying the Intersection Property.

Finally, if $\mathcal{H}_0$ is empty, $(X, \mathcal{H})$ is a quasi-median space by the equivalence of (v) and (vi) in Theorem 3, hence the existence of non-dictatorial strategy-proof social choice functions follows as in the proof of Proposition 5.1 below.

Proof of Proposition 4.1 For $l = 4$, the l-cycle is isomorphic to the 2-dimensional hypercube which is clearly not totally blocked. Thus, assume first that l is even and $l \geq 6$. For all j, denote by $H_j := \{x_j, x_{j+1}, ..., x_{j-1+l/2}\}$, where indices are understood modulo l throughout. The family $\{H_j, H_{j-1+l/2}, H_{j-2}\}$ is a critical family. This implies $H_j \geq^0 H_{j-1}$ for all j, since $H_{j-1} = (H_{j-1+l/2})^c$. From this, the total blockedness is immediate.

Now consider l odd with $l \geq 5$ (the 3-cycle corresponds to the unrestricted domain over three alternatives which has already been shown to be totally blocked). For all j, denote by $H_j^- = \{x_j, x_{j+1}, ..., x_{j-1+(l-1)/2}\}$ and by $H_j^+ = \{x_j, x_{j+1}, ..., x_{j-1+(l+1)/2}\}$. Criticality of of the pair $\{H_j^-, H_{j+(l-1)/2}^-\}$ implies $H_j^- \geq^0 H_{j-1}^+$ for all j. Furthermore, criticality of the family $\{H_j^+, H_{j-1+(l+1)/2}^+, H_{j+1+(l+1)/2}^-\}$ implies both $H_j^+ \geq^0 H_{j+1}^+$ and $H_j^+ \geq^0 H_j^-$ for all j. From this, the total blockedness is again immediate.

Proof of Fact 4.2 Suppose that voting by committees possesses the local dictator i. Let x and y be two distinct potential preference peaks in the corresponding subdomain, and consider a separating basic property H with $x \in H$ and $y \in H^c$. Since i can force the outcome to lie in H even when all other voters have their peak at y, one has $\{i\} \in \mathcal{W}_H$; symmetrically, one also obtains $\{i\} \in \mathcal{W}_{H^c}$.

Conversely, if $\{i\} \in \mathcal{W}_H$ and $\{i\} \in \mathcal{W}_{H^c}$ for some H and some i, choose $x \in H$, $y \in H^c$, and a subdomain $\mathcal{D}$ consisting of two single-peaked preferences with peak at x and y, respectively. Evidently, i is a local dictator.

Proof of Proposition 4.2 Let $F : \mathcal{S}^n \to X$ be onto and strategy-proof where $\mathcal{S}$ is a rich single-peaked domain on $(X, \mathcal{H})$. Also, let $Y \subseteq X$ be convex such that $(Y, \mathcal{H}|_Y)$ is totally blocked. Denote by $\mathcal{S}^Y$ the set of all preferences in $\mathcal{S}$ that have their peak in Y, and by $\mathcal{S}_Y$ the set of the restrictions to Y of the preferences in $\mathcal{S}^Y$. Define $F_Y : [\mathcal{S}_Y]^n \to Y$ as follows. For all $\succ_i \in \mathcal{S}_Y$,

$$F_Y(\succ_1, ..., \succ_n) := F(\succ_1', ..., \succ_n'),$$

where, for each i, $\succ_i'$ is any extension of $\succ_i$ to X such that $\succ_i' \in \mathcal{S}^Y$, i.e. such that $\succ_i'$ is single-peaked on X with the same peak as $\succ_i$. Since F satisfies peaks only, the definition of F_Y does not depend on the choice of the extension. Clearly, F_Y is strategy-proof on $\mathcal{S}_Y$ and its range is Y; furthermore, $\mathcal{S}_Y$ is a rich single-peaked domain on $(Y, \mathcal{H}|_Y)$. By assumption, $(Y, \mathcal{H}|_Y)$ is totally blocked, hence F_Y is dictatorial, by Theorem 1. But this implies that F possesses a local dictator, since the restriction of F to the subdomain $\mathcal{S}^Y$ coincides with F_Y.

For the proofs of Propositions 4.3 and 4.4, the following lemma is useful.

Lemma 3 *Suppose that two neighbours x and y are separated by two distinct issues, i.e. $x \in H \cap H'$ and $y \in H^c \cap (H')^c$ for two distinct basic properties H, H'. Then, $H \equiv H'$.*

Proof of Lemma 3 By symmetry, it suffices to show that $H \geq^0 H'$. Clearly, this holds if $H \cap (H')^c = \emptyset$. Thus, assume that $H \cap (H')^c = \{z_1, ..., z_k\}$. Since no z_j is between x and y, there exist G_j (not necessarily distinct) such that, for all j, $z_j \in G_j$ and $G_j^c \supseteq \{x, y\}$. By construction, we have $H \cap (H')^c \cap G_1^c \cap ... \cap G_k^c = \emptyset$, hence $\{H, (H')^c, G_1^c, ..., G_k^c\}$ contains a critical family $\mathcal{G}$. Since $H \cap G_1^c \cap ... \cap G_k^c$ and $(H')^c \cap G_1^c \cap ... \cap G_k^c$ are both non-empty (containing x and y, respectively), $\mathcal{G}$ must contain H and $(H')^c$, hence $H \geq^0 H'$.

Proof of Proposition 4.3 Let γ be the underlying graph, and assume, by way of contradiction, that there exists an odd cycle in γ, i.e. a closed path with $2n+1$ nodes.

Denote by $S = \{x_1, ..., x_{2n+1}\}$ the elements corresponding to these nodes. By the oddness of the cycle and the assumption that the property space is graphic, all basic properties (relativized to CoS) either contain n or $n+1$ consecutive cycle elements. Moreover, any two neighbours x and y are separated by at least two distinct issues (H, H^c) and $(\tilde{H}, \tilde{H}^c)$. By Lemma 3, $H \equiv \tilde{H}$. Denoting by H_j any basic property that contains n consecutive cycle elements with x_j as their middle point, we therefore have $H_j \equiv (H_{j+n})^c$, where indices are understood modulo $2n+1$. By the oddness of the cycle, we thus obtain $H \equiv H'$ for all basic properties, which immediately implies the total blockedness of CoS.

For the proof of Proposition 4.4, we need the following additional lemma.

Lemma 4 *Consider a subspace $(Y, \mathcal{H}|_Y)$ of $(X, \mathcal{H})$ and H, G such that $H \cap Y, G \cap Y \in \mathcal{H}|_Y$. If $H \cap Y \geq_Y G \cap Y$, then $H \geq G$, where $\geq_Y$ denotes the conditional entailment relation of $(Y, \mathcal{H}|_Y)$.*

Proof of Lemma 4 It suffices to show that if there exists a critical family in $\mathcal{H}|_Y$ that contains $H \cap Y$ and $G^c \cap Y$, then there exists a critical family in $\mathcal{H}$ that contains H and G^c. Thus, suppose that $\mathcal{G} := \{H \cap Y, G^c \cap Y, G_1 \cap Y, ..., G_l \cap Y\}$ is critical in $(Y, \mathcal{H}|_Y)$. If $H \cap G^c \cap G_1 \cap ... \cap G_l = \emptyset$, then evidently $\{H, G^c, G_1, ..., G_l\}$ is critical in $(X, \mathcal{H})$. Thus, assume that $H \cap G^c \cap G_1 \cap ... \cap G_l \neq \emptyset$. Since Y is convex, $Y = H_1 \cap ... \cap H_k$ for appropriate H_j. One has $H \cap G^c \cap G_1 \cap ... \cap G_l \cap H_1 \cap ... \cap H_k = \emptyset$, hence $\{H, G^c, G_1, ..., G_l, H_1, ..., H_k\}$ contains a critical family $\mathcal{G}'$, and by the assumed criticality of $\mathcal{G}$, $\mathcal{G}'$ must contain both H and G^c.

Proof of Proposition 4.4 Take any two basic properties H, H' of $(Z, \mathcal{H}|_Z)$. If both $H \cap Y$ and $H' \cap Y$ are elements of $\mathcal{H}|_Y$, then by the total blockedness of $(Y, \mathcal{H}|_Y)$ and Lemma 4, $H \equiv H'$ (in $(Z, \mathcal{H}|_Z)$).

Now suppose that $H \subseteq Z \setminus Y$ while $H' \cap Y \in \mathcal{H}|_Y$. Since $Z = Co(Y \cup \{x\})$, one has $x \in H$. Let y and z be two neighbours of x in Y. Since z is not between x and y, there exists G with $G \supseteq \{x, y\}$ and $z \notin G$. Thus, (H, H^c) and (G, G^c) are two distinct issues separating the neighbours x and z, hence by Lemma 3, $H \equiv G$. Furthermore, since $G \cap Y \in \mathcal{H}|_Y$, $G \equiv G^c$ by Lemma 4 and the total blockedness of $(Y, \mathcal{H}|_Y)$. Hence, by transitivity and complementation adaptedness, also $H \equiv H^c$. Moreover, since both $G \cap Y$ and $H' \cap Y$ are in $\mathcal{H}|_Y$, one has $G \equiv H'$, which shows that $H \equiv H'$.

Finally, if both $H \subseteq Z \setminus Y$ and $H' \subseteq Z \setminus Y$, we obtain $H \equiv H'$ by the above arguments using transitivity since there exists at least one G such that $G \cap Y \in \mathcal{H}|_Y$. Combining the three cases, we obtain the total blockedness of $(Z, \mathcal{H}|_Z)$, as desired.

Proof of Theorem 2 Obviously, (i) implies (ii). Thus, it suffices to show that (ii) implies (iii), and that (iii) implies (i).

"(ii) $\Rightarrow$ (iii)" We prove the claim by contraposition. Assume that $(X, \mathcal{H})$ is not quasi-unblocked. This means that there exists $G \in \mathcal{H}$ with $G \equiv G^c$ and some critical family $\mathcal{G}$ such that $(\mathcal{H}_{\equiv G} \cap \mathcal{G}) \supseteq \{H, H', H''\}$ for three distinct H, H', H''. By Theorem A, any strategy-proof $F : \mathcal{S}^n \to X$ takes the form of voting by committees. By Fact 4.1, $\mathcal{W}_H = \mathcal{W}_G$ for all $H \in \mathcal{H}_{\equiv G}$. By Lemma 1, applied to the critical family $\mathcal{G} \supseteq \{H, H', H''\}$, there exists i, such that $\{i\} \in \mathcal{W}_H$ for all $H \in \mathcal{H}_{\equiv G}$. Hence, i is a dictator on $\mathcal{H}_{\equiv G}$, which proves the claim.

"(iii) $\Rightarrow$ (i)" We will construct a consistent voting by quota rule, provided that $(X, \mathcal{H})$

is quasi-unblocked. Partition $\mathcal{H}$ as above, i.e.

$$\begin{aligned}\mathcal{H}_0 &:= \{H \in \mathcal{H} : H \equiv H^c\},\\ \mathcal{H}_1^+ &:= \{H \in \mathcal{H} : H > H^c\},\\ \mathcal{H}_1^- &:= \{H \in \mathcal{H} : H^c > H\},\\ \mathcal{H}_2 &:= \{H \in \mathcal{H} : \text{ neither } H \geq H^c \text{ nor } H^c \geq H\}.\end{aligned}$$

Furthermore, partition $\mathcal{H}_2$ according to Lemma 2c) into $\mathcal{H}_2^-$ and $\mathcal{H}_2^+$. Let n be odd, and define a voting by quota rule by setting

$$\begin{array}{lll}\mathcal{W}_H = \{W : \#W > 1/2 \cdot n\} & \text{if} & H \in \mathcal{H}_0,\\ \mathcal{W}_H = 2^N \setminus \{\emptyset\} & \text{if} & H \in \mathcal{H}_1^- \cup \mathcal{H}_2^-,\\ \mathcal{W}_H = \{N\} & \text{if} & H \in \mathcal{H}_1^+ \cup \mathcal{H}_2^+.\end{array}$$

Thus, the quotas correspond to $q_H = \frac{1}{2}$ for $H \in \mathcal{H}_0$ and $q_H = 1$ for $H \in \mathcal{H}_1^+ \cup \mathcal{H}_2^+$. Using the Intersection Property, we will show that this rule is consistent. Consider any critical family $\mathcal{G}$; we distinguish three cases.

Case 1: $\mathcal{G} \cap (\mathcal{H}_1^- \cup \mathcal{H}_2^-) \neq \emptyset$. If $G \in \mathcal{G} \cap \mathcal{H}_1^-$, then by Lemma 2a), $\mathcal{G} \setminus \{G\} \subseteq \mathcal{H}_1^+$, and the Intersection Property is clearly satisfied. Thus, suppose that there exists $H \in \mathcal{G} \cap \mathcal{H}_2^-$. By Lemma 2b), we must have $\mathcal{G} \cap \mathcal{H}_0 = \emptyset$, and by Lemma 2a), $\mathcal{G} \cap \mathcal{H}_1^- = \emptyset$. Hence, if there exists $H' \in \mathcal{G} \setminus \{H\}$ with $\mathcal{W}_{H'} \neq \{N\}$, we must have $H' \in \mathcal{H}_2^-$. But then $H \geq (H')^c$ contradicts the construction of $\mathcal{H}_2^-$ and $\mathcal{H}_2^+$ in Lemma 2c). Thus, if $H \in \mathcal{G} \cap \mathcal{H}_2^-$, one has $\mathcal{W}_{H'} = \{N\}$ for any other element $H' \in \mathcal{G}$, in which case the Intersection Property is satisfied.

Case 2: $\mathcal{G} \cap \mathcal{H}_0 \neq \emptyset$. First, observe that $G_1 \equiv G_2$ whenever $\{G_1, G_2\} \subseteq \mathcal{G} \cap \mathcal{H}_0$. Indeed, $G_1 \equiv G_2$ follows at once from $G_1 \geq G_2^c$, $G_2 \geq G_1^c$, $G_1 \equiv G_1^c$ and $G_2 \equiv G_2^c$. Thus, by quasi-unblockedness, $\mathcal{G}$ can contain at most two elements of $\mathcal{H}_0$. By Lemma 2b), for any $H \in \mathcal{G} \setminus \mathcal{H}_0$ one has $\mathcal{W}_H = \{N\}$. Hence, the Intersection Property is also satisfied in Case 2.

Case 3: If $\mathcal{G}$ does not meet $\mathcal{H}_0$, $\mathcal{H}_1^-$ and $\mathcal{H}_2^-$, then $\mathcal{G} \subseteq (\mathcal{H}_1^+ \cup \mathcal{H}_2^+)$, in which case the Intersection Property is trivially satisfied. This completes the proof of Theorem 2.

Proof of Fact 5.1 It is clear that $F_{\hat{x}}$ defines a unanimity rule. Conversely, under voting by committees, (5.1) implies $\mathcal{W}_H = \{N\}$ for any property H with $H \not\ni \hat{x}$; by (3.2), this implies $\mathcal{W}_H = 2^N \setminus \{\emptyset\}$ for all $H \ni \hat{x}$.

For the proofs of Proposition 5.1 and Theorem 3, we need the following lemma from Nehring (2004); for the sake of self-containedness, we reproduce its proof here. For any $x \in X$, denote by $\mathcal{H}_x := \{H \in \mathcal{H} : x \in H\}$.

Lemma 5 *$\hat{x} \in M(X)$ if and only if for any critical family $\mathcal{G}$, $\#(\mathcal{H}_{\hat{x}} \cap \mathcal{G}) \leq 1$.*

Proof of Lemma 5 Let $x \in M(X)$; we verify $\#(\mathcal{H}_x \cap \mathcal{G}) \leq 1$ by contradiction. Thus, assume that, for some critical family $\mathcal{G}$, $\mathcal{H}_x \cap \mathcal{G} \supseteq \{H_1, H_2\}$. Since $x \in H_1 \cap H_2$, there exits a $G \in \mathcal{G}$ different from H_1 and H_2. By criticality, one can choose $y \in \cap(\mathcal{G} \setminus \{H_1\})$ and $z \in \cap(\mathcal{G} \setminus \{H_2\})$. By construction, $[x,y] \subseteq H_2$, $[x,z] \subseteq H_1$ and $[y,z] \subseteq \cap(\mathcal{G} \setminus \{H_1, H_2\})$. But then $[x,y] \cap [x,z] \cap [y,z] \subseteq \cap\mathcal{G} = \emptyset$, contradicting the fact that $x \in M(X)$.

Conversely, suppose that $x \notin M(X)$, i.e. $[x,y] \cap [x,z] \cap [y,z] = \emptyset$ for some y, z. Define $\mathcal{H}_{xy} := \{H \in \mathcal{H} : \{x,y\} \subseteq H\}$, $\mathcal{H}_{xz} := \{H \in \mathcal{H} : \{x,z\} \subseteq H\}$ and $\mathcal{H}_{yz} :=$

$\{H \in \mathcal{H} : \{y,z\} \subseteq H\}$. By assumption, one has $(\cap\mathcal{H}_{xy}) \cap (\cap\mathcal{H}_{xz}) \cap (\cap\mathcal{H}_{yz}) = \emptyset$, hence $\mathcal{H}_{xy} \cup \mathcal{H}_{xz} \cup \mathcal{H}_{yz}$ contains a critical family $\mathcal{G}$. Any such critical family must contain H with $H \cap \{x,y,z\} = \{x,y\}$, H' with $H' \cap \{x,y,z\} = \{x,z\}$ and H'' with $H'' \cap \{x,y,z\} = \{y,z\}$. But this implies $\#(\mathcal{H}_x \cap \mathcal{G}) \geq 2$ since $x \in H \cap H'$.

Proof of Proposition 5.1 Let $F_{\hat{x}}$ be consistent and consider $\mathcal{H}_{\hat{x}}$, the family of all properties possessed by $\hat{x}$. Since $\mathcal{W}_H = 2^N \setminus \{\emptyset\}$ for all $H \in \mathcal{H}_{\hat{x}}$, the Intersection Property implies that $\#(\mathcal{H}_{\hat{x}} \cap \mathcal{G}) \leq 1$ for any critical family $\mathcal{G}$ (otherwise, if $H, H' \in \mathcal{H}_{\hat{x}} \cap \mathcal{G}$ with $H \neq H'$, one could choose $W \in \mathcal{W}_H$ and $W' \in \mathcal{W}_{H'}$ with $W \cap W' = \emptyset$, contradicting the assumed consistency). By Lemma 5, $\hat{x} \in M(X)$.

Conversely, Lemma 5 implies that for any median point $\hat{x} \in M(X)$, the unanimity rule $F_{\hat{x}}$ satisfies the Intersection Property.

To prove the last statement in Proposition 5.1, observe first that the outcome under voting by committees always lies in the convex hull of the voters' peaks since any basic property containing all voters' peaks gets unanimous support. The claim then follows at once from the fact that $\{i\} \in \mathcal{W}_H$ whenever $H \supseteq \{\hat{x}, x_i^*\}$.

Proof of Theorem 3 We first show the implication "(v) $\Rightarrow$ (vi)." Thus, suppose that for all $H \in \mathcal{H}$, $H \not\equiv H^c$. Partition $\mathcal{H}$ into $\mathcal{H}_1^-$, $\mathcal{H}_1^+$, $\mathcal{H}_2^-$ and $\mathcal{H}_2^+$ as above, where $\mathcal{H}_2^-$ and $\mathcal{H}_2^+$ are determined according to Lemma 2c). Then, any critical family $\mathcal{G}$ can meet $\mathcal{H}_1^- \cup \mathcal{H}_2^-$ at most once. Indeed, by Lemma 2a), $H \in \mathcal{G} \cap \mathcal{H}_1^-$ implies $\mathcal{G} \setminus \{H\} \subseteq \mathcal{H}_1^+$. Furthermore, if $\{H, H'\} \subseteq \mathcal{G} \cap \mathcal{H}_2^-$, one would obtain $H' \geq H^c$ which contradicts the construction of $\mathcal{H}_2^-$. But this implies that $\cap(\mathcal{H}_1^- \cup \mathcal{H}_2^-)$ is non-empty (otherwise it would contain a critical family), and by H3, it consists of a single element, say x. By Lemma 5, $x \in M(X)$.

Conversely, to verify "(vi) $\Rightarrow$ (v)," let $x \in M(X)$, and consider any $H \in \mathcal{H}_x$. Then, $H \geq^0 G$ implies $G \in \mathcal{H}_x$. Indeed, by definition, $H \geq^0 G$ means that $\{H, G^c\} \subseteq \mathcal{G}$ for some critical family $\mathcal{G}$. By Lemma 5, $\mathcal{G}$ contains at most one element of $\mathcal{H}_x$, hence $G^c \notin \mathcal{H}_x$, which implies $G \in \mathcal{H}_x$. This observation immediately implies $H \not\equiv H^c$.

The equivalence of (vi) and (iv) follows at once from Fact 5.1 and Proposition 5.1. The equivalence of (iii) and (iv) then follows from the observation that for $n = 2$ unanimity rules exhaust the class of locally non-dictatorial and strategy-proof social choice rules. The implications "(iv) $\Rightarrow$ (ii)" and "(ii) $\Rightarrow$ (i)" are evident. Thus, the proof is completed by verifying the implication "(i) $\Rightarrow$ (v)." This is done by contraposition. Thus, assume that H is blocked, i.e. $H \equiv H^c$. By Fact 4.1 this implies $\mathcal{W}_H = \mathcal{W}_{H^c}$ for any consistent committee structure. Under anonymity, this implies $q_H = q_{H^c} = \frac{1}{2}$, which is compatible with (3.2) only if the number of voters is odd.

Proof of Theorem 4 a) By Theorem 3, any quasi-median space admits at least one strategy-proof unanimity rule, and any such rule is neutral across issues and non-dictatorial.

Conversely, let $F : \mathcal{S}^n \to X$ be strategy-proof and neutral across issues. By Theorem B, F must be voting by committees satisfying the Intersection Property. We show by contraposition that if F is non-dictatorial, then $(X, \mathcal{H})$ must be a quasi-median space. Thus, suppose that $(X, \mathcal{H})$ is not a quasi-median space. By Theorem 3, there exists a basic property H that is blocked, i.e. $H \equiv H^c$. By Fact 4.1, this implies $\mathcal{W}_H = \mathcal{W}_{H^c}$, hence F is fully neutral, i.e. $\mathcal{W}_H = \mathcal{W}_0$ for all H and some fixed committee $\mathcal{W}_0$. Since $(X, \mathcal{H})$ is not a median space, there exists by NP, Proposition 4.1, a critical family $\mathcal{G}$ with at least three elements, say $\mathcal{G} \supseteq \{G_1, G_2, G_3\}$. By Lemma 1 above, $\{i\} \in \mathcal{W}_{G_3^c} = \mathcal{W}_0$, i.e. voter i is a dictator.

b) By NP, Proposition 4.1, median spaces are characterized by the property that all critical families have cardinality two. By (3.3) this implies that, e.g., issue-by-issue majority voting is consistent on any median space, and evidently, issue-by-issue majority voting is neutral, in particular neutral within issues.

Conversely, let $F : \mathcal{S}^n \to X$ be strategy-proof and neutral within issues. By Theorem B, F must be voting by committees satisfying the Intersection Property. We show by contraposition that if F is locally non-dictatorial, then $(X, \mathcal{H})$ must be a median space. Thus, suppose that $(X, \mathcal{H})$ is not a median space. Then there exists a critical family $\mathcal{G}$ with at least three elements, say $\mathcal{G} \supseteq \{G_1, G_2, G_3\}$, in particular, $G_j \geq G_k^c$ for distinct $j, k \in \{1, 2, 3\}$. By Fact 4.1, $\mathcal{W}_{G_j} \subseteq \mathcal{W}_{G_k^c}$ for distinct $j, k \in \{1, 2, 3\}$. Under neutrality within issues this implies at once that $\mathcal{W}$ assigns identical committees to G_1, G_2, G_3 and their respective complements. By Lemma 1 above, $\{i\} \in \mathcal{W}_{G_3^c}$, i.e. voter i is a local dictator.

c) As in part b), an underyling median space guarantees the existence of a fully neutral rule. The converse follows from part b) together with the observation that, under full neutrality, a local dictator must even be a global dictator.

Proof of Proposition 6.1 Suppose that $(X, \mathcal{H})$ is indecomposable. Then, for any $H, H' \in \mathcal{H}$, at least one of the following holds, $H' \geq H$, $H' \geq H^c$, $(H')^c \geq H$, or $(H')^c \geq H^c$. Indeed, otherwise the subfamilies $\mathcal{H}_1 := \{G \in \mathcal{H} : G \geq H, G \geq H^c, G^c \geq H, \text{ or } G^c \geq H^c\}$ and $\mathcal{H}_2 := \mathcal{H} \backslash \mathcal{H}_1$ form a decomposition, as is easily verified. The claim follows immediately from this observation using the complementation adaptedness of $\geq$ and Fact 4.1.

References

[1] Aswal, N., S. Chatterji and A. Sen (2003), Dictatorial Domains, *Economic Theory* **22**, 45-62.

[2] Barberá, S., F. Gul and E. Stacchetti (1993), Generalized Median Voter Schemes and Committees, *Journal of Economic Theory* **61**, 262-289.

[3] Barberá, S., J. Massò and A. Neme (1997), Voting under Constraints, *Journal of Economic Theory* **76**, 298-321.

[4] Barberá, S., H. Sonnenschein and L. Zhou (1991), Voting by Committees, *Econometrica* **59**, 595-609.

[5] Black, D. (1958), *The Theory of Committees and Elections*, Cambridge University Press, Cambridge.

[6] Bogomolnaia, A. (1999), A Characterization of Median Rules, *mimeo*.

[7] Border, K. and J. Jordan (1983), Straightforward Elections, Unanimity and Phantom Voters, *Review of Economic Studies* **50**, 153-170.

[8] Demange, G. (1982), Single Peaked Orders on a Tree, *Mathematical Social Sciences* **3**, 389-396.

[9] Moulin, H. (1980), On Strategy-Proofness and Single-Peakedness, *Public Choice* **35**, 437-455.

[10] Nehring, K. (1999), Diversity and the Geometry of Similarity, *mimeo*.

[11] Nehring, K. (2003), Arrow's Theorem as a Corollary, *Economics Letters* **80**, 379-382.

[12] Nehring, K. (2004), Social Aggregation without Veto, *mimeo*.

[13] Nehring, K. and C. Puppe (2002), Strategy-Proof Social Choice on Single-Peaked Domains: Possibility, Impossibility and the Space Between, *mimeo*.

[14] Nehring, K. and C. Puppe (2003), Efficient and Strategy-Proof Voting Rules: A Characterization, forthcoming *Games and Economic Behavior*.

[15] Nehring, K. and C. Puppe (2004), The Structure of Strategy-Proof Social Choice. Part I: General Characterization and Possibility Results on Median Spaces, forthcoming *Journal of Economic Theory*, downloadable at: http://www.wior.uni-karlsruhe.de/LS_Puppe/Personal/puppe_html

[16] Nehring, K. and C. Puppe (2005), Consistent Judgement Aggregation: A Characterization, *mimeo*.

[17] Schummer, J. and R. Vohra (2002), Strategy-Proof Location on a Network, *Journal of Economic Theory* **104**, 405-428.

[18] van de Vel, M. (1993), *Theory of Convex Structures*, North Holland, Amsterdam.