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Axiomatic districting

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Abstract We study the districting problem from an axiomatic point of view in a framework with two parties, deterministic voter preferences and geographical constraints. The axioms are normatively motivated and reflect a notion of fairness to voters. Our main result is an "impossibility" theorem demonstrating that all anonymous districting rules are necessarily complex in the sense that they either use information beyond the mere *number* of districts won by the parties, or they violate an appealing consistency requirement according to which an acceptable districting rule should induce an acceptable districting of appropriate subregions.

Keywords Districting, gerrymandering, normative political analysis.

JEL Classifications D72

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1 Introduction

The districting problem has received considerable attention recently, both from the political science and the economics viewpoint.[1] Much of the recent work has focused on strategic aspects and the incentives induced by different institutional designs on the political parties, legislators and voters (see, among others, Besley and Preston, 2007, Friedman and Holden, 2008, Gul and Pesendorfer, 2010). Other contributions have looked at the welfare implications of different redistricting policies (e.g. Coate and Knight, 2007). Finally, there is also a sizable literature on the computational aspects of the districting problem (see, e.g. Puppe and Tasnádi, 2008, and the references therein, and Ricca, Scozzari and Simeone, 2011, for a general overview of the operations research literature on the districting problem).

In contrast to these contributions, the present paper takes a *normative* point of view. We formulate desirable properties ("axioms"), and investigate which districting rules satisfy them. The axiomatic method allows one to endow the vast space of conceivable districting rules with useful additional structure: each combination of desirable properties characterizes a specific class of districting rules, and thereby helps one to assess their respective merits. Furthermore, one may hope that specific combinations of axioms single out a few, perhaps sometimes even a unique districting rule, thus reducing the space of possibilities. Finally, the axiomatic approach may reveal incompatibility of certain axioms by showing that *no* districting rule can satisfy certain combinations of desirable properties, thereby terminating a futile search.

In a framework with two parties and geographical constraints on the shape of districts, we propose a set of five simple axioms which are motivated by considerations of fairness to voters; importantly, and in contrast to other contributions in the literature, we assume that the distribution of supporters of each of the two parties is known. A *solution* assigns to each districting problem a set of acceptable districtings, i.e. partitions of the geographical region into equally sized districts. The first three axioms restrict the informational basis needed for the construction of acceptable ("fair") districtings. Essentially, they jointly amount to the requirement that the only information that may enter a fair districting rule is the *number* of districts won by the two parties, where a party "wins" a district if it receives the majority support in this district. The motivation for such a requirement is that, ultimately, voters care only about outcomes, i.e. the implemented policies, but these outcomes only depend on the distribution of seats in the parliament – through some political decision process that is not explicitly modeled here. Restricting the informational basis for the assessment of districting rules to the possible seat distributions they imply is also attractive from the viewpoint of managing the complexity of the districting problem, since evidently it greatly simplifies the issue. Our approach is thus "consequentialist" in the sense that the relative merits of a districting are measured only by the possible outcomes it produces. The destricting pro-

[1] See, e.g., Tasnádi (2011) for an overview.

cess as such does not matter. We emphasize that the geographical constraints nevertheless play an important role: they enter *indirectly* in the assessment of districtings since they influence the possible numbers of districts each party can win. For instance, a bias in the seat share in favor of one party may be acceptable if it is forced by the given geographical constraints, but not if it is avoidable by an alternative admissible districting.

The first two axioms apply only to the case in which the parliament has exactly two seats, and thus exactly two districts have to be formed. Concretely, the first axiom, *"two-district determinacy,"* requires that for any fixed two-district problem, a solution must not leave open the question whether a party receives both seats in the parliament or only one. In other words, if for a given distribution of party supporters, it is possible to form one districting under which either party wins one seat, and another districting under which one party wins both seats, then not both districtings can be acceptable. The second axiom, *"two-district uniformity,"* requires that a solution treats all two-district problems uniformly in the sense that it only depends on the set of the possible distributions of the number of districts won by each party. The motivation for this requirement is discussed in detail in Section 4 below. The third axiom, *"indifference,"* requires that if two different districtings induce the same seat shares in the parliament for a fixed problem, then either none or both are acceptable. The motivation behind this principle is that two districtings which induce the same distribution of seats in the parliament are indistinguishable in terms of final outcomes, hence either none or both should be considered to be fair.

Our fourth condition, the *"consistency"* axiom, requires that an acceptable districting should induce acceptable sub-districtings on all appropriate subregions. This axiom reflects the normative principle that a "fair" institution must be fair in every part (cf. Balinski and Young's *uniformity principle*, 2001), or more concretely in our context: a representation of voters via a districting is globally fair only if it is also locally fair. The consistency condition greatly simplifies the internal structure of the admissible districtings, too.

The fifth and final *"anonymity"* axiom requires that a solution should be invariant with respect to a re-labeling of parties. In our context, such anonymity requirement has a straightforward normative interpretation in terms of fairness since it amounts to an equal treatment of parties (and voters) *ex-ante.*

An important conceptual ingredient (and mathematical challenge) of our analysis is the presence of geographical constraints. We model this via an exogenously given collection of *admissible* districts from which a districting selects a subset that forms a partition of the entire region. We impose one restriction on the collection of admissible districts other than the standard requirement of equal population mass: that it be possible to move from one admissible districting to any other admissible districting via a sequence of intermediate districtings changing only two districts at each step.[2] This *"linked-*

[2] Since a districting forms a partition of the given region, it is evidently not possible to move from one districting to another districting by changing only *one* district.

ness" condition is satisfied by a large class of geographies. Except for a technical "no-ties" assumption, no other restriction is imposed on the collection of admissible districts, thus our approach is very general in this respect. In particular, the *absence* of geographical constraints can be modeled by taking *all* subregions of equal population mass as the collection of admissible districts (which gives rise to a linked geography). Moreover, restrictions that are frequently imposed on the shape of districts in practice, such as compactness or contiguity, can in principle be incorporated in our approach by an appropriate choice of admissible districts; for an explicit analysis of these and related issues, see e.g. Chambers and Miller (2010, 2013) and the references given there.

We prove that on all linked geographies, the first four of our axioms jointly characterize the districting rule which maximizes the number of districts that *one* party can win, given the distribution of individual votes (the "optimal gerrymandering rule"). Evidently, by generating a maximal number of winning districts for one of the two parties, the optimal gerrymandering rule violates the anonymity condition. As a corollary, we therefore obtain that no districting rule can satisfy all five axioms. The result also suggests that any acceptable districting rule must necessarily be complex: either it has a complex internal structure by violating the consistency principle, or it has to employ a complex informational basis in the sense that it depends on more than the mere *number* of districts won by each party.

The work closest to ours in the literature is Chambers (2008, 2009) who also takes an axiomatic approach. However, one of his central conditions is the requirement that the election outcome be *independent* of the way districts are formed ("gerrymandering-proofness"), and the main purpose of his analysis is to explore the consequences of this requirement (for a similar approach, see Bervoets and Merlin, 2012). By contrast, our focus is precisely on the issue how the districting influences the election outcome, and the aim of our analysis is to structure the vast space of possibilities by means of simple principles. In particular, geographical constraints which are absent in Chambers' model play an important role in our analysis.

The paper by Landau, Reid and Yershov (2009) also addresses the issue of "fair" districting. However, unlike our work their paper is concerned with the question of how to *implement* a fair solution to the districting problem by letting the parties themselves determine the boundaries of districts. Specifically, these authors propose a protocol similar in spirit to the well-known divide-and-choose procedure.

The districting rules that we consider depend among other things on the distribution of votes for each party in the population. One might argue, perhaps on grounds of some "absolute" notion of *ex ante* fairness, that a districting rule must not depend on voters' party preferences since these can change over time. From this perspective, the districting problem is not really an issue and it would seem that any districting which partitions the population in (roughly) equally sized subgroups should be acceptable. By contrast, in the present paper we are interested in a "relative" or *ex post* notion of fair districting, i.e. in the question of what would constitute an acceptable districting rule *given* the distribution of the supporters of each party in the population. This question seems particularly important for practical purposes since a districting policy can be successfully implemented only if it receives sufficient support by the *actual* legislative body.

2 The Framework

We assume that parties A and B compete in an electoral system consisting only of single member districts, where the representatives of each district are determined by plurality. The parties as well as the independent bodies face the following districting problem.

Definition 1 A **districting problem** is given by the structure $\Pi = (X, \mathcal{A}, \mu, \mu_A, \mu_B, t, G)$, where

- the voters are located within a subset X of the plane $\mathbb{R}^2$,
- $\mathcal{A}$ is the σ-algebra on X consisting of all districts that can be formed without geographical or any other type of constraints,
- the distribution of voters is given by a measure μ on $(X, \mathcal{A})$,
- the distributions of party A and party B supporters are given by measures μ_A and μ_B on $(X, \mathcal{A})$ such that $\mu = \mu_A + \mu_B$,
- t is the given number of seats in parliament,
- $G \subseteq \mathcal{A}$, also called *geography*, is a collection of admissible districts satisfying $\mu(g) = \mu(X)/t$ and
$$\mu_A(g) \neq \mu_B(g) \tag{1}$$
for all $g \in G$, and admitting a partitioning of X, i.e there exist mutually disjoint sets $g'_1, \ldots, g'_t \in G$ such that $\cup_{i=1}^t g'_i = X$.

Condition (1) excludes ties in the distribution of party supporters in all admissible districts to avoid the necessity of introducing tie-breaking rules. This condition is satisfied, for instance, if the set of voters is finite, μ, μ_A, μ_B are the counting measures and the district sizes are odd.

Definition 2 A **districting** for problem $\Pi = (X, \mathcal{A}, \mu, \mu_A, \mu_B, t, G)$ is a subset $D \subseteq G$ such that D forms a partition of X and $\#D = t$.

We shall denote by $\delta_A(D)$ and $\delta_B(D)$ the number of districts won by party A and party B under D, respectively. We write $\mathcal{D}_\Pi$ for the set of all districtings of problem Π and let $\delta_A(\mathcal{D}) = \{\delta_A(D) : D \in \mathcal{D}\}$ and $\delta_B(\mathcal{D}) = \{\delta_B(D) : D \in \mathcal{D}\}$ for any $\mathcal{D} \subseteq \mathcal{D}_\Pi$.

Definition 3 A **solution** F associates to each districting problem Π a non-empty set of chosen districtings $F_\Pi \subseteq \mathcal{D}_\Pi$.

3 Several Solutions

We now present a number of simple solution candidates. In all of the following definitions, $\Pi = (X, \mathcal{A}, \mu, \mu_A, \mu_B, t, G)$ denotes a generic districting problem. The first solution determines the optimal partisan gerrymandering from the viewpoint of party A.

Definition 4 The **optimal solution** O^A **for party** A determines the set of those districtings that maximize the number of winning districts for party A, i.e.

$$O^A_\Pi = \arg \max_{D \in \mathcal{D}_\Pi} \delta_A(D).$$

Evidently, in the absence of other objectives, O^A is the solution favored by party A supporters. The optimal solution O^B for party B is defined analogously. If we are referring to an optimal solution O, then we have either O^A or O^B in mind.

The next solution minimizes the difference in the number of districts won by the two parties. It has an obvious egalitarian spirit.

Definition 5 The **most equal solution** *ME* determines the districtings that minimize the difference of the seats won by the two parties, i.e.

$$ME_\Pi = \arg \min_{D \in \mathcal{D}_\Pi} \left| \delta_A(D) - \delta_B(D) \right|. \tag{2}$$

Clearly, depending on the distribution of votes in the population, an equal distribution of seats in the parliament may not be possible. The most equal solution aims to get as close as possible to equality in terms of the number of winning districts for the two parties.

The third solution maximizes the difference in the number of districts won by the two parties. The objective to maximize the winning margin of the ruling party could be motivated, for instance, by the desire to avoid too much political compromise.

Definition 6 The **most unequal solution** *MU* determines the districtings that maximize the difference of the seats won by the two parties, i.e.

$$MU_\Pi = \arg \max_{D \in \mathcal{D}_\Pi} \left| \delta_A(D) - \delta_B(D) \right|. \tag{3}$$

Fourth, we consider the solution that minimizes partisan bias. It has a clear motivation from the point of view of maximizing representation of the "people's will" in the sense that the share of the districts won by each party is as close as possible to its share of votes in the population.

Definition 7 The **least biased solution** *LB* determines the set of those districtings that minimize the absolute difference between shares in winning districts and shares in votes, i.e.

$$LB_\Pi = \arg \min_{D \in \mathcal{D}_\Pi} \left| \frac{\delta_A(D)}{t} - \frac{\mu_A(X)}{\mu(X)} \right| = \arg \min_{D \in \mathcal{D}_\Pi} \left| \frac{\delta_B(D)}{t} - \frac{\mu_B(X)}{\mu(X)} \right|. \tag{4}$$

Finally, we mention the trivial solution that associates to each problem the set of *all* admissible districtings.

Definition 8 The **complete solution** C associates with any districting problem the set of all possible districtings $\mathcal{D}_\Pi$.

4 Axioms

In this section, we formulate five simple axioms and argue that each has appeal from a normative (and sometimes also from a pragmatic) point of view. In the statements of the following axioms, F denotes a generic solution.

The case of two districts plays a fundamental role in our analysis. Note that by (1) it is not possible that a party can win both districts under one districting and lose both districts under another districting, i.e. if $t = 2$ then $\delta_A(\mathcal{D}_\Pi)$ (respectively, $\delta_B(\mathcal{D}_\Pi)$) cannot contain both 0 and 2. Our first axiom requires that a solution must in fact be "determinate" in the two-district case in the sense that it must not leave open the issue whether there is a draw between the two parties or a victory for one party. In other words, if a solution chooses a districting that results in a draw between the parties for a given problem it cannot choose another districting *for the same* problem that results in a victory for one party.

Two-district determinacy For all districting problems Π with $t = 2$, the sets $\delta_A(F_\Pi)$ and $\delta_B(F_\Pi)$ are singletons.[3]

The motivation for this axiom stems from our implicit assumption that voters do not care about the districtings as such, but only about the entailed shares of seats in the parliament, since it is the latter that influences final outcomes. Any indeterminacy in the distribution of seats in the parliament potentially influences the outcome and would thus introduce an element of *arbitrariness* of the final state of affairs. In the two-district case, such indeterminacy necessarily turns a (unanimous) victory of one party into a draw between the two parties, or vice versa. *Two-district determinacy* prevents this to occur.

Evidently, all solutions considered in Section 3 with the exception of the complete solution C satisfy two-district determinacy. Also observe that on the family of all two-district problems the most equal solution *ME* and the least biased solution *LB* coincide.[4]

Our next axiom requires that a solution behaves "uniformly" on the set of two-district problems in the sense that the solution must treat different

[3] Observe that overall determinacy, i.e. that $\delta_A(F_\Pi)$ and $\delta_B(F_\Pi)$ be singletons for *every* problem Π, is a strictly stronger requirement than *two-district determinacy*; for instance, the least biased solution satisfies *two-district determinacy* but can easily be shown to violate overall determinacy.

[4] To verify this, observe that if there exist admissible districtings $D, D' \in \mathcal{D}_\Pi$ with $\delta_A(D) = 2$ and $\delta_A(D') = 1$, then one must have $0.5 < \mu_A(X)/\mu(X) < 0.75$. Thus, D' must be chosen both by *ME* and *LB*.

two-district problems in the same way, provided they admit the same set of possible distributions of the number of districts won by each party.

Two-district uniformity For all districting problems Π and Π' with $t = 2$ such that $\delta_A(\mathcal{D}_\Pi) = \delta_A(\mathcal{D}_{\Pi'})$ (and therefore also $\delta_B(\mathcal{D}_\Pi) = \delta_B(\mathcal{D}_{\Pi'})$) we have $\delta_A(F_\Pi) = \delta_A(F_{\Pi'})$ (and therefore also $\delta_B(F_\Pi) = \delta_B(F_{\Pi'})$).

Even though it is imposed only in the two-district case, *two-district uniformity* is admittedly a strong requirement. It can be motivated by invoking again the assumption that voters care about districtings only via their influence on political outcomes. From this perspective, *two-district uniformity* states that if the *possible* political outcomes are the same in different two-district problems, then the *actual* outcome should also be the same. A violation of *two-district uniformity* would mean that characteristics other than the possible distributions of seat shares can influence the solution and hence the final outcome. But if these characteristics play no role in voters' preferences, it is not clear how one could justify such influence. To illustrate, consider two districting problems Π and Π' with $\delta_A(\mathcal{D}_\Pi) = \delta_A(\mathcal{D}_{\Pi'}) = \{1, 2\}$; thus, in either situation there exists one districting under which party A wins both districts and another districting which produces a draw between the two parties. Now assume that party A's share of votes in situation Π is in fact larger than its share of votes in situation Π', i.e. $\mu_A > \mu'_A$. Couldn't this give a good reason to select the districting under which A wins both seats in situation Π but the districting in which both parties receive one seat in situation Π', provided that the difference between μ_A and μ'_A is sufficiently large? But then, how large precisely is "sufficiently large"? Is x% enough? And wouldn't the threshold also have to depend on the absolute level of μ_A? *Two-district uniformity* answers these question by a very clearcut and simple recommendation: different treatment of different two-district situations, for instance on the grounds that one party has a larger share of votes in one of the situations, is justified only if the difference *manifests itself* in a difference of the possible number of seats in parliament that the parties can win. *Two-district uniformity* thus sets a high "threshold" for differential treatment of two-district situations. We emphasize therefore that all candidate solutions presented in Section 3 above satisfy this condition; for the least biased solution this follows from Footnote 4, for the other solutions it is evident.

A secondary motivation for *two-district uniformity* is to keep the complexity of a districting solution manageable. Indeed, any influence of characteristics different from the possible seat distribution in parliament – whether derived from the underlying distribution of party supporters or from geographical information – would considerably complicate the definition and implementation of a districting rule.

Our third axiom, imposed on districting problems of any size, has a motivation related to that of the two previous axioms. It states that if a possible districting induces the same distribution of the number of winning districts for each party than some districting chosen by a solution, it must be chosen by this solution as well.

Indifference For all districting problems Π, if $D \in F_\Pi$, $D' \in \mathcal{D}_\Pi$, $\delta_A(D) = \delta_A(D')$ and $\delta_B(D) = \delta_B(D')$, then $D' \in F_\Pi$.

The justification of the *indifference* axiom is straightforward under the intended notion of fairness to voters. If voters care only about final outcomes, and if final outcomes only depend on seat shares, then two districtings that entail the same seat distribution in parliament are undistinguishable in terms of final outcomes and have therefore to be treated equally. Evidently, all solutions presented above satisfy this condition.

The following consistency axiom plays a central role in our analysis. It requires that a solution to a problem should also deliver appropriate solutions to specific subproblems. Its spirit is very similar to the *uniformity principle* in Balinski and Young's (2001) theory of apportionment.

Prior to the definition of consistency we have to introduce specific subproblems of a districting problem. For any problem Π, any $D \in F_\Pi$ and any $D' \subseteq D$, let $Y = \cup_{d \in D'} d$ and define the subproblem $\Pi|_Y$ to be $(Y, \mathcal{A}|_Y, \mu|_Y, \mu_A|_Y, \mu_B|_Y, \#D', G|_Y)$, where $\mathcal{A}|_Y = \{A \cap Y : A \in \mathcal{A}\}$, $G|_Y = \{g \in G : g \subseteq Y\}$ and $\mu|_Y, \mu_A|_Y, \mu_B|_Y$ stand for the restrictions of measures μ, μ_A, μ_B to $(Y, \mathcal{A}|_Y)$.

Consistency For all districting problems Π, $D \in F_\Pi$ and $D' \subseteq D$ imply

$$D' \in F_{\Pi|_Y},$$

where $Y = \cup_{d \in D'} d$.

The motivation for imposing *consistency* in our context is as follows. Most federal countries have both federal and local legislatures, and in many of those countries the *same* districts are used for both, local and federal elections. The *consistency* axiom requires that a districting can be considered an acceptable global solution only if it also delivers acceptable solutions on all appropriate subregions.[5] In the intended interpretation of acceptability in terms of fairness to voters, *consistency* thus forbids to create a globally fair treatment of voters by equilibrating different locally unfair treatments. Moreover, it justifies using the *same* districts locally and globally – as is common practice in most countries. Finally, *consistency* may also be of practical value if regions decide to separate, or to increase political independence, since it would allow them to use the same districting as before.

The optimal and complete solutions satisfy *consistency*. This is evident for the complete solution. To verify it for the optimal solution suppose, by contradiction, that there would exist $D' \subset D \in O^A_\Pi$ such that $D' \notin O^A_{\Pi|_Y}$, where $Y = \cup_{d \in D'} d$. This would imply $\delta_A(D'' \cup (D \setminus D')) > \delta_A(D)$ for any $D'' \in O^A_{\Pi|_Y}$, a contradiction.

By contrast, the other solutions considered in Section 2 violate consistency. This can be verified by considering the districting problem Π with $t = 3$ shown in Fig. 1. It consists of 27 voters of which 11 are supporters of party

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[5] Clearly, this requirement has to be restricted to subregions that are *unions* of districts, since a given districting does in general not induce an admissible sub-districting on other subregions.

A (indicated by empty circles) and 16 are supporters of party B (indicated by solid circles), and four admissible districtings $D_1 = \{d_1, d_2, d_3\}$, $D_2 = \{d_1, d_4, d_5\}$, $D_3 = \{d_3, d_7, d_8\}$ and $D_4 = \{d_5, d_7, d_9\}$. Note that party A wins two out of the three districts in D_1 and D_2, respectively, and one of the three districts in D_3 and D_4, respectively. Consider the solution *ME* first. Since the difference in the number of winning districts for the two parties is one in all cases, we have $ME_{\Pi} = \{D_1, D_2, D_3, D_4\}$. Consider the districting $D_1 \in ME_{\Pi}$ and $Y = d_1 \cup d_2$. *Consistency* would require that the districting $\{d_1, d_2\}$ is among the chosen districtings if the solution is applied to the restricted problem on Y. But obviously, we have $ME_{\Pi|_Y} = \{\{d_7, d_8\}\}$, because the districting $\{d_7, d_8\}$ induces a draw between the winning districts on Y while the districting $\{d_1, d_2\}$ entails two winning districts for party A (and zero districts won by party B). Similarly, *MU* violates *consistency* with $D_3 \in MU_{\Pi}$ and $Y = d_7 \cup d_8$ since $MU_{\Pi} = \{D_1, D_2, D_3, D_4\}$ and $MU_{\Pi|_Y} = \{\{d_1, d_2\}\}$.

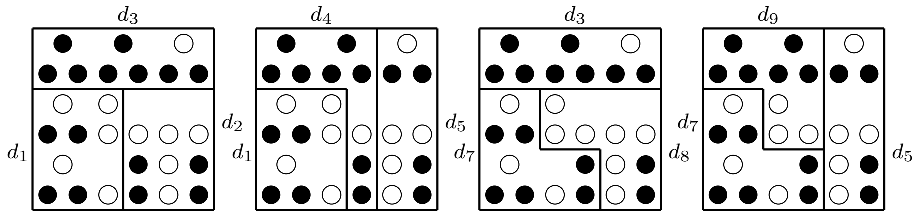

Fig. 1 *ME*, *MU* and *LB* violate consistency.

To verify, finally, that also *LB* violates *consistency* observe first that $LB_{\Pi} = \{D_3, D_4\}$ in Fig. 1. Consider $D_4 \in LB_{\Pi}$ and $Y = d_7 \cup d_9$. *Consistency* would require that the districting $\{d_7, d_9\}$ is among the districtings chosen by the solution on the restricted problem on Y. But it is easily seen that $LB_{\Pi|_Y} = \{\{d_1, d_4\}\}$, since the districting $\{d_1, d_4\}$ gives rise to a draw between the parties on Y which is closer to their respective relative shares of votes on Y. Thus the least biased solution also violates *consistency*.

Our final axiom expresses a very fundamental principle of fairness and equity in our context, namely the symmetric treatment of parties *ex ante*.

Anonymity Exchanging the distributions of party A and party B voters μ_A and μ_B does not change the set of chosen districtings: for all districting problems $\Pi = (X, \mathcal{A}, \mu, \mu_A, \mu_B, t, G)$,

$$D \in F_{(X,\mathcal{A},\mu,\mu_A,\mu_B,t,G)} \text{ if and only if } D \in F_{(X,\mathcal{A},\mu,\mu_B,\mu_A,t,G)}.$$

Note that this can also be interpreted as a requirement of anonymity with respect to voters *across* different parties; indeed, anonymity with respect to voters of the *same* party is already implicit in our definition of a districting problem since only the aggregate mass of parties' supporters matters and not their identity. It is easily seen that all solutions presented so far with exception of the optimal solution(s) satisfy the *anonymity* axiom.

Table 1 gives an overview of the axioms satisfied by the solutions introduced in Section 3.

Axioms \ Solutions	O	ME	MU	LB	C
Two-district determinacy	yes	yes	yes	yes	no
Two-district uniformity	yes	yes	yes	yes	yes
Indifference	yes	yes	yes	yes	yes
Consistency	yes	no	no	no	yes
Anonymity	no	yes	yes	yes	yes

Table 1 Properties of the five introduced solutions.

In the following we will show that for a large class of geographies no solution can satisfy all five axioms simultaneously. While we consider the anonymity condition to be an indispensable fairness requirement, our proof strategy is to show that the first four axioms characterize the optimal partisan gerrymandering solution O. Since this solution evidently violates *anonymity* the impossibility result follows.

5 A Characterization Result and an Impossibility

First, we consider districting problems with only two districts.

Lemma 1 *F satisfies two-district determinacy, two-district uniformity and indifference if and only if $F = O$, $F = ME$ or $F = MU$ for $t = 2$.*

Proof Observe that *two-district determinacy* and *two-district uniformity* jointly reduce the set of possible districting rules for $t = 2$ to O, ME and MU if only the number of winning districts matters (recall that $ME = LB$ on all two-district problems). Now *indifference* ensures that either *all* two-to-zero, *all* one-to-one, or *all* zero-to-two districtings admissible for problem Π have to be selected by solution F.

Finally, we have seen that O, ME and MU satisfy *two-district determinacy*, *two-district uniformity* and *indifference*, which completes the proof.

Consider a districting problem for $t = 3$ with the 9 admissible districts and the 3 resulting districtings shown in Fig. 2, in which party A voters are indicated by empty circles and party B voters by solid circles, μ equals the counting measure on $(X, \mathcal{A})$ and μ_A, μ_B are given by the respective number of party A and party B voters. It can be verified that, considering the districtings from left to right, we obtain 3 to 0, 2 to 1 and 1 to 2 winning districtings for party A, respectively. Thus, e.g. the optimal solution for party A would choose the first districting from the left, while the least biased solution would choose the middle districting.

The geography in the depicted problem is "thin" in the sense that all proper subproblems allow only one possible districting. Therefore, the *consistency* condition has no bite at all in this problem. In order to make use of the

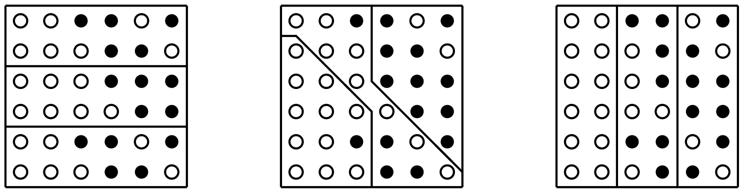

Fig. 2 Unlinked districtings.

consistency property, we will restrict the family of admissible geographies in the following way.

Definition 9 The geography G of a problem $\Pi = (X, \mathcal{A}, \mu, \mu_A, \mu_B, t, G)$ is *linked* if for any two possible districtings $D, D' \in \mathcal{D}_\Pi$ there exists a sequence $D_1, \ldots, D_k$ of districtings such that $D = D_1$, $\{D_2, \ldots, D_{k-1}\} \subseteq \mathcal{D}_\Pi$, $D' = D_k$, and $\# D_i \cap D_{i+1} = t - 2$ for all $i = 1, \ldots, k-1$.

In the appendix, we present a large and natural class of linked geographies, which arise from what we call *regular* districting problems. In a regular districting problem, μ is given by some finite measure that is absolutely continuous with respect to the Lebesgue measure, and the admissible districts are the bounded Borel sets whose boundary is a Jordan curve.

While the linkedness condition clearly limits the scope of our analysis, there is no hope in obtaining characterization results of the sort derived here without further assumptions on the family of geographies. Note also that under many specifications of the measure μ the unrestricted geography which admits *all* subsets of size $\mu(X)/t$ is linked (for instance, this holds if the set of voters is finite and μ is the counting measure).

Proposition 1 *If F equals O^A for $t = 2$ and F is consistent and indifferent, then $F = O^A$ for linked geographies.*

Proof Consider a districting problem $\Pi = (X, \mathcal{A}, \mu, \mu_A, \mu_B, t, G)$ with $t \geq 3$ and suppose that $F_\Pi \neq O^A_\Pi$ but F satisfies *consistency* and *indifference*. Since F_Π is not O^A_Π, there exist $D' \in O^A_\Pi$ and $D \in F_\Pi$ such that $\delta_A(D') > \delta_A(D)$ by *indifference*. Since Π has a linked geography there exists a sequence $D_1, \ldots, D_k$ of districtings such that $D' = D_1$, $\{D_2, \ldots, D_{k-1}\} \subseteq \mathcal{D}_\Pi$, $D = D_k$ and $\# D_i \cap D_{i+1} = t - 2$ for all $i = 1, \ldots, k-1$.

We claim that

$$|\delta_A(D_i) - \delta_A(D_{i+1})| \leq 1 \tag{5}$$

for all $i = 1, \ldots, k-1$, where D_i and D_{i+1} just differ in two districts. To verify (5) we shall denote the two pairs of different districts by d, d', e and e', where the first two districts belong to D_i while the latter two to D_{i+1}. Observe that $D_i \setminus \{d, d'\} = D_{i+1} \setminus \{e, e'\}$ by linkedness. Hence,

$$\begin{aligned} \delta_A(D_i) - \delta_A(D_{i+1}) &= \delta_A(\{d, d'\}) + \delta_A(D_i \setminus \{d, d'\}) - \delta_A(\{e, e'\}) - \\ &\quad \delta_A(D_{i+1} \setminus \{e, e'\}) \\ &= \delta_A(\{d, d'\}) - \delta_A(\{e, e'\}). \end{aligned} \tag{6}$$

By (1), $|\delta_A(\{d, d'\}) - \delta_A(\{e, e'\})| \leq 1$, which implies, taking (6) into consideration, (5).

Let $j^* \in \{2, \dots, k\}$ be the smallest index such that $\delta_A(D_{j^*}) = \delta_A(D_k)$. Since $D_k \in F_\Pi$ we have $D_{j^*} \in F_\Pi$ by *indifference*. Linkedness ensures that D_{j^*-1} and D_{j^*} just differ in two districts, which we shall denote by d, d', e and e', where the first two districts belong to D_{j^*-1} while the latter two to D_{j^*}. Furthermore, $D_{j^*-1} \setminus \{d, d'\} = D_{j^*} \setminus \{e, e'\}$ by linkedness. Let $Y = d \cup d' = e \cup e'$. Since F is consistent we have $\{e, e'\} \in F_{\Pi|_Y}$. Our assumption that F equals O^A for $t = 2$ implies $\delta_A(\{d, d'\}) \leq \delta_A(\{e, e'\})$. If $j^* = 2$, by *consistency*

$$\begin{aligned}\delta_A(D_1) > \delta_A(D_k) = \delta_A(D_2) &= \delta_A(\{e, e'\}) + \delta_A(D_2 \setminus \{e, e'\}) \\ &\geq \delta_A(\{d, d'\}) + \delta_A(D_1 \setminus \{d, d'\}) = \delta_A(D_1);\end{aligned}$$

a contradiction. Otherwise, suppose that $j^* > 2$. Then by *consistency* and the optimality of F on Y, $\delta_A(D_{j^*-1}) \leq \delta_A(D_{j^*})$. Moreover, $\delta_A(D_{j^*-1}) < \delta_A(D_{j^*})$ by the definition of j^*. Then by (5) and

$$\delta_A(D_1) > \delta_A(D_k) = \delta_A(D_{j^*}) > \delta_A(D_{j^*-1})$$

there exists a $j' \in \{2, \dots, j^* - 1\}$ such that $\delta_A(D_{j'}) = \delta_A(D_k)$. Clearly, $D_{j'} \in F_\Pi$ by *indifference*, contradicting the definition of j^*.[6]

Since neither the most equal or most unequal solutions satisfy *consistency* we cannot extend *ME* or *MU* for $t = 2$ to arbitrary t in the manner of Proposition 1. However, it might be the case that *ME* or *MU* for $t = 2$ can be extended to another consistent solution. The next proposition demonstrates that such an extension does not exist.

Proposition 2 *There does not exist a consistent and indifferent solution F that equals ME or MU for $t = 2$ even for linked geographies.*

Proof Suppose that there exists a solution F that satisfies *consistency* and *indifference* and that equals *ME* for $t = 2$. Consider the districting problem $\Pi = (X, \mathcal{A}, \mu, \mu_A, \mu_B, 3, G)$, where X consists of 27 voters, $\mathcal{A}$ equals the set of all subsets of X, μ is the counting measure, and $G = \{d_1, \dots, d_9\}$ is as shown in Fig. 3 in which party A supporters are indicated by empty circles and party B supporters by solid circles.

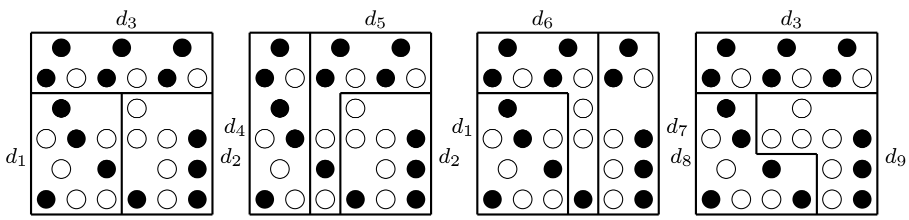

Fig. 3 *ME* and *MU* cannot be extended.

[6] We would like to thank Dezső Bednay for suggestions that improved our original proof.

We can see from Fig. 3 that the four possible districtings are $D_1 = \{d_1, d_2, d_3\}$, $D_2 = \{d_2, d_4, d_5\}$, $D_3 = \{d_1, d_6, d_7\}$ and $D_4 = \{d_3, d_8, d_9\}$. It can be checked that the given geography is linked. Since $\delta_A(D_1) = 2$ and $\delta_A(D_2) = \delta_A(D_3) = \delta_A(D_4) = 1$, either $\{D_1\} = F_\Pi$, $\{D_2, D_3, D_4\} = F_\Pi$ or $\{D_1, D_2, D_3, D_4\} = F_\Pi$ by *indifference*. First, consider the cases of $\{D_1\} = F_\Pi$ and $\{D_1, D_2, D_3, D_4\} = F_\Pi$. By *consistency*, $\{d_1, d_2\} \in F_{(X', \mathcal{A}', \mu', \mu'_A, \mu'_B, 2, G')}$, where $X' = d_1 \cup d_2$, $G' = \{d_1, d_2, d_8, d_9\}$ and $\mathcal{A}'$, μ', μ'_A, μ'_B denote the restrictions of $\mathcal{A}$, μ, μ_A, μ_B to X', respectively. However, $F_{(X', \mathcal{A}', \mu', \mu'_A, \mu'_B, 2, G')}$ should equal $\{d_8, d_9\}$ since $F = ME$ for $t = 2$; a contradiction. Second, consider the case of $\{D_2, D_3, D_4\} = F_\Pi$ and pick the case of D_3. By *consistency*, $\{d_6, d_7\} \in F_{(X'', \mathcal{A}'', \mu'', \mu''_A, \mu''_B, 2, G'')}$, where $X'' = d_6 \cup d_7$, $G'' = \{d_2, d_3, d_6, d_7\}$ and $\mathcal{A}''$, μ'', μ''_A, μ''_B denote the restrictions of $\mathcal{A}$, μ, μ_A, μ_B to X'', respectively. However, $F_{(X'', \mathcal{A}'', \mu'', \mu''_A, \mu''_B, 2, G'')}$ should equal $\{d_2, d_3\}$ since $F = ME$ for $t = 2$; a contradiction.

Now suppose that there exists a solution F that satisfies *consistency* and *indifference* and that equals MU for $t = 2$. Consider once again the problem shown in Fig. 3. First, consider the cases of $\{D_1\} = F_\Pi$ and $\{D_1, D_2, D_3, D_4\} = F_\Pi$. By *consistency*, $\{d_1, d_3\} \in F_{(X', \mathcal{A}', \mu', \mu'_A, \mu'_B, 2, G')}$, where $X' = d_1 \cup d_3$, $G' = \{d_1, d_3, d_4, d_5\}$ and $\mathcal{A}'$, μ', μ'_A, μ'_B denote the restrictions of $\mathcal{A}$, μ, μ_A, μ_B to X', respectively. However, $F_{(X', \mathcal{A}', \mu', \mu'_A, \mu'_B, 2, G')}$ should equal $\{d_4, d_5\}$ since $F = MU$ for $t = 2$; a contradiction. Second, consider the case of $\{D_2, D_3, D_4\} = F_\Pi$ and pick the case of D_4. By *consistency*, $\{d_8, d_9\} \in F_{(X'', \mathcal{A}'', \mu'', \mu''_A, \mu''_B, 2, G'')}$, where $X'' = d_8 \cup d_9$, $G'' = \{d_1, d_2, d_8, d_9\}$ and $\mathcal{A}''$, μ'', μ''_A, μ''_B denote the restrictions of $\mathcal{A}$, μ, μ_A, μ_B to X'', respectively. However, $F_{(X'', \mathcal{A}'', \mu'', \mu''_A, \mu''_B, 2, G'')}$ should equal $\{d_1, d_2\}$ since $F = MU$ for $t = 2$; a contradiction.

Our main result now follows from Lemma 1 and Propositions 1 and 2.

Theorem 1 *The optimal solution O is the only solution that satisfies two-district determinacy, two-district uniformity, indifference and consistency on linked geographies.*

We verify, on linked geographies, the tightness of Theorem 1, i.e. the independence of the axioms. First, the complete solution only violates *two-district determinacy*; second, ME, MU and LB just violate *consistency* (cf. Table 1 above).

Third, we investigate *indifference*. Consider the districting problem Π' given by Fig. 4 in which X' consists of 27 voters, $\mathcal{A}'$ equals the set of all subsets of X', μ' is the counting measure, and G' admit the districts shown in Fig. 4, where party A supporters are indicated by empty circles and party B supporters by solid circles. Observe that any two consecutive districtings in the sequence $D_1, \ldots, D_4$ only differ in two districts, and therefore, Π' has a linked geography. We shall denote by F the solution given by

$$F_\Pi = \begin{cases} \{D_4\} & \text{if } \Pi = \Pi', \\ O^A_\Pi & \text{if } \Pi' \text{ is not a subproblem of } \Pi \text{ and} \\ \{D_4\} \cup O^A_{\Pi|_{X \setminus X'}} & \text{if } \Pi' \text{ is a subproblem of } \Pi, \end{cases}$$

where the voters of problem Π are located within X and we say that Π' is a subproblem of Π if $\Pi' = \Pi|_{X'}$ and X' can be partitioned into three equally sized districts by picking three districts from the geography of problem Π. It can be verified that F satisfies *two-district determinacy*, *two-district uniformity* and *consistency*. Clearly, $F \neq O^A$ because of $\delta_A(D_1) = 3 > \delta_A(D_4) = 2$ and *indifference* is violated since otherwise $D_4 \in F_\Pi$ should imply $D_2 \in F_\Pi$.

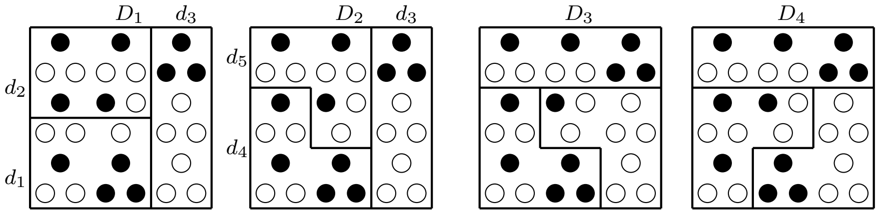

Fig. 4 Indifference is necessary.

Finally, to verify that *two-district uniformity* cannot be dropped from the list of conditions in Theorem 1 we are again considering problem Π' from Fig. 4 and are modifying solution F slightly. We shall denote the two-district subproblem of Π' on $X_1 = X' \setminus \{d_3\}$, which consists in choosing either districting $\{d_1, d_2\}$ or $\{d_4, d_5\}$, by Π_1. Define $\widehat{F}$ as follows,

$$\widehat{F}_\Pi = \begin{cases} \{D_2, D_4\} & \text{if } \Pi = \Pi', \\ O^A_\Pi & \text{if } \Pi' \text{ and } \Pi_1 \text{ are not a subproblems of } \Pi \\ \{D_2, D_4\} \cup O^A_{\Pi|_{X\setminus X'}} & \text{if } \Pi' \text{ is a subproblem of } \Pi, \\ \{\{d_4, d_5\}\} \cup O^A_{\Pi|_{X\setminus X_1}} & \text{if } \Pi' \text{ is not a subproblem of } \Pi \text{ but} \\ & \quad \Pi_1 \text{ is a subproblem of } \Pi. \end{cases}$$

It can be checked that $\widehat{F}$ satisfies *two-district determinacy*, *indifference* and *consistency*, but violates *two-district uniformity.*

Remark 1 Two-district determinacy is strictly weaker than overall determinacy[7] even in the presence of *two-district uniformity* and *consistency.*

This can verified by considering again the problem Π' defined in Fig. 4 and the following slightly modified solution $\widetilde{F}$:

$$\widetilde{F}_\Pi = \begin{cases} \{D_1, D_4\} & \text{if } \Pi = \Pi', \\ O^A_\Pi & \text{if } \Pi' \text{ is not a subproblem of } \Pi \text{ and} \\ \{D_1, D_4\} \cup O^A_{\Pi|_{X\setminus X'}} & \text{if } \Pi' \text{ is a subproblem of } \Pi, \end{cases}$$

where the voters of problem Π are located within X.

We obtain the following result as a simple corollary of Theorem 1.

Corollary 1 *There does not exist a two-district determinate, two-district uniform, indifferent, consistent and anonymous solution on linked geographies.*

[7] For a definition of overall determinacy see Footnote 3.

Appendix: Regular Districting Problems

We have already seen examples of linked geographies in Figures 1, 3 and 4. In this appendix we provide a natural and large class of further examples of districting problems with linked geographies.

A bounded subset A of $\mathbb{R}^2$ will be called *strictly connected* if its boundary ∂A is a Jordan curve (i.e. a non self-intersecting continuous loop). A subset A of a strictly connected set $B \subseteq \mathbb{R}^2$ *separates* B if $B \setminus A$ is not strictly connected. We call a continuous function $f : X \to \mathbb{R}$ *nowhere constant* if for any $x \in X$ and any neighborhood $N(x)$ of x there exists a $y \in N(x)$ such that $f(x) \neq f(y)$.

Definition 10 (Regular Districting Problems) A districting problem $\Pi = (X, \mathcal{A}, \mu, \mu_A, \mu_B, t, G)$ is called *regular* if

1. X is a bounded and strictly connected subset of $\mathbb{R}^2$,
2. $\mathcal{A}$ equals the set of Borel sets on X, i.e. following standard notation $\mathcal{A} = \mathcal{B}(X)$,
3. μ is a finite and absolutely continuous measure on $(X, \mathcal{B}(X))$ with respect to the Lebesgue measure,
4. G consists of all bounded, strictly connected and $\mu(X)/t$ sized subsets lying in $\mathcal{B}(X)$ and satisfying (1),
5. there exists a continuous nowhere constant function $f : X \to \mathbb{R}$ such that $\mu_A(C) = \int_C f(\omega) d\mu(\omega)$ for all $C \in \mathcal{B}(X)$, and
6. μ_B is given by $\mu_B(C) = \mu(C) - \mu_A(C)$ for all $C \in \mathcal{B}(X)$.

The fifth condition is a technical assumption to ensure that the districtings emerging in the proof of Lemma 3 below can be selected in a way that they satisfy (1). Specifically, we have the following lemma.

Lemma 2 *If we have two neighboring,*[8] *bounded, strictly connected and $\mu(X)/t$ sized sets $d, e \in \mathcal{B}(X)$ such that $\mu_A(d) = \mu(d)/2$ (i.e d violates (1)), then we can exchange territories between d and e in a way that the two resulting bounded, strictly connected and $\mu(X)/t$ sized sets $d', e' \in \mathcal{B}(X)$ satisfy (1).*

Proof Pick a point $x \in \partial d \cap \partial e$ from the relative interior of the common boundary of d and e. Since f is nowhere constant there exists a y arbitrarily close to x in the interior of d such that $f(y) \neq f(x)$. Assume that $f(y) > f(x)$. There exist a neighborhood $N_{\varepsilon_y}(y)$ of y and a neighborhood $N_{\varepsilon_x}(x)$ of x such that

$$\forall z \in N_{\varepsilon_y}(y) : \; f(z) > f(x) + \frac{2}{3}(f(y) - f(x)) \text{ and}$$
$$\forall z \in N_{\varepsilon_x}(x) : \; f(z) < f(x) + \frac{1}{3}(f(y) - f(x))$$

by continuity of f.

[8] We call two subsets of the plane neighboring if they share a common boundary of positive length.

By establishing a sufficiently thin connection between $N_{\varepsilon_y}(y)$ and $N_{\varepsilon_x}(x)$, which shall be assigned to e', and exchanging a subset of $N_{\varepsilon_y}(y)$ with a subset of $N_{\varepsilon_x}(x) \cap e$ in a way such that $\mu(d) = \mu(d') = \mu(e) = \mu(e')$, we can guarantee that $\mu_A(d') \neq \mu(d')/2$.[9]

Finally, the case of $f(y) < f(x)$ can be handled in an analogous way.

In the following, we write $D \sim D'$ if $D, D' \in \mathcal{D}_\Pi$ and there exists a sequence $D_1, \ldots, D_k$ of districtings such that $D = D_1$, $\{D_2, \ldots, D_{k-1}\} \subseteq \mathcal{D}_\Pi$, $D' = D_k$ and $\#D_i \cap D_{i+1} = t - 2$ for all $i = 1, \ldots, k-1$. It is easily verified that $\sim$ is an equivalence relation on the set of districtings.

Lemma 3 *The geographies of regular districting problems are linked.*

Proof Linkedness is clearly satisfied if $t = 1$ or $t = 2$. We show that the linkedness of the geographies of all regular districting problems for $t \leq n$ implies the linkedness of the geographies of all regular districting problems for $t = n + 1$. From this, Lemma 3 follows by induction.

Take two arbitrary districtings D and E of a districting problem with $t = n + 1$. We can pick a district $d \in D$ such that d and X have a non-degenerate curve as a common boundary, i.e. there exists a curve C of positive length such that $C \subseteq \partial d \cap \partial X$. We divide our proof into three steps.

Step 1: We show that there exists a districting $D' \sim D$ that contains a district d' which shares a common boundary of positive length with the boundary of X and which does not separate X.

If d itself does not separate X we are done. Thus, assume that d separates X. For simplicity, we start with the case in which d separates X into only two regions as shown in the picture on the left of Fig. 5.[10] By exchanging

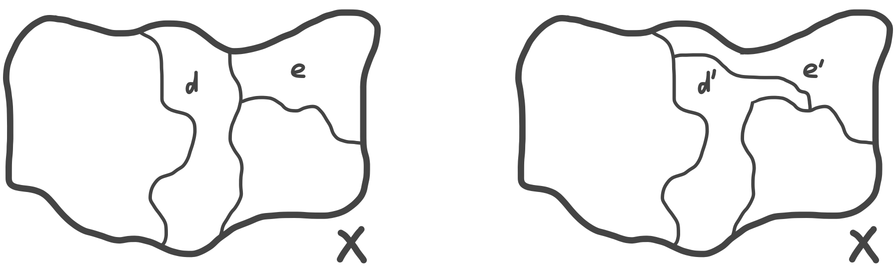

Fig. 5 d separates X into two regions.

territories between the two districts d and e, where e is a neighboring district

[9] If $\mu_A(e) \neq \mu(e)/2$, then $\mu_A(e') \neq \mu(e')/2$ can be guaranteed by exchanging sets of sufficiently small measure μ between d and e. In addition, if $\mu_A(e) = \mu(e)/2$ and $\mu_A(e') = \mu(e')/2$, then we can repeat the exchange of territories between e' and d' to ensure that both sets satisfy (1).

[10] Both pictures only show the two districts involved in a territorial exchange and not the entire districtings.

of d, as shown in the picture on the left of Fig. 5, we can arrive at districts d' and e' such that d' does not separate X.[11]

More generally, assume that d separates X, where the number of strictly disconnected regions of $X \setminus \{d\}$ equals $k \leq n$. We can find a district $e \in D$ and a unique boundary element $x \in \partial e$ such that $x \in \partial d \cap \partial X$ and such that ∂d and ∂e have a common curve of positive length starting from x. Hence, one can exchange territories between d and e so that for the resulting new districts d' and e' we have that d' separates X into at most $k-1$ strictly disconnected regions. Clearly, $D' = (D \setminus \{d, e\}) \cup \{d', e'\} \sim D$. Repeating the described bilateral territorial exchange $k-1$ times, we thus arrive at a districting D' that contains a district d' which shares a common boundary with X and which does not separate X.

By Step 1, we may thus assume that $d \in D$ shares a boundary of positive length with X and does not separate X.

Step 2: We establish that there exists a districting $E' \sim E$ containing a district $e \in E'$ such that e, d and X have a nondegenerate common boundary, $\mu(d \cap e) > 0$ and $d \cup e$ does not separate X.

Clearly, there exist a district $e \in E$ possessing a common boundary with d and X, and satisfying $\mu(d \cap e) > 0$.

Assume that e separates X, where the number of strictly disconnected regions of $X \setminus \{e\}$ equals $k \leq n$ (see Fig. 6 to the left for a situation with $k = 3$). Then $d^c \cap \partial e \cap \partial X \neq \emptyset$. We can find a district $e' \in E$ with a unique boundary element $x \in \partial e'$ satisfying $x \in d^c \cap \partial e \cap \partial X$ and that $\partial e \cap \partial e'$ has a common curve of positive length starting from x (as illustrated in the left hand side of Fig. 6). Hence, one can exchange territories between e and e' so that for the resulting new districts h and h' we have that $d \cap e \subset h$, h separates X into at most $k-1$ strictly disconnected regions (see the right hand side of Fig. 6). Clearly, $E' = (E \setminus \{e, e'\}) \cup \{h, h'\} \sim E$ and we can repeat the

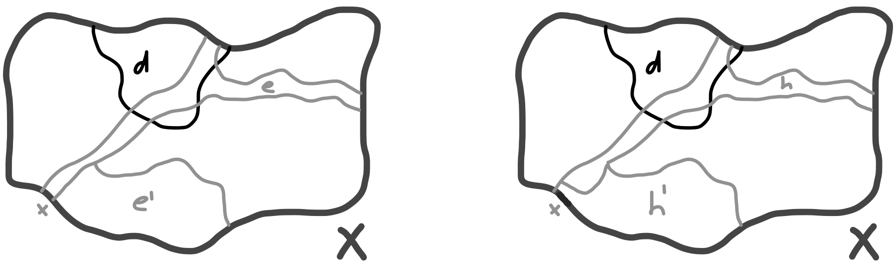

Fig. 6 Reducing the number of disconnected regions in E.

procedure to reduce the number of strictly disconnected regions by replacing

[11] It might happen that d' or e' violate (1) since we only took care of the shapes and sizes of the two districts. However, Lemma 2 ensures that through an appropriate territorial exchange between d' and e' we can also ensure (1). In what follows we will carry out all territorial exchanges between districts so as to satisfy (1) without explicitly mentioning Lemma 2 each time.

E and e with E' and h, respectively, until we arrive at a districting $E' \sim E$ containing a district e' that does not separate X and has a common boundary with d. Without loss of generality, we can thus replace e' and E' by e and E, respectively.

We still have to ensure that $d \cup e$ does not separate X. A situation in which $d \cup e$ separates X is shown in the picture on the left hand side of Fig. 7. In

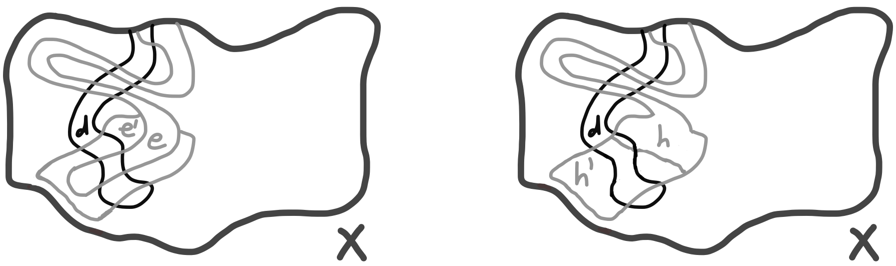

Fig. 7 Intertwined districts.

addition, the same picture contains (by the absolute continuity of μ) a possible neighboring district e' to e, which is drawn in a way such that $e \cup e'$ does not separate X, it covers an area from the separated regions and also an area within $d \cup e$. A possible exchange of territories which reduces the separated area by $d \cup e$ is illustrated in Fig. 7, where $d \cup h$ separates a smaller area than $d \cup e$.[12] Pick an arbitrary districting H of $X \setminus (e \cup e')$ into $n-1$ strictly connected districts and let $E' = H \cup \{h, h'\}$. Observe that $E \setminus \{e\} \sim H \cup \{e'\}$ by the induction hypothesis, $h \cup h' = e \cup e'$ by construction, and therefore $E \sim E'$. Replace e and E with h and E', respectively. After repeating the described territorial exchange finitely many times[13] one arrives at a district e and a districting E such that $d \cup e$ does not separate X and e still satisfies the other desired properties.

Step 3: Since $d \cup e$ does not separate X and μ is absolutely continuous, there exists a strictly connected set h such that $\mu(h) = 2\mu(X)/(n+1)$, $d \cup e \subset h$, $d' = h \setminus d \in G$ and $e' = h \setminus e \in G$ and h does not separate X (see Fig. 8). Let H be a districting of $Y = X \setminus h$ into $n-1$ strictly connected districts. Then $\Pi\,|_{Y \cup d'}$ and $\Pi\,|_{Y \cup e'}$ are regular districting problems, and therefore it follows by the induction hypothesis that $D \sim H \cup \{d, d'\}$ and $H \cup \{e, e'\} \sim E$. Clearly, $\{d, d'\} \sim \{e, e'\}$, which gives $H \cup \{d, d'\} \sim H \cup \{e, e'\}$. Finally, the statement of Lemma 3 follows from the transitivity of $\sim$.

[12] District e' in Fig. 7 is not drawn in the most efficient way in the sense that it is possible to draw e' such that it allows for a larger reduction of the separated areas. However, the purpose of Fig. 7 is only to illustrate the possibility of the reduction of separated areas.

[13] In fact the number of required iterations is at most $\lceil t\mu(Y)/\mu(X) \rceil + 1$, where Y stands for the area "intertwined" by $d \cup e$.

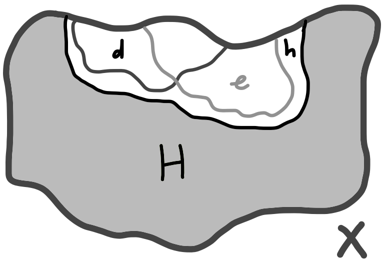

Fig. 8 Final step.

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