Nash Equilibria of Sealed-Bid Combinatorial Auctions

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Abstract

This paper characterizes the complete set of full-information, pure-strategy Nash equilibria of a class of sealed-bid combinatorial auctions. The class contains any auction that assigns bundles to maximize the sum of bids and that chooses bidder payments between the reported social opportunity cost for the bundle won and the bid amount. We rank equilibria of prevalent auctions in this class, finding that the equilibria of the pay-as-bid (PAB) auction are a subset of the equilibria of bidder-optimal core-selecting (BOCS) auctions, which are in turn a subset of the equilibria of the Vickrey auction. Any assignment and payments that generate individually rational payoffs can result from some equilibrium of the Vickrey and BOCS auctions, whereas possible equilibrium outcomes of the PAB auction are more limited.

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1. Introduction

Combinatorial or package auctions are increasingly used to sell heterogenous items because they give bidders the opportunity to express preferences involving substitutable or complementary items. The complex strategy space that includes bids for all bundles of items – 2^k bids when there are k distinct items in the auction – makes equilibria difficult to analyze. However, understanding the equilibria of combinatorial auctions can highlight the bidding incentives and allow comparison among different auction rules and outcomes.

Several studies derive Nash equilibria of combinatorial auctions. Most of them consider equilibria where a player bids some fixed constant below his true value (or zero) for every bundle, calling these strategies truthful, semi-sincere, profit-targeting, or truncation strategies (Bernheim and Whinston, 1986; Ausubel and Milgrom, 2002; Milgrom, 2004; Day and Raghavan, 2007; Day and Milgrom, 2008). Bernheim and Whinston (1986) characterize all of the Nash equilibria of menu auctions and provide a refinement that selects equilibria in truncation strategies. A special case of their model is the allocation of multiple, heterogenous goods using one of the auctions we consider – the pay-as-bid auction. Ausubel and Milgrom (2002) and Day and Milgrom (2008, 2013) find that these same Nash equilibria, which truncate by payoffs at the bidder-optimal frontier of the core, are equilibria of any core-selecting auction. Sano (2010) shows that these equilibria are also equilibria (in dominated strategies) of the combinatorial Vickrey auction (the VCG mechanism), and Sano (2013) generalizes this result by showing that a specific class of auctions – Vickrey-reserve auctions – all share these truncation Nash equilibria and outcomes, while auctions that choose payments below Vickrey payments do not.

Beck and Ott (2013) find Nash equilibria of core-selecting auctions with weak overbidding on all bundles that, like the truncation equilibria, involve payoffs in the true core of the auction game. Limited results about iterative combinatorial auctions also exist: Bichler et al. (2012) find ex-post equilibria of a modified combinatorial clock auction and Bikhchandani et al. (2011) prove that truthful bidding is an ex-post equilibrium of a particular ascending Vickrey auction. For sealed-bid combinatorial Vickrey auctions, it is well known that truthful bidding is a weakly dominant strategy. Additionally, Holzman and Monderer (2004) and Holzman et al. (2004) analyze strategies that satisfy a non-standard, recursive definition of ex-post equilibrium. For the special cases of single unit or multiple-unit Vickrey auctions, where bidders have weakly decreasing marginal values, Blume and Heidhues (2004) and Blume et al. (2009) characterize the sets of ex-post equilibria. However, aside from Bernheim and Whinston (1986), there has not been a full characterization of all of the Nash equilibria of core-selecting auctions or the Vickrey auction in a general setting with multiple, heterogenous goods. Also, the sets of equilibrium outcomes are only characterized for refinements (e.g. Bernheim and Whinston, 1986) and general comparisons of outcomes across auctions are missing.[1]

This paper provides a systematic analysis and comparison of *all* of the pure-strategy, full-information Nash equilibria of combinatorial auctions without refinements or restrictions on bidders' values or strategy spaces. We characterize the Nash equilibria of a class of combinatorial auctions that contains all core-selecting auctions and the Vickrey auction. Our characterization traces the incentives in all auctions back to two simple and intuitive conditions that emphasize strategic similarities and differences between these auctions, and that allow for straightforward comparisons between auctions. We apply this insight to the pay-as-bid and other core-selecting auctions and to the Vickrey auction to compare their sets of equilibria. The sets of equilibria are large, as is common in games with full information.

We find that the set of equilibria of the pay-as-bid auction is a (potentially strict) subset of that of any other core-selecting auction, including bidder-optimal core-selecting auctions. In fact, the equilibria of the pay-as-bid auction are equilibria of any auction (core-selecting or not) that chooses an optimal assignment and has payments that are bounded below

[1]For auctions with two bidders, Bernheim and Whinston (1986) find that their Nash equilibrium refinement selects outcomes that are equivalent to the outcomes of the Vickrey auction in the Nash equilibrium in weakly dominant strategies.

by those of the Vickrey auction. All bidder-optimal core-selecting auctions have the same equilibria and this set of equilibria is a (potentially strict) subset of that of the Vickrey auction. So the Vickrey auction admits the most equilibria, though most are in dominated strategies.

Any equilibrium bids in any of the considered auctions will lead to the same outcome as that of the Vickrey auction for the same bids. This implies that, in all of the equilibria of core-selecting auctions, the bidders report values such that the Vickrey payoffs are in the core with respect to the bids.

In addition to analyzing the Nash equilibria of these auctions, we characterize the range of resulting equilibrium outcomes - assignments, payments, and payoffs - for the pay-as-bid auction and for a second class of auctions that contains the bidder-optimal core-selecting auctions and the Vickrey auction. For this second class of auctions, we prove a type of folk theorem about the equilibrium outcomes. Any feasible assignment that does not leave valuable goods unassigned coupled with payments that correspond to individually rational payoffs are an outcome of a full-information Nash equilibrium of any of the auctions in the class. Thus, virtually any outcome is possible in equilibrium. The equilibrium outcomes of the pay-as-bid auction must satisfy a stronger necessary and sufficient condition than individual rationality, but we show that even the pay-as-bid auction can lead to inefficient assignments and payments below the Vickrey payments that result from truthful bidding. This wide range of outcomes suggests that conclusions based on particular Nash equilibria of the Vickrey auction or any core-selecting should be interpreted with caution.

The paper continues as follows. Section 2 presents the model. Section 3 characterizes the best responses and the Nash equilibria, and Section 4 analyzes the potential equilibrium outcomes in terms of assignments, payments, and payoffs. In Section 5, the results of the previous sections are used to derive insights into the seller's incentives and budget constraints in the respective extended games. Section 6 concludes. The tie-breaking rule and a possible mechanism for implementing it are discussed in Appendix A.

2. Model

We consider a setting with one seller, whom we denote 0, and a set of bidders $N = \{1,\ldots,n\}$, $n \geq 2$. The seller owns a set of goods $K = \{1,\ldots,k\}$, $k \geq 1$, which he does not value. Each bidder i has values $v_i : 2^K \rightarrow \mathbb{R}_+$ for bundle $y \in 2^K$. We normalize $v_i(\emptyset) = 0$ for all $i \in N$ and we denote the vector of these values $v = (v_1,\ldots,v_n)$. Bidders have quasilinear utility, so if bidder i wins bundle y and pays price p_i, then he gets a payoff of $v_i(y) - p_i$. The seller's payoff is the sum of the payments made by the bidders.

Bidders place bids $b = (b_1,\ldots,b_n)$, where the vector of bidder i's bids for all bundles is $b_i = (b_i(y))_{y\in 2^K}$. Let b_S and b_{-S} be the vectors of bids of groups $S \subseteq N$ and $N \setminus S$, respectively. Let $B = B_1 \times \cdots \times B_n$ where B_i denotes the set of feasible bids for bidder i and assume, unless otherwise noted, that $B_i = \mathbb{R}_+^{2^K}$ with the normalization $b_i(\emptyset) := 0$.

A sealed-bid combinatorial auction is a direct mechanism that assigns each bidder a bundle of goods $x_i(b)$ and a payment $p_i(b, x(b))$ based on bids. Let $x_0(b) \equiv K \setminus \{\cup_{i\in N} x_i(b)\}$ denote any unassigned items and $x(b) = (x_0(b),\ldots,x_n(b))$. The set of feasible assignments of goods $L \subseteq K$ is

$$X(L) = \{x = (x_0,\ldots,x_n)|x_i \in 2^L \ \forall i \in \{0,\ldots,n\}, x_i \cap x_j = \emptyset \ \forall i \neq j, \cup_{i=0}^n x_i = L\}.$$

The payment depends on the assignment as well as the bids because when randomizations are used to break ties, the payment will change with the realized assignment $x(b)$. We sometimes write $p(b) \equiv p(b, x(b))$ when the relationship with the assignment is clear. Apart from potential randomization for tie-breaking we only consider deterministic mechanisms.

Most of the auctions we consider are part of the class of core-selecting auctions, which use a concept from cooperative game theory as part of their design. To describe these auctions, we must first define the coalitional function w and the corresponding set of optimal assignments $\hat{X}$:

$$w(b) = \max_{x\in X(K)} \sum_{i\in N} b_i(x_i)$$

$$\hat{X}(b) = \arg\max_{x \in X(K)} \sum_{i \in N} b_i(x_i)$$

We call w the coalitional function because it is the maximum reported value that the coalition of the bidders and the seller can generate by trading. There may be multiple *optimal* (with respect to bids) assignments that achieve this maximum reported value, so $\hat{X}(b)$ need not be a singleton. Note that the optimal assignment need not be (true) value-maximizing. We call any value-maximizing assignment $x \in \hat{X}(v)$ *efficient* because, in our auction games, such an assignment corresponds to efficient payoffs.

To represent the value generated by the coalition of some subset of bidders $S \subseteq N$ and the seller or the value generated using only some subset of the seller's goods $L \subseteq K$, we use the notation:

$$w(b_S^L) = \max_{x \in X(L)} \sum_{i \in S} b_i(x_i)$$

$$\hat{X}(b_S^L) = \arg\max_{x \in X(L)} \sum_{i \in S} b_i(x_i)$$

Likewise, $w(b_{-S}^{-L})$ represents the value generated by bidders $N \setminus S$ and goods $K \setminus L$.

The core $\mathcal{C}(v)$ of a cooperative game consists of all feasible *payoffs* that are not blocked by any coalition (i.e., each group receives at least as much as it could achieve on its own so that it could not deviate and make all of its members better off). In our setting, since the seller owns all items, any group that does not include the seller cannot generate any value. Any coalition that contains the seller must receive at least what it could get from trading among its members. Therefore, the core consists of all payoff vectors $\pi = (\pi_0, \pi_1, \ldots, \pi_n)$ that satisfy the following constraints (where the first is feasibility and the rest assure no coalition can block the payoffs):

$$\begin{aligned}
\pi_0 + \textstyle\sum_{i \in N} \pi_i &\leq w(v) \\
\pi_i &\geq 0 \qquad \forall i \in N \\
\pi_0 &\geq 0 \\
\pi_0 + \textstyle\sum_{i \in S} \pi_i &\geq w(v_S) \quad \forall S \subseteq N
\end{aligned}$$

In our setting with a single seller, the core is always nonempty. The payoffs $\pi_0 = w(v)$ and $\pi_i = 0$ for all $i \in N$ always satisfy these constraints.

A core-selecting auction is a direct mechanism (inducing a non-cooperative game among bidders) that maps bids on bundles of goods to assignments and payments such that *reported payoffs* are in the core with respect to the bids. Denote bidder i's reported payoff as $\pi_i^r(b) = b_i(x_i(b)) - p_i(b, x(b))$ and the seller's payoff as $\pi_0^r(b) = \sum_{i \in N} p_i(b, x(b))$. Note that the seller actually receives these payments from the bidders, so $\pi_0^r(b) = \pi_0(b)$ is also the true payoff and revenues he receives. However, a bidder's true payoff resulting from bids b, $\pi_i(b) = v_i(x_i(b)) - p_i(b, x(b))$, need not equal his reported payoff $\pi_i^r(b)$.

Definition 1. *A* core-selecting auction *is a direct mechanism that chooses* $x(b)$ *and* $p(b, x(b))$ *such that:*

$$\pi^r(b) = (\pi_0^r(b), \pi_1^r(b), \ldots, \pi_n^r(b)) \in \mathcal{C}(b) \quad \forall b \in B$$

Many auctions fall into the class of core-selecting auctions. They all choose an optimal assignment $x(b) \in \hat{X}(b)$, which is necessitated by the first and last core inequalities. However, there may be infinitely many possible payments $p(b, x(b))$ that satisfy the core constraints for any given optimal assignment. Translating the constraints on payoffs into constraints on the assignment and payments yields:

$$
\begin{aligned}
& & \pi_0^r(b) + \sum_{i\in N} \pi_i^r(b) = w(b) \quad &\Leftrightarrow \quad x(b) \in \hat{X}(b) \qquad (1)\\
\forall i \in N: & & \pi_i^r(b) \geq 0 \quad &\Leftrightarrow \quad b_i(x_i(b)) \geq p_i(b, x(b))\\
& & \pi_0^r(b) \geq 0 \quad &\Leftrightarrow \quad \sum_{i\in N} p_i(b, x(b)) \geq 0\\
\forall S \subset N: & & \sum_{i\in S} \pi_i^r(b) \leq w(b) - w(b_{-S}) \quad &\Leftrightarrow \quad \sum_{i\in S} p_i(b, x(b)) \geq w(b_{-S}) - \sum_{i\notin S} b_i(x_i(b))
\end{aligned}
$$

The second to last constraint is implied by the last constraints: $p_j(b, x(b)) \geq w(b_{-j}) - \sum_{i\neq j} b_i(x_i(b)) \geq 0$ prevents negative payments (otherwise the seller and the bidders in $N \setminus \{j\}$ could do better) and implies $\sum_{i\in N} p_i(b, x(b)) \geq 0$. Also, the second constraint implies that losing bidders pay zero.

Some particular subclasses of core-selecting auctions that we consider are bidder-optimal core-selecting auctions, minimum-revenue core-selecting auctions, and pay-as-bid auctions. Bidder-optimal core-selecting auctions select payoffs on the bidder-Pareto-optimal frontier of the core.

Definition 2. *A* bidder-optimal core-selecting (BOCS) auction *is a core-selecting auction that chooses $x(b)$ and $p(b, x(b))$ such that, for all $b \in B$, there does not exist any $\hat{\pi}^r \in \mathcal{C}(b)$ such that $\hat{\pi}_i^r \geq \pi_i^r(b)$ for all $i \in N$ and the inequality is strict for at least one $i \in N$.*

There are generally many bidder-optimal payments, so a full specification of a particular BOCS auction requires a rule for choosing between them.[2] A subclass of BOCS auctions are those that minimize the seller's revenues.

Definition 3. *A* minimum-revenue core-selecting (MRCS) auction *is a core-selecting auction that chooses $x(b)$ and $p(b, x(b))$ such that, for all $b \in B$, the corresponding reported*

[2] One example of a payment rule is to choose the bidder-optimal payment vector that is closest in Euclidean distance to the Vickrey payments, as suggested by Day and Cramton (2012).

payoffs $\pi^r(b)$ *solve:*

$$\min_{\pi^r \in \mathcal{C}(b)} \pi_0^r$$

Again, there are usually many MRCS payments because the seller's revenue can be split in different ways among the bidders.[3] On the opposite end of the spectrum are auctions that maximize the seller's revenues among reported core payoffs by having each bidder pay his full bid for his winning bundle.

Definition 4. *A* pay-as-bid (PAB) auction *is a core-selecting auction that chooses* $x(b) \in \hat{X}(b)$ *and* $p_i(b, x(b)) = b_i(x_i(b))$.

Unlike in BOCS and MRCS auctions, in a PAB auction the payments are unique given a particular assignment. However, there are still multiple PAB auctions that differ in the way they choose among the optimal assignments when $\hat{X}(b)$ is not a singleton. In all auctions, we use a particular *tie-breaking rule* to decide between assignments in $\hat{X}(b)$. We assume that every auction breaks ties among $x \in \hat{X}(b)$ in favor of best responses of as many bidders as possible.[4] In other words, the tie-breaking rule chooses the $x \in \hat{X}(b)$ that satisfies conditions (I) as defined in the next section for as many $i \in N$ as possible, and among those it chooses among the x that satisfy (II) as defined in the next section for as many i as possible. If there is more than one such assignment, the tie is broken randomly. With the addition of this tie-breaking rule, the Vickrey and PAB auctions are unique. There are still multiple BOCS and MRCS auctions, but they differ only in the way they determine payments.

We use this particular tie-breaking rule because it creates the largest set of Nash equilibria (see Lemma 2 in Appendix A) and the aim is to characterize all possible Nash equilibria.

[3]One can transfer the idea of the Vickrey-nearest payments by minimizing the Euclidean distance between the Vickrey payments and those payments that minimize revenue in the core. The resulting payments may differ from the bidder-optimal Vickrey-nearest payments. Other reference rules that minimize distances to reference points have been suggested by Erdil and Klemperer (2010).

[4]Tie-breaking in favor of best responses means that if $|\hat{X}(b)| > 1$, the assignment x is chosen such that the number of bidders i for whom b_i is a best response is maximized.

In this way, we can characterize all Nash equilibria that are possible with some choice of tie-breaking rule. Some of the equilibria we find might not be Nash equilibria if ties were broken differently, but no additional equilibria would be added. In Appendix A we discuss properties and implementation of our tie-breaking rule, its relation to other tie-breaking rules, and implications of using other tie-breaking rules for our results.

We often make use of another kind of combinatorial auction that is not core-selecting: Vickrey or VCG auctions.

Definition 5. *A* Vickrey or VCG auction *chooses* $x(b) \in \hat{X}(b)$ *and* $p_i^V(b, x(b)) \equiv w(b_{-i}) - \sum_{j \neq i} b_j(x_j(b))$ *for all* $i \in N$*.*

A Vickrey auction charges each participant the opportunity cost that his presence imposes on the other bidders, or equivalently, the minimum bid necessary for winning his assigned bundle. Notice that one of the core constraints requires that each bidder's payment be at least as large as his Vickrey payment $p_i^V(b, x(b))$, so we call this the *Vickrey constraint*. We also refer to reported Vickrey payoffs as $\pi_i^{r,V}(b) \equiv b_i(x_i(b)) - p_i^V(b, x(b)) = w(b) - w(b_{-i})$. Notice that the reported Vickrey payoff, in contrast to the Vickrey payment, does not depend on the assignment.

The highest reported payoff a bidder can receive in any core allocation is his reported Vickrey payoff. If $\pi_i^r > w(b) - w(b_{-i})$, then

$$\pi_0^r + \sum_{j \neq i} \pi_j^r = w(b) - \pi_i^r < w(b) - w(b) + w(b_{-i}) = w(b_{-i})$$

a contradiction to $\pi^r \in \mathcal{C}(b)$. The reported Vickrey payoff is always achievable in the core by any single bidder because the payoff profile $\pi_0^r = w(b_{-i})$, $\pi_i^r = w(b) - w(b_{-i})$, and $\pi_j^r = 0$ for all $j \neq i$, always lies in the core. However, it may not be possible in a core allocation for all bidders to simultaneously receive their reported Vickrey payoffs. Thus, when the full reported Vickrey payoff vector $\pi^{r,V}(b) = (w(b) - \sum_{i \in N} \pi_i^{r,V}(b), \pi_1^{r,V}(b), \ldots, \pi_n^{r,V}(b))$ lies in the core, it is the unique bidder-optimal allocation and the unique outcome of any BOCS auction. When it is not in the core, there are multiple bidder-optimal allocations, all of which give the seller a larger payoff than $\pi_0^{r,V}(b)$.[5] The bidder-pessimal core allocation is always unique and is the outcome of the PAB auction: $(\pi_0^r(b), \pi_1^r(b), \ldots, \pi_n^r(b)) = (w(b), 0, \ldots, 0)$.

Some of our results will hold for the following class of auctions that contains all of the previously defined auctions.

Definition 6. *The class of auctions $\mathcal{A}$ contains any sealed-bid auction $(x(b), p(b, x(b)))$ that, for every $b \in B$, selects $x(b) \in \hat{X}(b)$, breaks ties in favor of best responses, and chooses payments $p_i(b, x(b)) \in [p_i^V(b, x(b)), b_i(x_i(b))]$ for all $i \in N$.*

Any auction that chooses an optimal assignment and that requires payments be no more than bids and no less than the minimum bid necessary to win the assigned bundle is in class $\mathcal{A}$. Ignoring the tie-breaking rule, $\mathcal{A}$ consists of the Vickrey-reserve auctions defined by Sano (2013) as efficient and individually rational auctions with payments above Vickrey payments.[6] These conditions are satisfied by every core-selecting auction, including the PAB auction and all BOCS auctions. However, this class is more general because the core constraints place more restrictions on payments. For example, the Vickrey auction is in $\mathcal{A}$ because it has payments that are always equal to the lower bound.

3. Nash Equilibria

In this section, we characterize all of the pure-strategy, full-information, Nash equilibria of the Vickrey auction and of all core-selecting auctions. The equilibria are similar because a bidder facing a certain profile of bids from his opponents has the same maximum payoff in all of these auctions. This maximum payoff results from winning something he would have won by bidding truthfully and paying his Vickrey payment for it. Moreover, in all of these auctions, there is a common best-reply bid that guarantees this maximum payoff.

[5] See Ausubel and Milgrom (2002), Theorem 6, and Ausubel and Milgrom (2006), Theorem 5.

[6] Sano (2013) calls these auctions efficient because they choose an optimal assignment based on bids and he calls them individually rational because payments are below bid amounts.

The following conditions are essential to our main results on Nash equilibria:

(I) $x(b) \in \hat{X}(v_i, b_{-i})$

(II) $p_i(b, x(b)) = p_i^V(b, x(b)) = w(b_{-i}) - \sum_{j \neq i} b_j(x_j(b))$

(II′) $w(b) = w(b_{-i})$

Condition (I) ensures that bidder i wins what he would win if he reported his values truthfully, holding the others' bids fixed: $x(b^*) \in \arg\max_{x \in X}[v_i(x_i) + \sum_{j \in N \setminus \{i\}} b_j^*(x_j)]$. Condition (II) ensures that bidder i pays the least possible for the bundle he wins because payments are bounded below by Vickrey payments in all auctions that we consider. Condition (II′) is the special case of (II) when $p_i(b^*, x(b^*)) = b_i^*(x_i(b^*))$. This condition will apply to the PAB auction because it selects payments equal to winning bids.

The conditions are at the heart of our analyses, because we will show that in any of the considered auctions, a bid b that satisfies conditions (I) and (II) gives bidder i the highest payoff that he can hope for given the bids b_{-i} of his opponents. Moreover, we will show that in any of the auctions, by choosing an appropriate bid, he can achieve this best payoff. Any Nash equilibrium consists of mutual best responses and, therefore, must fulfill the conditions for any bidder. Such mutual best responses exist.

Before we consider Nash equilibria we analyze best responses and show that the auctions in $\mathcal{A}$ have incentives in common in the sense that they share best responses.

Theorem 1. *Consider any bids profile b_{-i}.*

(a) *For every auction in $\mathcal{A}$, b_i is an element of i's best response correspondence if and only if (I) and (II) hold.*

(b) *In the PAB auction, b_i is a best response if and only if (I) and (II′) hold.*

(c) *If b_i is a best response in the PAB auction, then b_i is a best response in any auction in $\mathcal{A}$.*

Proof: (a) Consider bidder i and fix all other bids b_{-i}. Bidder i cannot only guarantee himself any bundle $y \in 2^K$ but can also guarantee he pays the lowest possible payment, conditional on b_{-i} and on winning y, by bidding $b_i(y) = w(b_{-i}) - w(b_{-i}^{-y})$ and bidding zero on all other bundles. This bid is just high enough to win y, subject to tie breaking: $\hat{X}(b) = \hat{X}(b_{-i}) \cup \left\{\{y\} \times \hat{X}(b_{-i}^{-y})\right\}$.[7] With this bid, the lower bound on his payment is $p_i^V(b, x(b)) = w(b_{-i}) - \sum_{j\neq i} b_j(x_j(b)) = [b_i(y) + w(b_{-i}^{-y})] - \sum_{j\neq i} b_j(x_j(b)) = b_i(y)$. (Note that $w(b_{-i}^{-y}) = \max_{x_{-i}\in X(K\setminus y)} \sum_{j\neq i} b_j(x_j) = \sum_{j\neq i} b_j(x_j(b))$ because bidder i wins y in at least one optimal assignment.) Therefore, the lower and upper bounds on bidder i's payment are the same and in every such auction, bidder i pays his bid. The lower bound p_i^V depends on i's bid only through i's assignment, so he is indeed paying the lowest possible payment, conditional on b_{-i} and on winning y.

Since bidder i has the ability, given b_{-i}, to win any bundle at the lower bound on his payment, he has no profitable deviation from a bid profile b^* if and only if $x_i(b^*) \in \arg\max_{y\in K} v_i(y) - [w(b_{-i}^*) - w(b_{-i}^{*-y})]$ and $p_i(b^*, x(b^*)) = p_i^V(b^*, x(b^*))$:

$$x_i(b^*) \in \arg\max_{y\in K} \left[v_i(y) - [w(b_{-i}^*) - w(b_{-i}^{*-y})]\right] \iff$$
$$x_i(b^*) \in \arg\max_{y\in K} \left[v_i(y) + w(b_{-i}^{*-y})\right] \iff$$
$$x_i(b^*) \in \arg\max_{y\in K} \left[v_i(y) + \max_{x\in X(K\setminus y)} \sum_{j\neq i} b_j^*(x_j)\right] \iff$$
$$x(b^*) \in \arg\max_{x\in X(K)} \left[v_i(x_i) + \sum_{j\neq i} b_j^*(x_j)\right] = \hat{X}(v_i, b_{-i}^*)$$

(b) The PAB auction has payment rule $p_i(b, x(b)) = b_i(x_i(b))$, so for the PAB auction (II) holds if and only if (II′) holds. The result then follows directly from (a) because the PAB auction is in $\mathcal{A}$.

(c) If (II′) holds for any given b_{-i} and b_i, then $p_i^V(b, x(b)) = b_i(x_i(b)) = w(b_{-i}) -$

[7] Technically, a slightly higher bid may be required if the tie-breaking rule would not choose to assign y to bidder i. However, this will never be true for the best y for bidder i because we break ties in favor of best responses. This implies he must be able to win at least one of the bundles that maximize his payoff with the bid $b_i(y) = w(b_{-i}) - w(b_{-i}^{-y})$.

$\sum_{j \neq i} b_j(x_j(b))$, and $p_i(b, x(b)) = p_i^V(b, x(b)) = b_i(x_i(b))$ satisfies (II) for any auction in $\mathcal{A}$. ■

Bernheim and Whinston (1986) find that a bidder always has a truncation strategy among his best responses in the PAB auction, Day and Milgrom (2008) extend the result to any core-selecting auction, Sano (2013) extends it to all auctions in $\mathcal{A}$, and we extend it in part (c) to all best responses in the PAB auction and to all auctions in $\mathcal{A}$. Beck and Ott (2013) find that a bidder always has an overbidding strategy among his best responses in BOCS auctions. The strategy they use in their proof fulfills condition (II′). Thus, in all auctions in $\mathcal{A}$, bidders have a best response in truncation strategies and in overbidding strategies.[8]

This leads to our main theorem.

Theorem 2. *Consider any auction in $\mathcal{A}$. Then b^* is a Nash equilibrium if and only if (I) and (II) hold for all $i \in N$. Such a pure-strategy Nash equilibrium b^* always exists.*

Proof: From Theorem 1(a) it follows directly that no bidder has a profitable deviation and b^* is a Nash equilibrium if and only if (I) $x(b^*) \in \hat{X}(v_i, b^*_{-i})$ and (II) $p_i(b^*, x(b^*)) = p_i^V(b^*, x(b^*))$ for all $i \in N$.

Proof of existence: We will show that the following bids b^* satisfy conditions (I) and (II) for all $i \in N$. Define bids b^* according to:

$$
\begin{aligned}
b_i^*(y) &= v_i(y) \qquad \forall i \in N, \ \forall y \subset K \\
b_i^*(K) &= w(v) \qquad \forall i \in N
\end{aligned}
$$

Breaking ties in favor of best responses in an equilibrium means favoring the assignment that satisfies (I) for all $i \in N$, so a bidder can only be assigned K if $v_i(K) = w(v)$. Since

[8] Note that in core-selecting auctions both types of strategies may be dominated or undominated. Beck and Ott (2013) contains settings in which overbidding strategies are undominated, and settings in which truncation strategies are dominated in BOCS auctions. In PAB auctions, overbidding strategies are dominated and truncation strategies may be undominated.

bids for all bundles except K are truthful, $v_i(x_i(b^*))+\sum_{j\neq i} b^*_j(x_j(b^*)) = \sum_{j\in N} b^*_j(x_j(b^*)) = w(b^*) = w(v) = w(v_i, b^*_{-i})$. Thus, $x(b^*)$ generates value $w(v_i, b^*_{-i})$ and condition (I), $x(b^*) \in \hat{X}(v_i, b^*_{-i})$, holds for all $i \in N$.

Furthermore, by construction, $w(b^*) = w(v) = w(b^*_{-i})$ for all $i \in N$. This implies $p^V_i(b^*, x(b^*)) = b_i(x_i(b^*))$, which collapses the payment interval to a single point for all bidders and, thereby, ensures condition (II) holds for all $i \in N$. ■

Theorem 2 states not only that conditions (I) and (II) are sufficient but that they are also necessary for an equilibrium. Condition (II) is necessary and sufficient because Vickrey payments are the lowest payments that are possible while qualifying as a winning bidder of a bundle, and because they are achievable by an appropriate bid. Condition (I) is sufficient because winning a bundle $x \in \hat{X}(v_i, b_{-i})$ maximizes the payoff, given that the payment equals the Vickrey payment. Condition (I) is necessary because a bidder who pays his Vickrey payment is indifferent between all bids with that a bundle $x \in \hat{X}(v_i, b_{-i})$ is chosen but strictly prefers $x \in \hat{X}(v_i, b_{-i})$ over any $x \notin \hat{X}(v_i, b_{-i})$.

Given most valuation profiles, there exist infinitely many equilibria of the auctions in Theorem 2. However, these auctions do not necessarily have all of the same equilibria. Condition (II) depends on the auction's payment rule, so different bids will satisfy (II) for different auctions. The next theorem compares the equilibrium strategies of some common combinatorial auctions.

Theorem 3.

(a) *b^* is a Nash equilibrium of the PAB auction if and only if (I) and (II′) hold for all $i \in N$.*

(b) *b^* is a Nash equilibrium of the Vickrey auction if and only if (I) holds for all $i \in N$.*

(c) *All BOCS auctions have the same Nash equilibria.*

Proof: **(a)** The PAB auction has payment rule $p_i(b, x(b)) = b_i(x_i(b))$, so for the PAB auction (II) holds if and only if (II′) holds. The result then follows directly from Theorem

2.[9]

(b) The Vickrey auction's payment rule dictates that (II) holds for all $b \in B$. Thus, condition (II) is unnecessary and the result follows directly from Theorem 2.

(c) According to condition (II), for some b^* to be an equilibrium of a BOCS auction, it must result in *all* bidders receiving their reported Vickrey payoffs. This can happen only if $\pi^{r,V}(b^*) \in \mathcal{C}(b^*)$, in which case it is the unique bidder-optimal allocation given b^* and every BOCS auction chooses the same payments. Since all BOCS auctions choose the assignment in the same way for a given bid profile, condition (I) holds for some b^* and some BOCS auction if and only if it holds for that b^* and all BOCS auctions. Therefore, if b^* is an equilibrium of some BOCS auction it must be a Nash equilibrium of all BOCS auctions. ■

In the PAB auction, the bid $b_i(x_i(b))$ for the item that bidder i wins is decisive both for winning and for his payment. Thus, he must bid just high enough to win the bundle in equilibrium. In BOCS auctions, equilibrium bids can be higher because the auction rules will not always force bidders to pay their full bids. Both a bidder's winning bid $b_i(x_i(b))$ and losing bids $b_i(y)$ for bundles $y \neq x_i(b)$ can influence his payment in a BOCS auction. In the Vickrey auction, bidder i's bid determines only which bundle he wins and has no impact on his payment. (He is indifferent between all bids with that $x \in \hat{X}(v_i, b_{-i})$ is chosen.) For this reason, the Vickrey auction admits the most equilibria and the PAB auction the least. Any Nash equilibrium of the PAB auction is also a Nash equilibrium of any auction from Theorem 2 because bids satisfying (II′) have a core that is a single point: $\mathcal{C}(b) = (w(b), 0, \ldots, 0)$, meaning the only possible core payments are the Vickrey payments.

Bidding both above and below true values for bundles is possible in equilibrium in all of these auctions. In the PAB auction, overbidding occurs only on items that are not won in equilibrium, while in the Vickrey auction and in BOCS auctions bidders may also

[9]Part (a) was first proven by Bernheim and Whinston (1986), but has been adjusted to our model and notation.

overbid on the bundle they win. Underbidding also occurs in equilibrium in all auctions. Strategies that involve overbidding are weakly dominated in the PAB auction and in the Vickrey auction and strategies that involve underbidding are weakly dominated in the Vickrey auction. However, both over- and underbidding may be part of undominated equilibrium strategies in minimum-revenue core-selecting auctions (Beck and Ott, 2013).

Theorem 4. *Denote by* $NE^A(v)$ *the set of Nash equilibria in auction* A *for a given profile of values* v*. Denote the PAB auction and the Vickrey auction by* $A = PAB$ *and* $A = V$*, respectively.*

(a) $NE^{PAB}(v) \subseteq NE^A(v) \subseteq NE^V(v)$ *for all* v *and for any* $A \in \mathcal{A}$*.*

(b) *For any BOCS auction* A*, there exist* v *such that the subsets in (a) are strict.*

Proof: **(a)** By Theorem 2 b^* is an equilibrium in any auction in $\mathcal{A}$ if and only if conditions (I) and (II) hold. For given bids, condition (I) means the same in any auction in $\mathcal{A}$.

By Theorem 3(a), b^* is an equilibrium in the PAB auction if and only if conditions (I) and (II′) hold. Thus, if b^* fulfills (I) in the PAB auction, b^* fulfills (I) in any auction in $\mathcal{A}$. If b^* fulfills (II′), then $w(b^*) = w(b^*_{-i})$ for all $i \in N$. But if $w(b^*) = w(b^*_{-i})$, then b^* fulfills (II) for any auction in $\mathcal{A}$, because then $p_i(b^*, x(b^*)) = p_i^V(b^*, x(b^*)) = w(b^*_{-i}) - \sum_{j \neq i} b_j^*(x_j(b^*)) = w(b^*_{-i}) - (w(b^*) - b_i^*(x_i^*(b))) = b_i^*$ and any auction in $\mathcal{A}$, which by definition choose $p_i(b) \in [p_i^V(b, x(b)), b_i(x_i(b))]$, must choose $p_i(b^*) = b_i^*(x_i^*(b)) = p_i^V(b^*, x(b^*))$.

By Theorem 3(b), b^* is an equilibrium in the Vickrey auction if and only if condition (I) holds. If b^* fulfills (I) and (II) in any auction in $\mathcal{A}$, it fulfills (I) and is an equilibrium in the Vickrey auction.

(b) To prove that the subsets may be strict for certain value profiles, consider the following two examples where the set of goods is $K = \{A, B\}$.

	A	B	AB
v_1	1	0	1
v_2	0	1	1
v_3	0	0	1

Truthful bidding is always an equilibrium in the Vickrey auction, but it is not an equilibrium of the PAB auction for this example because condition (II′) is not satisfied: $w(v) = 2 > 1 = w(v_{-1})$. It is also not an equilibrium of any BOCS auction because $p_1+p_2 = 1$ implies condition (II) cannot hold for both 1 and 2. Either $p_1 > w(v_{-1}) - b_2(x_2(v)) = 0$ or $p_2 > w(v_{-2}) - b_1(x_1(v)) = 0$.

For the values shown above, the following bids form an equilibrium of every BOCS auction because the unique BOCS payments $p_1 = 0$ and $p_2 = 1$ satisfy condition (II). However, these bids are not an equilibrium of the PAB auction because $w(b) = 2 > 1 = w(b_{-1})$.

	A	B	AB
b_1	1	0	2
b_2	0	1	1
b_3	0	0	1

■

This theorem implies that the PAB auction has the smallest set of Nash equilibria of any core-selecting auction, but also that analysis of the equilibria of the PAB auction will shed light on all other core-selecting auctions because they share those equilibria.

Moreover, the set of equilibria of the PAB auction is the largest set that all auctions in $\mathcal{A}$ have in common. If condition (II′) holds in the PAB auction then also condition (II) holds for any auction in $\mathcal{A}$, because $\sum_{i \in N} b_i^*(x_i(b^*)) = w(b^*) = w(b_{-i}^*)$ if and only if $b_i^*(x_i(b^*)) = w(b_{-i}^*) - \sum_{j \in N \setminus \{i\}} b_j^*(x_j(b^*)) = p_i^V(x(b), b)$ and, thus, any auction that always chooses payments in $[p_i^V(x(b), b), b_i(x_i(b))]$ needs to choose $p_i^V(x(b), b) = b_i(x_i(b))$.

Corollary 1. *All auctions in $\mathcal{A}$ have Nash equilibria in common. b^* is an equilibrium of every auction in $\mathcal{A}$ if and only if (I) and (II′) hold for all $i \in N$.*

This corollary covers several results from the literature. The equilibria in "truthful" strategies in the PAB menu auction discovered by Bernheim and Whinston (1986) appear in Ausubel and Milgrom (2002) and Day and Milgrom (2008) as equilibria in "profit-targeting strategies," "semi-sincere," or "truncation strategies" for combinatorial BOCS auctions and for all combinatorial core-selecting auctions. Sano (2013) then proved that they are equilibria in any auction in $\mathcal{A}$. We now know that *any* equilibrium of the PAB auction will be an equilibrium of any auction in $\mathcal{A}$, and we will see that the outcome of such an equilibrium will be identical in any auction (see Theorem 5).

Theorem 4 shows the relationship between the equilibria of specific auctions for given value profiles. In particular, though these auctions have equilibria with similar properties, the set of equilibria of the PAB auction can be strictly smaller than that of BOCS auctions, and likewise for BOCS auctions and the Vickrey auction. However, to understand whether the exclusion of some equilibria matters, we must consider the outcomes of these equilibria and not just the strategies themselves. The equilibrium outcomes are the subject of the next section.

4. Equilibrium Assignments, Payments, and Payoffs

In any auction in $\mathcal{A}$, equilibrium payments are equal to the Vickrey payments $p_i^V(b, x(b))$. Moreover, all of these auctions choose the assignment in the same way. Therefore, whenever they share an equilibrium b^*, they share the equilibrium outcome (assignment, payments, and payoffs), and the outcome equals that of the Vickrey auction rules applied to b^*.[10]

Theorem 5. *The assignment $x(b^*)$ and the payments $p_i(b^*, x(b^*))$ from any Nash equilibrium b^* of an auction in $\mathcal{A}$ equal the assignment and payments calculated by applying the*

[10] Note that for given b_{-i}, i's best response gives him the same payoff in any of the auctions in $\mathcal{A}$ (by Theorem 1 and the arguments from the proof of Theorem 5).

Vickrey auction rules to b^*.

Proof: By Theorem 2, b^* satisfies conditions (I) and (II). Thus, $p_i(b^*, x(b^*)) = p_i^V(b^*, x(b^*))$ by (II) and the assignment is the same by (I) (up to the result of a possible randomization to break ties). ■

This result allows to easily exclude that a profile of bids b be an equilibrium in an auction in $\mathcal{A}$ without any further information on v. If b results in different payments in the considered auction and in the Vickrey auction applied to b, then b is not an equilibrium of the considered auction.[11]

The theorem implies that in any full-information Nash equilibrium of any core-selecting auction, the reported Vickrey payoffs will be in the reported core $\pi^{r,V}(b^*) \in \mathcal{C}(b^*)$ (forming the unique bidder-Pareto-optimal point in the core) and the outcome will be equivalent to that of the Vickrey auction, given bids b^*. Furthermore, in any equilibrium the bidders will earn their true Vickrey payoffs $v_i(x_i^*) - p_i^V(b^*, x(b^*))$, which may be neither in the reported core $\mathcal{C}(b)$ nor in the core with respect to the true values $\mathcal{C}(v)$. However, one of the reasons core-selecting auctions were proposed in the first place was to correct some weaknesses of the Vickrey auction that occur precisely when the Vickrey payoffs are not in the core $\mathcal{C}(v)$ (see Ausubel and Milgrom, 2006).

We now study the equilibrium outcomes of the auctions in $\mathcal{A}$ in more detail. For the PAB auction and a class of auctions that contains Vickrey and BOCS auctions we provide necessary and sufficient conditions for assignment-payment pairs to be supportable by some equilibrium from Section 3. For ease of notation, we assume for the remainder of this section that bidders' values satisfy free disposal: $v_i(y) \leq v_i(y')$ for all $i \in N$ and all $y \subseteq y' \subseteq K$. All of the results will hold without free disposal, but the third condition in Theorems 6 and 7 must be modified to reflect the possibility that smaller sets of items yield larger values.

[11] The reverse is not true as can be seen for example from the bids $b_i(y) = 0$ for all y and all i, which lead to the same outcome in any auction but which for generic v aren't equilibria.

To characterize the equilibrium outcomes, we will make use of the following lemma that reduces the equilibria of the PAB auction to a simpler bidding structure, using outcome equivalence.

Lemma 1. *If b^* is an equilibrium of the PAB auction, then $\tilde{b}$ as defined below is also an equilibrium of the PAB auction and $\tilde{b}$ leads to the same assignment and payments as b^*.*

$$\tilde{b}_i(x_i(b^*)) = b_i^*(x_i(b^*)) \quad \forall i \in N$$
$$\tilde{b}_i(K) = w(b^*) \quad \forall i \in N$$
$$\tilde{b}_i(y) = 0 \quad \forall i \in N, \forall y \notin \{x_i(b^*), K\}$$

Proof: We will show that $\tilde{b}$ satisfies conditions (I) and (II′) for all $i \in N$, so that by Theorem 3, it is a Nash equilibrium of the PAB auction. Notice that, by construction, there are n or $n+1$ assignments that maximize the reported surplus: $x(b^*)$ and assigning all of the goods to any single bidder. All of these assignments generate the same value, $w(b^*)$. So $w(b^*) = w(\tilde{b})$ and $w(b^*) = w(\tilde{b}_{-i})$ for all $i \in N$, which satisfies condition (II′).

To prove condition (I), first note that $w(v_i, b_{-i}^*) = \max_{x \in X(K)} \left[\sum_{j \neq i} b_j^*(x_j) + v_i(x_i)\right] \geq \max_{x \in X(K), x_j \neq K \forall j \neq i} \left[\sum_{j \neq i} b_j^*(x_j) + v_i(x_i)\right] \geq \max_{x \in X(K), x_j \neq K \forall j \neq i} \left[\sum_{j \neq i} \tilde{b}_j(x_j) + v_i(x_i)\right] \geq \sum_{j \neq i} \tilde{b}_j(x_j(b^*)) + v_i(x_i(b^*)) = w(v_i, b_{-i}^*)$. The first inequality holds because the maximum over a smaller set is weakly smaller. The second inequality holds because $\tilde{b}_j(y) \leq b_j^*(y)$ for all $y \neq K$. The third inequality holds because $x(b^*)$ is one feasible assignment.

Then, $w(v_i, \tilde{b}_{-i}) = \max\left\{\max_{x \in X(K), x_j \neq K \forall j \neq i} \left[\sum_{j \neq i} \tilde{b}_j(x_j) + v_i(x_i)\right], \max_{j \neq i} \tilde{b}_j(K)\right\} = \max\left\{w(v_i, b_{-i}^*), w(b^*)\right\} = w(v_i, b_{-i}^*)$. The second equality holds by the above inequalities and the third holds because we must have $v_i(x_i(b^*)) \geq b_i^*(x_i(b^*))$ in any equilibrium of the PAB auction. Therefore, we know by b^* being an equilibrium that $x(b^*)$ generates value $w(v_i, b_{-i}^*) = w(v_i, \tilde{b}_{-i})$, so $x(b^*) \in \hat{X}(v_i, \tilde{b}_{-i})$ and is an optimal assignment given $\tilde{b}$. The tie-breaking rule then implies $x(\tilde{b}) = x(b^*) \in \hat{X}(v_i, \tilde{b}_{-i})$, satisfying condition (I) for all $i \in N$. Moreover, this proves that $\tilde{b}$ leads to the same assignment as b^*, which implies the payments are the same by construction. ■

This lemma allows us to reduce attention to equilibria with the simple structure above, in which bidders place nonzero bids only for the bundle they win and the bundle of all items in the auction. The potential profitable deviations with this simple structure are much fewer than with a full set of nonzero bids, which helps us characterize the following necessary and sufficient condition for equilibrium outcomes of the PAB auction.

Theorem 6. *There exists a Nash equilibrium b^* of the PAB auction that results in outcome $x = x(b^*)$ and $p = p(b^*, x(b^*))$ if and only if the following conditions hold:*

(1) $x \in X(K)$

(2) $p_i \in [0, v_i(x_i)] \quad \forall i \in N$

(3) $\sum_{j \in S} p_j \geq \max_{i \in N \setminus S} [v_i(x_i \cup x_0 \cup (\cup_{j \in S} x_j)) - v_i(x_i)] \quad \forall S \subset N$

Proof: By Lemma 1, (x, p) will be the outcome of some equilibrium of the PAB auction if and only if it is the outcome of an equilibrium of the form given in the lemma. Therefore, we will restrict attention to equilibria b^* of the form for all $i \in N$:

$$b_i^*(y) = \begin{cases} p_i & \text{if } y = x_i \\ \sum_{j \in N} p_j & \text{if } y = K \\ 0 & \text{if } y \notin \{x_i, K\} \end{cases}$$

By construction of these bids, the winning assignment is x and the prices paid are the bids for that assignment, p, if and only if $x \in X(K)$ and $p_i \geq 0$ for all $i \in N$. Now we must show that this is an equilibrium if and only if the other conditions in the theorem hold.

By Theorem 3, b^* is an equilibrium of the PAB auction if and only if $x \in \hat{X}(v_i, b^*_{-i})$ for all $i \in N$ and $w(b^*) = w(b^*_{-i})$ for all $i \in N$. The second condition is satisfied because each bidder places a bid of $w(b^*) = \sum_{j \in N} p_j$ for the bundle of everything, K, so $w(b^*_{-i}) = w(b^*)$ for all $i \in N$. Therefore, we need only to prove $x \in \hat{X}(v_i, b^*_{-i})$ for all $i \in N$ if and only if $p_i \leq v_i(x_i)$ for all $i \in N$ and (3) holds. Given the structure of b^*_{-i} and the fact that values satisfy free disposal, bidder i can only win subsets of $\{x_0, x_1, \ldots, x_n\}$ or nothing. He prefers to win x_i instead of nothing if and only if $v_i(x_i) \geq p_i$. Bidder i prefers to win x_i instead of some subset of bundles $\{x_j | j \in S\} \cup \{x_0\} \cup \{x_i\}$ if and only if $v_i(x_i) + \sum_{j \in S} p_j \geq v_i(\cup_{j \in S} x_j \cup x_0 \cup x_i)$, which is equivalent to (3) if we require it for all i and all $S \subseteq N \setminus \{i\}$. ■

According to condition (3), successful bidders in the PAB auction have to pay at least any opponent's willingness to pay for their assigned bundles. In particular, the sum of payments in any equilibrium assignment are bounded below by the values of losing bidders for any union of assigned bundles; the lower bound may be higher because successful bidders' might also desire additional assigned bundles if payments are too low.

Not every assignment is possible in some equilibrium of the PAB auction because condition (3) must hold. In particular, the value of the assignment must exceed the minimum sum of payments mentioned in the previous paragraph.However, the condition from this theorem certainly allows inefficient assignments. It only considers deviations in which a single bidder takes all bundles from some subset of the other bidders, which does not recognize any possibility for coordination among bidders' values. Two bidders might together be able to outbid a winning bidder, but as long as they both bid zero for what they want, neither bidder by himself has a profitable deviation.

Denote the set of true payoff vectors in Nash equilibria of the PAB auction by:

$$\Pi^{PAB}(v) = \{(\textstyle\sum_{i \in N} p_i, v_1(x_1) - p_1, \ldots, v_n(x_n) - p_n) | (x, p) \text{ satisfies (1)–(3) of Thm. 6}\}$$

Corollary 2. *The equilibrium revenues from the PAB auction may be below the Vickrey auction revenues from truthful bidding:*

$$\exists v \textit{ such that } \min\{\pi_0 | \pi = (\pi_0, \ldots, \pi_n) \in \Pi^{PAB}(v)\} < \pi_0^V(v).$$

Proof: The following is an example in which bidder 1 wins AB and pays 1, which is less than his Vickrey payment with respect to the true values of 2.

	A	B	AB		A	B	AB
v_1	2	2	4	b_1	0	0	1
v_2	1	0	1	b_2	0	0	1
v_3	0	1	1	b_3	0	0	1

■

In fact, even the revenues in efficient equilibria of the PAB auction may be higher or lower than the truthful Vickrey revenues. The low revenue equilibrium in the example in the proof of Corollary 2 is efficient. For an example in which revenues in efficient equilibria of the PAB auction are always above the Vickrey revenues from truthful bidding, consider three bidders with values for bundles A, B, and AB of $v = ((2,0,2),(0,2,2),(0,0,3))$. The Vickrey revenue is $\pi_0^V(v) = 2$ but, in the PAB auction, any efficient equilibrium must generate revenue of at least 3. With these values, any equilibrium of the PAB auction improves the seller's revenue compared to the dominant-strategy equilibrium of the Vickrey auction.

BOCS auctions can admit even worse outcomes than the PAB auction in terms of efficiency and revenues. Thus, using the PAB auction instead of a BOCS auction eliminates some undesirable Nash equilibria without introducing any new equilibria. The following theorem illustrates this point through a type of folk theorem: any individually rational outcome that does not leave valuable items unassigned can occur in an equilibrium of a class of auctions that contains any BOCS auction and the Vickrey auction.

Denote by $p_i^{BOCS}(b, x(b))$ the payments in some specific BOCS auction when bids b are submitted.

Definition 7. *The class of auctions $\mathcal{B}$ contains any sealed-bid auction $(x(b), p(b, x(b)))$ that, for every $b \in B$, selects $x(b) \in \hat{X}(b)$, breaks ties in favor of best responses, and chooses payments $p_i(b, x(b)) \in [p_i^V(b, x(b)), p_i^A(b, x(b))]$ for all $i \in N$ and for some BOCS* *auction A.*

Theorem 7. *Given any auction in class $\mathcal{B}$, there exists a Nash equilibrium b^* that results in outcome $x = x(b^*)$ and $p = p(b^*, x(b^*))$ if and only if the following conditions hold:*

(1) $x \in X(K)$

(2) $p_i \in [0, v_i(x_i)] \quad \forall i \in N$

(3) $v_i(x_0 \cup x_i) = v_i(x_i) \quad \forall i \in N$

Proof: Note that $\mathcal{B} \subset \mathcal{A}$. So, any equilibrium of any auction in $\mathcal{B}$ must satisfy (I) and (II) for all $i \in N$ by Theorem 2. Satisfying (I) for all $i \in N$ necessitates that the assignment x be feasible and that the unassigned items have no incremental value for any bidder, so (1) and (3) hold. Satisfying condition (II) for all $i \in N$ means $p_i(b^*, x(b^*)) = w(b^*_{-i}) - \sum_{j \neq i} b^*_j(x_j(b^*)) \in [0, v_i(x_i(b^*))]$ and condition (2) holds.

To show that (1)–(3) imply the existence of such an equilibrium for any auction $A \in \mathcal{B}$, note that if (1)–(3) imply the existence of such an equilibrium in the BOCS auctions, then (1)–(3) imply the existence of such an equilibrium in any auction $A \in \mathcal{B}$. This holds because by Theorem 5, in any equilibrium b^* of the BOCS auction it must be $x^{BOCS}(b^*) = x^V(b^*)$ and $p^{BOCS}(b^*, x(b^*)) = p^V(b^*, x(b^*))$, and, thus, by definition of $A \in \mathcal{B}$, $x^A(b^*) = x^V(b^*)$ and $p^A(b^*, x(b^*)) = p^V(b^*, x(b^*))$, implying that b^* satisfies (I) and (II) in auction A.

Consider any feasible assignment $x \in X(K)$ such that $v_i(x_0 \cup x_i) = v_i(x_i)$ $\forall i \in N$. Choose some $M > w(v)$. Let ω be the number of winning bidders in x. Relabel the bidders such that the winning bidders are labeled $1, \ldots, \omega$ and let $W = \{1, \ldots, \omega\}$. Then, for all $p_i \in [0, v_i(x_i)]$, the following is a Nash equilibrium of every BOCS auction and of the Vickrey auction with $x(b^*) = x$ and $p(b^*, x(b^*)) = p$:

$$\begin{aligned}
b_i^*(x_i) &= M && \forall i \in \{1 \dots, \omega\} \\
b_i^*(x_i \cup x_{i+1}) &= M + p_{i+1} && \forall i \in \{1, \dots, \omega - 1\} \\
b_\omega^*(x_\omega \cup x_1) &= M + p_1 && \\
b_i^*(y) &= 0 && \forall y \notin \{x_i, x_i \cup x_{i+1}\} \text{ for } i \in \{1, \dots, \omega - 1\}, \\
& && \forall y \notin \{x_\omega, x_\omega \cup x_1\} \text{ for } i = \omega, \\
& && \forall y \text{ for } i \in \{\omega + 1, \dots, n\}
\end{aligned}$$

With these bids, the unique optimal assignment is $x(b^*) = x$. Since $M > w(v) \geq v_i(y)$ for all $y \in 2^K$ and for all $i \in N$, $w(v_i, b_{-i}^*) = v_i(x_i \cup x_0) + (\omega - 1)M = v_i(x_i) + (\omega - 1)M$ for all $i \in W$ and $w(v_i, b_{-i}^*) = v_i(x_0) + \omega M = \omega M$ for all $i \notin W$. Therefore, $x \in \hat{X}(v_i, b_{-i}^*)$ for all $i \in N$ and b^* satisfies condition (I). So b^* is an equilibrium of the Vickrey auction by Theorem 3.

To prove that b^* is an equilibrium of every BOCS auction, we have to show condition (II) also holds or, equivalently, that Vickrey payments $p_i^V(b^*, x_i) = w(b_{-i}^*) - \sum_{j \neq i} b_j^*(x_j)$ satisfy the core constraints (in which case they will be the payments chosen by every BOCS auction). Note that $w(b^*) = \omega M$, $\sum_{i \in N \setminus S} b_i^*(x_i) = (\omega - |S \cap W|)M$, and $w(b_{-S}^*) = (\omega - |S \cap W|)M + \sum_{i \in S} p_i$ for all $S \subset N$. Thus, $p_i^V(b^*, x_i) = (\omega - |\{i\} \cap W|)M + p_i - (\omega - |\{i\} \cap W|)M = p_i$. The core constraints are satisfied because $\sum_{i \in S} p_i^V(b^*, x_i) = \sum_{i \in S} p_i = (\omega - |S \cap W|)M + \sum_{i \in S} p_i - (\omega - |S \cap W|)M = w(b_{-S}^*) - \sum_{i \in N \setminus S} b_i^*(x_i)$ for all $S \subset N$. Thus, condition (II) holds for all $i \in N$. ■

These results on payoffs apply equally to BOCS auctions and the Vickrey auction. So even though the Vickrey auction has a larger number of equilibria by Theorem 4, its extra equilibria do not add any outcomes that are not already being achieved in an equilibrium of every BOCS auction.

The construction of the equilibrium in this proof uses bids that may seem implausibly large because $M > w(v)$. The use of such large bids is a convenient construction that works for all assignments. However, many assignments can be implemented with more reasonable bids. For example, the efficient assignment and any individually rational payments can be implemented with the same structure as above, except replacing $b_i^*(x_i) = M$ with $b_i^*(x_i) = v_i(x_i)$.

Equilibria may involve colluding, harming, or discouraging bidding. Winning bidders may bid high on the bundles they win, thereby reducing the payments of the other winners. They may also bid as low as possible on the items they win and/or high on bundles they do not win, e.g. on the full package K, to increase the payments of the other winners. A bidder may bid very high to discourage others from bidding. Such strategies or equilibria might appear attractive in BOCS auctions or even in the Vickrey auction, in which deviating from the dominant strategy to bid truthfully reduces the own expected payoff, depending on the environment in which the auction takes place, for example in recurring auctions.

The conditions in Theorem 7 do not impose any restrictions on payoff profiles, that is, any combination of payoffs is possible in equilibrium, and the gains from trade can be extremely unequally distributed.

Corollary 3. *For any $x \in X(K)$, $\pi_{-0} \in [0, v_1(x_1)] \times [0, v_2(x_2)] \times \ldots \times [0, v_n(x_n)]$ are the bidders' payoffs in some Nash equilibrium of any auction in $\mathcal{B}$. The seller's equilibrium payoff may take any value in $[0, w(v)]$ and each bidder i's equilibrium payoff may take any value in $[0, v_i(K)]$.*

While the upper bound on revenues in the Vickrey and BOCS auctions is the same as that of the PAB auction, the lowest equilibrium revenue in the PAB auction is generally higher than that in the Vickrey and BOCS auctions. This occurs because the Vickrey and BOCS auctions allow more bid profiles to satisfy (I) and (II), and choose lower revenues than the PAB auction for any given bids. Uncompetitive equilibria in which the revenue is zero because losing bidders do not place positive bids do not generally exist in the PAB auction, but do always exist in the Vickrey and BOCS auctions. The payoffs from truthful bidding in the Vickrey auction are also always an outcome of some equilibrium of all BOCS auctions, but this is not always true for the PAB auction.[12]

Corollary 4. *The payoff vector $\pi^V(v)$ resulting from truthful bidding by all $i \in N$ in the Vickrey auction is in $\Pi^A(v)$ for any auction $A \in \mathcal{B}$.*

Proof: In the class of equilibria in the proof of Theorem 7, when $p_i = p_i^V(v, x(v))$, which is feasible because $p_i^V(v, x(v)) \leq v_i(x_i(v))$, the equilibrium payoffs equal those from truthful bidding in the Vickrey auction. ■

While the Vickrey auction has a unique equilibrium in undominated strategies, $b = v$, core-selecting auctions generally have many such equilibria. For example, Beck and Ott (2013) show for a BOCS auction that a bid that involves bidding strictly above the true value may be an undominated strategy.

The final theorem summarizes the relationships between potential equilibrium payoffs.[13] Remember, that any BOCS auction and the Vickrey auction is in the class $\mathcal{B}$.

Theorem 8. *Denote by $\Pi^A(v)$ the set of payoffs in Nash equilibria in auction A for a given profile of values v. Denote the PAB auction by $A = PAB$.*

(a) $\Pi^{PAB}(v) \subseteq \Pi^A(v) \subseteq \Pi^B(v)$ *for all v and for any $A \in \mathcal{A}$ and any $B \in \mathcal{B}$.*

(b) *There exist v such that $\Pi^{PAB}(v) \subset \Pi^B(v)$ for any $B \in \mathcal{B}$.*

Proof: **(a)** By Theorem 4(a), $\text{NE}^{PAB}(v) \subseteq \text{NE}^A(v) \subseteq \text{NE}^V(v)$ for all v and for any $A \in \mathcal{A}$. Combining this with Theorem 5, we find that any equilibrium in the PAB auction is an equilibrium with the same outcome in any auction in $\mathcal{A}$, and any equilibrium in any auction in $\mathcal{A}$ is an equilibrium with the same outcome in the Vickrey auction. Remember

[12] See the example after Corollary 2 with $v = ((2,0,2),(0,2,2),(0,0,3))$.

[13] The second subset will often be strict. For example, $\Pi^{PAB}(v) \subset \Pi^B(v)$ for $B \in \mathcal{B}$ whenever $v_i(K) > v_i(y)$ for all $y \subset K$ and for at least one bidder i, because in this case the revenue in the PAB auction is strictly positive.

that the Vickrey auction is in class $\mathcal{B}$ and that by Theorem 7 any auction in class $\mathcal{B}$ has the same set of equilibrium outcomes. Therefore, $\Pi^{PAB}(v) \subseteq \Pi^{A}(v) \subseteq \Pi^{B}(v)$.

(b) The following example shows that there exists v such that $\Pi^{PAB}(v) \subset \Pi^{B}(v)$ for any $B \in \mathcal{B}$, using that the Vickrey auction is in class $\mathcal{B}$ and that by Theorem 7 any auction in class $\mathcal{B}$ has the same set of equilibrium outcomes.

	A	B	AB	$p_i^V(v)$	$\pi_i^V(v)$
v_1	5	0	5	1	4
v_2	0	5	5	1	4
v_3	0	0	6	0	0

The payoffs $\pi_i^V(v)$ in the Vickrey auction equilibrium with truthful bids $b = v$ are not possible in any equilibrium of the PAB auction. Achieving these payoffs in the PAB auction requires payments and, thus, bids of $b_1(A) = b_2(B) = 1$. (They must win A and B, respectively, because no other assignment can generate a positive payoff for both bidders 1 and 2.) However, with such low bids, bidder 3 would have a profitable deviation to bid more than 2 on AB. ■

Any equilibrium of the PAB auction is an equilibrium with the same outcome in any auction in $\mathcal{A}$ (see Corollary 1 and Theorem 5). Because the PAB auction is in class $\mathcal{A}$, this implies the following corollary.

Corollary 5. *All auctions in $\mathcal{A}$ have Nash equilibrium outcomes in common. $(x(b^*), p(x(b^*), b^*)$ is an equilibrium outcome of every auction in $\mathcal{A}$ if and only if it is an equilibrium outcome of the PAB auction.*

5. Extensions on the Seller's Incentives and Budget Constraints

We discuss implications of our previous analyses for extended games in which the seller might manipulate the auction outcome after having collected the bids, an extension in which the seller sets secret reserve prices, and an extension in which the bidders have to take individual binding budget constraints into account. The insights follow straightforwardly from our previous results and analyses. Proofs are given in B.

5.1. The Seller's Commitment to the Auction Rules and Secret Reserve Prices

In the PAB auction, knowing the bids, the seller maximizes his revenue by choosing the optimal assignment. In other auctions, like the Vickrey auction or different core-selecting auctions, the seller may be able to manipulate the outcome to his advantage by disqualifying bidders (e.g. Ausubel and Milgrom, 2002), asking bidders to reduce bids (e.g. Beck and Ott, 2009; Lamy, 2010), holding back items like a monopolist might want to do, or switching to the revenue maximizing tie-breaking rule in auctions in which the revenue might differ depending on how ties are broken (the PAB auction, MRCS auctions, and the Vickrey auction are not among these auctions, as shown in the proof of Theorem 11). However, there are equilibria that circumvent such incentives to manipulate.

Consider the two-stage game in which bidders simultaneously submit their sealed bids, the seller opens the bids, and then decides whether to exert any of the aforementioned manipulations and to apply the auction rules to the manipulated bids, or to follow the announced assignment and payment rule without manipulations.

Theorem 9. *In any auction in $\mathcal{A}$ there is a Nash equilibrium such that the seller, after collecting the bids, has no incentives to disqualify bidders or bids, to ask bidders to reduce their bids, to hold back items from sale, or to alter the tie-breaking rule.*

With another extension of our auction games we address secret reserve prices. Consider the extended auction game in which the seller sets secret but binding reserve prices $b_0(y) \geq 0$ for all $y \in 2^K$. Because the bidders do not learn the reserve prices, the extended game is one of simultaneous decisions.[14] The assignment rule assigns as before, with the exception

[14]In any subgame perfect equilibrium in the sequential game with full information in which the seller sets reserves first, he would clearly set reserves equal to the values of the winning bidders in the efficient assignments and earn the revenue $\pi_0 = w(v)$ in all considered auctions.

that it assigns the bundle not to i but to the seller if $b_0(x_i(b)) > b_i(x_i(b))$. Payments are determined as $\max\{p_i(b, x_i(b)), b_0(x_i(b))\}$ if $b_0(x_i(b)) \leq b_i(x_i(b))$, and zero otherwise, where $p_i(b, x_i(b))$ is the payment of the respective auction as defined in the previous sections.

Theorem 10. *If the seller sets secret reserve prices, the sets of Nash equilibria in all auctions in $\mathcal{A}$ are the same and equal to NE^{PAB} augmented by reserves $b_0(x_i(b)) = p_i(b, x(b))$ for all assigned bundles and arbitrary reserves for the other bundles.*

Both results in this section base upon the insight that all auctions have the Nash equilibria and the Nash equilibrium outcomes of the PAB auction in common.

5.2. Budget Constraints

Assume that each bidder has a fixed finite budget $c_i > 0$. This budget impacts his payoff as follows. A bidder i's payoff is zero, if he does not win any item, i's payoff is $v_i(y) - p_i$ if he wins bundle y and pays price $p_i \leq c_i$, and his payoff is $-\infty$ if he wins a bundle y at a price $p_i > c_i$.

Bidders' budgets that do not allow them to pay up to their value may impact their bidding behavior and, thereby, the efficiency of the auction.[15] For example, in the Vickrey auction, truthful bidding is no longer a weakly dominant strategy for bidders with $v_i(y) > c_i$ for some y. However, in any auction in class $\mathcal{B}$, which includes the Vickrey auction and BOCS auctions, there exist Nash equilibria that circumvent the problem.

Corollary 6. *Any auction in class $\mathcal{B}$ has efficient Nash equilibria in which positive budget constraints are not binding, that is, in which $p_i(b, x(b)) < c_i$ for all i.*

This is a direct consequence of Theorem 7, which states that for any v these auctions have Nash equilibrium outcomes with an efficient assignment and payments strictly below c_i for any $c_i > 0$ for all i.

[15] In BOCS auctions with incomplete information, even budgets that allow to pay up to the value may impact the bidding behavior. For example, bidders with such budget constraints would not want to stick with their equilibrium strategies with overbidding in the independent private values setting in Beck and Ott (2013).

6. Conclusion

This paper shows that all auctions that choose an optimal assignment given bids and have payments bounded above by bids and below by reported Vickrey payments share equilibria. This class of auctions includes all core-selecting auctions and the Vickrey auction. Moreover, even the equilibria that are not shared among them have similar properties – each bidder wins what he would have won had he bid truthfully and pays his Vickrey payment, given the bids of the others. Differences in incentives in the auctions result from the different requirements on bids to achieve Vickrey payments that result from the payment rules.

The set of outcomes that result from the equilibria of the Vickrey auction and all BOCS auctions is the same and includes almost everything. Every feasible assignment that doesn't leave valuable goods unassigned, including giving all items to *bidders* who don't value them, is possible and can be accompanied by any payments that make the payoffs individually rational. This means the seller can get zero revenue or capture all of the surplus in equilibrium. The equilibrium outcomes of the PAB auction must satisfy a stronger condition, but still include many inefficient assignments and can include payments that are outside of the core and even lower than the Vickrey payments with truthful bidding. The worst case revenue in the PAB auction is at least as high as in any other considered auction.

Knowing the full set Nash equilibria and equilibrium outcomes is important to judge the role of specific refinements. In how far does a specific refinement indeed refine with respect to equilibria and outcomes? What equilibria are rejected by a refinement? The wide range of Nash equilibrium outcomes suggests that conclusions based on particular Nash equilibria should be interpreted with caution. Given that the usual refinement in the Vickrey auction is the focus on undominated strategies while in the PAB and BOCS auctions the focus is on truncation strategies , it might be fruitful to find one refinement that covers the underlying ideas and that can be applied to all auctions.[16] Our paper provides the basis for such analyses by describing the universal set from which refinements select.

There are many types of equilibria. In particular in Vickrey and BRCS auctions, winners may use their unsuccessful bids to influence payments (of others only in the Vickrey auction of oneself and others in the BRCS auctions). In some equilibria of Vickrey or BOCS auctions, aggressive winning bids deter bids by others. In Vickrey auctions, winning complementing bidders may even collude by submitting high bids that reduce each other's payments.[17] All our auctions have equilibria in which high unsuccessful bids drive up the payments or bids of others. In other equilibria, all bidders determine their bids such that any successful bid guarantees to reach a profit target. Results on existence of best responses in overbidding and in profit-targeting strategies indicate that at least Vickrey and BOCS auctions also have equilibria in which both types of strategies occur. All of our auctions also have equilibria in which positive bids are submitted only for the bundle of the full set of items, thereby effectively reducing the auction to a single unit (the bundle of all items).[18] Depending on the setting, different of these equilibria may be more or less plausible. For example, in repeated settings colluding by repeatedly playing low-payment equilibria is likely; in other settings bidders might outside the auction game profit from hurting others and therefore use best responses that involve high payments by others. Not losing sight of such equilibria is another contribution of this paper.

[16] For example, Beck and Ott (2013) find equilibria in undominated strategies in BOCS auctions that aren't truncation strategies but involve bidding more than a bundle's value. In PAB auctions, bid shading by the same amount for all bundles as in truncation equilibria is not necessary for equilibria in undominated strategies.

[17] For example if there are three bidders 1, 2, and 3, and two items, A and B, and bids are $b_i = (b_{iA}, b_{iB}, b_{iAB})$, by bidding $b_1 = (x, 0, x)$, $b_2 = (0, x, x)$, $b_3 = (0, 0, y)$ the complementing bidders 1 and 2 win and pay zero if $y \geq x$.

[18] PAB, BOCS, and Vickrey auctions all have Bayesian equilibria of this type for any number of bidders or items (Bernheim and Whinston, 1986; Beck and Ott, 2013; Holzman and Monderer, 2004).

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A. Properties and Implementation of the Tie-Breaking Rule

Equilibria in combinatorial auctions usually depend on the tie-breaking rule. We applied tie-breaking in favor of best responses.

First, we prove our claim that tie-breaking in favor of best responses maximizes the set of equilibria (Lemma 2). Then, we show that tie-breaking in favor of a higher grade of efficiency results in the same sets of equilibria as tie-breaking in favor of best responses (Lemma 3).[19] Finally, we show that the tie-breaking rule can be implemented by an extended game in which bidders, next to their bids, name their desired bundle in case of a tie (Theorem 11).

Our tie-breaking rule, by choosing in any tie an assignment that fulfills (I) and (II) if such assignment exists, assures that a bidder can achieve his best payoff. Consider a different tie-breaking rule. If b evokes a tie and b_i is a best response with our tie-breaking rule, but the alternate rule assigns positive probability to assignments that our tie-breaking rule assigns probability zero, then b_i is not a best response. This leads to the following lemma.

Lemma 2. *For any auction in $\mathcal{A}$, the set of Nash equilibria contains all Nash equilibria of an auction with the same assignment and pricing rules but with another tie-breaking rule. In the joint equilibria, the alternate tie-breaking rule chooses among outcomes that have positive probability under tie-breaking in favor of best responses.*

Proof: Denote tie-breaking in favor of best responses by TB and any alternate tie-breaking rule by TB'. If conditions (I) and (II) hold for some $x \in \hat{X}(b)$ for all $i \in N$, and TB is applied, then b is a Nash equilibrium by Theorem 2. Thus, if b is not an equilibrium with TB, then there is no $x \in \hat{X}(b)$ such that (I) and (II) hold. We show that if $x \in \hat{X}(b)$ is chosen with positive probability by TB' and there is an i for whom (I) or (II) do not hold for x and b, then b is not an equilibrium with TB'.

[19] Bernheim and Whinston (1986, Footnote 5, p. 5) already state that any equilibrium in the PAB auction necessarily maximizes the grade of efficiency among tied assignments.

This bidder i can strictly improve by changing b_i such that (I) holds, and (II) holds or is not met by a sufficiently small amount. By the proof of Theorem 1, given b_{-i}, $\pi_i^V((v_i, b_{-i}), x(v_i, b_{-i}))$ is i's maximum payoff in any auction in $\mathcal{A}$, which is feasible unter TB but might not be feasible under TB', and he receives this maximum payoff if and only if (I) and (II) hold. Thus, if either (I) or (II) do not hold, then $\pi_i(b, x(b)) < \pi_i^V((v_i, b_{-i}), x(v_i, b_{-i}))$. If i submits a bid b_i that is a best response under TB, $\pi_i^V((v_i, b_{-i}), x(v_i, b_{-i}))$ is i's virtual payoff in one of the tied assignments, denoted by $\hat{x}$. Denote his virtually successful bid by $b_i(\hat{x}_i)$. By submitting $b_i'(y) = b_i(y)$ for all $y \neq \hat{x}_i$ and $b_i'(\hat{x}_i) = b_i(\hat{x}_i) + \varepsilon$ he will achieve a payoff in $[\pi_i^V((v_i, b_{-i}), x(v_i, b_{-i})) - \varepsilon, \pi_i^V((v_i, b_{-i}), x(v_i, b_{-i}))]$ in any auction that chooses payments in $[p_i^V(b, x(b)), b_i(x_i(b))]$ because the increased bid makes $\hat{x}$ the unique assignment in $\hat{X}$, such that i is assigned $\hat{x}_i$, and because the unilateral increase does not influence the payment's lower bound $p_i^V(b, \hat{x})$ and sets the upper bound to $b_i(\hat{x}_i)$. For sufficiently small ε we get $\pi_i(b, x(b)) < \pi_i^V((v_i, b_{-i}), x(v_i, b_{-i})) - \varepsilon$, a strict improvement. ■

Some of our auctions will never make use of condition (II) when breaking ties. In the Vickrey auction, condition (II) is fulfilled for any outcome by definition of the auction rules. In the PAB auction, if there is one tied outcome that fulfills conditions (I) and (II), $w(b) = w(b_{-i})$, for bidder i, then other tied assignments that fulfill condition (I) obviously need to fulfill condition (II) because it is independent of the assignment. In any auction that provides the same revenue for all $x \in \hat{X}(b)$, if conditions (I) and (II) hold for all i for one $x \in \hat{X}(b)$, then they hold for all $x \in \hat{X}(b)$ that fulfill condition (I) for all i. That is, bidders are indifferent between $x \in \hat{X}(b)$ that fulfill condition (I) if one of these x also fulfills (II), because then necessarily all such x fulfill (II). As just shown, the Vickrey and the PAB auction have this property. MRCS auctions have this property as will be shown in the proof of Theorem 11, and BOCS auctions have this property if the payment rule chooses the same $(\pi_1^r(b), \ldots, \pi_n^r(b))$ for all $\hat{x}(b) \in \hat{X}(b)$. In these auctions, our tie-breaking rule and random tie-breaking among x that satisfy condition (I) for all i return the same sets of equilibria.

A common tie-breaking rule is tie-breaking in favor of highest efficiency, which is used, e.g., by Bernheim and Whinston (1986). For tie-breaking in favor of underlying efficiency we find that with respect to equilibria it is equivalent to tie-breaking in favor of best responses: it returns the same set of equilibria.

Lemma 3. *Consider any equilibrium b in any auction with assignment and pricing rules as in $\mathcal{A}$, but with any tie-breaking rule. Then, for all x that are chosen with positive probability, $x \in \arg\max_{x \in \hat{X}(b)} \sum_{i \in N} v_i(x_i)$.*

Proof: By Lemma 2, if b is an equilibrium with some tie-breaking rule, then it is an equilibrium with tie-breaking in favor of best responses, and (I) and (II) hold for all i for all assignments that are chosen with positive probability. Thus, $x \in \hat{X}(b)$ satisfies $x \in \hat{X}(v_i, b_{-i})$ for all i in any equilibrium b. If $x \in \hat{X}(b)$ satisfies $x \in \hat{X}(v_i, b_{-i})$ for all i then $x \in \arg\max_{x \in \hat{X}(b)} \sum_{i \in N} v_i(x_i)$.[20] Assume to the contrary that there are two assignments $x^1, x^2 \in \hat{X}(b)$ for which (a) $x^1 \in \hat{X}(v_i, b_{-i})$ for all i (and there may exist i such that $x^2 \notin \hat{X}(v_i, b_{-i})$) but (b) $\sum_{i \in N} v_i(x_i^1) < \sum_{i \in N} v_i(x_i^2)$.

By (a) it is $v_i(x_i^1) + \sum_{j \neq i} b_j(x_j^1(b)) \geq v_i(x_i^2) + \sum_{j \neq i} b_j(x_j^2(b))$ for all i. Summing up we get $\sum_{i \in N} \left(v_i(x_i^1) + \sum_{j \neq i} b_j(x_j^1(b))\right) \geq \sum_{i \in N} \left(v_i(x_i^2) + \sum_{j \neq i} b_j(x_j^2(b))\right)$. Rearranging, simplifying terms, and using (b) we get

$$\sum_{i \in N} \big(\sum_{j \neq i} b_j(x_j^1(b)) - \sum_{j \neq i} b_j(x_j^2(b)) \big) = (n-1) \sum_{i \in N} \big(b_i(x_i^1) - b_i(x_i^2) \big)$$
$$\geq \sum_{i \in N} \big(v_i(x_i^2) - v_i(x_i^1) \big) > 0.$$

But it is $\sum_{i \in N} b_i(x_i^1) = \sum_{i \in N} b_i(x_i^2)$ because $x^1, x^2 \in \hat{X}(b)$, a contradiction. ■

[20] The reverse need not hold, but then the bids do not form an equilibrium. An example with two bidders and three bundles A, B, and AB in which $x^1, x^2 \in \arg\max x \in \hat{X}(b) \sum_{i \in N} v_i(x_i)$ but $x^2 \notin \hat{X}(v_1, b_{-1})$ is $v_1 = (v_1(A), v_1(B), v_1(AB)) = (2, 1, 2) = v_2$, $b_1 = (0, 1, 1)$, and $b_2 = (1, 2, 2)$. Here, $\hat{X}(b) = \{(A, B), (B, A), (\emptyset, AB)\}$ with highest degree of efficiency among these at $x^1 = (A, B)$ or $x^2 = (B, A)$, but $\hat{X}(v_1, b_2) = \{(A, B)\}$ and $\hat{X}(b_1, v_2) = \{(B, A)\}$. In this example, tie-breaking in favor of best responses chooses (A, B) or (B, A), and the bids do not form an equilibrium.

Lemma 3 shows the identity of equilibria with tie-breaking in favor of best-responses and with tie-breaking in favor of the underlying efficiency. By Lemma 2, tie-breaking in favor of best responses maximizes the set of equilibria over all tie-breaking rules. Thus, any equilibrium under any tie-breaking rule selects an assignment with maximum sum of underlying values among the tied assignments. In general, other tie-breaking rules might eliminate some equilibria. In particular, if there is a unique efficient (value-maximizing) assignment, applying the tie-breaking rule to choose the assignment with minimum sum of underlying values will destroy all efficient pure-strategy equilibria in the PAB auction, and in BOCS auctions if the true Vickrey payoff vector is not in the true core. In the Vickrey auction, truthful bidding is an equilibrium independent of the tie-breaking rule and the same holds for BOCS auctions if the true Vickrey payoff vector is in the true core.

The two tie-breaking rules in Lemma 3 can be implemented based on information provided by the bidders in an equilibrium of an extended game. Consider the following normal-form game that is composed of the auction and a tie-breaking game. The bidders $i \in N$ simultaneously report bids b_i': $2^K \to \mathbf{R}^{2^K}$ and a set $R_i(\hat{X}(b)) \subseteq \hat{X}(b)$ of assignments among tied assignments for any feasible bids profile b. The seller's strategy is to report a set $R_0(\hat{X}(b)) \subseteq \hat{X}(b)$ among that he randomly chooses the auction's assignment (that is, he could manipulate tie-breaking but he is bound to stick to the auction's rules of choosing an optimal assignment and its payment rule). We show that for any Nash equilibrium b^* of the PAB, MRCS, or Vickrey auctions, this extended game has Nash equilibria in which bidders bid b^* and report all their preferred assignments $R_i^*(\hat{X}(b)) = \{x | x \in \arg\max_{\hat{x} \in \hat{X}(b)} v_i(\hat{x}_i) - p_i(\hat{x})\}$ among tied assignments for any b, for b^* there is at least one assignment on which all bidders agree, the seller randomizes over the assignments that all bidders prefer or randomizes over all assignments if no such mutually reported assignment exists, and the chosen assignment corresponds to an assignment that might have been chosen under tie-breaking in favor of best responses.[21]

[21] The result of Theorem 11 holds for all auctions in $\mathcal{A}$ if the seller is not a player, but is bound to choose

Theorem 11. *For any Nash equilibrium b^* of the PAB, MRCS, or Vickrey auctions, the extended game implements tie-breaking in favor of a high degree of efficiency in a Nash equilibrium.*

Proof: Consider any of the auctions in Theorem 2. Assume bidders report any equilibrium bid profile b^* and $R_i^*(\hat{X}(b)) = \{x | x \in \arg\max_{\hat{x} \in \hat{X}(b)} v_i(\hat{x}_i) - p_i(\hat{x})\}$ for any b, and the seller chooses randomly from $\cap_{i \in N} R_i(\hat{X}(b))$ if $| \cap_{i \in N} R_i(\hat{X}(b))| \geq 1$ and randomly from $\hat{X}(b)$, otherwise. We show that no player has an incentive to deviate.

By Lemma 3, tie-breaking in favor of a high degree of efficiency is equivalent to tie-breaking in favor of best responses in equilibrium. Thus, in any equilibrium b^* of the auction $x(b^*) \in \hat{X}(v_i, b^*_{-i})$ for all i. By condition (II), any bidder pays his Vickrey payment and receives his true Vickrey payoff given b^*_{-i}, the maximum payoff he could get, independent of the tie-breaking rule. Thus, he has no incentive to deviate from reporting the full set $R_i(\hat{X}(b^*)) = \{x | x \in \arg\max_{\hat{x} \in \hat{X}(b^*)} v_i(\hat{x}_i) - p_i(\hat{x})\}$ if the other bidders also do so, because $x(b^*) \in \hat{X}(v_i, b^*_{-i})$, he is indifferent between all his reported assignments, $| \cap_{i \in N} R_i^*| \geq 1$, and some assignment for that $x(b^*) \in \hat{X}(v_i,)$ holds for all i is chosen by the seller.

The seller does not deviate, because in PAB, MRCS, or Vickrey auctions he is indifferent between all tied assignments. In the PAB auction, the revenue in all tied optimal assignments is $\pi_0 = w(b)$. In MRCS auctions, the revenue is minimized subject to the core constraints, which can be written as $\pi_0^r(b) + \sum_{i \in N} \pi_i^r(b) = w(b)$, $\forall i \in N: \ \pi_i^r(b) \geq 0$, $\pi_0^r(b) \geq 0$, $\forall S \subset N: \ \sum_{i \in S} \pi_i^r(b) \leq w(b) - w(b_{-S})$. The right-hand side of each of these constraints is identical for each $\hat{x}(b) \in \hat{X}(b)$. Thus, the $\pi_0^r(b)$ that solves the problem of maximizing $\sum_{i \in N} \pi_i^r(b) = w(b) - \pi_0^r(b)$ is identical for all optimal assignments $\hat{x}(b) \in \hat{X}(b)$ given b.[22] In the Vickrey auction, the revenue in all tied assignments is $\pi_0^r(b) = \sum_{i \in N}(w(b_{-i}) - w_{-i}(b)) = \sum_{i \in N} w(b_{-i}) - (n-1)w(b)$. ■

from the bidder's preferred assignments $\cap_{i \in N} R_i(\hat{X}(b))$ when this set is not empty.

[22] The same holds for BOCS auctions if the payment rule chooses the same $(\pi_1^r(b), \ldots, \pi_n^r(b))$ for all $\hat{x}(b) \in \hat{X}(b)$.

B. Proofs of Section 5

Theorem 9. *In any auction in $\mathcal{A}$ there is a Nash equilibrium such that the seller, after collecting the bids, has no incentives to disqualify bidders or bids, to ask bidders to reduce their bids, to hold back items from sale, or to alter the tie-breaking rule.*

Proof: The seller's payoff is his revenue and, given any bids profile b, $w(b)$ is in any auction in $\mathcal{A}$ the maximum revenue the seller can extract. Disqualifying bidders or bids, reducing bids, or holding back items from sale decrease $w(b)$ and altering the tie-breaking rule does not influence $w(b)$. Thus, in any equilibrium in which the seller earns $w(b)$, he cannot take advantage of such manipulations. In the PAB auction he earns $w(b)$ (see also Bernheim and Whinston, 1986). Every Nash equilibrium in the PAB auction is also a Nash equilibrium in any auction in $\mathcal{A}$ by Theorem 1 with the same payoffs for all bidders and the seller by Theorem 5. Thus, at least in these equilibria that the auctions have in common, such manipulations are not profitable for the seller. ■

Theorem 10. *If the seller sets secret reserve prices, the sets of Nash equilibria in all auctions in $\mathcal{A}$ are the same and equal to NE^{PAB} augmented by reserves $b_0(x_i(b)) = p_i(b, x(b))$ for all assigned bundles and arbitrary reserves for the other bundles.*

Proof: Secret reserves make the game a game of simultaneous decision about reserves by the seller and bids by the bidders. If $b \in \mathrm{NE}^{PAB}(v)$ and $b_0(x_i(b)) = p_i(b, x(b)) = b_i(x_i(b))$ for all assigned bundles and arbitrary reserves for the other bundles, no bidder or the seller would deviate in any considered auction. We have already shown that b is an equilibrium in any auction in $\mathcal{A}$ such that no bidder wants to deviate if there are no secret reserves, and the secret reserves given above do not add profitable deviations. The seller earns $w(b)$ in this equilibrium and, therefore, cannot increase his revenue. In the other equilibria with respect to bidders' strategies, in which (II) holds but not (II$'$), there is i such that $w(b) > w(b_{-i})$, and the seller would raise the reserve price for the item that bidder i wins to $b_i(x_i(b))$, thereby increasing his revenue. ■