CHOOSING AND RANKING. LET'S BE LOGICAL ABOUT IT

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Abstract

A variety of problems of social choice theory are examined from a unifying point of view that relies on the logic of degrees of belief. Belief is here a social attribute, its degrees being measured by the fraction of individuals that share a given opinion. Different known methods and some new ones are obtained depending on which concepts are considered and which logical implications are assumed between them. Divergences between different methods arise especially when individuals do not express a comparison between every pair of options.

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As its name says, the main subject matter of social choice theory is choosing among several options so as to best fulfil the existing individual preferences about them. Besides choosing, social choice theory is often concerned also with ranking the candidate options into an order of social priority. Obviously, every ranking method automatically entails a choosing method. On the other hand, successive choosing is a natural ranking method. However, when one looks for a ranking, one is constrained by the condition of transitivity, whereas choosing is not really concerned with it. Now, it is well known that collective preferences may lack transitivity even when all individual preferences have this property. In such situations, enforcing transitivity may enter into conflict with getting the best choice at the top of the ranking.

In other words, a good ranking method might not be so good as a choosing method. This idea lies somehow behind the customary distinction between choice functions and ranking functions [*21, 1*].

This article exposes the existence and nature of this conflict by adopting a point of view where choosing and ranking appear as particular cases of a general theory. The unifying framework is provided by a general method which has been given in [*8*] for deciding about several logically constrained issues that are the matter of certain degrees of belief. In the case of social choice theory, we are dealing with the collective degrees of belief that are obtained by counting how many individuals share a given opinion.

As we will see, depending on which constraints are adopted one obtains a variety of rules for either choosing or ranking. These rules will include some well-known ones, such as plurality, maximin, Schulze's method of paths and approval voting. On the other hand, we will also obtain some new rules, such as a new Condorcet rule that we call the "comprehensive prominence method" and a new method for dealing with approval-disapproval-preferential voting. Anyway, the used method has the virtue of revealing the precise logic behind each of these rules.

The above-mentioned conflict between choosing and ranking occurs mainly when the degrees of belief about preferences are incomplete, i. e. when individuals need not express a comparison between every pair of options (see for instance § 5.4.5).

The subtitle of this article is just a way of emphasizing that we will be crucially based on propositional logic and the logic of degrees of belief. By no means does it mean that other approaches and methods are "illogical". In particular, our approach will leave out such methods as Borda, Condorcet-Kemény-Young, and ranked pairs, each of which has its own rationale and good properties.

1 General framework

In this section we summarize the general method given in [*8*]. Its aim is to revise the existing degrees of belief about several logically constrained issues and to arrive at consistent decisions about them.

1.1 The issues under consideration are represented by a set of basic logical propositions together with the corresponding negations. This set of propositions —their negations included— will be denoted as $\mathit{\Pi}$, and the negation of p will be denoted as $\overline{p}$. The elements of $\mathit{\Pi}$ are referred to as **literals.**

A system of degrees of belief is represented by a mapping w from Π to the interval $[0,1]$. We refer to such a mapping as a **valuation,** and the image of $p \in \Pi$ by a particular valuation w will be denoted as w_p or $w(p)$. A valuation w is called **balanced** when $w_p + w_{\overline{p}}$ is equal to 1 for any $p \in \Pi$. The truth assignments of classical logic are balanced valuations with all-or-none values, that is either 0 or 1. In contrast, degrees of belief can take fractional values. Besides, they need not be balanced: $w_p + w_{\overline{p}}$ may be less than 1 (lack of information) or even greater than 1 (presence of contradiction). As we will see, the latter case can arise because of the logical implications contained in the constraints.

1.2 By a **decision** we mean a mapping whereby each proposition p in Π is assigned one of the three following possibilities: 'accepted', 'rejected' or 'undecided', with the restriction that p is accepted if and only if $\overline{p}$ is rejected, and that p is undecided if and only if $\overline{p}$ is undecided. A decision can be seen as a balanced valuation with values in $\{0, \frac{1}{2}, 1\}$, where these three values mean respectively 'rejected', 'undecided' and 'accepted'.

Every valuation w gives rise to a decision in the following way, that depends on a parameter η in the interval $0 \leq \eta \leq 1$: For any $p \in \Pi$,

$$p \text{ is accepted and } \overline{p} \text{ is rejected} \quad \text{whenever} \quad w_p - w_{\overline{p}} > \eta, \tag{1}$$

$$p \text{ and } \overline{p} \text{ are left undecided} \quad \text{whenever} \quad |\, w_p - w_{\overline{p}} \,| \leq \eta. \tag{2}$$

We refer to such a decision as the **decision of margin** η according to the valuation w. In the case $\eta = 0$ we call it the **basic decision** according to w. In tune with these definitions, the difference $w_p - w_{\overline{p}}$ is called the **acceptability** of p according to w. If the valuation w is balanced, then the basic decision criterion is equivalent to the majority rule, namely accepting p and rejecting $\overline{p}$ whenever $w_p > \frac{1}{2}$.

1.3 The logical constraints between issues are referred to as the **doctrine.** They are specified by a set of compound propositions that are required to be true. This entails a series of material implications between literals that are conveniently codified by rewriting the set of those constraints in **conjunctive normal form,** i. e. in the form

$$\Phi(\mathcal{D}) := \bigwedge_{C \in \mathcal{D}} \left(\bigvee_{p \in C} p \right), \tag{3}$$

where $\mathcal{D}$ stands for a certain collection of subsets of Π. Each expression within parentheses in the preceding formula —or equivalently the corresponding set $C \subset \Pi$— is called a **clause.**

The conjunctive normal form of a doctrine is not unique. Generally speaking, this can make a difference for the procedure that we are about to introduce. This ambiguity is eliminated by resorting to the **Blake canonical form,** that consists of all the prime clauses of the doctrine under consideration; a clause being prime means that no proper subset of it is still entailed by the doctrine. However, for many doctrines one is ensured to get the same results with other conjunctive normal forms made of prime clauses; this happens whenever the form under consideration has a property that we call **disjoint-resolvability.** For these and other technical matters we refer the reader to [8 : § 3.2]. Anyway, the doctrine, that from now on we are assimilating to the set $\mathcal{D}$, is required to satisfy the following conditions: (D1) It is satisfiable. (D2) It does not contain unit clauses, i. e. clauses with a single literal. (D3) It explicitly contains the *tertium non datur* clause $p \vee \overline{p}$ for any $p \in \mathit{\Pi}$. (D4) It is made of prime clauses.

1.4 A clause being true means that at least one of its literals is true; in other words, if all of its literals but one are known to be false, then the remaining one must be true. Therefore, the doctrine associated with (3) provides the following implications:

$$p \leftarrow \bigwedge_{\substack{\alpha \in C \\ \alpha \neq p}} \overline{\alpha}, \tag{4}$$

for any $C \in \mathcal{D}$ such that $p \in C$. Each of these implications is a possible source of belief in p. In this connection, it makes sense to apply the classical rule that the conclusion p should be believed at least as the weakest of the premises $\overline{\alpha}$. This rule requires the right-hand side of (4) to be satisfiable, which is ensured because C is prime. This leads to the following procedure for revising any given degrees of belief v about $\mathit{\Pi}$: every $p \in \mathit{\Pi}$ should be believed at least in the new degree v'_p defined by

$$v'_p = \max_{\substack{C \in \mathcal{D} \\ C \ni p}} \min_{\substack{\alpha \in C \\ \alpha \neq p}} v_{\overline{\alpha}}, \tag{5}$$

One easily checks that this formula has the following consequences:

Lemma 1.1 ([8 : Lem. 3.1])**.** *The transformation $v \mapsto v'$ has the following properties:*

(a) *It is continuous.*

(b) *$v \leq w$ implies $v' \leq w'$.*

(c) *$v \leq v'$.*

(d) *The image set of v' is contained in that of v.*

As soon as we accept v' as new degrees of belief, it makes sense to repeat the same operation with v replaced by v', thus obtaining a still higher valuation v'', and so on. By proceeding in this way, one obtains a non-decreasing sequence of valuations $v^{(n)}$ $(n = 0, 1, 2, \dots)$ with the property that all of them take values in the same finite set. Obviously, this implies that this sequence will eventually reach an invariant state v^*. This eventual valuation is, by definition, the **upper revised valuation**.

1.5 The main properties of the upper revised valuation are collected in the following statements:

Theorem 1.2 (Basic facts [8: Thm. 3.2]). *The transformation $v \mapsto v^*$ has the following properties:*

(a) *It is continuous.*

(b) *$v \leq w$ implies $v^* \leq w^*$.*

(c) *$v \leq v^*$.*

(d) *The image set of v^* is contained in that of v.*

Theorem 1.3 (Characterization [8: Thm. 3.3]). *The upper revised valuation v^* is the lowest of the valuations w that lie above v and are consistent with the doctrine in the sense of satisfying the equation $w' = w$.*

Theorem 1.4 (Consistency of the associated decisions [8: Cor. 3.7]). *For any η in the interval $0 \leq \eta \leq 1$, the decision of margin η associated with the upper revised valuation is always definitely consistent with the doctrine in the following sense: for each clause $C \in \mathcal{D}$ and every $p \in C$, one has the following implication: if α is rejected for every $\alpha \in C \setminus \{p\}$, then p is accepted.*

Theorem 1.5 (Respect for consistent majority decisions [8: Thm. 3.9]). *Assume that every $p \in \Pi$ satisfies either $v_p > \frac{1}{2} > v_{\overline{p}}$ or, contrarily, $v_{\overline{p}} > \frac{1}{2} > v_p$. Assume also that the basic decision associated with v (which contains no undecidedness) is consistent with the doctrine. In this case, the basic decision associated with the upper revised valuation v^* is the same.*

Theorem 1.6 (Respect for unanimity [8: Thm. 3.11]). *Assume that v is an aggregate of consistent truth assignments. In this case, having $v_p = 1$ implies that p is accepted by the basic decision associated with the upper revised valuation v^*.*

Theorem 1.7 (Monotonicity [8: Thm. 3.28 and Cor. 3.29]). *Assume that the valuation v is modified into a new one $\widetilde{v}$ such that*

$$\widetilde{v}_p > v_p, \qquad \widetilde{v}_q = v_q, \quad \forall q \in \Pi \setminus \{p\}. \tag{6}$$

In this case, the acceptability of p either increases or stays constant:

$$\widetilde{v}^*_p - \widetilde{v}^*_{\overline{p}} \;\geq\; v^*_p - v^*_{\overline{p}}. \tag{7}$$

As a consequence, if p is accepted [resp. not rejected] in the decision of margin η associated with v^*, then it is also accepted [resp. not rejected] in the decision of margin η associated with $\widetilde{v}^*$.

1.6 Having the equality $v^*_p = v'_p$ for the Blake canonical form —or for any disjoint-resolvable prime conjunctive normal form— guarantees that the degree of belief v^*_p does not derive from unsatisfiable conjunctions [*8*: § 3.5]. If that equality holds no matter the initial valuation v, we say that the doctrine under consideration is **unquestionable for p.** Sometimes, the equality can be guaranteed only under certain special circumstances. In particular, it can happen that it holds whenever p is accepted according to v^*; in that case we say that the doctrine is **unquestionable for p when accepted.** Sufficient conditions for ensuring such properties are given in [*8*: Thm. 3.18, Cor. 3.19].

1.7 The following fact will be useful for computations:

Lemma 1.8. *The successive valuations $v^{(n)}$ satisfy the following formulas for any $n \geq 1$:*

$$v^{(n)}_p \;=\; \max\big(\, v_p,\; \max_{\substack{C\in\mathfrak{D}\\ C\ni p\\ C\neq\{p,\overline{p}\}}} \; \min_{\substack{\alpha\in C\\ \alpha\neq p}} \; v^{(n-1)}_{\overline{\alpha}} \,\big). \tag{8}$$

Proof. By definition, $v^{(n)}$ is obtained from $v^{(n-1)}$ The resulting expression is the same as (8) except that the right-hand side shows $v^{(n-1)}_p$ instead of v_p. by the transformation $v \mapsto v'$ defined by (5). In order to obtain formula (8) it suffices to apply repeatedly the two following facts, which are easily checked by induction: (i) If a_n $(n \geq 0)$ satisfies $a_n = \max(a_{n-1}, b_{n-1})$ $(n \geq 1)$, where b_n $(n \geq 0)$ is a non-decreasing sequence, then $a_n = \max(a_0, b_{n-1})$ for any $n \geq 1$; (ii) If a_n and b_n $(n \geq 0)$ are non-decreasing sequences, then the sequences $\max(a_n, b_n)$ and $\min(a_n, b_n)$ are also non-decreasing. □

1.8 For our purposes, $\overline{p}$ need not be the exact semantic negation of p. Instead, quite often it is more appropriate to look at $\overline{p}$ as the opposite, or antithesis, of p. This may seem to conflict with the excluded-middle principle $p \vee \overline{p}$. However, this principle somehow loses its character just as fractional valuations come in. In fact, its role in connection with the latter is only through the excluded-middle clauses that we systematically include in the Blake canonical form; and this has only the following two effects: (a) providing the trivial implications $p \rightarrow p$ and $\overline{p} \rightarrow \overline{p}$, through which the revised degrees of belief become larger than or equal to the original ones; and (b) forbidding any implication of the form $\overline{p} \wedge p \wedge \chi \rightarrow t$, which would be a gratuitous source of belief (this effect occurs because clauses are restricted to be prime, which prevents them from containing $p \vee \overline{p}$).

Anyway, the belief in $\overline{p}$ is not the lack of belief in p, but it should have its own reasons. This agrees with the general views of [*14:* see for instance p. 12]. Besides, it fully agrees also with the traditional views of the adversarial system of justice.

In order to apply the preceding method to a particular matter, one must specify the main issues at stake as well as the existing logical implications between them. As we will see, the notions of preference and suitability for choice allow for several views about which logical implications are associated with them. As a result we will obtain several alternative models or theories for choosing and ranking.

2 Preferences

In the sequel we will be dealing all the time with a finite set of options. This set will be denoted as A. A system of preferences about the members of A is usually formalized as a binary relation on A that complies with certain properties, typically including antisymmetry, completeness and transitivity (see for instance [*12*]). Having said that, nowadays it is well established that preferences are sometimes not transitive (see for instance [*19*], or [*12* : § 1.3]). So, we will take the view that transitivity is a special way of having preferences about things. On the other hand, antisymmetry and completeness admit of certain alternatives depending on whether indifference and/or lack of opinion are allowed into consideration; as we will see next, however, these alternatives become unnecessary when fractional degrees of belief are used.

Describing preferences by means of propositional logic requires considering all of the propositions p_{xy}: 'x is preferable to y', where x and y are different from each other (allowing for $x = y$ leads to useless distinctions).

From our point of view, the main principle associated with the notion of preference is that $\overline{p}_{xy}$ can be identified with p_{yx}, or equivalently, that

$$\overline{p}_{xy} \leftrightarrow p_{yx}, \qquad \text{for any two different } x, y \in A. \tag{9}$$

In the all-or-none framework of classical logic, this double implication embodies a limitation to complete strict preferences, leaving no place for definite indifference (x and y 'equally good') nor for incompleteness (lack of information about the comparison between x and y).

However, in the context of degrees of belief, both definite indifference and incompleteness can be suitably modelled within the logical constraint (9). In fact, incompleteness can be described by putting $v(p_{xy}) = v(p_{yx}) = 0$ which means a full lack of belief in p_{xy} as well as in p_{yx}. On the other hand, definite indifference can be described by putting $v(p_{xy}) = v(p_{yx}) = \frac{1}{2}$, i. e. by splitting the unit of belief into equal amounts for the contrary preferences p_{xy} and p_{yx}.

So, *from now on we identify* $\overline{p}_{xy}$ *with* p_{yx}.

The collective degrees of belief about the propositions p_{xy} are given by

$$v(p_{xy}) = \sum_k \alpha_k \, v^k(p_{xy}), \tag{10}$$

where α_k are the relative frequencies or weights of the individual opinions v^k. In preferential voting, the individual opinions are usually expressed in the form of a ranking, that is, a list of options in order of preference, possibly truncated or with ties. In order to translate this information into paired comparisons, we use the following interpretation:

(a) When x and y are both in the list and x is ranked above y (without a tie), we certainly take $v^k(p_{xy}) = 1$ and $v^k(p_{xy}) = 0$.

(b) When x and y are both in the list and x is ranked as good as y, we take $v^k(p_{xy}) = v^k(p_{xy}) = \frac{1}{2}$.

(c) When x is in the list and y is not in it, we take $v^k(p_{xy}) = 1$ and $v^k(p_{xy}) = 0$.

(d) When neither x nor y are in the list, we take $v^k(p_{xy}) = v^k(p_{xy}) = 0$.

Instead of rule (d), one can consider the possibility of using the following alternative:

(d′) When neither x nor y are in the list, we interpret that they are considered equally good (or equally bad), so we proceed as in (b).

This amounts to complete each truncated ranking by appending to it all the missing options tied to each other, which brings the problem to the complete case. Generally speaking, however, this interpretation can be criticized in that the added information might not be really meant by the voter.

The table that collects the numbers $v(p_{xy})$ given by (10) for all ordered pairs xy will be called the Llull matrix of the vote. The elements of this matrix satisfy the inequality

$$v(p_{xy}) + v(p_{yx}) \leq 1, \tag{11}$$

which is inherited from the component valuations v^k. When (11) holds with the equality sign, it means that every individual expressed a comparison (a preference or a tie) about every pair of options. We will refer to this situation as the **complete** case.

3 Transitivity

Let us begin by considering the constraint of transitivity. So, here we take the view that preferences satisfy the implications $p_{xy} \wedge p_{yz} \rightarrow p_{xz}$. On account of (9), they are logically equivalent to the following clauses:

$$p_{xy} \vee p_{yz} \vee p_{zx}, \qquad \text{for any pairwise different } x, y, z \in A \tag{12}$$

(where x, y, z have been relabelled). The doctrine that is made of these clauses will be referred to as the **transitivity doctrine.**

One can see that the preceding clauses lead to the same upper revised valuation as the corresponding full Blake canonical form, and that the doctrine is unquestionable in the sense mentioned in § 1.6:

Proposition 3.1. *The conjunctive normal form formed by the clauses* (12) *is disjoint-resolvable and therefore* $*$*-equivalent to the corresponding Blake canonical form. This doctrine is unquestionable for every* p_{xy}.

Scheme of the proof. The Blake canonical form, with the *tertium non datur* clauses included, consists of all clauses of the form

$$p_{x_0x_1} \vee p_{x_1x_2} \vee \ldots \vee p_{x_{n-1}x_n} \vee p_{x_nx_0}, \tag{13}$$

with $n \geq 1$ and all x_i $(0 \leq i \leq n)$ pairwise different (which restricts n to be less than or equal to the number of elements of A). These clauses are easily derived from (12) by successive concatenation, and suitably ordered this derivation will use only disjoint resolution.

The unquestionability is easily obtained through condition (a) of [8 : Thm. 3.18], which amounts to the following fact: if two cycles without repetitions contain opposed links, then suppressing both of these links and putting together all the others results in a set L of links with the following property: for any link contained in L there exists a cycle without repetitions that is included in L and contains that link. The reader will easily convince himself —maybe by means of some drawings— that this is really a fact. □

The properties of disjoint resolvability and unquestionability allow to express v^* directly as the result of the one-step transformation associated with the Blake canonical form, namely:

$$v^*(p_{xy}) \;=\; \text{Max}\;\min\big(v(p_{x_0x_1}), v(p_{x_1x_2}), \ldots, v(p_{x_{n-1}x_n})\big), \tag{14}$$

where the Max operator considers all paths $x_0x_1\ldots x_n$ of length $n \geq 1$ from $x_0 = x$ to $x_n = y$ with all x_i pairwise different.

In this case, our general method corresponds to the method introduced in 1997 by Markus Schulze [*15*, *16*; *18*: p. 228–232], which is sometimes called the method of paths. In the way that we have introduced it, it is clearly a method for ranking all the candidates. Having said that, later on (§ 5.4) we will see that in the complete case its winners are quite in agreement with a doctrine that does not include transitivity but aims only at choosing the most prominent option.

As a ranking method, the method of paths complies with the following extension of the Condorcet principle introduced in 1973 by John H. Smith [*17*: § 5]: Assume that the set of candidates is partitioned in two classes X and Y such that for each member of X and every member of Y there are more than half of the individual votes where the former is preferred to the latter; in that case, the social ranking should also prefer each member of X to any member of Y. The proof can be found in [*16*: § 4.7] (see also [*6*: § 10; *7*: Thm. 8.1]). Another interesting property of the method of paths is clone consistency, also known as independence of clones, which refers to the effect of replacing a single option c by a set C of several options similar to c; for more details we refer the reader to [*15*: § 5.4; *16*: § 4.6] as well as [*6*: § 11; *7*: Thm. 8.2].

Moreover, it has also been shown [*6, 7*] that this method can be extended to a continuous rating method that allows to sense the closeness of two candidates at the same time that it allows to recognise certain situations that are quite opposite to a tie.

4 Supremacy

Instead of constraining the binary preferences between several options to form a total order, one can require only the existence of a supreme option, i. e. an option that is preferred to any other. Such a constraint is specified by the *disjunctive* normal form

$$\bigvee_{x\in A}\;\bigwedge_{\substack{y\in A\\ y\neq x}} p_{xy}. \tag{15}$$

Instead of directly bringing (15) into conjunctive normal form, one can arrive at a much shorter conjunctive formulation by considering the propositions s_x : 'x is preferred to any other member of A' ($x \in A$), which are related to the p_{xy} by the double implications

$$s_x \;\leftrightarrow\; \bigwedge_{\substack{y\in A\\ y\neq x}} p_{xy}. \tag{16}$$

By proceeding in this way, we are led to consider the set of propositions $\Pi = \{\, s_x \mid x \in A \,\} \cup \{\, \overline{s}_x \mid x \in A \,\} \cup \{\, p_{xy} \mid x, y \in A,\, x \neq y \,\}$ together with the doctrine formed by the following clauses:

$$s_x \;\vee\; \bigvee_{\substack{y\in A\\ y\neq x}} p_{yx}, \qquad \text{for any } x \in A; \tag{17}$$

$$\overline{s}_x \;\vee\; p_{xy}, \qquad \text{for any two different } x, y \in A; \tag{18}$$

$$\bigvee_{x\in A} s_x. \tag{19}$$

We will refer to it as the **supremacy doctrine**.

Again one can see that the preceding clauses lead to the same upper revised valuation as the corresponding full Blake canonical form:

Proposition 4.1. *The conjunctive normal form formed by the clauses* (17–19) *is disjoint-resolvable and therefore $*$-equivalent to the corresponding Blake canonical form.*

Scheme of the proof. We will limit ourselves to indicating that one can arrive at the Blake canonical form through disjoint resolution by means of the following procedure: First, each clause of the form (18) is combined by disjoint resolution with the clause of the same form where x and y are interchanged with each other (recall that we identify $\overline{p}_{xy}$ with p_{yx}). This produces the clauses

$$\overline{s}_x \;\vee\; \overline{s}_y, \qquad \text{for any two different } x, y \in A. \tag{20}$$

Second, one successively applies disjoint resolution to combine clause (19) with one or more clauses of the form (18), each of them corresponding to a different x and admitting any $y \neq x$. This produces all clauses of the form

$$\bigvee_{x\in X} s_x \;\vee\; \bigvee_{x\in A\setminus X} p_{xf(x)}, \tag{21}$$

where X is any subset of A, and f is any mapping from $A \setminus X$ to A with $f(x) \neq x$. The interested reader can go over the rather tedious task of checking that no further resolution is possible. $\square$

Proposition 4.2. *The supremacy doctrine is unquestionable for $\overline{s}_x$, i. e. it satisfies $v^*(\overline{s}_x) = v'(\overline{s}_x)$, and it is also unquestionable for s_x when accepted, i. e. it satisfies $v^*(s_x) = v'(s_x)$ whenever $v^*(s_x) > v^*(\overline{s}_x)$.*

Proof. The proof is long but quite mechanical. The unquestionability for $\overline{s}_x$ is obtained by checking that condition (a) of [8 : Thm. 3.18] is satisfied for any pair of clauses C, C' and any literal q satisfying in opposition, one of which containing $\overline{s}_x$. Finally, the unquestionability for s_x when accepted is obtained by checking that either condition (a) or condition (b$'$) of [8 : Cor. 3.19] is satisfied in the analogous situation for s_x instead of $\overline{s}_x$. $\square$

The clauses (20) assert that one cannot have two supreme options. Applying the definite consistency theorem (Theorem 1.4) to these clauses ensures the following fact: When s_x is accepted, then x is the only option with this property. On the other hand, the same theorem applied to (19) ensures the following fact: When s_z is rejected for any $z \neq x$, then s_x is accepted. By taking into account that a proposition need not be accepted or rejected but it can be left undecided, the preceding statement is equivalent to the following one: When s_z is not accepted for any $z \in A$, then s_z is undecided for more than one $z \in A$. In the sequel, an option for which s_x is not rejected will be called a **supremacy winner.**

The one-step revision transformation $v \mapsto v'$ reads as follows: For any $x, y \in A$:

$$v'(s_x) \;=\; \max\Big(\, v(s_x),\, \min_{y\neq x} v(\overline{s}_y),\, \min_{y\neq x} v(p_{xy}) \,\Big), \tag{22}$$

$$v'(\overline{s}_x) \;=\; \max\Big(\, v(\overline{s}_x),\, \max_{y\neq x} v(p_{yx}) \,\Big), \tag{23}$$

$$v'(p_{xy}) \;=\; \max\Big(\, v(p_{xy}),\, v(s_x),\, \min\big(v(\overline{s}_y),\, \min_{\substack{z\neq x\\ z\neq y}} v(p_{yz})\big) \Big). \tag{24}$$

We will use the following notations:

$$\sigma_x = \min_{y\neq x}\, v(p_{xy}), \quad \tau_x = \max_{y\neq x}\, v(p_{yx}), \quad \sigma_{xy} = \min_{\substack{z\neq x\\ z\neq y}}\, v(p_{xz}). \tag{25}$$

These definitions immediately imply that

$$\sigma_x \;\leq\; v(p_{xy}) \;\leq\; \tau_y, \qquad \text{whenever } x \neq y. \tag{26}$$

$$\sigma_{xy} \;\leq\; v(p_{xz}) \;\leq\; \tau_z, \qquad \text{whenever } z \notin \{x, y\}. \tag{27}$$

In the sequel we will have to look at the successive valuations $v^{(n)}$ that are obtained by iterating the transformation (22–24) starting from $v^{(0)} = v$. Later on, it will be convenient to allow n to take negative values by putting $v^{(n)}(p_{xy}) = 0$ for $n < 0$. We will also make use of the quantities analogous to those of (25) with $v^{(n)}$ substituted for v. These quantities will be denoted by $\tau_x^{(n)}, \sigma_x^{(n)}, \sigma_{xy}^{(n)}$. Obviously, they satisfy inequalities analogous to (26–27).

4.1 The minimax rule

Assume that our choice must be based solely on the Llull matrix $(v(p_{xy}))$. In principle, this matrix does not give (direct) information about the (collective) degrees of belief for s_x and $\overline{s}_x$. So, it makes sense to take

$$v(s_x) \;=\; v(\overline{s}_x) \;=\; 0, \quad \text{for any } x \in A. \tag{28}$$

Proposition 4.3. *For the initial values* (28), *the supremacy doctrine gives*

$$\begin{aligned} v^*(s_x) \;&=\; v''(s_x) \;&&=\; \min_{z \neq x} \tau_z, && (29)\\ v^*(\overline{s}_x) \;&=\; v'(\overline{s}_x) \;&&=\; \tau_x, && (30)\\ v^*(p_{xy}) \;&=\; v'''(p_{xy}) \;&&=\; \max\,(v(p_{xy}),\, \min_{z \neq x} \tau_z). && (31) \end{aligned}$$

Proof. Let us introduce the initial values (28) in (22–24). Starting from $v^{(n)}(\overline{s}_x)$, we successively obtain:

$$\begin{aligned} v^{(n)}(\overline{s}_x) \;&=\; \tau_x^{(n-1)}, && (32)\\ v^{(n)}(s_x) \;&=\; \max\big(\, \min_{y \neq x} \tau_y^{(n-2)},\, \sigma_x^{(n-1)}\,\big), && (33)\\ v^{(n)}(p_{xy}) \;&=\; \max\Big(\, v(p_{xy}),\, \min_{z \neq x} \tau_z^{(n-3)},\, \sigma_x^{(n-2)},\, \min\big(\tau_y^{(n-2)},\, \sigma_{yx}^{(n-1)})\big)\,\Big). && (34) \end{aligned}$$

Let us now plug (34) into the definition of $\tau_x^{(n)}$. By making use of the inequalities (26–27) and their n-th counterparts, one easily arrives at the inequality $\tau_x^{(n)} \leq \max(\tau_x, \tau_x^{(n-2)}, \tau_x^{(n-3)})$. By induction, it follows that $\tau_x^{(n)} \leq \tau_x$ for $n \geq 0$. Since we also know that $\tau_x^{(n)}$ is not decreasing, we get

$$\tau_x^{(n)} \;=\; \tau_x, \qquad \text{for } n \geq 0. \tag{35}$$

Finally, by plugging this result into (32–34) and making use of the inequalities of the type (26–27), one arrives at the conclusion that

$$v^{(n)}(\bar{s}_x) \;=\; \tau_x, \qquad \text{for } n \geq 1; \tag{36}$$

$$v^{(n)}(s_x) \;=\; \min_{z \neq x} \tau_z, \qquad \text{for } n \geq 2; \tag{37}$$

$$v^{(n)}(p_{xy}) \;=\; \max\big(v(p_{xy}),\, \min_{z \neq x} \tau_z\big), \qquad \text{for } n \geq 3. \quad \square \tag{38}$$

Corollary 4.4. *For the initial values* (28)*, the supremacy winners are the options* $x \in A$ *that minimize* $\tau_x = \max_{y \neq x} v(p_{yx})$ *.*

We refer to this rule as the **minimax** rule. In the voting literature, this term is sometimes used with a variety of meanings, in which case the preceding rule is specifically known as "pairwise opposition". In the complete case ($v(p_{xy}) + v(p_{yx}) = 1$) it coincides with the **maximin** rule, i. e. choosing the option $x \in A$ that maximizes $\sigma_x = \min_{y \neq x} v(p_{xy})$ [*18*: p. 212–213], which complies with Condorcet's majority principle (see § 5.1). In the general incomplete case, however, the minimax rule does not comply with Condorcet's principle.

4.2 The plurality rule.

When the Llull matrix comes from preferential voting in the sense that every vote is an ordered list (possibly restricted to a subset of most preferred options), then the preceding treatment admits of a serious objection. In fact, in that case it is natural to adopt certain specific values as initial degrees of (collective) belief in s_x and $\bar{s}_x$, namely and specifically, the fraction f_x of votes where x is placed at the top of the list, and that of those where some other option is placed at the top:

$$v(s_x) \;=\; f_x, \quad v(\bar{s}_x) \;=\; \bar{f}_x, \quad \text{for any } x \in A, \tag{39}$$

where $\bar{f}_x = \sum_{y \neq x} f_y$. Since s_x cannot be true for two different options —clause (20)— in the event of a vote that ties k options at the top, it makes sense to count it as $1/k$-th of a vote for each of the top-placed options. In the sequel we will refer to f_x as the **plurality fraction** of x, and $\bar{f}_x$ will be called the **antiplurality fraction** of x. The values of f_x cannot be read from the Llull matrix except in very few special cases. However, they are easily obtained from the votes themselves. Using this additional information should lead to better grounded results. Yet we get something rather unexpected:

Proposition 4.5. *For the initial values* (39), *the supremacy doctrine gives*

$$v^*(s_x) \;=\; v'(s_x) \;=\; \min_{z\neq x} \bar f_z, \tag{40}$$

$$v^*(\overline{s}_x) \;=\; v(\overline{s}_x) \;=\; \bar f_x, \tag{41}$$

$$v^*(p_{xy}) \;=\; v''(p_{xy}) \;=\; \max\big(v(p_{xy}),\, \min_{z\neq x} \bar f_z\big). \tag{42}$$

Proof. Let us begin by noticing that the plurality and antipurality fractions f_x and $\bar f_x$ are related to the entries of the Llull matrix in the following way:

$$f_x \;\le\; v(p_{xy}) \;\le\; \bar f_y. \tag{43}$$

This is an immediate consequence of the definitions when the votes are strict rankings: If x is placed at the top, then x is preferred to any other option y; on the other hand, if x is preferred to y, then the top option cannot be y. A vote that ties $k\ge 2$ options at the top contributes also to the three terms of (43) in agreement with the stated inequalities; in particular, if both x and y are placed at the top, the respective contributions are $1/k \le 1/2 \le (k-1)/k$. From (43) and (25) it follows that

$$f_x \;\le\; \sigma_x \;\le\; \tau_y \;\le\; \bar f_y, \qquad \text{whenever } x\neq y. \tag{44}$$

$$f_x \;\le\; \sigma_{xy} \;\le\; \tau_z \;\le\; \bar f_z, \qquad \text{whenever } z\notin\{x,y\}. \tag{45}$$

Let us introduce the initial values (39) in (22–24). We get (for any $n\ge 0$):

$$v^{(n)}(s_x) \;=\; \max\big(\, f_x,\, \min_{y\neq x} v^{(n-1)}(\overline{s}_y),\, \sigma_x^{(n-1)}\,\big), \tag{46}$$

$$v^{(n)}(\overline{s}_x) \;=\; \max\big(\, \bar f_x,\, \tau_x^{(n-1)}\,\big), \tag{47}$$

$$v^{(n)}(p_{xy}) \;=\; \max\Big(\, v(p_{xy}),\, v^{(n-1)}(s_x),\, \min\big(v^{(n-1)}(\overline{s}_y),\, \sigma_{yx}^{(n-1)}\big)\,\Big). \tag{48}$$

The equalities (40–42) will be obtained by showing that one has the following equalities and inequalities:

$$v^{(n)}(\overline{s}_x) \;=\; \bar f_x, \qquad \text{for } n\ge 0; \tag{49}$$

$$v^{(n)}(s_x) \;=\; \begin{cases} f_x, & \text{for } n=0,\\ \min_{y\neq x} \bar f_y, & \text{for } n\ge 1; \end{cases} \tag{50}$$

$$v^{(n)}(p_{xy}) \;\le\; \max\big(v(p_{xy}),\, \min_{z\neq x} \bar f_z\big), \qquad \text{with equality for } n\ge 2; \tag{51}$$

$$\tau_x^{(n)} \;\le\; \bar f_x, \qquad \text{for } n\ge 0. \tag{52}$$

These equalities and inequalities will be proved by induction. For $n = 0$, (49) and (50) are true because of (39), (51) is true by definition, and (52) is true because of (44). Let us now go from $n-1$ to n: (49_n) is an immediate consequence of introducing (52_{n-1}) into (47). (50_n) is obtained by introducing (49_{n-1}) into (46) and noticing on the one hand that $f_x \leq \sigma_x \leq \sigma_x^{(n-1)}$ (for any $n \geq 1$), and on the other hand that $\sigma_x^{(n-1)} \leq \tau_y^{(n-1)} \leq \bar{f}_y$ for any $y \neq x$; here we have used (44), the $(n-1)$-th counterpart of inequality (26), and (52_{n-1}). (51_n) is obtained by introducing (49_{n-1}) and (50_{n-1}) into (48) and noticing that, on the one hand, $v^{(n-1)}(s_x) \leq \min_{y\neq x} \bar{f}_y$, with equality for $n \geq 2$, and on the other hand, $\min(\bar{f}_y, \sigma_{yx}^{(n-1)}) \leq \min_{z\neq x} \bar{f}_z$ because of the $(n-1)$-th counterpart of the inequality (27) as well as (52_{n-1}). Finally, (52_n) follows from (51_n) by making use of the definition of $\tau_x^{(n)}$ and the second inequality of (43):

$$\tau_x^{(n)} = \max_{y\neq x} v^{(n)}(p_{yx}) = \max\big(\max_{y\neq x} v(p_{yx}),\, \min_{z\neq y} \bar{f}_z\big) \leq \bar{f}_x. \quad \square$$

Corollary 4.6. *For the initial values* (39)*, the supremacy winners are the plurality winners, i. e. the options $x \in A$ that maximize the plurality fraction f_x.*

Proof. Recall that we have defined a supremacy winner as an option $x \in A$ for with s_x is not rejected, i. e. such that $v^*(s_x) \geq v^*(\overline{s}_x)$. In view of (40–41), this is equivalent to say that x minimizes $\bar{f}_x$. Finally, since $\bar{f}_x = \sum_{z\neq x} f_z = (\sum_{z\in A} f_z) - f_x$, minimizing $\bar{f}_x$ is equivalent to maximizing the plurality fraction f_x. $\square$

This is quite embarrassing: We started from a method that in the complete case complies with the Condorcet principle, thus ruling out the quite objectionable plurality rule, and now, by adding more information, we have fallen back into the plurality rule! A little reflection shows that in order to avoid the main drawback of the plurality rule one should not look for supremacy, but for something slightly different. In fact, the main objection against the plurality rule —raised by Borda in his seminal paper of 1770–84 [*13* : ch. 5]— is that one can have a majority of voters for which the plurality winner is the *worst* option, which is certainly quite undesirable. Notice that what matters here is the opposition between 'best' and 'worst', whereas the supremacy doctrine has to do with the opposition between 'best' and 'not best'.

5 Prominence.

In order to properly deal with the 'best-worst' opposition, one is led to replace supremacy by a weaker concept whose connection to preferences requires only that 'best' implies the presence of that concept and 'worst' (instead of 'not best') implies the lack of it. This concept could be viewed as a sort of tempered supremacy. We will refer to it as 'prominence'. More properly speaking, and using the notation t_x to represent the proposition 'x is prominent', the connection of this concept to preferences is given by the following two implications: if x is preferred to any other option, then x is prominent: $\bigwedge_{y\neq x} p_{xy} \to t_x$; if every option other than x is preferred to x, then x is not prominent: $\bigwedge_{y\neq x} p_{yx} \to \bar{t}_x$. In conjunctive normal form these implications read as follows:

$$t_x \vee \bigvee_{y\neq x} p_{yx}, \qquad \text{for any } x \in A; \tag{53}$$

$$\bar{t}_x \vee \bigvee_{y\neq x} p_{xy}, \qquad \text{for any } x \in A; \tag{54}$$

Concerning the initial values for $v(t_x)$ and $v(\bar{t}_x)$, we will take simply

$$v(t_x) \;=\; v(\bar{t}_x) \;=\; 0, \qquad \text{for any } x \in A. \tag{55}$$

One could argue that in the case of preferential voting one should proceed in a different way: In accordance with the implication $\bigwedge_{y\neq x} p_{xy} \to t_x$ —contained in (53)— every top placing of x is a piece of evidence in favour of t_x. Similarly, every last placing of x is a piece of evidence in favour of $\bar{t}_x$ —by the implication contained in (54)—. So, one should take

$$v(t_x) \;=\; f_x, \quad v(\bar{t}_x) \;=\; \ell_x, \qquad \text{for any } x \in A, \tag{56}$$

where ℓ_x denotes the fraction of votes where x is placed last (a vote that ties k options at the bottom being counted as $1/k$-th of a vote for each of the bottom-placed options). In the supremacy doctrine a similar change in the initial values led to an entirely different result. Here, however, the initial values (56) lead to the same result as (55). This happens because instead of the second inequality of (43) here we have the following one: $\ell_y \leq v(p_{xy})$. This inequality, together with the first inequality of (43), has the following consequence: no matter whether we start from (55) or from (56), we get $v'(t_x) \geq \min_{y\neq x} v(p_{xy}) \geq f_x$ as well as $v'(\bar{t}_x) \geq \min_{y\neq x} v(p_{yx}) \geq \ell_x$. In fact, the initial values (56) are based on the implications $\bigwedge_{y\neq x} p_{xy} \to t_x$ and

$\bigwedge_{y\neq x} p_{yx} \to \bar{t}_x$, so they are doing part of the job that will be one anyway by the revision transformation. If the doctrine includes (53) but not (54) —as it will be the case in § 5.2— then the preceding considerations hold only with respect to the first equality of (56).

5.1 The Condorcet principle.

The implication $\bigwedge_{y\neq x} p_{xy} \to t_x$ that is coded in clause (53) is akin to the celebrated **Concorcet principle.** This principle has the two following versions:

C *Condorcet principle (majority version).* If an option x has the property that $v(p_{xy}) > \frac{1}{2}$ for any $y \neq x$, then x must be chosen as the winner.

C′ *Condorcet principle (margin version).* If an option x has the property that $v(p_{xy}) > v(p_{yx})$ for any $y \neq x$, then x must be chosen as the winner.

In the complete case $v(p_{xy}) + v(p_{xy}) = 1$ (where the Condorcet principle was originally proposed) these two conditions are equivalent to each other. Generally speaking, however, condition C is weaker than C′ (which makes the former more compatible with other desirable properties, as it was remarked in [7: § 1.4]).

On the other hand, the implication $\bigwedge_{y\neq x} p_{yx} \to \bar{t}_x$ coded in clause (54) corresponds to the dual statement that is usually referred to as the "Condorcet loser criterion", also with two versions: the majority one requiring $v(p_{yx}) > \frac{1}{2}$ for any $y \neq x$, and the margin one requiring $v(p_{yx}) > v(p_{xy})$ for any $y \neq x$. In both versions, the conclusion is that x must then be deemed the loser.

In the sequel, the term **Condorcet winner** [resp. **loser**] will be understood in the majority sense, i. e. to denote an option x with the property that $v(p_{xy}) > \frac{1}{2}$ [resp. $v(p_{yx}) > \frac{1}{2}$] for any $y \neq x$.

A main difference between our point of view and that of the Condorcet principle is that the latter, in both versions C and C′, looks at whether a certain particular situation happens in the initial (collective) degrees of belief v. It it does not happen, then no conclusion is arrived at. In contrast, our method will take the implication $\bigwedge_{y\neq x} p_{xy} \to t_x$, i. e. the clause (53), as a guide for revising those initial degrees of belief so as to arrive at a conclusion consistent with that implication. In accordance with the definite consistency theorem (Theorem 1.4), we will have the following property similar to C′: If an option x satisfies $v^*(p_{xy}) > v^*(p_{yx})$ for any $y \neq x$, then it satisfies also

$v^*(t_x) > v^*(\bar{t}_x)$, i. e. x is accepted as a prominent option.

This property will be satisfied whenever the doctrine contains the clause (53). Nevertheless, the specific results, for instance, which options are accepted as prominent ones, or the values of $v^*(t_x)$ and $v^*(\bar{t}_x)$, will depend on which other clauses are present in the doctrine.

Another main difference between our point of view and that of the Condorcet principle is that the latter, also in both versions C and C$'$, aims at finding out the winner, i. e. choosing a single option, whereas here we aim, in principle, at finding out all prominent options. In fact, in contrast to the supremacy doctrine, the clauses (53–54) allow for the possibility of having several prominent options or having none of them. To the effect of making a single choice, we will consider two different approaches. The first one, followed in § 5.2–5.3, is simply to select the option(s) x for which the proposition t_x gets a highest acceptability, which corresponds to deciding by a large margin. The second approach, is to impose existence and uniqueness as part of the doctrine.

As we will see in § 5.4, imposing existence and uniqueness motivates a more comprehensive prominence doctrine that will satisfy not only the majority version of the Condorcet principle, but also certain generalizations of it.

5.2 The maximin rule.

In this section we show that the well-known maximin rule [*18* : p. 212–213], i. e. selecting the $x \in A$ that maximizes $\sigma_x = \min_{y \neq x} v(p_{xy})$, corresponds exactly to keeping only the clauses (53) and applying the highest acceptability approach.

Often attributed to Simpson (1969) and Kramer (1977), in actual fact the maximin rule appears already in Duncan Black's celebrated work of 1958 [*2* : (i) in p. 208 of 3rd ed]. The term 'maximin' that we are using is taken from [*18*]. Having said that, all of these authors limited their attention to the complete case $v(p_{xy}) + v(p_{yx}) = 1$, where maximizing $\sigma_x = \min_{y \neq x} v(p_{xy})$ is equivalent to minimizing $\tau_x = \max_{y \neq x} v(p_{yx})$. In the general case, however, one must distinguish between these two rules, that we call respectively 'maximin' and 'minimax' (see § 4.1).

Proposition 5.1. *The conjunctive normal form formed by the clauses* (53) *is disjoint-resolvable and therefore* $*$*-equivalent to the corresponding Blake canonical form.*

Scheme of the proof. The only possibility for producing new clauses is combining pairs of clauses of the form (53), which results in the following ones:

$$t_x \vee t_y \vee \bigvee_{\substack{z\neq x\\ z\neq y}} p_{zx} \vee \bigvee_{\substack{z\neq x\\ z\neq y}} p_{zy}, \quad \text{for any two different } x,y \in A. \tag{57}$$

One can check that no further resolution is possible. □

The one-step revision transformation $v \mapsto v'$ takes the following form: For any $x, y \in A$:

$$v'(t_x) \;=\; \max\Big(\, v(t_x),\, \min_{y\neq x} v(p_{xy}) \,\Big), \tag{58}$$

$$v'(\bar{t}_x) \;=\; v(\bar{t}_x), \tag{59}$$

$$v'(p_{xy}) \;=\; \max\Big(\, v(p_{xy}),\, \min\big(v(t_x),\, \min_{\substack{z\neq x\\ z\neq y}} v(p_{zx})\big) \,\Big), \tag{60}$$

Proposition 5.2. *For initial valuations satisfying* (55)*, the doctrine* (53) *is unquestionable for all of its propositions and it gives*

$$v^*(t_x) \;=\; v'(t_x) \;=\; \sigma_x, \tag{61}$$

$$v^*(\bar{t}_x) \;=\; v(\bar{t}_x) \;=\; 0, \tag{62}$$

$$v^*(p_{xy}) \;=\; v(p_{xy}), \tag{63}$$

where $\sigma_x = \min_{y\neq x} v(p_{xy})$*. Therefore, the most prominent option is the* $x \in A$ *that maximizes* σ_x*.*

Proof. Equalities (61–63) are easily obtained by making use of Lemma 1.8 and the inequality $v(p_{xy}) \geq \sigma_x$. The unquestionability statement is simply a consequence of having obtained $v^* = v'$. □

5.3 Symmetric prominence.

In this section, we consider the doctrine that includes both (53) and (54) and we choose the option that maximizes the (revised) acceptability of t_x. We refer to it as the symmetric prominence method. As a consequence of including also the clauses (54), the winner need not be the same as the maximin one. What is more, we will see that a Condorcet winner need not be the symmetric prominence winner. However, in this doctrine, a Condorcet winner is always accepted as a prominent option. Besides, the fact that this doctrine is symmetric under negation ensures also that a Condorcet loser is always rejected as a prominent option.

Proposition 5.3. *The conjunctive normal form formed by the clauses* (53–54) *is disjoint-resolvable and therefore* $*$*-equivalent to the corresponding Blake canonical form.*

Scheme of the proof. Once again, we will limit ourselves to indicating a path that allows to arrive at the Blake canonical form through disjoint resolution. First, we combine pairs of clauses of the form (53), which results in (57). Second, we combine pairs of clauses of the form (54) to obtain

$$\bar{t}_x \vee \bar{t}_y \vee \bigvee_{\substack{z\neq x\\ z\neq y}} p_{xz} \vee \bigvee_{\substack{z\neq x\\ z\neq y}} p_{yz}, \quad \text{for any two different } x, y \in A. \tag{64}$$

One can check that no further resolution is possible. □

The one-step revision transformation $v \mapsto v'$ takes the following form: For any $x, y \in A$:

$$v'(t_x) = \max\Big(v(t_x),\, \min_{y\neq x} v(p_{xy}) \Big), \tag{65}$$

$$v'(\bar{t}_x) = \max\Big(v(\bar{t}_x),\, \min_{y\neq x} v(p_{yx}) \Big), \tag{66}$$

$$v'(p_{xy}) = \max\Big(v(p_{xy}),\, \min\big(v(t_x),\, \min_{\substack{z\neq x\\ z\neq y}} v(p_{zx})\big),\, \min\big(v(\bar{t}_y),\, \min_{\substack{z\neq x\\ z\neq y}} v(p_{yz})\big) \Big), \tag{67}$$

Proposition 5.4. *For initial valuations satisfying* (55)*, the symmetric prominence doctrine is unquestionable for all of its propositions and it gives*

$$v^*(t_x) = v'(t_x) = \sigma_x, \tag{68}$$

$$v^*(\bar{t}_x) = v'(\bar{t}_x) = \rho_x, \tag{69}$$

$$v^*(p_{xy}) = v(p_{xy}), \tag{70}$$

where $\sigma_x = \min_{y\neq x} v(p_{xy})$*, and* $\rho_x = \min_{y\neq x} v(p_{yx})$*.*

Proof. Equalities (68–70) are easily obtained by making use of Lemma 1.8 and the inequalities $v(p_{xy}) \geq \sigma_x$, $v(p_{xy}) \geq \rho_y$. The unquestionability statement is simply a consequence of having obtained $v^* = v'$. □

Corollary 5.5. *Whenever there is a Condorcet winner, the symmetric prominence method accepts it as a prominent option.*

Proof. It suffices to notice that x being a Condorcet winner implies $\sigma_x > \frac{1}{2} > \rho_x$. $\square$

The symmetric prominence doctrine, formed by clauses (53) and (54) is symmetric under negation, i. e. the substitution that interchanges t_x and $\bar{t}_x$ as well as p_{xy} and p_{yx}. As a consequence, the preceding proposition is accompanied here by the following one:

Proposition 5.6. *Whenever there is a Condorcet loser, i. e. an option x such that $v(p_{yx}) > \frac{1}{2}$ for any $y \neq x$, the symmetric prominence method rejects it as a prominent option.*

Remark. In spite of Corollary 5.5, the Condorcet winner can differ from the symmetric prominence winner, i. e. the option whose prominence gets a highest acceptability. A simple example is the following: 3 $a \succ b \succ c$, 2 $b \succ c \succ a$, with the following Llull matrix[1]

a	3	3
2	b	5
2	0	c

, (71)

where one easily checks that the Condorcet winner a gets $v^*(t_a) - v^*(\bar{t}_a) = 3 - 2 = 1$, but $v^*(t_b) - v^*(\bar{t}_b) = 2 - 0 = 2$.

5.4 Comprehensive prominence

5.4.1 To the effect of making a single choice, one can go for supplementing the symmetric prominence doctrine (53–54) with two additional clauses postulating the existence and uniqueness of a prominent option, namely: $\bigvee_{z\in A} t_z$ (existence), and $t_x \to \bar{t}_y$ (uniqueness). Let us write down all of these clauses together:

$$\text{(53)} \qquad t_x \vee \bigvee_{y\neq x} p_{yx}, \qquad \text{for any } x \in A; \tag{72}$$

$$\text{(54)} \qquad \bar{t}_x \vee \bigvee_{y\neq x} p_{xy}, \qquad \text{for any } x \in A; \tag{73}$$

$$\bigvee_{z\in A} t_z\,; \tag{74}$$

$$\bar{t}_x \vee \bar{t}_y, \qquad \text{for any two different } x, y \in A. \tag{75}$$

[1]Since we deal only with p_{xy} for $x \neq y$, we use the diagonal cells for specifying the simultaneous labelling of rows and columns by the members of A. The cell in row x and column y gives the value of $v(p_{xy})$.

Let see which clauses derive from (72–75). As before, we begin by combining pairs of clauses of the form (72), which leads to (57):

$$(57) \quad t_x \vee t_y \vee \bigvee_{\substack{z \neq x \\ z \neq y}} p_{zx} \vee \bigvee_{\substack{z \neq x \\ z \neq y}} p_{zy}, \quad \text{for any two different } x, y \in A. \tag{76}$$

One can now combine (73) and (74). This results in

$$\bigvee_{z \neq x} t_z \vee \bigvee_{z \neq x} p_{xz}, \qquad \text{for any } x \in A. \tag{77}$$

On the other hand, one can also combine clauses (72) and (75), which gives

$$\bar{t}_y \vee \bigvee_{z \neq x} p_{zx}, \qquad \text{for any two different } x, y \in A. \tag{78}$$

Notice that, in the special case of having only two options, (76), (77) and (78) coincide respectively with (74), (72) and (73). Finally, for more than two options one can combine (78) with itself, which leads to

$$\bar{t}_y \vee \bigvee_{\substack{z \neq x \\ z \neq x'}} p_{zx} \vee \bigvee_{\substack{z \neq x \\ z \neq x'}} p_{zu}, \qquad \text{for any three different } x, x', y \in A. \tag{79}$$

In contrast to the preceding sections, here we have not been able to stay with disjoint resolution. The problem lies in the derivation of $(79)_{x,x',y}$ from $(78)_{x,y}$ and $(78)_{x',y}$. Therefore, (72–75) is **not** guaranteed to be $*$-equivalent to the corresponding Blake canonical form, namely (72–79). In such a situation, the standard course of action would be using the Blake canonical form. However, when looking at the rationale behind the clauses that have been obtained, one sees that they are contained in a more comprehensive doctrine that seems quite reasonable and worth being adopted in its full generality.

This doctrine, that we will refer to as that of **comprehensive prominence,** is made up by the following clauses, two of which are indexed by arbitrary non-empty subsets of A:

$$\bigvee_{r \in X} t_r \vee \bigvee_{\substack{r \in X \\ s \notin X}} p_{sr}, \qquad \text{for any non-empty } X \subseteq A; \tag{80}$$

$$\bar{t}_y \vee \bigvee_{\substack{r \in X \\ s \notin X}} p_{sr}, \qquad \text{for any non-empty } X \subseteq A, \text{ and any } y \notin X; \tag{81}$$

$$\bar{t}_x \vee \bar{t}_y, \qquad \text{for any two different } x, y \in A; \tag{82}$$

Clauses (80) and (81) are saying the following: If there exists a non-empty $X \subseteq A$ such that every $r \in X$ is preferred to any $s \notin X$, then X contains at least one prominent option, whereas $A \setminus X$ contains none.

One easily sees that clauses (72), (76), (77) and (74) are particular cases of (80) (in particular, the existence clause corresponds to the case $X = A$). On the other hand, (73), (78) and (79) are particular cases of (81). More specifically, the only difference between (80–82) and (72–79) is that the latter is restricted to subsets X of size $|X| = 1, 2, N-1, N$, where $N = |A|$. Therefore, both doctrines are different from each other when $N \geq 5$. In this case, (80–82) contains clauses that cannot be derived from (72–75).

The fact that (80) and (81) are indexed by all possible subsets of A makes things rather involved. However, we will see that the results are interesting enough.

Proposition 5.7. *The conjunctive normal form formed by the clauses* (80–82) *is already in Blake canonical form. This doctrine is unquestionable for $\bar{t}_x$, i. e. it satisfies $v^*(\bar{t}_x) = v'(\bar{t}_x)$, and it is also unquestionable for t_x when accepted, i. e. it satisfies $v^*(t_x) = v'(t_x)$ whenever $v^*(t_x) > v^*(\bar{t}_x)$.*

Proof. The proof is long but quite mechanical. The first statement requires checking that all the would-be resolutions are absorbed by some clause already present. The unquestionability for $\bar{t}_x$ is obtained by checking that condition (a) of [8: Thm. 3.18] is satisfied for any pair of clauses C, C' and any literal q satisfying in opposition, one of which containing $\bar{t}_x$. Finally, the unquestionability for t_x when accepted is obtained by checking that either condition (a) or condition (b′) of [8: Cor. 3.19] is satisfied in the analogous situation for t_x instead of $\bar{t}_x$. □

The one-step revision transformation $v \mapsto v'$ can be written in the following form: For any $x, y \in A$,

$$v'(t_x) = \max\Big(v(t_x),\ \max_{\substack{X\subseteq A\\ X\ni x}} \min\big(\min_{\substack{r\in X\\ r\neq x}} v(\bar{t}_r),\ \min_{\substack{r\in X\\ s\notin X}} v(p_{rs})\big)\Big), \tag{83}$$

$$v'(\bar{t}_y) = \max\Big(v(\bar{t}_y),\ \max_{r\neq y} v(t_r),\ \max_{\substack{\emptyset\neq X\subseteq A\\ X\not\ni y}} \min_{\substack{r\in X\\ s\notin X}} v(p_{rs})\Big), \tag{84}$$

$$v'(p_{yx}) = \max\Bigg(v(p_{yx}),\ \max_{\substack{X\subseteq A\\ X\ni x\\ X\not\ni y}} \min\Big(\max\big(\min_{r\in X} v(\bar{t}_r), \max_{s\notin X} v(t_s)\big),\ \min_{\substack{r\in X\\ s\notin X\\ rs\neq xy}} v(p_{rs})\Big)\Bigg), \tag{85}$$

where the operators max and min should be understood as giving respectively the values 0 and 1 whenever they are applied to an empty set. Recall that our aim is to iterate this transformation starting from an initial valuation satisfying (55).

Corollary 5.8. *The following equality holds whenever t_x is accepted:*

$$v^*(t_x) \;=\; v'(t_x) \;=\; \min_{s\neq x} v(p_{xs}). \tag{86}$$

On the other hand, the following one holds for any y:

$$v^*(\bar{t}_y) \;=\; v'(\bar{t}_y) \;=\; \max_{\emptyset\neq X\subseteq A\setminus\{y\}} \; \min_{\substack{r\in X\\ s\notin X}} v(p_{rs}). \tag{87}$$

Proof. It follows from Proposition 5.7 on account of the formulas (83–84) and the initial values (55). □

As in the supremacy doctrine, the definite consistency theorem (Theorem 1.4) applied to (75) and (74) guarantees that: (i) when t_x is accepted, then x is the only option with this property, and (ii) when t_x is not accepted for any $x \in A$, then this proposition is undecided for more than one $x \in A$. In the sequel an option x for which t_x is not rejected will be called a **comprehensive prominence winner**.

Remark. Notice also that x is the unique comprehensive prominence winner as soon as $v'(t_x) > v'(\bar{t}_x)$. In fact, starting from this inequality, the unquestionability of $\bar{t}_x$ and the general fact that $v^* \geq v'$ allow to derive that $v^*(t_x) > v^*(\bar{t}_x)$.

Proposition 5.9. *An option x is a comprehensive prominence winner if and only if it minimizes $v^*(\bar{t}_x)$.*

Proof. Assume that x is a comprehensive prominence winner. In order to see that it minimizes $v^*(\bar{t}_x)$ it suffices to notice that the following inequalities hold for any $y \neq x$:

$$v^*(\bar{t}_y) \;\geq\; v^*(t_x) \;\geq\; v^*(\bar{t}_x), \tag{88}$$

where the first one holds because of the consistency of v^* with the uniqueness clause (75).

Assume now that x minimizes $v^*(\bar{t}_x)$. In order to see that it is a comprehensive prominence winner, it suffices to notice that

$$v^*(t_x) \;\geq\; \min_{z\neq x} v^*(\bar{t}_z) \;\geq\; \min_{z} v^*(\bar{t}_z) \;=\; v^*(\bar{t}_x), \tag{89}$$

where the first inequality holds because of the consistency of v^* with the existence clause (74). □

5.4.2 The comprehensive prominence doctrine has good properties in connection with majority-dominant sets. A set $S \subseteq A$ is said to be **majority-dominant** when one has $v(p_{xy}) > \frac{1}{2}$ for every $x \in S$ and $y \notin S$. If both S and T are majority-dominant, then one must have either $S \subseteq T$ or $T \subseteq S$. Otherwise it would be incompatible with (11). As a consequence, there is always a unique **minimal majority-dominant set** M**.** The case of a Condorcet winner is simply that where the minimal majority-dominant set consists of a single option.

The notion of minimal majority-dominant set was introduced by Benjamin Ward in 1961 [*20*], and again by Irving John Good in 1971 [*11*]. This set is often called the Smith set (see for instance [*18*: p. 154]), in reference to a subsequent work of John H. Smith [*17*]; however, the latter was not especially interested in locating the social winner in the minimal majority-dominant set, but only in the social binary preferences associated with a general majority-dominant set.

Proposition 5.10. *The minimal majority-dominant set M has the following properties:*

$$v^*(\bar{t}_y) > \tfrac{1}{2}, \quad \textit{for any } y \notin M, \tag{90}$$

$$v^*(\bar{t}_x) \le \tfrac{1}{2}, \quad \textit{for any } x \in M, \tag{91}$$

$$v^*(t_y) \le \tfrac{1}{2}, \quad \textit{for any } y \notin M. \tag{92}$$

As a consequence, any comprehensive prominence winner is ensured to belong to M.

Proof. The inequality (90) follows readily from (87) by taking $X = M$.

In view of (87), in order to prove (91) we must show that

$$\min_{\substack{r\in X\\ s\notin X}} v(p_{rs}) \le \tfrac{1}{2}, \quad \text{whenever } x \in M \text{ and } \emptyset \ne X \subseteq A \setminus \{x\}. \tag{93}$$

For $X \not\subset M$, this inequality, and even the corresponding strict one, is ensured to hold because there exists $z \in X \setminus M$ such that $v(p_{xz}) > 1/2$ and therefore $v(p_{zx}) < 1/2$. On the other hand, for $X \subset M$, the minimality of M guarantees the existence of $z \in X$ and $y \in M \setminus X$ such that $v(p_{zy}) \le 1/2$, which also ensures (93).

The inequality (92) follows easily from (91) because of the consistency of v^* with the clause (82).

Finally, (90) together with (92) says that t_y is rejected for any $y \notin M$. Therefore, the comprehensive prominence winner(s) must belong to M. $\square$

Remark. When M consists of a single option x, i. e. in the case of x being the Condorcet winner, the strict inclusion $X \subset M$ is not possible for a non-empty set, so (91) holds with the strict inequality that obtained in the case $X \not\subset M$. As a consequence, (92) holds also with a strict inequality.

Proposition 5.11. *If $S \subseteq A$ is a majority-dominant set, then*

$$v^*(p_{yx}) \;\le\; \tfrac{1}{2}, \quad \textit{whenever } x \in S \textit{ and } y \notin S. \tag{94}$$

As a consequence, the comprehensive prominence method accepts p_{xy} for any $x \in S$ and $y \notin S$, and this decision is unquestionable.

Proof. In order to establish (94), we will base ourselves on equation (85) and Lemma 1.8, which allow us to write

$$v^*(p_{yx}) = \max\Big(v(p_{yx}),\, \max_{\substack{X\subseteq A\\ X\ni x\\ X\not\ni y}} \min\Big(\max\big(\min_{r\in X} v^*(\bar t_r), \max_{s\notin X} v^*(t_s)\big),\, \min_{\substack{r\in X\\ s\notin X\\ rs\neq xy}} v^*(p_{rs})\Big)\Big).$$

Now, the consistency of v^* with the clause (82) implies that $v^*(t_s) \le v^*(\bar t_r)$ for any $s \neq r$. Therefore,

$$v^*(p_{yx}) \;=\; \max\Big(\, v(p_{yx}),\, \max_{\substack{X\subseteq A\\ X\ni x\\ X\not\ni y}} \min\big(\min_{r\in X} v^*(\bar t_r),\, \min_{\substack{r\in X\\ s\notin X\\ rs\neq xy}} v^*(p_{rs})\big)\Big). \tag{95}$$

Let us assume that $x \in S$ and $y \notin S$. Since we know that $v(p_{yx}) < \frac{1}{2}$, (94) will be established if we are able to show that

$$\min\big(\min_{r\in X} v^*(\bar t_r),\, \min_{\substack{r\in X\\ s\notin X\\ rs\neq xy}} v^*(p_{rs})\big) \;\le\; \tfrac{1}{2}, \quad \text{whenever } X \ni x \text{ and } X \not\ni y. \tag{96}$$

In order to obtain this property, we will distinguish two possibilities depending on whether or not X intersects the minimal majority-dominant set M considered in the preceding proposition. Let us begin by considering the case $X \cap M \neq \emptyset$. In this case, (96) holds because of (91). In the especial case that $x \in M$, this argument covers all of the sets X considered in (95–96) (because X is restricted to contain x). Therefore, (94) is by now established for $S = M$. This fact allows us to fix the pending case $X \cap M = \emptyset$. Indeed, in this case the already obtained result guarantees that $v^*(p_{xs}) \le \frac{1}{2}$ for every $s \in M$, which values are included in the left-hand side of (96) (since $y \notin S$ implies $xs \neq xy$).

Since $v^*(p_{xy}) \ge v(p_{xy}) > \frac{1}{2}$, having obtained (94) ensures that p_{xy} is accepted for any $x \in S$ and $y \notin S$. On the other hand, since we have not only $v^*(p_{xy}) > v^*(p_{yx})$, but even $v(p_{xy}) > v^*(p_{yx})$, we can be sure that this decision does not rely on belief derived from unsatisfiable conjunctions. □

Remark. The above-remarked fact that (91) holds as a strict inequality whenever x is the Condorcet winner entails that (94) holds also as a strict inequality whenever x is the Condorcet winner and $y \neq x$.

5.4.3 A bit unexpectedly, the comprehensive prominence winner is often the same as the maximin one (§ 5.1):

Proposition 5.12. *Whenever there is a unique comprehensive prominence winner, then there is also a unique maximin winner, and they coincide with each other.*

Proof. As it has been remarked, x being the unique comprehensive prominence winner is equivalent to say that t_x is accepted, i. e. $v^*(t_x) > v^*(\bar{t}_x)$. According to Cor. 5.8, this translates into the first of the next two inequalities:

$$\min_{s\neq x} v(p_{xs}) \; > \; \max_{X\subseteq A\setminus\{x\}} \; \min_{\substack{r\in X\\ s\notin X}} v(p_{rs}) \; \geq \; \min_{s\neq y} v(p_{ys}). \tag{97}$$

The second of these inequalities is easily seen to hold for any $y \neq x$: it suffices to consider $X = \{y\}$ in the central expression. Therefore, we get $\sigma_x > \sigma_y$ for any $y \neq x$, i. e. x is the unique maximin winner. □

However, the converse is not true: It may happen that there is a unique maximin winner but there is not a unique prominence winner. Not only that, a unique maximin winner may even be rejected as a prominent option. Example: 1 $a\succ b\succ c\succ d$, 1 $a\succ b\succ d\succ c$, 2 $b\succ c\succ a\succ d$, 1 $b\succ c\succ d\succ a$, 1 $c\succ a\succ d\succ b$, 1 $d\succ a\succ b\succ c$, 2 $d\succ c\succ a\succ b$. The Llull matrix is

a	6	3	5
3	*b*	6	5
6	3	*c*	5
4	4	4	*d*

, (98)

which shows that the maximin winner d is defeated by any other option! The possibility of such situations has been pointed out as the main drawback of the maximin method [*18*: p. 212–213]. In contrast, the comprehensive prominence method definitely rejects d it as a prominent choice, and it remains undecided between the other three options. This is quite in agreement with Proposition 5.10. In fact, here the minimal majority-dominant set is clearly $M = \{a, b, c\}$.

Let us assume that 0.75 of the first vote of the preceding example changes from $a \succ b \succ c \succ d$ to $a \succ b \succ d \succ c$. The Llull matrix becomes then

a	6	3	5
3	b	6	5
6	3	c	4.25
4	4	4.75	d

. (99)

In this case, the minimal majority-dominant set is not $\{a, b, c\}$ but the whole of $A = \{a, b, c, d\}$. Even so, however, the set of comprehensive prominence winners still reduces to $\{a, b, c\}$. As it will be shown in the next result, this has to do with the fact that this set has the property of maximizing the quantity $\sigma_X = \min_{x \in X,\, y \notin X} v(p_{xy})$. From now on, a proper subset of A with this property will be called a **maximin set.**

Proposition 5.13. *If the set of comprehensive prominence winners is not the whole of A, then it is contained in the intersection of all the maximin sets.*

Proof. It suffices to show that the following implication holds for any maximin set: If $z \notin X$ then z is not a comprehensive prominence winner. This is a consequence of Corollary 5.8 and Proposition 5.9. In fact, since X maximizes $\sigma_X = \min_{x \in X, y \notin X} v(p_{xy})$, (87) allows to derive that any $z \notin X$ satisfies $v^*(\bar{t}_z) \geq v^*(\bar{t}_x)$ for any $x \neq z$. On the other hand, if z were a comprehensive prominence winner, then Proposition 5.9 ensures that $v^*(\bar{t}_z) < v^*(\bar{t}_x)$ for any x that is not such a winner, thus obtaining a contradiction. □

In the way that we have defined it in § 5.4.1, the set of comprehensive prominence winners is never empty. Therefore, the preceding proposition has the following consequence:

Corollary 5.14. *It the maximin sets have an empty intersection, then the comprehensive prominence winner is undecided between the whole of A.*

5.4.4 Let us perturb example (98) so that the Condorcet cycle $a \succ b \succ c \succ a$ becomes uneven. The Llull matrix could take, for instance, the following value:

a	6	$3{+}2\epsilon$	5
3	b	$6{-}\epsilon$	5
$6{-}2\epsilon$	$3{+}\epsilon$	c	5
4	4	4	d

, (100)

with $\epsilon > 0$. A bit unexpectedly, the comprehensive prominence method does not select a as the unique winner (which corresponds to breaking the cycle by the weakest link): for $0 \leq \epsilon < \frac{1}{2}$, i. e. when the victories within the cycle are stronger than those outside it, the result is still an undecidedness between a, b, c.

In this example one gets $v^*(t_x) = v''(t_x) = 4$ for $x = a, b, c$. These values derive from unsatisfiable conjunctions. In fact, they derive through the concatenation of $t_x \leftarrow \bigwedge_{r \neq x} \bar{t}_r$ with $\bar{t}_r \leftarrow \bigwedge_{s \neq d} p_{ds}$ for $r = a, b, c$ as well as $\bar{t}_d \leftarrow \bigwedge_{s \neq d} p_{sd}$. This concatenation amounts to the implication $t_x \leftarrow \bigwedge_{s \neq d} (p_{sd} \wedge p_{ds})$, whose right-hand side negates the *tertium non datur* clauses $p_{sd} \vee p_{ds}$. So the undecidedness between a, b, c as comprehensive prominence winners is questionable. According to Proposition 5.7, what is unquestionable is the rejection of d.

This suggests that in the event of undecidedness one should restrict the attention to all the undecided winners and start again the comprehensive prominence algorithm from the corresponding restriction of the original Llull matrix. Generally speaking, this progressive elimination could involve several rounds. We will refer to this procedure as the **refined comprehensive prominence method**. In the preceding example, this procedure selects a as a single winner.

5.4.5 In the incomplete case, one easily finds examples where the comprehensive prominence winner does not coincide with the transitivity one. For instance: $1\ a \succ b \succ c$, $1\ b \succ c \succ a$, $2\ c \succ a \succ b$, $1\ a$, $2\ b$; the ranking produced by the transitivity doctrine is $a \succ b \succ c$, whereas the comprehensive prominence winner is b.

In contrast, in the complete case, there is a strong experimental evidence that the transitivity winners are always included among the refined comprehensive prominence winners. On the other hand, one easily finds examples where this inclusion is strict. A proof of the stated inclusion is lacking. A weaker fact, proved by Schulze [*15*: § 4.8] is the following: In the complete case, the transitivity winners are included in the union of all the maximin sets (which union Schulze calls the MinMax set). When there is only one transitivity winner, then Schulze's proof is easily adapted to show that this winner is contained in the intersection of all maximin sets.

6 Goodness

Generally speaking, x being preferred to any other option does not imply x being good: in fact, x could be the lesser of several evils. In other words, what we have called supremacy does not imply goodness; as a consequence, since supremacy does imply prominence, the latter cannot either be identified with goodness. In fact, goodness has an absolute character: it does not rely on comparing an option to another, but it makes sense for every option by itself. On the other hand, preference is related to goodness in the following way: if x is considered good and y is considered bad, then x is preferred to y.

So, the doctrine that relates goodness to preference is made of the following clauses, where g_x denotes the proposition 'x is good': $g_x \wedge \overline{g}_y \rightarrow p_{xy}$. In conjunctive normal form:

$$\overline{g}_x \vee p_{xy} \vee g_y, \qquad \text{for any two different } x, y \in A; \tag{101}$$

This doctrine will be referred to as the **goodness doctrine**. Clearly, it is symmetric under negation.

Notice that, similarly to § 5.2, here we are admitting the possibility of having several good options as well as having none of them. To the effect of making a choice, it makes sense to select the option(s) x for which the proposition g_x gets a highest acceptability, i. e. a highest value of the difference $v^*(g_x) - v^*(\overline{g}_x)$. Notice however, that this number could be negative for all options, which means that no good option is found; in this case we are choosing the lesser of the evils. On the other hand, it can also happen that we find several good options. In this case, choosing the one(s) with highest acceptability corresponds to deciding by a margin (§ 1.2).

We will refer to this procedure as the **goodness method,** and to its winners as the **goodness winners.**

Proposition 6.1. *The conjunctive normal form formed by the clauses* (101) *is disjoint-resolvable and therefore $*$-equivalent to the corresponding Blake canonical form. This doctrine is unquestionable for any of the propositions g_x and $\overline{g}_x$.*

Scheme of the proof. The Blake canonical form consists of all clauses of the form

$$\overline{g}_{x_0} \vee p_{x_0x_1} \vee p_{x_1x_2} \vee \ldots \vee p_{x_{n-1}x_n} \vee g_{x_n}, \tag{102}$$

with $n \geq 1$ and all x_i $(0 \leq i \leq n)$ pairwise different (which restricts n to be less than or equal to the number of elements of A). The clauses (102) are easily derived from (101) by successive concatenation, and suitably ordered this derivation will use only disjoint resolution.

The unquestionability g_x and $\overline{g}_x$ is easily obtained through condition (a) of [*8*: Thm. 3.18]. For g_x —the case of $\overline{g}_x$ is analogous by symmetry— this condition requires the following: for any clause C of the form (102) that includes g_x (i. e. $x_n = x$), and any other clause C' of the form (102), if C and C' contain respectively q and $\overline{q}$, then the would-be resolution $C \underset{q}{\vee} C'$ contains a third clause C_1 of the form (102) that still includes g_x. The reader will easily become convinced that it is so, both in the case where $q = \overline{g}_y$ ($y = x_0$) and in the case where $q = p_{ab}$. □

The one-step revision transformation $v \mapsto v'$ takes the following form: For any $x, y \in A$:

$$v'(g_x) \;=\; \max\Big(\, v(g_x),\, \max_{y \neq x} \min\big(v(g_y), v(p_{xy})\big)\,\Big), \tag{103}$$

$$v'(\overline{g}_x) \;=\; \max\Big(\, v(\overline{g}_x),\, \max_{y \neq x} \min\big(v(\overline{g}_y), v(p_{yx})\big)\,\Big), \tag{104}$$

$$v'(p_{xy}) \;=\; \max\Big(\, v(p_{xy}),\, \min\big(v(g_x), v(\overline{g}_y)\big)\,\Big). \tag{105}$$

The properties of disjoint resolvability and unquestionability allow to express v^* directly as the result of the one-step transformation associated with the Blake canonical form, namely:

$$v^*(g_x) \;=\; \mathrm{Max}\, \min\big(\, v(p_{x_0x_1}), v(p_{x_1x_2}), \dots, v(p_{x_{n-1}x_n}), v(g_{x_n})\,\big), \tag{106}$$

$$v^*(\overline{g}_x) \;=\; \mathrm{Max}\, \min\big(\, v(\overline{g}_{x_0}), v(p_{x_0x_1}), v(p_{x_1x_2}), \dots, v(p_{x_{n-1}x_n})\,\big), \tag{107}$$

$$v^*(p_{xy}) \;=\; \max\Big(\, v(p_{xy}),\, \min\big(v^*(g_x), v^*(\overline{g}_y)\big)\,\Big), \tag{108}$$

where the Max operator of (106–107) considers here all paths $x_0x_1 \dots x_n$ of length $n \geq 0$ from $x_0 = x$ to $x_n = y$ with all x_i pairwise different.

6.1 Approval-disapproval voting.

In approval voting each voter is asked for a list of approved options [5]. Clearly, the fraction of voters who approve a given option x can be seen as the collective degree of belief in the goodness of x, i. e. as the value of $v(g_x)$. The standard approval-voting rule for choosing an option x is simply taking the one that maximizes $v(g_x)$.

Now, from the point of view of this article, in order to take a decision about g_x we should consider also the support for $\overline{g}_x$. Properly speaking, this requires that voters specifically pronounce themselves about it. This idea is considered in [9], whose CAV rule chooses the option that maximizes the difference $v(g_x) - v(\overline{g}_x)$.

In this section we assume that the votes contain no direct information about binary preferences. This amounts to having $v(p_{xy}) = 0$ for any x and y. In this case, (106–108) reduce to $v^*(g_x) = v(g_x)$, $v^*(\overline{g}_x) = v(\overline{g}_x)$ and $v^*(p_{xy}) = \min(v(g_x), v(\overline{g}_y))$. Therefore, the acceptability of g_x is simply the difference $v(g_x) - v(\overline{g}_x)$. So *the goodness method fully coincides in this case with the CAV rule.*

This rule coincides with that of standard approval voting, i. e. rating the options by $v(g_x)$, in the following two cases: (i) $v(\overline{g}_x) = 0$, (ii) $v(\overline{g}_x) = 1 - v(g_x)$, which correspond respectively to interpreting that (i) non-approved options are not necessarily disapproved, or contrarily, that (ii) all non-approved options are disapproved [completeness]. In the case of interpretation (ii) an option is accepted as a good one if and only if $v(g_x) > \frac{1}{2}$, i. e. if it is approved by a majority.

6.2 Approval-disapproval-preferential voting.

Let us consider now the case where the individual votes give information not only about approval or disapproval, but also about binary preferences that are not a consequence of approval and disapproval. For instance, besides saying that x and y are both approved (or both disapproved), a voter can add the information that he prefers y to x. As we will see, this added information may lead to revising the degrees of belief about the goodness or badness of the different options. For instance, it might happen that x is approved more often than y but at the same time y is preferred to x more often than x is preferred to y.

With more or less generality, such forms of voting have been considered by several authors (see [*5* : ch. 3], [*10*]). A real example is the 2006 Public Choice Society election [*3*], whose actual ballots are listed in [*7* : §3.3].

The PAV procedure proposed in [*4* ; *5* : § 3.3], gives priority to the approval information, which decides the winner unless several candidates are approved by a majority; in this case, the attention is restricted to the set A^* of these majority-approved candidates, and the preferential information about them is used to single out, if possible, their Condorcet winner; if this is not possible, then the attention is restricted to the minimal majority dominant subset of A^* —i. e. the smallest subset M^* of A^* with the property that each $x \in M^*$ preferred to any $y \in A^* \setminus M^*$— and the winner is selected from M^* by looking again at the approval score.

Though certainly reasonable, this procedure alternates between approval and preferential information in a categorical way that does not seem fully justified. In particular, it is not difficult to set up examples where a tiny preference margin between two majority-approved candidates may select a candidate approved by a small majority instead of another one that was approved by a very large majority. For instance, for the profile

$$(1-\varepsilon)/2 : a \succ b\,|, \quad (1-\varepsilon)/2 : b \mid a, \quad \varepsilon : a \mid b, \tag{109}$$

with a small value of $\varepsilon > 0$, one has $v(g_a) = (1+\varepsilon)/2$, $v(g_b) = 1-\varepsilon$, and $v(p_{ab}) = (1+\varepsilon)/2$, so that the PAV procedure gives the victory to a on the basis of a nearly vanishing margin of preference, in spite of the fact that the approval of b is almost unanimous whereas that of a is near to only half the vote.

In contrast, our method carefully gauges the interplay between both kinds of information in accordance with the doctrine under consideration. The goodness doctrine that we are considering in this section contains only the clauses (101) and their derivates (102). More particularly, it does not include the transitivity of preferences. Having said that, it is interesting to see that (102) shows that having a chain of preferences from x_n to x_0, i. e. having $p_{x_n x_{n-1}} \wedge \ldots \wedge p_{x_1 x_0} \wedge p_{x_1 x_0}$, implies $g_{x_n} \vee \overline{g}_{x_0}$ just as well as the direct preference $p_{x_n x_0}$.

In accordance with the doctrine (101), our proposal to deal with combined approval and preference information is to iterate the transformation (103–105) until invariance, and then select the option x with a highest value of $v^*(g_x) - v^*(\overline{g}_x)$, which we have already called the goodness winner.

For the profile (109) one gets $v^*(g_a) - v^*(\overline{g}_a) = \varepsilon$ and $v^*(g_b) - v^*(\overline{g}_b) = (1-\varepsilon)/2$; therefore, for small values of ε the goodness winner is b. So the goodness method does not let a slight margin of preference to prevail over a big difference in approval. In other cases, however, preferences can overturn an initial difference in approval. Such a phenomenon occurs for instance in the following example:

$$5 : a \mid b \succ c, \quad 4 : b \succ c \mid a, \quad 3 : c \mid a \succ b, \quad 1 : a \succ c \mid b. \tag{110}$$

The values of $v(g_x) - v(\overline{g}_x)$ for $x = a, b, c$ are respectively $-1, -5, 3$; so initially —without taking into account the preferential information— the only approved candidate is c. After revision, however, the values of $v^*(g_x) - v^*(\overline{g}_x)$ are respectively $1, -1, -1$; so the final decision rejects the goodness/approval of c and chooses a as the only approved candidate.

The following result establishes a highly desirable property:

Proposition 6.2. *When applied to approval-disapproval-preferential voting, the goodness method is monotonic in the following sense: Assume that some* *votes are modified by raising x to a better position (with no other change). Such a modification can never decrease the acceptability of g_x, nor can it increase the acceptability of g_y for any $y \neq x$.*

Proof. Raising x decomposes in several cases: (i) $v(g_x)$ increases; (ii) $v(\overline{g}_x)$ decreases; (iii) $v(p_{xy})$ increases; (iv) $v(p_{yx})$ decreases. Using (106–107), one easily checks that each of these changes cannot have but the following effects: $v^*(g_x)$ does not decrease; $v^*(\overline{g}_x)$ does not increase; $v^*(g_y)$ does not increase; $v^*(\overline{g}_y)$ does not decrease. □

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