Impartial award of a prize

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Abstract

A group of peers must choose one of them to receive a prize, when everyone cares only about winning, not about who gets the prize if someone else. An award rule is *impartial* if my message never influences whether or not I win the prize.

If each agent nominates a *single* (other) *agent*, and nominations are anonymous, impartiality is feasible provided not all agents can nominate every other agent. Set the agents on the nodes of a fixed tree, allow node i to nominate someone on the same side of i as the median of the tree, and give the prize to the median of the actual votes.

If agents report an *ordering* of potential winners, there exist fairly simple impartial award rules distributing the decision power more equally than in the tree rules above. Partition the agents in two or more *districts*, each of size at least three; agents nominate first someone in their own district, and a *local winner* is one who reaches a fixed quota of nominations; then all non local winners vote to select one of the local winners.

Key words: impartiality, no dummy, no discrimination, median voting on a tree, qualified majority.

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1 Introduction

1.1 Impartial decision making

The possibility of an impartial judgment is a cornerstone of modern theories of justice, from Harsanyi's impartial observer [7], to Rawls' veil of ignorance [14], and Sen's transpositional objectivity [17]. In the more mundane context of committees and elections, impartial evaluations are a desirable but elusive ingredient of group decision making. When individual opinions are aggregated into a collective outcome, an agent may be tempted to corrupt her valuable disinterested opinion (which influences the final decision) to serve her selfish preferences. Thus corrupted, the profile of messages may yield a suboptimal decision. Avoiding such *conflicts of interest* is a tall order, illustrated by the familiar challenge of *peer evaluation,* a central institution in many communities of experts.

We argue that conflicts of interest can sometimes be eliminated in those collective decision problems where each participant's disinterested opinion is *disconnected* from her selfish interest. As suggested by Sen (who proposes a taxonomy of social choice problems according to this distinction: [16]), we think of an agent as endowed with two orderings over outcomes, one representing her honest opinion/views, the other her selfish interest. If these two orderings overlap, it is typically impossible to elicit honest opinions from selfish agents: the Gibbard-Satterthwaite Theorem [6] gives a formal statement of this general fact. However if these orderings bear on different aspects of the final outcome, then we may be able to design an aggregation rule such that Jones' message is by design *irrelevant* to her selfish welfare, and conveys information only about her disinterested opinion. Then we can speak of an *impartial* decision rule: participants have no incentives to corrupt the report of their opinions, and the noise from selfish preferences is filtered out of the final outcome.

Here we study impartial decision rules to *award a prize* among peers. A group of agents must choose one of them to receive a prize or undertake a task. The agents must rely on their own judgments to allocate the prize. The key assumption is that the prize/task is a purely private commodity: I selfishly care about whether I receive it or not, but in the latter case who gets the prize is a matter of selfish indifference to me, about which I can contribute my disinterested opinion. Impartiality requires that whether or not I get the prize is completely independent of my own message, while allowing this message to influence who gets it if not me. To repeat the important point in the previous paragraph, an impartial rule ignores selfish preferences entirely: if some agents relish a certain task, while others view it as a painful chore, the profile of opinions about who is the best suited for the job (if not oneself) completely overrides these irrelevant selfish concerns.

We show that with four or more participants there are many impartial award rules, and identify two families of particular interest because individual messages convey disinterested opinions in the two most natural formats. In the *nomination rules,* every agent nominates a single other agent, and ballots are anonymous. In our *median nomination rules* agents are the nodes of a fixed tree, and the prize goes to the median of their ballots on this tree. The second family consists of *direct voting rules* where, as in standard voting, each participant reports a complete ordering of the other agents, describing her top choice for winner, her second choice and so on. Our *partition voting rules* resemble a two-rounds election with a fixed partition of the agents in districts: a first vote in each district aims at selecting a local representative, next the winner is selected among the local representatives.

1.2 Overview of the results

We can always ensure impartiality by ignoring individual messages entirely. Just like in the search for strategyproof voting rules (more below on the connection between impartiality and strategyproofness), the hard question is whether this property is compatible with other basic normative requirements. The nature of the problem precludes an entirely symmetric treatment of all participants. Indeed, for a standard voting rule, anonymity is easily achieved at the expense of neutrality, by using a fixed ordering of the candidates to break ties. But in our model anonymity and neutrality are one and the same so this familiar trick does not work. We will use weaker tests than anonymity to evaluate the equity performance of an award rule.

One such test, in the spirit of the familiar citizen sovereignty in Social Choice, is *Non Discrimination:* everyone wins the prize at *some* profile of messages. Another small step toward ensuring equal influence of the participants is the *No Dummy* axiom, requiring that everyone has *some* influence on the outcome at some profile of messages.

It is neither straightforward nor very hard to construct impartial award rules where every participant can win and her message matters sometime. In Section 2 we propose a family of rules adapted from [9] where each agent has exactly two messages. However in these rules individual messages have no clear interpretation in terms of the underlying opinions. Our main contribution is to propose impartial award rules allowing the participants to express disinterested opinions about the award, both in the nomination and in the direct voting format.

1.2.1 Median nomination rules

Each agent i nominates another agent j as the awardee. To sustain this interpretation, we require *Monotonicity*: if j wins at a profile of votes where i's vote is for $k, k \neq j$, then j still wins when i becomes a supporter of j. We also impose a strong symmetry property, *Anonymous ballots*: if two agents exchange their ballots, the winner does not change. This is one way to equalize the influence of each agent over the final outcome.

We promptly run into a simple impossibility result: an impartial nomination rule with anonymous ballots must be constant, i.e., always choose the same winner (Proposition 3 in Section 3). However, if we judiciously restrict the set of agents that each agent can nominate, it is possible to construct attractive impartial, monotonic nomination rules with anonymous ballots.

Here is an example. Assume an odd number of agents, and arrange them as $1, 2, \ldots, n$ by increasing seniority (or some other exogenous criterion, like wealth, location on a line, etc.). Pick an arbitrary agent $i^0, i^0 \neq 1, 2, n$. Ask every agent i less senior than i^0 to nominate someone more senior than himself, while every agent i more senior than i^0, as well as i^0, must nominate someone less senior than herself; then give the prize to the median of the reported votes. This rule is clearly impartial, monotonic, and nominations are anonymous. No one is a dummy but the rule discriminates against at least one agent. Let i^m be the median seniority level: if $i^0 = i^m$, the most senior agent cannot win; if $i^0 = i^m - k$, the $k + 1$ most senior agents cannot win; ditto for the k youngest if $i^0 = i^m + k$.

Theorem 1 in Section 3 proposes a large class of similarly restricted nomination rules, where the pattern of restrictions is derived from a *tree* of which the vertices are the agents themselves. If i^m is the median vertex of the tree, an agent i other than i^m can only nominate those agents j located on the same side of i as i^m (i.e., i is not on the path joining i^m and j); the median i^m can nominate the agents located on a suitably chosen side of i^m. The winner is the median of the profile of nominations. This *median rule* is impartial, monotonic and anonymous. If the tree is neither a path nor a star, no one is a dummy and no one is discriminated against. Furthermore, an agent i *influences* another agent j (in the sense that i's vote is sometimes critical for j's win) if and only if i can nominate j.

1.2.2 Partition voting rules

We partition N in two or more *districts* , each containing at least three agents. In each district k we choose a majority quota q_k, and in district 1 we choose a *default* agent i^*.

In the first step we run local elections in each district, barring self votes; agent i is a *local winner* if she gets at least q_k votes in her district. If there is no local winner, i^* wins the prize; otherwise, in a second step, all non-local winners vote (by means of a standard voting rule) to choose the winner among the local winners. To guarantee impartiality, a special provision is needed in district 1: there an agent i must garner at least q_1 votes without counting i^*'s vote.

We interpret the two-step process just described as a direct voting rule where each agent reports a full ranking of other agents, and this ranking determines her vote in each step: she supports her top choice within her district, then her top choice among local winners.

The grouping of candidates in districts and two-tier structure of the decision process is often natural: think of awarding a scientific prize among scholars from different fields of research. Our rules check first in each field for a strong candidate gathering on her name an absolute majority of that field, then almost everyone is involved in the selection of the overall winner among these strong field specialists.

Note that ballots are no longer anonymous. However we gain *Full Influence*: every agent influences every other agent. Additional properties satisfied are *Monotonicity* and *Unanimity*: an agent who is the top choice of all other agents wins the prize.

Theorem 2 in Section 4 makes this construction and results precise. Also discussed are several variants of the partition rules, where in the local election a *relative* quota of 2 or more is enough to win, and/or where the default winner is left outside of any district, and wins only if there is no local winner in any district.

Finally Proposition 4 proposes a class of more complex rules in the same vein, where each participant can be *pivotal* over any pair of other agents: for any triple i, j, k, at some profile of messages i's ballot can change j's win to k's win. These *cross-partition rules* actually use the nomination format. An agent is allowed to nominate any other agent (in her district or outside), and the second step is carried out using these same nominations.

1.3 Related literature

It is possible to view our model and results as a contribution to the sizeable literature[1] on strategyproof allocation of private goods in Arrow-Debreu economies, going back to Hurwicz's seminal work. Fix the set of agents, each with strictly monotonic preferences, and an impartial award rule f where M^i is the set of agent i's messages. Attach to each admissible preference R_i of agent i an arbitrary message m_i in M^i. At a preference profile R, give *all* the resources to the agent $f(m)$, where m is the profile of messages corresponding to R: an efficient and strategyproof (direct revelation) mechanism obtains. This is the point of view adopted by Kato and Ohseto in [8], [9], where they also discuss the No Discrimination (under a different name) and No Dummy properties, and construct an impartial rule with those two properties (presented in the next section).

The point of these papers is to disprove an earlier conjecture in [20]: with four or more agents, strategyproofness and efficiency do not imply, as Zhou conjectured, that one of them is never allocated anything. Observe that if the mechanism gives by design all the resources to a single winner, strategyproofness cannot be interpreted in the usual way to mean the elicitation of sincere preferences, for these preferences are entirely clear! Instead it amounts to our notion of Impartiality, a property of interest in other contexts as well.

The concept of impartial decision making appears first in [3], exploring the division of cash within a group of partners. Each partner cares selfishly about his share, not about the distribution among others of the money he does not get. Partners report their subjective opinion about the *relative* contributions of the *other* partners to the pot of cash, and one's report has no impact on one's final share. With four or more partners, there exist symmetric

[1] See in particular [4], [15], [20], [18], [19].

impartial rules with the additional property that if there is a *consensual division* compatible with all reports, it is implemented. The prize award problem is clearly the indivisible counterpart of the "division of a dollar" problem, however the rules there are not directly comparable to those in Theorems 1, 2 below.

Alon et al. [1] discuss a model of "selection from the selectors" conceptually related to the present paper. Each agent can "approve" of an arbitrary subset of other agents (abstention is allowed), and the rule must choose a fixed number of winners (e.g., a single one). Their Theorem 3.1 shows that an impartial mechanism must sometimes select a winner whom nobody approves of, while not everyone abstains. This result does not apply to our model, because it depends crucially on the possibility to abstain. Yet the same feature appears in our median nomination rules where the winner may be nominated by no one at all.

Peer ranking is a related but more difficult problem than the award of a prize, where the n agents must collectively rank themselves.[2] In [2], [12], each participant reports an ordinal ranking of all other agents, and these rankings are aggregated into a social ranking of the whole group. These papers (as well as those on peer ratings discussed in the footnote) do not examine the strategic incentives to misreporting one's opinion for the sake of improving one's ranking. Our approach applies to peer ranking provided selfish preferences bear only on one's own rank. This makes sense because managers (or sport fans, or alumni) have a vested selfish interest in the relative standing of their own company (team, school), but typically not in those companies (teams, schools) ranked higher or lower. Impartiality means that my message has no influence on my rank (but it typically influences the ranking of others). Work in progress shows that it is possible to construct impartial ranking rules, but equalizing the influence of participants over the final outcome appears more difficult than in the simpler award problem.

2 The model

We fix N, the set of agents of cardinality $|N| = n \geq 2$. Let M^i be agent i's message space, and M^N be the Cartesian product of these spaces, with

[2] Not to be confused with peer *ratings* where each agent provides numerical scores for all participants, including self, and the rule ignores self evaluations to compute final scores ([11], [13]).

generic element $m = (m_i)_{i \in N}$.

Notational conventions: $M^{N \diagdown i}$ denotes the Cartesian product of the message spaces of agents other than i, with generic element m_{-i}; the profile obtained from $m \in M^N$ when i's message becomes m'_i is denoted $(m|^i m'_i)$.

Definition 1 *An award rule is a triple (N, M^N, f), where f maps M^N into N: given a profile of messages $m = (m_i)_{i \in N}$, it chooses a winner $f(m) \in N$.*

We consider three requirements:

- **Impartiality:** for all $i \in N, m_i, m'_i \in M^i$, and all $m_{-i} \in M^{N \diagdown i}$:

$$[f(m|^i m_i) = i] \Leftrightarrow [f(m|^i m'_i) = i].$$

- **No Discrimination:** for all $i \in N$, there exists $m \in M^N$ such that $f(m) = i$.

- **No Dummy:** for all $i \in N$, there exist $m_i, m'_i \in M^i$ and $m_{-i} \in M^{N \diagdown i}$ such that $f(m|^i m_i) \neq f(m|^i m'_i)$.

Under Impartiality we do not model selfish preferences about receiving the prize or not (as explained in Subsection 1.1). The general formulation of Definition 1 does not model either the disinterested opinions of an agent about who should win if not himself. This will be done by the rules presented in the next two sections. In this section we establish some basic possibility and impossibility results about impartial rules, captured by the two axioms above.

No Dummy is related to the notion of a dummy in cooperative game theory, and No Discrimination is similar to Citizen Sovereignty in classic social choice. Both axioms are mild fairness requirements.

There is a sharp threshold for the size of the agent set which admits award rules satisfying the above three properties. For 4 or more agents, such rules exist, whereas for 2 or 3 agents Impartiality is incompatible with *each one* of No Discrimination and No Dummy. This is a consequence of the following proposition, which is essentially due to Kato and Ohseto [9] who state it in the context of pure exchange economies (see Subsection 1.3).

Proposition 1
i) If $n \leq 3$, an impartial award rule is either constant (the same agent wins no matter what), or has one agent choosing the winner among the other two.

ii) If $n \geq 4$, there are impartial award rules meeting No Discrimination and No Dummy.

Proof *i)* The case $n = 2$ is very easy, and left to the reader. Now set $N = \{1, 2, 3\}$. If the impartial rule f is not constant, up to relabeling the agents there exist m_2, m_3 and a partition of M^1 in two nonempty parts $M^1(2)$ and $M^1(3)$, such that

$$f(m_1, m_2, m_3) = j \text{ if } m_1 \in M^1(j),\ j = 2, 3.$$

Note that for all $m_1 \in M^1(2)$ and $m_3' \in M^3$ we cannot have $f(m_1, m_2, m_3') = 1$, otherwise by Impartiality $f(m_1', m_2, m_3') = 1$ as well for any $m_1' \in M^1(3)$, a contradiction because $f(m_1', m_2, m_3) = 3$. Impartiality rules out $f(m_1, m_2, m_3') = 3$ as well, so we have $f(M^1(2) \times \{m_2\} \times M^3) = 2$; one more application of the property gives $f(M^1(2) \times M^2 \times M^3) = 2$, and a symmetrical argument gives $f(M^1(3) \times M^2 \times M^3) = 3$.

ii) We describe an award rule g^n for $n \geq 4$, adapted from Kato and Ohseto [9]. Set $N = N_0 \cup \{i^*\}$, and arrange $N_0 = \{1, 2, \ldots, n-1\}$ in this order along a circle (so their numbers are taken modulo $n-1$). Let $M^i = \{0, 1\}$ for all $i \in N$, and for a given $m \in M^N$ denote $supp_0(m) = \{i \in N_0 | m_i = 1\}$. The winner is determined as follows. If $supp_0(m) = \{i\}$ then $i-1$ wins if $m_{i^*} = 0$, and $i+1$ wins if $m_{i^*} = 1$. If $supp_0(m) = \{i, i+1\}$ then i wins if $m_{i^*} = 0$, and $i+1$ wins if $m_{i^*} = 1$. In all other cases, i^* wins. It is easy to check that this rule is impartial and satisfies No Discrimination and No Dummy.■

In the rule g^n, agent i's message bears no relation to his opinion about who should win in $N \diagdown i$, therefore it has no intuitive interpretation except as a strategy in the fanciful game generated by the rule. Another problem is that g^n is heavily biased (for large n) in favor of the default agent i^*.[3]

It is interesting to note that the possibility part of Proposition 1 was established using the smallest conceivable message spaces: $|M^i| = 2$ for all i. If we restrict attention to the case $n = 4$ and binary message spaces ($|M^i| = 2$ for all i), then our three axioms are met by a canonical rule. Define g^4:

$$g^4(\cdot, 1, 0, 0) = g^4(\cdot, 0, 1, 1) = 1;\ g^4(0, \cdot, 1, 0) = g^4(1, \cdot, 0, 1) = 2$$

$$g^4(1, 0, \cdot, 0) = g^4(0, 1, \cdot, 1) = 3;\ g^4(0, 0, 0, \cdot) = g^4(1, 1, 1, \cdot) = 4$$

[3] We have constructed another class of rules based on a circular pattern and binary messages, called *circular veto rules*. These rules place the default agent on the circle and minimize his special role, while retaining the three properties stated above. But intuition is still lacking, so we do not present the rather complicated construction here.

See Figure 1. Notice that the rule gives the "same" power to all in the sense that every agent is chosen at 4 of the 16 profiles of messages, and is decisive (changes the outcome by switching message) at the other 12.

Proposition 2 *For $n = 4$ the rule g^4 is the only impartial rule (up to relabeling agents or messages) with binary message spaces, and meeting No Discrimination and No Dummy.*

Proof Let $f : \{0,1\}^4 \to \{1,2,3,4\}$ be a rule satisfying the three axioms. Impartiality implies that the profiles at which a given agent i wins come in pairs of the form $(m|^i 0), (m|^i 1)$; call such a pair an i-edge. The rule f amounts then to a partition of $\{0,1\}^4$ that consists of, say, a_i i-edges, $i = 1,\ldots,4$. We clearly have $\sum_{i=1}^4 a_i = 8$. Furthermore, each a_i must be even. To see this, consider the partition induced on the set of 8 profiles m with $m_i = 0$, which consists of a_i singletons and some number of j-edges for $j \neq i$; this determines the parity of a_i. Since No Discrimination requires $a_i > 0$ for all i, the only possibility is $a_1 = a_2 = a_3 = a_4 = 2$. Now, let us return to the partition induced on the set of 8 profiles m with, say, $m_4 = 0$, which consists of 2 singletons and some j-edges for $j = 1,2,3$. We check a few cases as to how this partition may look. If the two singletons are adjacent, i.e., differ only in coordinate j, then j must be a dummy. If the two singletons differ in exactly two coordinates, there is no way to complete the partition. Finally, if the two singletons are antipodal, there are exactly two ways to complete the partition of these 8 profiles. For agent 4 not to be a dummy, we must use one way here and the other for the 8 profiles with $m_4 = 1$. This gives g^4 up to relabeling.■

We end this section by introducing some additional properties of award rules, pertaining to the ability of agents to affect other agents' winning. Given an award rule and three distinct agents i, j, j', we say that agent i is *pivotal* for the pair j, j' if for some profile $m_{-i} \in M^{N \setminus i}$ there exist $m_i, m'_i \in M^i$ such that

$$f(m_i, m_{-i}) = j, f(m'_i, m_{-i}) = j'. \tag{1}$$

For two distinct agents i, j we say that i *influences* j if there exists j' so that i is pivotal for j, j'. These notions lead to the following properties of an award rule (N, M^N, f), taking it closer to a symmetric distribution of the decision power:

- **Full Pivots:** for all distinct $i, j, j' \in N$ agent i is pivotal for j, j'.
- **Full Influence:** for all distinct $i, j \in N$, agent i influences j.

Clearly, for $n \geq 4$ Full Pivots is stronger than Full Influence, which in turn is a common strengthening of No Discrimination and No Dummy. Among the rules defined above, the canonical rule g^4 in Proposition 2 has Full Pivots. For general $n \geq 4$, the Kato-Ohseto rules g^n have Full Influence but not Full Pivots (for example, the default agent i^* is pivotal for a pair j, j' exactly when these two are neighbors or at distance two from each other).

3 Median nomination rules

The simplest way to let i's message convey his disinterested opinion is to have him nominate one of the other agents, thus expressing support for awarding the prize to that agent, in the event that i himself does not win.

Definition 2 *A nomination rule φ is an award rule where $M^i = N \diagdown i$ for all i.*

The properties defined in the previous section for general award rules remain relevant and desirable. In addition we consider three properties specific to nomination rules:

- **Monotonicity:** for all distinct $i, j \in N$ and all $x \in M^N$

$$\varphi(x) = i \Rightarrow \varphi(x|^j i) = i.$$

- **Unanimity:** for all $i \in N$ and all $x \in M^N$

$$[x_j = i \text{ for all } j \in N \diagdown i] \Rightarrow \varphi(x) = i.$$

- **Anonymous Ballots:** for all $x, y \in M^N$

$$[\text{for all } i \ |\{j \in N \diagdown i | x_j = i\}| = |\{j \in N \diagdown i | y_j = i\}|] \Rightarrow \varphi(x) = \varphi(y).$$

These three axioms require that the nomination rule φ reflect the wishes expressed by the voters. They demand, respectively, that additional votes for the winner cannot hurt her, that a unanimous support is decisive, and that only the number of votes for each agent matters. For brevity, we refer to rules satisfying the latter property as *anonymous* rules.

We observe that Unanimity implies No Discrimination, and for impartial and monotonic nomination rules the two are equivalent.

The Anonymity requirement leads to the following impossibility result.

Proposition 3 *The only impartial and anonymous nomination rules are the constant rules: for some i, $\varphi(x) = i$ for all $x \in M^N$.*

Proof For each $x \in M^N$ define its profile of scores $\delta(x) = s = (s_1, \ldots, s_n)$ where s_i is the number of votes for i in x. Then $\delta(M^N)$ is the set

$$S = \{(s_1, \ldots, s_n) \in \{0, 1, \ldots, n-1\}^n | \sum_{i=1}^{n} s_i = n\}.$$

Anonymity says that our rule is defined directly on S, so we will abuse notation and write it as $\varphi(s)$. We show next that if $s, s' \in S$ and $s_k = s'_k$ for some agent k, then $\varphi(s) = k \Leftrightarrow \varphi(s') = k$. Since s' is obtained from s by a redistribution of the scores of agents in $N \diagdown k$, it suffices to prove this in the case when it is obtained by the transfer of one vote from i to i', for some $i, i' \in N \diagdown k$. Assume first that $\max_{j \in N \diagdown \{i,i',k\}} s_j \leq n-2$. Then there exists $x \in M^N$ with $\delta(x) = s$ and $x_k = i$ (this can be shown, for example, by using a harem version of the marriage lemma). When k switches his vote to $x'_k = i'$, we have $\delta(x') = s'$, and Impartiality gives $\varphi(s) = k \Leftrightarrow \varphi(s') = k$. It remains to handle the case where, for some $j \in N \diagdown \{i, i', k\}$, we have $s_j = s'_j = n-1$ and $s_i = s'_{i'} = 1$. Now we can get from s to s' in two steps: first transfer one vote from j to i', then transfer one vote from i to j. To each of these steps, the analysis of the previous case applies, and the conclusion follows. So for every agent, winning only depends on his own score.

Now consider the winner, say agent 1, at $s^* = (1, 1, \ldots, 1)$. Since any other agent can have any given score while 1 has the score 1, other agents can never win.■

We construct now a rich class of nomination rules with the desired properties, in which however an agent can not necessarily nominate every other agent: the message space M^i is only a subset of $N \diagdown i$.

Definition 3 *A restricted nomination rule φ is an award rule such that $\varnothing \neq M^i \subseteq N \diagdown i$ for all i.*

The definitions of Monotonicity, Unanimity and Anonymous Ballots are adapted by restricting attention to agents j who can nominate i.

Our construction of restricted nomination rules satisfying all properties above works for $n \geq 5$. To build the intuition for these rules, we give the simplest one for five agents.

Arrange the agents on the tree Γ depicted in Figure 2a. Define:

$$M^i = N \diagdown i \text{ for } i = 1, 2, 5;\ M^3 = \{4, 5\};\ M^4 = \{1, 2, 3\}$$

and set $\varphi(x)$ to be the median of $x_1, \ldots, x_5$ with respect to Γ.[4]

Monotonicity, Anonymity and No Dummy are clear. For instance, agent 3 is pivotal for 4 and 5, because if $1, 2$ nominate 5 while $4, 5$ nominate 3, 3's ballot determines the winner.

Any agent can be the winner: agent 1 wins if he is nominated by $2, 4$ and 5 (and only then); agent 2 (resp. 5) is similarly elected if and only if he is supported by $1, 4, 5$ (resp. $1, 2, 3$); there are more options resulting in agent 3 or 4 winning. In particular 3 can win when nobody nominates him.

The rule is impartial because, for instance, 3 is the median of $x_1, \ldots, x_5$, if and only if there are at most two ballots for 1, for 2, and for 4 and 5 combined. Agent 3's ballot cannot change any of these facts.

Our large family of impartial, monotonic, anonymous nomination rules generalizes this example to arbitrary trees, while retaining the idea of restricting the set of potential nominees of a given agent, then selecting the median of the profile of ballots.

If Γ is a tree on the set of nodes N (each agent appears in exactly one node, so we speak indifferently of an agent or a node), and $i \in N$ is a given node, we write $ad(i)$ for the set of nodes j adjacent to i (i.e., $ij \in \Gamma$). Thus $|ad(i)|$ is the degree of node i. Given $j \in ad(i)$, we write $\Delta(i; j)$ for the subset of nodes k such that j is weakly between i and k; the corresponding *adjacent* subtree rooted at j and away from i is written $\Gamma(i; j)$. Note that the sets $\Delta(i; j), j \in ad(i)$, partition $N \diagdown i$.

For any odd $T = 2s + 1$ and any list of nodes $k_1, k_2, \ldots, k_T$ in Γ with possible repetitions, the *median* $med\{k_1, k_2, \ldots, k_T\}$ is the node i uniquely defined by the property that for all $j \in ad(i)$, $|\{t | k_t \in \Delta(i; j)\}| \leq s$.

If n is odd, denote by i^m the *median* node of Γ, defined by

$$\text{for all } j \in ad(i^m), |\Delta(i; j)\}| \leq \frac{n-1}{2}. \qquad (2)$$

If n is even, we still define the median of Γ by property (2), keeping in mind that it may not exist, and is unique if it exists.

[4]This means that for each side of $\varphi(x)$, the total number of nominations of agents on that side is a minority (in this case, at most 2).

Consider the set $ad^{mx}(i) = \arg\max_{j \in ad(i)} |\Delta(i;j)|$ of nodes adjacent to i with the largest adjacent subtree. Irrespective of the parity of n, observe that if at node i the set $ad^{mx}(i)$ is not a singleton, then i must be the median of Γ. Therefore for every non-median node i, $ad^{mx}(i)$ is a singleton; it corresponds to the adjacent subtree containing the median of Γ if such median exists.

A *leaf* of Γ is a node of degree 1. A tree Γ is called a *path* if it has only 2 leaves (so all nodes are aligned). It is called a *star* if it has $n-1$ leaves (all adjacent to the center of the star). Note that a tree Γ is neither a path nor a star if and only if it has at least one node of degree 3 or more, and at least two nodes of degree 2 or more.

Theorem 1 *Fix N, $n \geq 5$, and a tree Γ on N that is neither a path nor a star. Set $M^i = \Delta(i; ad^{mx}(i))$ if i is not a median, and if i^m is the median, $M^{i^m} = \Delta(i^m; j^*)$ where $j^* \in ad^{mx}(i^m)$.*

i) Assume n is odd so the median i^m exists. If the degree of i^m is 3 or more, for any choice of j^* the mapping

$$M^N \ni x \to \varphi(x) = med\{x_i : i \in N\} \tag{3}$$

is an impartial, monotonic, unanimous and anonymous restricted nomination rule; each agent i influences all agents $j \in M^i$, and only them. The same is true if the degree of i^m is 2, provided we choose j^* such that the ***other*** *adjacent set $\Delta(i^m; j)$ contains at least one node of degree 3 or more.*

ii) Assume n is even. Then we can choose a fixed ballot $i_0 \in N$, and, if a median i^m exists, choose $j^* \in ad^{mx}(i^m)$ such that

$$M^N \ni x \to \varphi(x) = med\{i_0, x_i : i \in N\} \tag{4}$$

has all the above properties.

Note that No Discrimination and No Dummy are immediate consequences of the above properties.

Proof Recall that if n is odd, $n = 2p+1, p \geq 2$, then $\varphi(x) = i$ means

$$|\{k | x_k \in \Delta(i;j)\}| \leq p \text{ for all } j \in ad(i).$$

And if $n = 2p, p \geq 3$, the definition is identical except that if $\Delta(i;j)$ contains i_0, and only then, the corresponding inequality is strengthened to $|\{k | x_k \in \Delta(i;j)\}| \leq p-1$.

These definitions imply at once Monotonicity and Anonymous Ballots. To prove Impartiality, notice that $x_i \in M^i = \Delta(i;j)$ for some $j \in ad(i)$. Therefore by changing his message agent i cannot change the number $|\{k|x_k \in \Delta(i;j')\}|$ for any $j' \in ad(i)$, so he cannot affect the satisfaction of the above inequalities for i.

Let $G(i) = \{j|i \in M^j\}$. To prove Unanimity, it suffices to show that $|G(i)| \geq p+1$ for all i in the odd case, and likewise in the even case except that $|G(i)|$ may be p for $i = i_0$. Indeed, assuming these facts, if all members of $G(i)$ nominate i then agent i gets a majority of votes (including the fixed ballot when relevant), and therefore i wins. The above facts about $|G(i)|$ will be proved below.

Regarding influences, we note that if $j \notin M^i$ then it follows from the definition of the message spaces that $M^i \subseteq M^j$, and so i cannot influence j by the argument given above for Impartiality. It remains to show that if $j \in M^i$ then i influences j. This will be proved below, along with the facts about $|G(i)|$, separately for the odd and even cases.

Case 1: $n = 2p+1$, $p \geq 2$.

Claim 1 $|G(i)| \geq p+1$ *for all* i.

Proof For $i = i^m$ we have $G(i) = N \diagdown i$, so we may assume that $i \in \Delta(i^m; j)$ for some j.

The facts that $N \setminus (\Delta(i^m; j) \cup \{i^m\}) \subseteq G(i)$ and $|\Delta(i^m; j)| \leq p$ imply that $|G(i)| \geq p$. We need one more element, which we are assured to get if either $|\Delta(i^m; j)| \leq p-1$, or $j = j^*$ (in which case $i^m \in G(i)$), or $\Delta(i^m; j)$ contains a node of degree 3 or more (here we can use a leaf of Γ in $\Delta(i^m; j)$ which is not i). The provision for choosing j^* stated in the theorem guarantees that one of these cases will always hold.■

For a non-median node k, denote by k' the unique node in $ad^{mx}(k)$; in other words, k' is the neighbor of k in the direction of i^m (possibly $k' = i^m$).

Claim 2 *For any* $k \neq i^m$, *every agent in* $G(k) \cap G(k')$ *is pivotal for the pair* k, k'.

Proof Let $i \in G(k) \cap G(k')$. We construct a profile $x_{-i} \in M^{N \diagdown i}$ where i is pivotal for k, k' by nominating one or the other. First, we consider the set $G(k) \diagdown \{i\}$ which, by Claim 1, has at least p members. We take some p-element subset S, making sure that if $i^m \in G(k) \diagdown \{i\}$ then $i^m \in S$, and let all members of S nominate k in x_{-i}. We let all p members of $T = N \diagdown (S \cup \{i\})$ nominate i^m in x_{-i}, except that if $i^m \in T$ then he nominates j^* (note that, when this exception occurs, the nodes j^*, i^m, k', k appear along a path in this order). Clearly x_{-i} makes i pivotal for k, k'.■

We complete the proof that i influences j whenever $j \in M^i$ by observing that we can always find k, k' so that $j \in \{k, k'\} \subseteq M^i$.

Case 2: $n = 2p$, $p \geq 3$.

Subcase 2.1: There is a median i^m in Γ.

We arbitrarily pick some j^* in $ad^{mx}(i^m)$ and some node i_0 of Γ as the fixed ballot, with only one provision: if i^m has degree 3, with two of its adjacent sets of size $p-1$ each, then j^* is in one of them and i_0 is a leaf in the other.

Claim 3 $|G(i)| \geq p+1$ *for all* $i \neq i_0$, *and* $|G(i_0)| \geq p$.

This is proved by a straightforward adaptation of the accounting done for Claim 1, we omit the details. Furthermore, Claim 2 remains true as in the odd case. The only adaptation required in its proof concerns the sizes of the sets S and T. If $i_0 \in \Delta(k'; k)$ then we take $|S| = p-1, |T| = p$; if $i_0 \in \Delta(k; k')$ then we take $|S| = p, |T| = p-1$. We complete the proof exactly as in Case 1.

Subcase 2.2: Γ has no median.

Then there exists a (unique) pair of adjacent nodes i_1, i_2 such that $|\Delta(i_1; i_2)| = |\Delta(i_2; i_1)| = p$. These two sets partition N. We choose the fixed ballot to be a leaf in one of them, making sure that the other contains a node of degree 3 or more.

It is easy to check that Claim 3 is valid in this subcase, too. Now, for *every* node k, the node k' is well defined; note that if k is one of i_1, i_2 then k' is the other one. Furthermore, Claim 2 remains true with the same proof as adapted in the previous subcase, except that now if $k \in \Delta(i_1; i_2)$ (resp. $k \in \Delta(i_2; i_1)$) then i_1 (resp. i_2) plays the role of i^m in that proof. We complete the proof as above.■

We note that in the median rules, although agent i influences all agents in M^i, he may not be pivotal for all pairs of such agents. A simple example is the 5-node tree discussed above (Figure 2a), where the pair $1, 2$ is contained in M^4 and M^5, but neither 4 nor 5 is pivotal for $1, 2$.

Applying the above rules to a tree that is a path or a star would fall short of our goals: in a path, one of the end-nodes is discriminated against (as in the median seniority rule presented in the introduction); in a star, the center-node may nominate only one other agent, and thus is a dummy.

Note that for $n = 4$, there is no tree satisfying the premises of the theorem. For $n = 5$, the only such tree has already been discussed (see Figure 2a).

With six agents, we have four possible trees, depicted in Figures 2, 3. In Figure 2b there is no median, so $M^3 = \{4,5,6\}$ and $M^4 = \{1,2,3\}$; we must choose $i_0 = 6$. In Figure 2c the median is 4, we can choose $M^4 = \{2,3\}$ and $i_0 = 6$, or $M^4 = \{5,6\}$ and $i_0 = 2$. In Figure 3a the median is 1, we must have $M^1 = \{2,3\}$, and the choice of i_0 is arbitrary. In Figure 3b there is no median, and $M^1 = \{4,5,6\}$, $M^4 = \{1,2,3\}$; we can choose for i_0 an arbitrary leaf.

Within the class of median rules, there is a tradeoff between the goal of providing to *each* agent the opportunity to nominate at least a substantial fraction of other agents, and that of maximizing the *total* number of options across all participants. Formally, this amounts to maximize either $\min_N |M^i|$ or $\sum_N |M^i|$.

At one end consider a tree close to a star: it has a center 1, a branch $1-2-3$, and all other agents connected only to agent 1 (Figure 4a). Agent 1 is the median. We have $M^i = N \diagdown i$ for $3 \leq i \leq n$, $M^1 = \{2,3\}$, $M^2 = N \backslash \{2,3\}$. Here $\sum_N |M^i| = n^2 - 2n + 2$ is within $\frac{1}{n}$-th of the optimal $n(n-1)$. On the other hand agent 1 has very little influence (but he wins often): $\min_N |M^i| = 2$ is only about $\frac{4}{n}$-th of the optimum $\frac{n}{2}$.

At the other end, consider the "almost linear" tree of Figure 4b: 2 and 3 are leaves adjacent to 1, after which the other agents are aligned. Here for $n = 2p+1$ odd, Theorem 1 suggests the message sets $M^i = N \diagdown i$ for $i = 2,3$, $M^1 = \{4,\ldots,n\}$, $M^i = \{i+1,\ldots,n\}$ for $4 \leq i \leq p+1$, $M^i = \{1,\ldots,i-1\}$ for $p+2 \leq i \leq n$. For n even, the situation is similar. Here $\min_N |M^i|$ is about 50% of the optimum unconstrained choice, but $\sum_N |M^i|$ is about 25% short of the optimum.

One obvious defect of median rules is the asymmetric treatment of participants along the tree. Typically, agents in the periphery can nominate more of their peers but win less often compared to the centrally placed agents. Another drawback is that an agent cannot always nominate his preferred winner. This critique is accentuated by the observation that a node of degree 3 or more may win even without being nominated by anyone, as was pointed out in the discussion of the 5-node tree in Figure 2a.[5] In such a case, for all we know from the agents' nominations, the winner may be the worst agent in the opinion of everyone else!

[5] In the even case, this may occur also with a node of degree 2 that serves as the fixed ballot. But we can avoid this undesirable effect of the fixed ballot by always placing it at a leaf.

The above critique is daunting if the agents are placed at the nodes of a tree in an arbitrary fashion, just in order to apply the median rule. But in some contexts the placement on the tree may represent some objective characteristics of the agents. As an example, suppose that a committee comprised of representatives of several cities has to select one of them to receive a certain cultural facility free of charge. If our cities are located on a tree, the median rule can be defended as a sensible aggregation method, even if it sometimes awards the facility to a city that no one nominated. Moreover, if the tree has an objective meaning such as in the above example, it is reasonable to assume that the agents' disinterested opinions are single-peaked on the tree.[6] When this is the case, a node of degree 3 or more cannot be anyone's worst outcome, and so the selection of a unanimously worst agent is certainly precluded.

We note finally that there is a wider class of generalized median rules based on a tree Γ. For an agent i we can let M^i be any of the adjacent subtrees, not necessarily the largest. Also, we can use several fixed ballots, and take the median of all ballots, fixed and live (provided their total number is odd). These rules will always be impartial, monotonic and anonymous; the other properties in Theorem 1 may or may not hold, depending on our choices. It may be possible to characterize this class of rules and/or the subclass featured in the theorem among all restricted nomination rules by suitable combinations of axioms. We leave this for further study.

Another open question, prompted by the above critique of median rules, is this: Does there exist an impartial (unrestricted) nomination rule satisfying our basic axioms, and guaranteeing that the winner is one of the nominees?

4 Partition voting rules

In standard voting rules, where the sets of voters and candidates are disjoint, every voter reports a ranking of the candidates. The natural analog in our context is for every agent i to report a linear ordering of $N \diagdown i$. Let $L(N \diagdown i)$ denote the set of all such orderings.

[6]Under the single-peaked assumption, when i nominates j this means not only that j is his top choice, but also that for any two distinct $k, l \in N \diagdown i$ such that k is between j and l in the tree, k is ranked above l in i's opinion. See [5] for a treatment of single-peaked preferences on a tree.

Definition 4 *A direct voting rule F is an award rule where $M^i = L(N \diagdown i)$ for all i.*

The properties defined in Section 2 for general award rules apply of course to direct voting rules. In order to adapt some of the properties of nomination rules to direct voting rules, we introduce some notation and terminology. A profile of messages is written $\succ$, with $k \succ_j l$ meaning that k is ranked above l by agent j. For two orderings $\succ_j, \succ'_j$ in M^j, we say that $\succ'_j$ is obtained from $\succ_j$ by lifting agent i, if i is ranked higher in $\succ'_j$ than in $\succ_j$, and the restrictions of $\succ_j, \succ'_j$ to $N \diagdown \{i, j\}$ coincide.

- **Monotonicity:** for all distinct $i, j \in N$ and all $\succ \in M^N$

$$[F(\succ) = i \text{ and } \succ'_j \text{ is obtained from } \succ_j \text{ by lifting } i] \Rightarrow F(\succ |^j \succ'_j) = i.$$

- **Unanimity:** for all $i \in N$ and all $\succ \in M^N$

$$[i \succ_j k \text{ for all distinct } j, k \in N \diagdown i] \Rightarrow F(\succ) = i.$$

These two axioms have the same intuitive appeal as their counterparts in the nomination format. Here, too, Unanimity implies No Discrimination, but the converse does not hold even for impartial and monotonic direct voting rules. Note that there is no natural analog of Anonymous Ballots for direct voting rules, because different agents have disjoint message spaces.

We proceed to construct a large class of direct voting rules with the desired properties. The building blocks of the partition voting rules are the following q-*absolute* quota rules $\mathcal{I}^a(N, q)$ and q-*relative* quota rules $\mathcal{I}^r(N, q)$. Let $M^i = N \diagdown i$ as in a nomination rule, and for a given profile of messages, let $s_i(X)$ be the number of votes that i gets from voters in X. We define a q-absolute winner and q-relative winner as follows:

$$i \text{ wins in } \mathcal{I}^a(N, q) \overset{def}{\Leftrightarrow} s_i(N \diagdown i) \geq q, \tag{5}$$

$$i \text{ wins in } \mathcal{I}^r(N, q) \overset{def}{\Leftrightarrow} s_i(N \diagdown i) \geq s_j(N \diagdown \{i, j\}) + q \text{ for all } j \neq i. \tag{6}$$

We let the reader check that there is at most one q-absolute winner if $\frac{n}{2} < q \leq n - 1$, and at most one q-relative winner if $2 \leq q \leq n - 1$. On the other hand there may be no (absolute or relative) winner. Thus $\mathcal{I}^\varepsilon(N, q), \varepsilon = a, r$ are not nomination rules in the sense of Definition 2. We still call them quota rules, keeping in mind they can fail to identify a "winner".

Note that q-relative quota rules with $q \geq n-3$ are just absolute quota rules in disguise: $\mathcal{I}^r(N, n-3) = \mathcal{I}^a(N, n-2)$ and $\mathcal{I}^r(N, n-2) = \mathcal{I}^r(N, n-1) = \mathcal{I}^a(N, n-1)$. Taking this into account, as well as the above conditions for uniqueness, we will confine attention to absolute quotas in the range $\frac{n}{2} < q \leq n-1$ and relative quotas in the range $2 \leq q \leq n-4$. In particular, for $n = 3, 4, 5$ all quota rules are absolute.

We describe first our partition rules as two-step voting processes.

Definition 5 *Partition rules* $\mathcal{P}[(N_k, \varepsilon_k, q_k)_{k=1}^K, i^*, (F_X)]$:
Assume $n \geq 7$ and fix a partition $N = \cup_{k=1}^K N_k \cup \{i^*\}$ with $K \geq 2$, such that $|N_k| = n_k \geq 3$ for all k. We refer to $\overline{N}_1 = N_1 \cup \{i^*\}, N_2, \ldots, N_K$ as districts $1, 2, \ldots, K$, respectively, and to i^*, a special member of district 1, as the default agent. Choose for each k a type $\varepsilon_k \in \{a, r\}$ and a quota q_k so that either $[\varepsilon_k = a$ and $\frac{n_k}{2} < q_k \leq n_k - 1]$ or $[\varepsilon_k = r$ and $2 \leq q_k \leq n_k - 4]$. For every subset X of N containing at most one member of each district, such that $|X| \geq 2$, choose a standard voting rule F_X with candidate set X and voter set $N \diagdown X$.[7]

Step 1: Run $\mathcal{I}^{\varepsilon_k}(N_k, q_k)$ in each district k, $2 \leq k \leq K$; call $i \in N_k$ a local winner if she wins. In district 1, run a variant of the corresponding rule, in which agent i^* participates as a candidate but not as a voter; namely, call $i \in \overline{N}_1$ a local winner when

$$\text{if } \varepsilon_1 = a : \; s_i(N_1 \diagdown \{i\}) \geq q_1, \tag{7}$$

$$\text{if } \varepsilon_1 = r : \; s_i(N_1 \diagdown \{i\}) \geq s_j(N_1 \diagdown \{i, j\}) + q_1 \text{ for all } j \in \overline{N}_1 \diagdown \{i\}. \tag{8}$$

Let X be the set of local winners in the various districts. If $X = \varnothing$ then i^* wins. If $X = \{i\}$ then i wins. If $|X| \geq 2$ we go to

Step 2: Using the standard voting rule F_X the members of $N \diagdown X$ determine the winner within X.

In the direct voting formulation of the above partition rule, every agent i reports a linear ordering $\succ_i$ of $N \diagdown i$. We adapt Definition 5 in the obvious way: if $i \in N_k$, his top ranked agent in $N_k \diagdown \{i\}$ is his vote in Step 1. In Step 2, each $i \in N \diagdown X$ uses the restriction of $\succ_i$ to X.

[7]By this we mean that F_X maps profiles of linear orderings of X by the members of $N \diagdown X$, into X. We assume that these rules are monotonic, unanimous, and no voter is a dummy. Typical examples of such rules include plurality, Borda, Copeland and many others, where ties are broken by a fixed priority ordering of X.

Theorem 2 *Assume $n \geq 7$ and consider a partition voting rule of the form $\mathcal{P}[(N_k, \varepsilon_k, q_k)_{k=1}^K, i^*, (F_X)]$. This rule is impartial, unanimous and has Full Influence. The rule is monotonic if and only if it uses an absolute quota in district 1 (i.e., $\varepsilon_1 = a$).*

Proof We start with Impartiality. First recall there is at most one local winner in each district. This was said earlier for any rule $\mathcal{I}^{\varepsilon_k}(N_k, q_k)$; it works similarly for the case of district 1 where the rule is slightly different. Suppose agent $i, i \neq i^*$, wins the prize. By changing his report he cannot affect the fact that he is the unique local winner in his district, or alter the set of local winners in other districts, and his victory in the vote among local winners, if such arises, is also independent of his own message. Suppose next i^* wins. If i^* is the local winner in district 1 the above argument applies. If i^* wins by default, i.e., there is no local winner anywhere, he cannot move to create a local winner in district 1 because properties (7),(8) do not depend upon his message.

Checking Unanimity is easy: if all agents but i report i as their top choice then i is a local winner in his district and is the top choice of every voter in Step 2, if any; hence he wins the prize.

In the first step an agent $i \neq i^*$ can influence anyone in the same district. But the second step guarantees Full Influence. Indeed, given a pair $i \neq j$, we can arrange the vote in Step 1 so that there are exactly two local winners: j and some $j' \in N \diagdown \{i, j\}$. Then in Step 2 i is a non-dummy voter, so he can change the winner from j to j'.

Regarding Monotonicity, suppose first that agent $i \neq i^*$ wins at a profile of messages. Notice that if i wins in $\mathcal{I}^{\varepsilon_k}(N_k, q_k), k \geq 2$, she still does after moving up in some agent's ordering: inequalities (5),(6) are preserved. Furthermore, the change has no effect on local elections in other districts so the set X of local winners remains the same and i's election among them is similarly preserved. The same argument applies to $i \in \overline{N}_1$ who is a local winner in the sense of (7),(8). But the case of winning by default is more subtle. Assume that i^* wins because there is no local winner anywhere. When i^* is lifted, the only agent who may become an absolute local winner is i^* herself; whether or not this happens i^* remains the winner. This establishes Monotonicity when $\varepsilon_1 = a$. By contrast, an agent $i \in N_1$ may become a relative local winner as a result of i^*'s lifting, violating Monotonicity when $\varepsilon_1 = r$. Indeed, suppose that the initial scores are $q_1 + 1$ for i, 2 for some $j \in N_1 \diagdown \{i\}$ (none of whose votes is cast by i), and 1 each for $n_1 - q_1 - 3$ other agents in

N_1; all other agents in district 1, including i^*, get no votes (our assumption that $q_1 \leq n_1 - 4$ makes this construction possible). Then no agent satisfies (8), but when one of j's supporters switches to i^*, agent i becomes a relative local winner.■

Notice that our partition rules fail Full Pivots because an agent cannot be pivotal for a pair j, j' in a different district than his own. When i, j, j' are in the same N_k, agent i is pivotal for j, j' if and only if we use an absolute quota of $\frac{n_k+1}{2}$ or a relative quota of 2 in that district. In district 1 we also have i pivotal for j, i^* for every $i \neq j$ in N_1. Finally, every agent i is pivotal for j, j' when these two are in different districts. Thus, if the number of agents increases while the district sizes remain bounded, every agent is pivotal for all but a negligible fraction of pairs of agents.

We criticized the median rules because the winner may not be a nominee, and may even be the worst outcome for everyone else. The partition voting rules have the same problem, because the default agent can win without gathering any vote in the first step of the voting process. Thus as direct voting rules, they fail the following natural property:

- **Negative Unanimity:** for all $i \in N$ and all $\succ \in M^N$

$$[k \succ_j i \text{ for all distinct } j, k \in N \diagdown i] \Rightarrow F(\succ) \neq i.$$

One simple remedy is to modify partition voting rules by choosing two default agents i^*, j^*, in two different districts, and when there is no local winner anywhere, we run a standard voting rule in $N \diagdown \{i^*, j^*\}$ to choose one of the two (we omit the details). But this is not satisfactory because now we may have to choose one of the two worst outcomes in the eyes of $N \diagdown \{i^*, j^*\}$, etc..

Another way to downplay this critique of partition rules arises if we can assert that the default option in their definition occurs very rarely (and hence so do the violations of Negative Unanimity); in other words, if we can guarantee a high probability that a local winner exists. Under the familiar *impartial culture* assumption each agent reports any one of his messages with equal probability, independently across agents. Then if the number of agents increases but the size of each district remains bounded, the probability of at least one local winner under any relative or absolute quota rule goes to one: this is because the probability of no local winner in a given district is bounded away from one, the number of districts increases, and the existence of a local winner is independent across districts.[8]

A simple variant of the partition rules sets the default agent i^* outside the partition in districts. Then we choose a quota rule $\mathcal{I}^{\varepsilon_k}(N_k, q_k)$ in each district, with no special provision for any district. In the first step an agent i in N_k casts a vote[9] either for someone in $N_k \diagdown \{i\}$ or for i^*. If there is no local winner in any district i^* wins, otherwise the non local winners (including i^*) vote to award the prize to one of the local winners.

This voting rule is impartial, unanimous and has Full Influence. It is monotonic if and only if absolute quotas are used in all districts. Its obvious defect is that the role of the default agent is much more important than in the rules of Definition 5. Yet if we are allowed to adjust the district structure and the choice of quota rules in large populations, we may make i^*'s winning chances roughly equal to everyone else's.

4.1 Partition rules in the nomination format

The partition rules may be adapted to become (unrestricted) nomination rules in the sense of Definition 2. We use a partition into districts with quota rules as above, but now an agent may nominate *any* other agent, not necessarily in his district. Step 1 takes place as before, counting only intra-district votes. In Step 2, if needed, the tie among local winners is broken by plurality, using the original nominations as votes (but counting only those going from non local winners to local winners). The properties of partition rules listed in Theorem 2 also hold in this nomination format.[10] The disadvantage of framing partition rules as nomination rules is that votes

[8]If, on the other hand, we keep the number of districts bounded as the population increases, then the probability of at least one local winner *cannot* go to one, regardless of the choice of district sizes and quota rules. Indeed, in a large district we maximize decisiveness (i.e., the probability of there being a local winner in that district) by using a relative quota of 2. But even under this rule, decisiveness goes to zero as the size of the district goes to infinity (this non-obvious and perhaps non-intuitive fact may be deduced from results on random allocations, see [10]).

[9]As usual, in the direct voting format this means that we identify i's top choice in $(N_k \diagdown \{i\}) \cup \{i^*\}$.

[10]For Full Influence, we have to avoid a situation where a local winner in a large district is assured to win any tie break on the strength of his local votes. There are several ways to do this: keep the district sizes balanced, or use low quotas in large districts, or ignore local votes in the tie break. For Monotonicity we need to use absolute quotas in all districts.

are often "wasted", because they cannot be transferred from a local favorite to a global one as in the direct voting formulation.

Nevertheless, it is interesting that many of our desirable properties are compatible even in the simple framework of nomination rules. In this framework we now have two escape routes from the impossibility of impartial and anonymous nomination rules (Proposition 3). Median nomination rules obtain when we relax the fullness of nomination opportunities, but still allow most agents to nominate most of their peers. Partition nomination rules emerge when we relax the anonymity of ballots, but still treat ballots anonymously within each district in the local elections.

The only desirable property on our list that is not achieved by any of the above families of rules is Full Pivots. We present now a more complicated variant of partition rules, also in the nomination format, which meets Full Pivots in addition to all other properties in Theorem 2.

Definition 6 *Cross-partition rules* $\mathcal{CP}[(N_k, \Pi_k, \Lambda_k)_{k=1}^K, i^*, \succ_p]$:
Assume $n \geq 13$, and partition the agents into $K \geq 3$ districts N_k of nearly equal size[11] (at least 4 each), and the singleton $\{i^*\}$. For each $k = 1, \ldots, K$, let Π_k be a partition of N_k into two components of nearly equal size, and let Λ_k be another such partition, so that the two are orthogonal (in the sense that each component of one intersects each component of the other). Let $\succ_p$ be a fixed ordering of N.

Step 1: Every agent i nominates another agent in $N \diagdown i$. Call agent $i \in N_k$ an *outer hero* if he gets the votes of an entire component of Π_l for every l other than $k-1$ and k, and of an entire component of Λ_{k-1} (with subscripts taken modulo K). Call agent $i \in N_k$ an *inner hero* if he gets the votes of all members of $N_k \diagdown \{i\}$.

Define the set of *eligible agents* as follows. If there exists at least one outer hero, it is the set of outer heroes. If there is no outer hero, it is the set of inner heroes. If there is no eligible agent, i^* wins. If there is exactly one, he wins. Otherwise, we go to

Step 2: We break the tie between the eligible agents by two criteria. First, we count the votes each eligible agent got in Step 1 from non-eligible agents. Then we break any remaining tie for the highest count according to the priority ordering $\succ_p$.

Proposition 4 *Assume* $n \geq 13$. *The cross-partition rules of the form*

[11] By this we mean that the sizes differ by at most one.

$\mathcal{CP}[(N_k, \Pi_k, \Lambda_k)_{k=1}^K, i^*, \succ_p]$ *are impartial, monotonic, and satisfy Full Pivots.*

Proof We note first some useful facts about heroes. Suppose there is an outer hero in district k. Then there can be no heroes outside N_k: outer heroes are ruled out by the orthogonality of the Π_l, Λ_l, and inner heroes are impossible since an outer hero needs the votes of a full component (hence at least two votes) from every other district. On the other hand, in district k itself there may be one other outer hero and/or a local hero. But, by definition, such a local hero is not considered eligible.

Impartiality for i^* is obvious: if she wins, this is determined in Step 1, in which her vote plays no role. Next, note that for agent i in N_k, being eligible does not depend upon her own vote. This is clear if she is an outer hero, and if she is an inner hero it is because, regardless of her vote, there can be no outer hero outside N_k. Moreover, an eligible agent cannot change the set of eligible agents: this is clear for an inner hero, and for an outer hero it follows because there is no outer hero outside her district. Together these two facts establish Impartiality.

For Monotonicity, suppose initially i wins, then j changes his vote to i. If $i = i^*$, the eligible set remains empty after the change and we are done. Assume $i \neq i^*$ from now on. Whether i was initially an inner or an outer hero (or both), this remains true after the change and no other agent becomes a hero. If i was an outer hero, he was either the only eligible agent or one of two. Whether or not the change disqualifies the other one (if any), he remains the winner. If i was an inner hero, there was no outer hero to start with; the change could make i an outer hero as well, in which case i wins at once because he is the sole outer hero. Alternatively, the change could disqualify an inner hero in j's district, say j'. In this case, consider the effect of the change on Step 2: agent j' turns from candidate to voter, and his vote may go to an opponent of i, but i now gets the vote of j which guarantees that he still wins. Finally, if the eligible set does not change, agent i remains the winner.

To show Full Pivots, consider a triple i, j, j'. Assume first that one of j, j' is i^*, say $j = i^*$. Then i is pivotal for i^*, j' at a profile where there are no heroes other than j', and the status of j' as a hero (inner if i, j' are in the same district, outer otherwise) depends on the vote of i. Next assume that j, j' are in the same district N_k. Then we can have both of them outer heroes, and make sure that they get nearly the same total number of votes from agents in the other districts (this relies on the assumption that the components of Π_l, Λ_l are of nearly equal size). Now, if i is in a district other than N_k, he can be pivotal by disqualifying j or j', and otherwise he can be pivotal in Step 2. Finally, if j, j' are in different districts, we can make them the only two inner heroes. Now, if i is in one of their districts he can be pivotal by disqualifying j or j', and otherwise he can be pivotal in Step 2.[12] ■

Note that, as in partition rules, one can replace the local unanimity requirement for an inner hero by an absolute quota q that is slightly higher than half the size of the district (enough to prevent the coexistence of an outer hero and an inner hero in distinct districts). But a more decisive relative quota won't work, because it cannot prevent that coexistence.

With the inevitable exception of Anonymous Ballots, the cross-partition rules satisfy all desirable properties defined in this paper (those not explicitly mentioned in Proposition 4 are consequences of the listed properties). Yet these rules seem too complex to qualify as practical award rules.

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[12] To justify the last statement, we have to rule out a situation where j, say, is assured to win the tie break by virtue of his local votes. The near equality of district sizes certainly suffices to achieve this. But note that we made this assumption to keep the presentation simple; a much weaker condition forbidding a district to hold a majority of all agents would suffice, as would the alternative ways mentioned in footnote 10.

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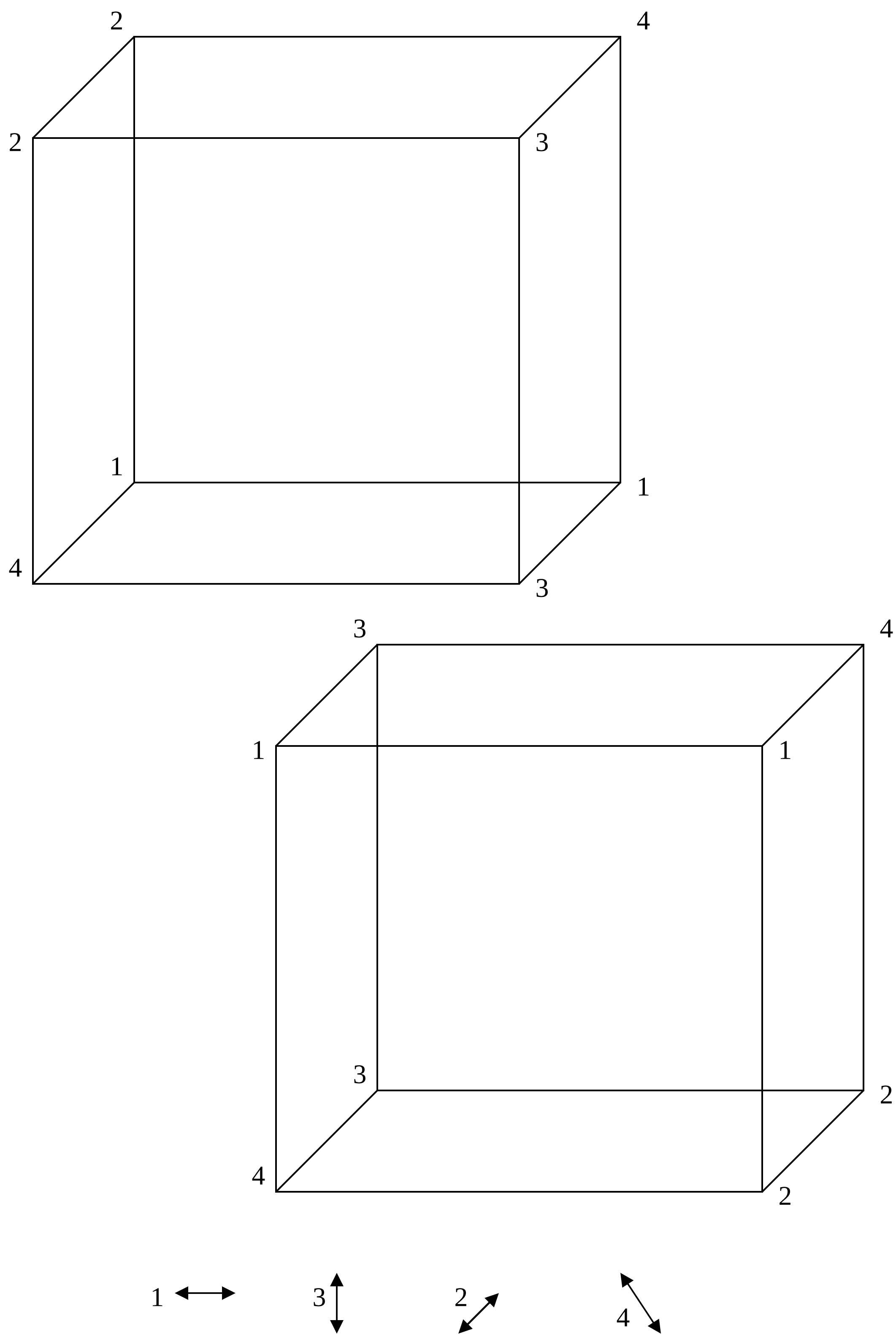
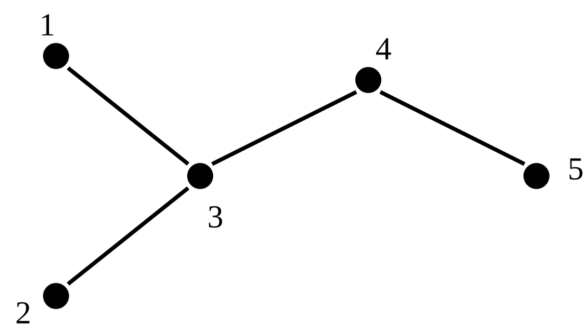


Figure 1

Figure 2a



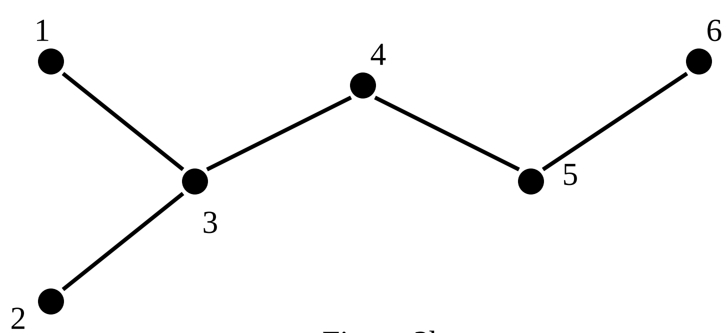


Figure 2b

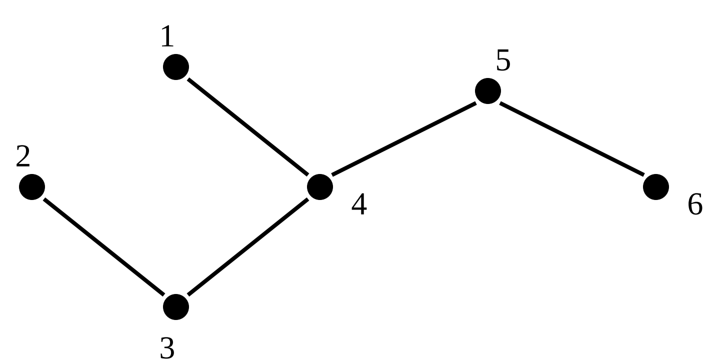


Figure 2c

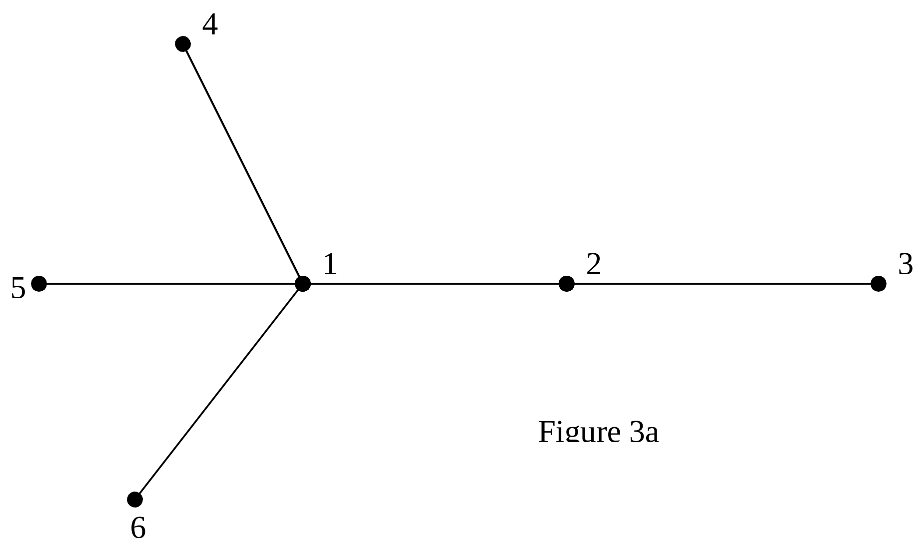


Figure 3a

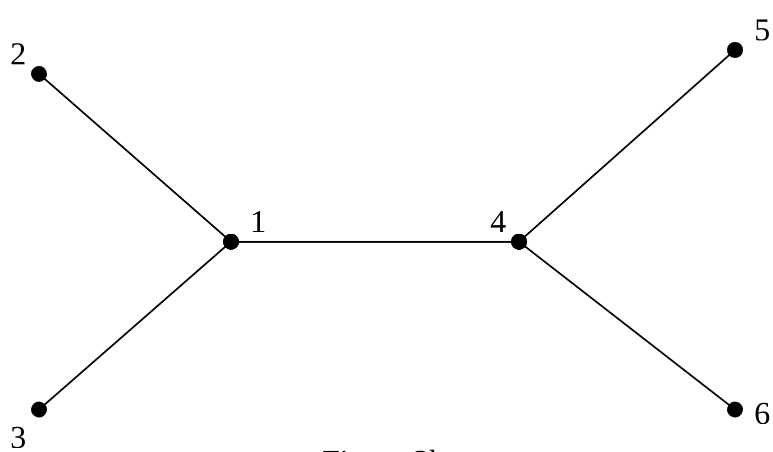


Figure 3b

Figure 4a

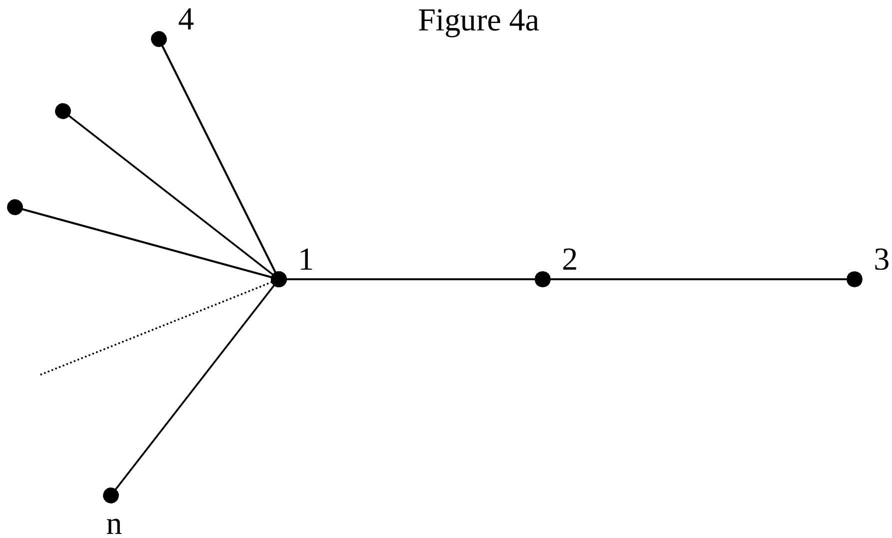


Figure 4b