Choosing Collectively Optimal Sets of Alternatives Based on the Condorcet Criterion

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Motivation

Holding weekly research seminars in a department.

A	B	C	D	E
Tue.	Tue.	Thu.	Thu.	Tue.
Mon.	Wed.	Wed.	Fri.	Mon.
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Related Work

- Proportional Representation
- Condorcet Committees

Notations

- n voters
- a set of p candidates X
- preference profile $P = \langle \succ_1, \ldots, \succ_n \rangle$

θ-Winning Sets

Definition

For $Y \subseteq X$, $z \in X \setminus Y$, and $0 < \theta \leq 1$
Y θ-covers z if

$$\#\{i \in N \mid \exists y \in Y \text{ such that } y \succ_i z\} > \theta n.$$

(A proportion at least θ of the voters prefers *some* alternative of Y to z).

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Y is a θ-winning set if $\forall z \in X \setminus Y$, Y θ-covers z.

Given P, θ, and k

$$D(P, \theta, k) = \{Y, Y \text{ is a } \theta\text{-winning set}, |Y| \leq k\}$$

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We may

- fix θ and minimize k
- fix k and maximize θ

Example

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a	b	d
c	c	a
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$D(P_1, \frac{1}{2}, 1) = \emptyset$
$D(P_1, \frac{1}{2}, 2) = \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, d\}, \{c, d\}\}$

Particular Cases

- $\theta = \frac{1}{2}, k = 1$
  If P has a Condorcet winner c
  then $D(P, \frac{1}{2}, 1) = \{\{c\}\}$
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  $\{x\}$ is a θ^*-winning set iff x is a winner for the maximin voting rule
- $\forall Y \in D(P, 1, k)$
  Y contains every candidate ranked first by some voter

CWS: not a tournament solution

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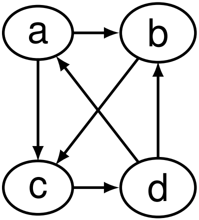
a	b	d
c	c	a
d	d	b
b	a	c



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$\{a, b\}$ is a CWS

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Condorcet Dimension

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Condorcet dimension of a profile P:
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- If P has a Condorcet winner then $\dim_C(P) = 1$.
- We have seen that $\dim_C(P_1) = 2$

P_1

$\succ_1$	$\succ_2$	$\succ_3$
a	b	d
c	c	a
d	d	b
b	a	c

A profile of dimension 3

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
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10	6	7	8	9	15	11	12	13	14	5	1	2	3	4
11	12	13	14	15	1	2	3	4	5	6	7	8	9	10
12	13	14	15	11	2	3	4	5	1	7	8	9	10	6
13	14	15	11	12	3	4	5	1	2	8	9	10	6	7
14	15	11	12	13	4	5	1	2	3	9	10	6	7	8
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Not CWS:

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Not CWS:

- $\{1, 2\} \prec 5$

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Not CWS:

- $\{1, 2\} \prec 5$
- $\{1, 3\} \prec 11$

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- $\{1, 2\} \prec 5$
- $\{1, 3\} \prec 11$
- $\{1, 6\} \prec 5$
- etc.

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CWS:

- $\{1, 6, 11\}$
- $\{1, 3, 6\}$
- etc.

High dimension profiles?

- finding P such that $\dim_C(P) = 1$ or $\dim_C(P) = 2$ is trivial.
- $\dim_C(P) = 3$ needs more work(previous slide).
- *we could not find a profile of dimension 4 or more*

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Question

Does there exist a profile of dimension k for any k?

Probabilistic approach

- n voters
- $m = |X|$ candidates
- generate profiles randomly with a uniform distribution (impartial culture)

Proposition

$\{a, b\} \subseteq X$ is CWS with probability $\geq 1 - me^{-n/24}$

Hint: with probability $\frac{2}{3}$ in any given vote, either a or b is ranked above c, therefore the expected number of votes where a or b beats c is $\frac{2n}{3}$. By Chernoff bound, the probability that a or b is ranked above c in at least $\frac{n}{2}$ votes is at most $e^{-n/24}$. Therefore the probability that $\{a, b\}$ is not a CWS is at most $me^{-n/24}$.

Experimental results (1)

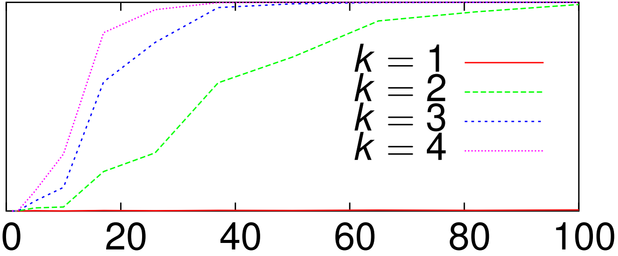

Figure: probability that a fixed set of size k is a Condorcet winning set as a function of n, for a 30-candidate election

Important remark: dominating sets are CWS

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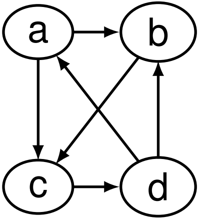
$\{a, b\}, \{a, c\}, \{a, d\}, \{b, d\}, \{c, d\}$



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An upper bound on the dimension

Proposition

For any profile P with n voters (n odd) we have $\dim_C(P) \leq \lceil \log_2 m \rceil$.

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Proof.

- n odd $\Rightarrow$ the majority graph is a tournament
- dominating sets of the majority graph are CWS.
- Megiddo and Vishkin (1988): a tournament has a dominating set of size $\lceil \log_2 m \rceil$.

$\square$

Complexity

CONDORCET DIMENSION: compute $\dim_C(P)$.

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Yes

- enumerate all subsets of size $\leq K$
- $\rightarrow \mathrm{poly}(n, m)m^K$
- polynomial ($\in$ P)

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No

- enumerate all subsets of size $\leq \lceil \log_2 m \rceil$
- $\rightarrow \mathrm{poly}(n, m)m^{\log m}$
- quasi-polynomial ($\in$ QP)

θ-Winning Sets for $\theta \neq \frac{1}{2}$

$\theta = \frac{1}{2}$, $k \geq 2$,

- every pair is with high probability a CWS.
  $\Rightarrow$ fixing $\theta = \frac{1}{2}$ and minimizing k is not interesting.
- fix k and use $\theta = \frac{k}{k+1}$

Experimental Results (2)

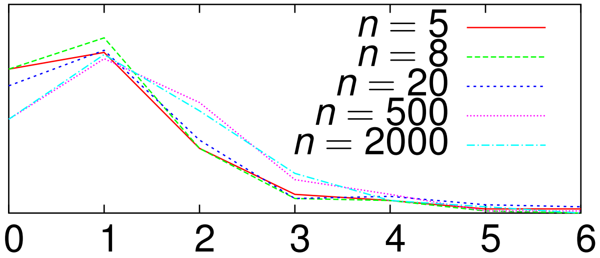

Figure: Empirical distribution of the number of $\frac{2}{3}$-winning sets of size 2 for 20 candidates

Experimental Results (3)

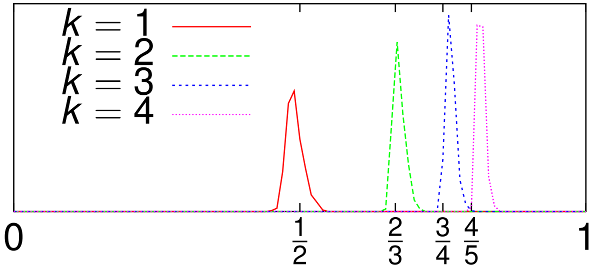

Figure: Empirical distribution of $\theta(k)$ for $m = 30$ and $n = 100$

Related Work (1)

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Proportional representation

Chamberlin and Courant (1983):
choose the highest-ranking alternative from the given set in each vote, but use the Borda score as a basis.

A set Y receives $\max_{y \in Y} s_B(y; i)$ points from a voter i and the winning committee of size k is the k-element set of candidates with the highest score.

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Procaccia *et al.* (2008): computing a winning committee of size k is **NP**-hard.

Lu and Boutilier (2011): trade-off between committee size and quality of representation; computation of optimal sets.

Related Work (2)

Condorcet committees: “conjunctive” sets

Gehrlein (1985): $Y \subseteq X$ is a Condorcet committee if for *every* alternative y in Y and every alternative x in $X \setminus Y$, a majority of voters prefers y to x.

$\neq$ *CWS*: *disjunctive interpretation of sets*

Related Work (2)

Condorcet committees, continued

Ratliff (2003): generalizes Dodgson and Kemeny to sets of alternatives.

Fishburn (1981): defines preference relations on *sets* of alternatives and looks for a subset that beats any subset in a pairwise election.

Kaymak and Sanver (2003): under which conditions on the extension function can a Condorcet committee in the sense of Fishburn be derived from preferences over single alternatives?

Can Condorcet committees be also CWSs?
Depends on the extension function.

For "standard" extension functions: no.

Conclusion

Reconciliating both approaches

- disjunctive interpretation (as in proportional representation)
- satisfies the Condorcet criterion (like Condorcet committees)

Question

Are there profiles of Condorcet dimension 4 or more?