Some Aggregative Theory of Dichotomous Evaluations (PROVISIONAL AND INCOMPLETE VERSION)

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Abstract

Dichotomous preferences have been investigated in voting theory, usually by connecting them with the approval voting method, to which they are naturally related. The paper develops their aggregative theory in the not so well explored direction of collective preference functions. It shows that Arrow's impossibility theorem holds in the twice dichotomous case (both the individual and collective preferences are dichotomous). Then, it derives a possibility result in the simply dichotomous case (only individual preferences are) by characterizing the approval voting, or equivalently plurality voting, collective preference function. This generalizes Arrow's and May's results for majority voting on two alternatives. Finally, the paper makes a step towards dichotomous evaluations of the judgmental sort, thus drawing a connection with the recently developed theory of judgment aggregation.

1 Introduction

Dichotomous preference orderings are those orderings which have at most two indifference classes. They have specific, essentially favourable aggregative properties that have long been noted in voting theory. They bear a natural connection with an attractive rule, *approval voting*, whereby individuals cast votes for as many candidates as they wish, giving no more than one vote to each of them, and those candidates with the greatest vote total are elected. A salient consequence of dichotomous preferences, which Inada (1969) first highlighted, is that the set of Condorcet winners is never empty, and as Brams and Fishburn (1977) further showed, approval voting on this preference domain exactly selects the set of Condorcet winners, which the plurality voting rule does not always do. Another fact of significance is that this peculiar domain makes it easier for some voting rules to be non-manipulable. This observation again favours approval voting, which Brams and Fishburn (1977) demonstrated to be less open to voters' strategic manipulations than any other non-ranking voting system, including the plurality rule. Much of what can be said for approval voting under dichotomous preferences extends to non-dichotomous ones when the election has two stages with a run-off (Brams and Fishburn, 1981). Since these early findings, more has been said on the voting theory of dichotomous preferences; see the review in Brams and Fishburn (2002) and the recent explorations by Vorsatz (2007, 2008).

The present paper approaches the aggregation of dichotomous preference from the angle of social welfare functions, or *collective preference functions*, as we will rather say in order to avoid the welfaristic connotations that the former expression irrelevantly brings out. This side of the topic has been somewhat neglected compared with the voting side just explained, perhaps on the view that everything that can be expressed in the language of collective choice functions, which is appropriate to define voting rules, can equivalently be formulated in the language of collective preference functions, and vice-versa. No such straightforward equivalence holds, as our results actually demonstrate.

We distinguish between the *simply dichotomous* case, in which only individual preferences are dichotomous, and the *twice dichotomous* case, in which collective preferences also are. The latter is plagued with an aggregative impossibility that is dramatized in Theorem 1, a version of Arrow's (1963) for this unusual context. Surprisingly, dictatorship follows from the Independence of Irrelevant Alternatives and Weak Pareto conditions despite the fact that no cyclical profile of individual preferences is available for the proof. Another unexpected feature is that the twice dichotomous case does not reduce to a simply dichotomous situation in which a collective choice function would replace the collective preference function. The markedly different consequences - impossibility on the one side, possibility on the other - illustrates the discrepancy that we just claimed between the two frameworks.

By contrast, the simply dichomotous case permits elegant possibilities, including approval voting, which Proposition 1 characterizes in terms of Independence of Irrelevance Alternatives, Anonymity and the Strong Pareto Principle. Such a list is reminiscent of characterizations of approving voting in terms of choice functions as in Vorsatz (2007), and there is at this juncture no discrepancy anymore between the two frameworks. To interpret the positive result further, it is useful to notice that approval and plurality voting share the same collective preference function on a dichotomous preference domain. Arrow's impossibility theorem does not apply when there are only two alternatives, and the classic way of showing that is to check that the collective preference function of plurality voting (equivalently here, majority voting,) does satisfy the conditions of the theorem (see Arrow, 1963, ch. V, and May, 1952). One may then see Proposition 1 as extending this straightforward possibility theorem to the more general context of two indifference classes, with the same voting example serving as a touchstone. An open question, however, is what would result from replacing the Strong Pareto Principle with Weak Pareto, which is the unanimity condition actually used in the straightforward theorem.

The final part of the paper draws some tentative connections between the present analysis of dichotomous preferences and the recently developed theory of judgment aggregation...

2 Definitions and aggregative conditions

As usual, a *weak preference ordering* R means a binary relation that is transitive, reflexive and complete; equivalently R has an asymmetric part P and a symmetric part I, which satisfy the PP, PI, IP, II variants of transitivity. The statements xPy, xIy, xRy have their standard readings, i.e., "x is strictly preferred to y", "x is indifferent to y", "x is strictly preferred or indifferent to y". An *indifference class for* R is one of the equivalence classes generated by I, i.e., a set of the form $\{x \in X : xIx_0\}$ for some fixed $x_0 \in X$.

As a particular case of R, a *dichotomous weak preference ordering* has one or two indifference classes. It satisfies the PP variant of transitivity vacuously and (for a sufficient number of elements) the PI, IP and II variants non-vacuously. It can be also be described in terms of its indifference classes directly. If R has two indifference classes, we denote by H and L the higher and lower one, respectively, and if R has only one indifference class, a case of complete indifference, we denote this class by C. The obvious translation rules are:

$$\begin{aligned} xPy &\Leftrightarrow x \in H, y \in L \\ xIy &\Leftrightarrow \text{either } x, y \in H \text{ or } x, y \in L \text{ or } x, y \in C. \end{aligned}$$

Define $\mathcal{O}$ to be the set of all weak preference orderings on X, and $\mathcal{D} \subset \mathcal{O}$ to be the set of dichotomous weak preference orderings on X.

Given a set of social alternatives X with $\mid X \mid \geq 3$ and a finite set of $n \geq 2$ individuals, we define a *collective preference function* to be a mapping

$$F : (R_1, ..., R_n) \longmapsto R.$$

The paper investigates collective preference functions for dichotomous preferences. There will be two cases, i.e., $F : \mathcal{D}^n \to \mathcal{D}$ and $F : \mathcal{D}^n \to \mathcal{O}$, which we refer to as *twice dichotomous* and *simply dichotomous* respectively. Section 3 is concerned with the former case, and section 4 with the latter.

The following is a list of properties that F may satisfy. We start with those which Arrow (1963) and followers imposed axiomatically on "social welfare functions" and then introduce some slightly less familiar ones.

Condition 1 *Independence of irrelevant alternatives (IIA): For all* $(R_1, ..., R_n)$, $(R'_1, ..., R'_n)$ *and all* $x, y \in X$*, if* $xR_iy \Leftrightarrow xR'_iy$ *and* $yR_ix \Leftrightarrow yR'_ix$ *for all* $i = 1, ..., n$*, then* $xRy \Leftrightarrow xR'y$.

Condition 2 *Weak Pareto (WP): For all* $(R_1, ..., R_n)$ *and all* $x, y \in X$*, if* xP_iy *for all* $i = 1, ..., n$*, then* xPy*.*

Condition 3 *Dictatorship (D): There is* $j = 1, ..., n$ *such that for all* $(R_1, ..., R_n)$ *and all* $x, y \in X$*, if* xP_jy*, then* xPy*.*

Condition 4 *Anonymity (A):For all* $(R_1, ..., R_n)$ *and all permutations* σ *of* $\{1, ..., n\}$*,* $F(R_1, ..., R_n) = F(R_{\sigma(1)}, ..., R_{\sigma(n)})$*.*

Condition 5 *Neutrality (N): For all* $(R_1, ..., R_n)$*,* $(R'_1, ..., R'_n)$ *and all* $x, y, z, w \in X$*, if* $xR_iy \Leftrightarrow zR'_iw$ *and* $yR_ix \Leftrightarrow wR'_iz$ *for all* $i = 1, ..., n$*, then* $xRy \Leftrightarrow zR'w$*.*

Condition 6 *Pareto Indifference (PI): For all* $(R_1, ..., R_n)$ *and all* $x, y \in X$*, if* xI_iy *for all* $i = 1, ..., n$*, then* xIy*.*

Condition 7 *Strict Pareto (SP): For all* $(R_1, ..., R_n)$ *and all* $x, y \in X$*, if* xR_iy *for all* $i = 1, ..., n$ *and* xP_iy *for some* i*, then* xPy*.*

Condition 8 *Pareto Preference (PP): For all* $(R_1, ..., R_n)$ *and all* $x, y \in X$*, if* xR_iy *for all* $i = 1, ..., n$*, then* xRy*.*

Condition 9 *Positive Responsiveness 1 (PR1): For all* $(R_1, ..., R_n)$*,* $(R'_1, ..., R'_n)$ *and all* $x, y \in X$*, if* $xP_iy \Rightarrow xP'_iy$ *and* $xI_iy \Rightarrow xR'_iy$ *for all* $i = 1, ..., n$*, and* yP_ix *and* xR'_iy*, or* xI_iy *and* xP_iy*, for some* i*, then* $xPy \Rightarrow xP'y$*.*

Condition 10 *Positive Responsiveness 2 (PR2): For all* $(R_1, ..., R_n)$*,* $(R'_1, ..., R'_n)$ *and all* $x, y \in X$*, if* $xP_iy \Rightarrow xP'_iy$ *and* $xI_iy \Rightarrow xR'_iy$ *for all* $i = 1, ..., n$*, and* yP_jx *and* xR'_jy*, or* xI_jy *and* xP_jy*, for some* i*, then* $xRy \Rightarrow xP'y$*.*

Condition 11 *Positive Responsiveness 3 (PR3): For all* $(R_1, ..., R_n)$*,* $(R'_1, ..., R'_n)$ *and all* $x, y \in X$*, if* $xP_iy \Rightarrow xP'_iy$ *and* $xI_iy \Rightarrow xR'_iy$ *for all* $i = 1, ..., n$*, and* yP_jx *and* xR'_jy *for some* i*, then* $xRy \Rightarrow xP'y$*.*

Some obvious implications hold between the conditions: $N \Rightarrow$*IIA*, *SP*$\Rightarrow$*WP*,*(PR2)*$\Rightarrow$*(PR1)*, *(PR2)*$\Rightarrow$*(PR3)*. The *Strong Pareto Principle* (*SPP*) is *SP*&*PI*.

3 Aggregating dichotomous preferences: an impossibility theorem

Arrow (1963) proved his famous impossibility theorem - i.e., that *IIA* and *WP* together entail D - for collective preference functions $F : \mathcal{O}^n \to \mathcal{O}$. It has been either extended or shown not to carry through to many F with a different domain or a different range, but to the best of our knowledge, the twice dichotomous case, i.e., $F : \mathcal{D}^n \to \mathcal{D}$, has been left aside entirely; this may be checked from Le Breton and Weymark's (2011) authoritative survey. The reason is presumably that the preferences under consideration cannot support that part of the proofs which makes use of PP-transitivity. In particular, it is impossible to apply the *free triple property*, a convenient sufficient condition for a domain of preferences to be dictatorial when F satisfies *IIA* and *WP*.

However, as we demonstrate, Arrow's negative conclusion also holds in the twice dichotomous case.

Notation 12 *Instead of ($xR_iy \Leftrightarrow xR'_iy$ and $yR_ix \Leftrightarrow yR'_ix$) and of ($xRy \Leftrightarrow xR'y$ and $yRx \Leftrightarrow yR'x$), we write , we write $xR_iy \approx xR'_iy$ and $xRy \approx xR'y$.*

Theorem 13 *If a collective preference function $F : \mathcal{D}^n \to \mathcal{D}$ satisfies IIA and WP, it satisfies D.*

The proof goes through three lemmas.

Lemma 14 *If a collective preference function $F : \mathcal{D}^n \to \mathcal{D}$ satisfies IIA and WP, it satisfies PI and N.*

Proof. Consider $(R_1, ..., R_n)$ and $x, y \in X$ s.t. xI_iy for all $i = 1, ..., n$. Take $z \neq x, y$ and $(R'_1, ..., R'_n)$ s.t. xP'_iz and xI'_iy for all $i = 1, ..., n$. Then, $xP'z$ and $yP'z$ by *WP*, $xI'y$ since F has range $\mathcal{D}$, and xIy follows by *IIA*. This completes the derivation of *PI*.

To derive *N*, consider first the case of four distinct $x, y, z, w \in X$. By assumption, $(R_1, ..., R_n)$ and $(R'_1, ..., R'_n)$ are s.t. $xR_iy \approx zR'_iw$ for all $i = 1, ..., n$. Take $(\overline{R}_1, ..., \overline{R}_n)$ s.t. $xR_iy \approx x\overline{R}_iy$ for all $i = 1, ..., n$, and s.t. $x\overline{I}_iz$ and $y\overline{I}_iw$ for all $i = 1, ..., n$. Thus, by construction, $z\overline{R}_iw \approx zR'_iw$ for all $i = 1, ..., n$. Suppose that xRy. Then, $x\overline{R}y$ follows from *IIA*, $z\overline{R}w$ from *PI*, and finally $zR'w$ from *IIA*.

Related proofs take care of the two cases in which there are three distinct elements among $x, y, z, w \in X$, $x \neq y$, and the position of the common element is the same in the two pairs, i.e., $x = z$ or $y = w$. Now, suppose that the common element changes position, i.e., $x = w$ or $y = z$. We give a proof for the former case. By assumption, $(R_1, ..., R_n)$ and $(R'_1, ..., R'_n)$ are s.t. $xR_iy \approx zR'_ix$ for all $i = 1, ..., n$. Take $(\overline{R}_1, ..., \overline{R}_n)$ s.t. $xR_iy \approx z\overline{R}_iy$ for all $i = 1, ..., n$. From one of the cases with unchanged positions, $xRy \Leftrightarrow z\overline{R}y$. By construction, $zR'_ix \approx z\overline{R}_iy$ for all $i = 1, ..., n$, so from the other case, $zR'x \Leftrightarrow z\overline{R}y$, and finally $xRy \Leftrightarrow zR'x$.

If there are three distinct elements among $x, y, z, w \in X$, and $x = y$, or $z = w$, *N* reduces to *PI*.

If there are two distinct elements, say x and y, which do not exchange positions, *N* reduces to *IIA*. Otherwise, suppose that $(R_1, ..., R_n)$ and $(R'_1, ..., R'_n)$ are s.t. $xR_iy \approx yR'_ix$ for all $i = 1, ..., n$. Take $z \neq x, y$ and $(\overline{R}_1, ..., \overline{R}_n)$ s.t. $xR_iy \approx x\overline{R}_iz$ for all $i = 1, ..., n$. It follows that $xRy \Leftrightarrow x\overline{R}z$. Take $(\overline{\overline{R}}_1, ..., \overline{\overline{R}}_n)$ s.t. $yR'_ix \approx y\overline{\overline{R}}_iz$ for all $i = 1, ..., n$. It follows that $yR'x \Leftrightarrow y\overline{\overline{R}}z$. Now, by construction, $x\overline{R}_iz \approx y\overline{\overline{R}}_iz$ for all $i = 1, ..., n$, whence $x\overline{R}z \Leftrightarrow y\overline{\overline{R}}z$. Combining the equivalences, one gets $xRy \Leftrightarrow yR'x$, as desired. ∎

Lemma 15 *If a collective preference function $F : \mathcal{D}^n \to \mathcal{D}$ satisfies N and WP, it satisfies PP and PR1.*

Proof. Consider $(R_1, ..., R_n)$ and $x, y \in X$ s.t. xR_iy for all $i = 1, ..., n$. Take $z \neq x, y$ and $(R'_1, ..., R'_n)$ s.t., for all $i = 1, ..., n$, xP'_iz and $xR_iy \approx xR'_iy$. Then, $xP'z$ by *WP*, so $xR'y$ since F has range $\mathcal{D}$, and xRy follows by *IIA*. This completes the proof of *PP*.

To derive *PR1*, we first assume that $(R_1, ..., R_n)$, $(R'_1, ..., R'_n)$ and x, y meet the antecedent condition without any full reversal of strict preference, i.e., without any j s.t. yP_jx and xP'_jy.

Take $z \neq x, y$ and $(\overline{R}_1, ..., \overline{R}_n)$ so defined: for all $i = 1, ..., n$,

- if xP_iy and xP'_iy, then $x\overline{P}_iy\overline{I}_iz$; if yP_ix and yP'_ix, then $z\overline{I}_iy\overline{P}_ix$; if xI_iy and xI'_iy, then $x\overline{I}_iy\overline{I}_iz$;
- if yP_ix and xI'_iy, then $z\overline{P}_ix\overline{I}_iy$; if xI_iy and xP'_iy, then $z\overline{I}_ix\overline{P}_iy$.

Thus, for all $i = 1, ..., n$, $xR_iy \approx x\overline{R}_iz$, $z\overline{R}_iy$, and $xR'_iy \approx x\overline{R}_iy$. Suppose that xPy. Then, $x\overline{P}z$ by *N*, and because *PP* entails that $z\overline{R}y$, it follows that $x\overline{P}y$, hence $xP'y$ by *IIA*. This completes the proof of *PR1* in the case just considered.

Now, assume that $(R_1, ..., R_n)$, $(R'_1, ..., R'_n)$ and x, y meet the antecedent of *PR1* in full generality. Take $(R''_1, ..., R''_n)$ so defined: for all $i = 1, ..., n$,

- if yP_ix and xP'_iy, then xI''_iy;
- otherwise, $xR''_iy \approx xR'_iy$.

Suppose that xPy. By the particular case just proved, $xP''y$, and again by this case, $xP'y$, which completes the proof of *PR1*. ■

A group $M \subseteq N = \{1, ..., n\}$ is said to be *semi-decisive* on the pair $(x, y) \in X^2$ if, for all $(R_1, ..., R_n)$,

$$xP_iy, i \in M, \text{ and } yR_ix, i \in N \setminus M \implies xPy.$$

Define $V \subseteq 2^N$ to be the set of all groups that are semi-decisive on some pair in X^2.

Lemma 16 *If a collective preference function $F : \mathcal{D}^n \to \mathcal{D}$ satisfies IIA, WP and PP, there exist j and $(x, y) \in X^2$ such that $\{j\}$ is semi-decisive on (x, y).*

Proof. The set V is non-empty because $N \in V$ in virtue of *WP*. Since N is finite, there exists a smallest group M^* in V, which cannot be $\varnothing$ in virtue of *PP*. Suppose that $\mid M^* \mid \geq 2$. We can then partition M^* into two non-empty groups M^*_1, M^*_2, and given that M^* is semi-decisive on a pair $(x, y) \in X^2$, take $z \neq x, y$ and $(R_1, ..., R_n)$ with the following properties:

- xP_iyI_iz for all $i \in M^*_1$;
- zI_ixP_iy for all $i \in M^*_2$;
- yP_ixI_iz for all $i \in N \setminus M^*$.

It follows, first, that xPy because $M^* = M_1^* \cup M_2^*$ is semi-decisive on (x, y), and second, that yRz because (in virtue of *IIA*) zPy would mean that M_2^* is semi-decisive on (z, y), contradicting the minimality of M^*. The two conclusions entail that xPz, but this would mean (in virtue of *IIA*) that M_1^* is semi-decisive on (x, z), again contradicting the minimality of M^*. Hence, M^* is a singleton, as was to be shown. ■

Proof. (End) By N, the individual j of Lemma 3 is semi-decisive on any pair in X^2, and by *PR1*, it is a dictator, i.e., it satisfies D. ■

Inspection of the proof shows that the complete indifference ordering $CI \in \mathcal{D}$ does not occur as an auxiliary profile unless$\mid X \mid= 3$. Thus, if we put $\mathcal{D}^-$ $= \mathcal{D} \setminus \{CI\}$, we have an immediate extension of the theorem.

Remark 17 *If* $\mid X \mid\geq 4$, *the theorem also holds for* $F : (\mathcal{D}^-)^n \rightarrow \mathcal{D}$.

We have used the range $\mathcal{D}$ of F twice in the proof, i.e., to derive the Pareto conditions *PI* and *PP*, which would be logically independent of *IIA* and *WP* in an unrestricted Arrovian framework. A collective preference function to be introduced now makes it clear that the negative conclusion of the theorem does not hold when $\mathcal{D}$ is replaced by $\mathcal{O}$. It corresponds to the approval voting rule, which is normally defined in terms of a collective choice function.

For any profile $(R_1, ..., R_n)$ in $\mathcal{O}$ or $\mathcal{D}$, we put $N(xP_iy) = \{i : xP_iy\}$ and $n(xP_iy) = |\{i : xP_iy\}|$ (similarly for R_i, I_i), and $N(x \in H_i) = \{i : x \in H_i\}$ and $n(x \in H_i) = |\{i : x \in H_i\}|$ (similarly for L_i, C_i).

Definition 18 *F is approval voting* if, for all* $(R_1, ..., R_n) \in \mathcal{D}^n$, $xPy \Leftrightarrow n(x \in H_i) > n(y \in H_i)$.

With $\mid X \mid\geq 3$, the range of approval voting* can only be $\mathcal{O}$, and since it satisfies *IIA* and *WP*, it illustrates that the simply dichotomous case is open to non-dictatorial possibilities, contrary to the twice dichotomous case.

We may now redefine the collective preference function in such a way that it becomes twice dichotomous. In this case, the dictatorial conclusion of the theorem entails that one of the two conditions *IIA* and *WP* is violated. For any profile $(R_1, ..., R_n)$ in $\mathcal{D}^n$, we put

$$Max(R_1, ..., R_n) = \{x \in X : N(x \in H_i) \geq N(y \in H_i), \forall y \in X\}.$$

Definition 19 *F is approval voting** if, for every* $(R_1, ..., R_n) \in \mathcal{D}^n$,

- *if* $Max(R_1, ..., R_n) \neq X$, *then* $H = Max(R_1, ..., R_n)$ *and* $L = X \diagdown Max(R_1, ..., R_n)$,
- *if* $Max(R_1, ..., R_n) = X$, *then* $R = CI$.

By construction, approval voting ** has range $\mathcal{D}$. It satisfies WP, but not IIA, as the following 3-alternative, 2-individual profiles (R_1, R_2), (R_1', R_2') illustrate: xP_1yI_1z, zP_2yI_2x and $xI_1'zP_1'y$, $zP_2'yI_2'x$. This leads to $x, y \in H, y \in L$, so xPy, and to $z \in H', x, y \in L'$, so xIy.

It is trivial, but useful to notice that plurality voting has the same collective preference function as approval voting* when individual preferences are dichotomous. By the standard definition (Arrow, 1963, p. 58, and May, 1952), *majority voting* requires that for all $(R_1, ..., R_n)$ in the domain, xRy if

$$n(xR_iy) \geq n(yR_ix), \text{ or equivalently } n(xP_iy) \geq n(yP_ix),$$

and this can be taken to define the collective preference function of *plurality voting* when there are more than two alternatives. Now, if we restrict the domain to be $\mathcal{D}^n$, the definition becomes equivalent to that just given for approval voting*.

4 Aggregating dichotomous preferences: possibility results

The aim of this section is to characterize approval voting* in terms of some of the above conditions. Specifically:

Proposition 20 *A collective preference function $F : \mathcal{D}^n \to \mathcal{O}$ is approval voting* if and only if it satisfies IIA, SPP and A.*

Only the sufficiency part requires a proof. It relies on two lemmas.

Lemma 21 *If $F : \mathcal{D}^n \to \mathcal{O}$ satisfies IIA, WP and PI, it satisfies N.*

Proof. Same as the proof of N in Lemma 1. ■

Lemma 22 *If $F : \mathcal{D}^n \to \mathcal{O}$ satisfies N and SP, it satisfies PR2.*

Proof. By Lemma 2, N and *SP* entail *PR1*, which has the same antecedent as, and a weaker consequent than, *PR2*. In order to complete the proof, it is enough to derive the consequent under the form $xIy \Rightarrow xP'y$.

Assume that $(R_1, ..., R_n)$, $(R'_1, ..., R'_n)$ and x, y meet the antecedent without any full reversal of strict preference, and take $z \neq x, y$ and $(\overline{R}_1, ..., \overline{R}_n)$ as in the corresponding part of the proof of *PR1*.
It again follows that for all $i = 1, ..., n$, $xR_iy \approx x\overline{R}_iz$, $z\overline{R}_iy$, and $xR'_iy \approx x\overline{R}_iy$. We also note by inspecting the possibilities that for some i, $z\overline{P}_iy$. Now, suppose that xIy. Then, $x\overline{I}z$ by N, and because *SP* entails that $z\overline{P}y$, it follows that $x\overline{P}y$, hence $xP'y$ by *IIA*.

If $(R_1, ..., R_n)$, $(R'_1, ..., R'_n)$ and x, y meet the antecedent of *PR1* in full generality, take $(R''_1, ..., R''_n)$ as in the the remaining part of the proof of *PR1*. Suppose that xIy. By the case just proved, $xP''y$, and again by this case, $xP'y$. ■
Proof. (End) Suppose that there is some $F : \mathcal{D}^n \to \mathcal{O}$ that is not approval voting*. Then, there are $(R_1, ..., R_n) \in \mathcal{D}^n$ and $x, y \in X$ s.t. either (i) $n(x \in H_i) = n(y \in H_i)$ and xPy, or (ii) $n(x \in H_i) > n(y \in H_i)$ and yRx.

In case (i), there are three groups of individuals, i.e., $N(xP_iy), N(yP_ix), N(xI_iy)$ with $n(xP_iy) = n(xP_iy)$. The first two groups are non-empty by *PI*. We may take a permutation σ that interchanges them and leaves the third group unchanged; by A, the resulting profile $(R_{\sigma(1)}, ..., R_{\sigma(n)})$ has the collective preference $xP_\sigma y$. Now, observing that for all $i = 1, ..., n$, $xR_i y \approx yR_{\sigma(i)}x$, we apply N to the profile to get the contradiction that $yP_\sigma x$.

In case (ii), the three groups of individuals $N(xP_iy), N(yP_ix), N(xI_iy)$ are s.t. $n_1 = n(xP_iy) > n_2 = n(xP_iy)$. The second group is non-empty by *SP*, and from the inequality, the first group also is. Take a permutation σ that interchanges n_2 individuals in $N(xP_iy)$ with those in $N(yP_ix)$ and leaves the position of any others unchanged; by A, the resulting profile $(R_{\sigma(1)}, ..., R_{\sigma(n)})$ has the collective preference $yR_\sigma x$. Now, modify this profile into $(R'_1, ..., R'_n)$ by putting yP'_ix if $i = \sigma(i)$ is any of the remaining $n_1 - n_2$ individuals of $N(xP_iy)$ and leaving any other individual's preference the same. Given this reinforcement of strict preference for y, *PR2* entails that $yP'x$. However, $(R'_1, ..., R'_n)$ also modifies $(R_1, ..., R_n)$ in such a way that N entails that $xR'y$, a contradiction. ■

Proposition 1 should be compared with Vorsatz's (2007, Theorem 1) characterization of approval voting for the simply dichotomous case, as a social choice function satisfying Anonymity, Neutrality, Strategyproofness and Strict Monotonicity. The conditions in the two results are put on different objects, which make them difficult to compare in detail, but they are heuristically related, as Vorsatz's Neutrality corresponds with *IIA* and *PI* in the present list, and his Strict Monotonicity corresponds to a condition *PR3*, which slightly weakens *PR2*, here obtained from N and *SP*. By our choice of framework, we go farther into the Arrovian foundations of approval voting*, Proposition 1 being a positive counterpart to Theorem 1.

A comparison is also in order with Arrow's (1963, ch. V) "possibility theorem" for two alternatives and May's (1952) derivative characterization of majority voting under the same assumption. Both results involve checking that majority voting satisfies *IIA* and *WP*, and we really use the same rule here since, as the last section pointed out, it coincides with approval voting*. The added value of Proposition 1 compared with these classic early findings is that it places the cardinality restriction on the equivalence classes available to the individuals, and not bluntly on the alternatives they evaluate.

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