The Doctrinal Paradox, the Discursive Dilemma, and Logical Aggregation Theory[1]

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Abstract

Judgment aggregation theory, or rather, as we conceive of it here, logical aggregation theory generalizes social choice theory by having the aggregation rule bear on judgments of all kinds instead of merely preference judgments. It derives from Kornhauser and Sager's doctrinal paradox and Pettit's discursive dilemma, two problems that we distinguish emphatically here. The current theory has developed from the discursive dilemma, rather than the doctrinal paradox, and the final aim of the paper is to give the latter its own theoretical development, along the lines of Dietrich and Mongin's recent technical work. However, the paper also aims at reviewing the main existing results, starting from the first impossibility theorem proved by List and Pettit. It provides a uniform logical framework in which the whole of theory can be stated and its theorems can be compared with each other. The account goes through three historical steps: the scattered early results on the independence axiom, the collective achievement of the canonical theorem which provided the theory with its specific method of analysis; and finally the recent extension mentioned above to the doctrinal paradox.

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1 Introduction

Contemporary aggregation theories have their roots in mathematical analyses of voting, developed in France from the end of the 18th century, as well

[1]The present English paper has evolved from an earlier French paper co-authored with Franz Dietrich ("Un bilan interprétatif de la théorie de l'agrégation logique", *Revue d'économie politique*, vol. 120, 2010, n°6). Many thanks to him for allowing this author to present this new version. Thanks also for their comments to Mikaël Cozic, Daniel Eckert, Itay Fainmesser, Jim Joyce, Lewis Kornhauser, Gabriella Pigozzi, Rohit Parikh, Roberto Serrano, Jonathan Zvesper, and the participants to the many conferences or seminars where versions or variants of this paper were given.

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as in the technical formulations of utilitarianism and its rarified variant, welfare economics, which were to follow in Great Britain from the 18th century to the middle of the 20th century. Classical and then neo-classical economists set great store by this second source, but were entirely unaware of the first, until Arrow masterfully orchestrated their reconciliation in *Social Choice and Individual Values* (1951). The title of his work fixed the accepted name of the theory it propounds, of *social choice*, a rather inaccurate name, because social choice theory starts with preference and not choice, as its fundamental concept, and it bears on all types of collectivity, the whole of society being just one particular instance. With the no less improperly named "social welfare" function, which is defined from individual to collective preferences, the Arrovian theory develops a formalism that can cover all of the aggregation problems that the two historical traditions, the French and the British, offered in isolation. Indeed, the notion of preference, individual or collective, can tend either towards the side of the utility function, which "represents" preferences according to economists' contemporary conception, or towards the side of choice, which "reveals" preferences according to the same conception. Voting is merely a particular kind of choice; it is in this way that Arrow and his successors were able to connect Bentham with Condorcet.

However, as considerable as that step of generalization might have been, it was still not enough, since the preferences of individuals or of the collectivity between two states of affairs do not exhaust the judgments they could make about those states, and it is just as legitimate to look at the aggregation of other kinds of judgments. "Bob prefers a long monthly meeting to short weekly meetings" can be paraphrased by saying that Bob judges one such meeting to be preferable to the others, and this makes us notice that preference is a special case in several ways. It is a *comparative* judgment made from the *evaluative* point of view that is specific to *preferability.* Concerning the same states of affairs, Bob can form other judgments, either absolute or again comparative: a long monthly meeting is tiring; more tiring than short weekly meetings; successful; more successful than short weekly meetings; and so on. It is even inevitable that Bob will form judgments other than of preference, since like any form of evaluative judgment, they require factual judgments to have already been made. When several Bobs express themselves concerning both of these levels of judgment, should the theory of aggregation only consider the final one without examining the intermediary one? Such a restriction would appear counter-productive, yet it is imposed in social choice theory, which does not admit judgments other than of preference.

A new theory, called *judgment aggregation theory*, overcomes this limitation. Its formalism allows it to represent individual and collective judgments of any kind, and thus to attack an entire class of neglected aggregation problems. The contributors to this theory add one step of generalization to that made by Arrow and his disciples in their time. From this group, they retain the axiomatic method of investigation. Indeed, they posit on individual and collective judgments certain normative properties which parallel the constraints imposed on preferences, and define a collective judgment function, going from admissible profiles of individual judgments to admissible collective judgments, which is the formal analogue of the social welfare function. After listing properties that the collective judgment function could satisfy, they study collections of these properties via impossibility and possibility theorems, which are reminiscent of the Arrovian ones.

There are already so many and diverse such axiomatic studies that it is impossible to account for them without fixing an angle of attack. In line with the specialist work of its two authors, this interpretive account underlines the logical side of individual and collective judgments and sets out the new theory from this vantage point. Accordingly, we will refer to it as *logical aggregation theory*. As well as being more informative, this name also has the advantage of marking a clean separation with the preceding theory of probabilistic aggregation. In a broad conception of judgments, which is in fact the common sense one, subjective certainty is not inherent to them, and the statements that express them may not have the full force of the values "true" and "false", the only ones considered by standard logic. If today's theory were concerned with this conception, it should include the theory of probabilistic aggregation; but that is far from the case, the former having been created without the support of the latter, and both still being unaware of each other apart from a few exceptions.[1]

As a matter of fact, it is logic to which the new theory has turned for technical help. Essentially, it assumes that the individual or collective subject who is making a judgment contemplates the proposition associated with it, and either affirms it in giving it the value "true" or denies it by giving it the value "false". The theory also allows the subject not to make any judgment, in which case he attributes no truth value to the proposition. That sketch is fleshed out by introducing a symbolic language with a formula for each proposition, and then, as in any logical work, arranging one way or another the semantic and syntactic components of the analysis. Propositional logic suffices even for some advanced results, but we will push this article in the direction of less elementary logics.

[1] The theory of probabilistic aggregation goes back to the 60s with the work of Stone, Raiffa and Winkler. The main results were obtained quickly, so that the survey by Genest and Zidekh (1986) remains up to date. In McConway (1981), the theory takes on the axiomatic approach of social choice theory, whereas in Lehrer and Wagner (1981), it takes the form of a theory of consensus applicable to scientific activity.

It is with List and Pettit (2002) that an aggregation theory incorporated – for the first time rigorously – the logical analysis of judgment that we just sketched. Before that, two American legal theorists, Kornhauser and Sager (1986, 1993) had outlined an aggregative conception of judgment, but only taken in the judiciary, and not the broader philosophical, sense. They showed that collegiate courts were subject to the so-called *doctrinal paradox*: individually coherent opinions of judges can lead the court as a group to inconsistency. In a seminal article that prepares the formalism later introduced with List, Pettit (2001) reexamines Kornhauser and Sager's paradox and, judging it too specific, propose reformulating it as a universal problem that he calls the *discursive dilemma*. We will investigate afresh this semantic step and show – it is the guiding insight of the present article – that there was no need to transform the doctrinal paradox into the discursive dilemma in order to treat it analytically. It can be used as a departure point for a branch of aggregation theory whose results are yet more general than those of the main trunk. We follow here Dietrich and Mongin (2010) without reproducing their full technical exposition.

Without List and Pettit being aware of it, a French scholar of social mathematics, Guilbaud (1952), had already set about generalizing from preference to judgment. Inspired by Arrow, who had just published his book, but also by Condorcet, whose work he was to help rescue from oblivion, Guilbaud reformulated the former's theory of aggregation as the latter would have done, that is: not only for relations of preference, but for all sorts of "opinions". (Guilbaud prefers the term, from Condorcet, to "judgment", which he still uses sometimes.) Opposed to the Bourbakianism that then dominated French mathematics, Guilbaud rejected the axiomatic method and even eschewed general proofs, which makes it complicated to evaluate the extent of his contribution, but the trend today is to see him as a direct forerunner of logical aggregation theory.[1]

If we must find a first source for the current work, Condorcet is the only choice, with his *Essai sur l'application de l'analyse à la probabilité des décisions à rendre à la pluralité des voix* (1785) and his other treatises or articles on mathematical politics. His abiding method is to treat a preference as the accepting or rejecting of certain propositions. The voter who prefers A to B, B to C, and A to C, accepts "A is preferable to B", "B is preferable to C", "A is preferable to C", and rejects the contrary propositions. One would think, under Arrow's influence, that Condorcet only describes preference

[1] Monjardet (2003) singles out in Guilbaud a theorem, and its proof, still of the Arrovian style, but Eckert and Monjardet (2009) attribute to him a theorem and a proof already of the judgment aggregation style, and this reading is more faithful. As for purely technical matters, one glimpses in Guilbaud the first use of filters and ultrafilters, notions which had just begun to enter the mathematics of his time. Since Kirman and Sondermann (1972), aggregative theories borrow frequently from this technique.

orderings in a roundabout way, but that is far from the case. He starts with propositions and their supposed logical connections, and he only attributes the ordering property to the preference relation in virtue of these logical connections. The greater generality of his reasoning is better seen in the jury theorem than in the voting paradox, but the latter is still representative. He analyzes it by writing that the propositions chosen by the majority constitute an "inconsistent system" (*Essai*, p. LV-LVI), and this suggests that logical coherence, and not preferability, is the crux of the matter. Reread in this manner, which is exactly Guilbaud's, Condorcet becomes the distant precursor of the doctrinal paradox, of the discursive dilemma and of all the ongoing research.[2]

This article consists of a long circular development around the doctrinal paradox. Section 2 presents the paradox, returning to the initial judiciary source, then compares it to its reinterpretation as a discursive dilemma, which motivates the formal framework set up in section 3. From there, we present a series of impossibility results. That of List and Pettit (Theorem 1) imposes the questionable axiom of *systematicity* on the collective judgment function, whereas those of section 4, due to Pauly and van Hees, Dietrich, Mongin, and Nehring and Puppe (Theorems 2–5), only require *independence*, a normatively more defensible axiom that is close to the famous Arrovian one of independence of irrelevant alternatives. Section 5 sets out, with improvements, Dietrich's general logic, which overcomes the limitations of the preceding results, which are all formulated in logics that are still too specific. Using this tool, section 6 states the theorems that best structure the field today and can, because of this, be considered canonical. They have as their mathematical object the *agenda*, i.e., the set of logical formulas standing for the propositions about which the individuals and the group express opinions. The conditions placed on this object turn any collective judgment function that is subjected to certain axioms into a degenerate rule, such as dictatorship or oligarchy. Importantly, these agenda conditions are not only sufficient, but also necessary for the axiomatized function to degenerate, so the theorems state possibilities no less than impossibilities. Each researcher in the field has made some contribution here, but we will focus on Dokow and Holzman's (Theorems 6 and 7). Section 7 returns to the doctrinal paradox to give that its own theoretical development, along the lines of Dietrich and Mongin (Theorems 8 and 9).

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[2] Others scholars, like Granger (1956) and Black (1958), were involved in rediscovering Condorcet, but their reading is the standard Arrovian one in terms of preferences.

Regrettably, this paper will touch on Condorcet only in connection with the paradox of voting. The jury theorem would deserve an extended separate discussion, so large is the literature now devoted to it. Some writers have related it to logical aggregation theory; see Bovens and Rabinowicz (2004), Pigozzi (2006), Hartmann, Pigozzi and Sprenger (2010), Hartmann and Sprenger (forthcoming).

2 From the doctrinal paradox to the discursive dilemma

Early forerunners aside, logical aggregation theory originates in the analysis of the legal institution laid out by Kornhauser and Sager (1986, 1993; see also Kornhauser, 1992). From this analysis, logical aggregation theorists only really retained the *doctrinal paradox*, and today use it merely to introduce and motivate their theorems informally. We restore here the initial judiciary problem by distinguishing it carefully from the transformations it underwent in Pettit (2001) and List and Pettit (2002). Presenting it as a *discursive dilemma*, these authors opened the way to the theory that will be reviewed in the next sections, but - we will argue - also swept aside some significant conceptual concerns.

The doctrinal paradox occurs for the first time in Kornhauser (1992) but becomes central only in Kornhauser and Sager (1993), where it is illustrated by the following - by now famous, and even a little tired - example. A plaintiff P, has brought a civil suit against a defendant D, alleging a breach of contract between them. The court is composed of three judges A, B and C, who must, based on contract law, determine whether or not D owes damages to P, a decision represented by the logical formulas d or $\neg d$. The case brings up two issues, i.e., whether the contract was valid or not in the first instance (v or $\neg v$), and whether D was or not in breach of it (b and $\neg b$), and the law decides for all possible responses, stipulating that D must pay damages to P if, and only if, both issues are answered in the affirmative. Suppose that the judges' deliberations lead them to the following responses and conclusions:

A	v	$\neg b$	$\neg d$
B	$\neg v$	b	$\neg d$
C	v	b	d

If the court rules directly on the case using simple majority voting, it will arrive at the conclusion $\neg d$, against the plaintiff. However it can, still using the same voting rule, first decide on the two issues, and then draw a conclusion about the case based on the law, and this will return the answers a and b, hence finally d, in favour of the plaintiff.

To avoid misunderstandings, it is worth adding that the actual US contract law is more complex than is said here. Kornhauser and Sager also examine genuine cases, but they are too intricate to be so strikingly paradoxical, and the authors therefore conceived of the present toy example, which came somewhat late in their joint work. Their ultimate target is to investigate how the law changes when judiciary decisions have a collective form. Concretely, the only collegiate courts in the American legal system are the appellate courts of the States and of the Union. Everyone has heard of the Supreme Court, whose nine judges reach their decisions about federal cases through deliberation and - sometimes but not always - explicit voting. Less well known are the State appellate courts, often composed of three judges, and above them, the State supreme courts, which operate similarly.

Beside being exemplified, the doctrinal paradox has been defined in the abstract. This requires some relevant legal concepts to be introduced first, as in Kornhauser (1992), and we review them here sketchily. A *case* brought before a court is subjected by this court to a *characterization*, which amounts to defining what part of the law, if any, is relevant to it. The judges' inquiry leads them to delineate the *legal doctrine*, which, once applied to the case, will provide its complete resolution. The doctrine relies on an admixture of statutes and common law, depending on the case at hand; given the Anglo-American tradition, the former will prevail in criminal matters, and the latter in civil matters. The more jurisprudence is involved, the closer the judges's activity to law-making, and the more entangled their deliberation. Logically, the doctrine does two things at once, i.e., it fixes the *issues* that the case presents, and it translates possible decisions on these issues into decisions on the case. This is captured by assuming that there are unambiguous questions - to be answered by yes or no - for both the case and each issue, and that the legal doctrine dictates an answer of the first type once all answers of the second type are collected. The last part of the scheme is subject to interpretations. Standard presentations of the judiciary example associate the logical formula $d \longleftrightarrow v \wedge b$ with the legal doctrine, and represent the judges' deliberations as deductive inferences made with the help of this extra premiss. Such a modelling is natural and convenient, but the legal theorists' writings seem to point in the direction of a less summary logical treatment of the way relate to the legal doctrine.

The preceding concepts are sufficient only if there is a single judge. If the court is collegial, one has also to describe how individual answers are aggregated into a final judgment. Kornhauser and Sager contemplate two possibilities. The first has the court record directly *the individual answers about the case* and apply to them some collective decision-making procedure, like simple majority voting. In the second, *the individual answers about each issue* are recorded, and the collective decision-making procedure is applied to each of these separately, after which the answer about the case follows according to the legal doctrine. The doctrinal paradox arises any time that the first method, which is *case-by-case*, does not yield the same result as the second, which is *issue-by-issue*. That is the authoritative definition in Kornhauser (1992, p. 453, where it appears for the first time), Kornhauser and Sager (1993, p. 10-12) and subsequent legal theorists.[3]

[3] Post and Salop's (1991–1992) work seems partly independent of Kornhauser and

There is an interesting contrast between Condorcet's voting paradox and Kornhauser and Sager's. The former does not already indicate where to search for solutions, but the latter does, since it is defined precisely in terms of their discord. It therefore has a structural quality which brings it already close to an impossibility theorem. Furthermore, being abstract, it allows for more than one interpretation. One may say that there are colleagues who decide simultaneously, but this is not the only possible view, nor is it the most interesting from the legal perspective. In the above example, the three judges might have sitted apart, each arriving at a decision for himself, whereas a fourth judge, involved after them, would ask how to make the best of the jurisprudence thus created. Can he only retain the answers on the case, or should he make use of the answers on the issues? As we read them, Kornhauser and Sager initially concerned themselves with the collective functioning of courts primarily from the angle of their *diachronic* consistency. If, once they had discovered the paradox, they focussed on *synchronic* consistency, we believe that this is simply for intellectual convenience. Of the two problems, the first is more important than the second, because it affects all courts, collective as well as individual, and all the more when common law tends to outweigh statutes in determining the law. However, the first problem being also more difficult, it was good policy to start with the second.[4]

If there is anything paradoxical in the clash between the case-by-case and issue-by-issue methods, it is because each can rely on a solid normative argument. By deciding case-by-case, the court fully respects the deliberations of individual judges, right up until the decisions they would make, were they alone in adjudicating the case. By deciding issue-by-issue, the court guarantees that its decision is based on the same type of reasons - those allowed by the legal doctrine - as the judges' individual decisions. According to Kornhauser and Sager, "where the doctrinal paradox arises, judgment and reason are immediately and inexorably pulled apart" (1993, p. 25). By "reason", they classically mean one's ability to justify conclusions using logic. As they assume that each judge exercises this capacity competently, the question is whether it holds at the group's level, and the issue-by-issue method arguably ensures that it does. By "judgment", they mean a conclusion obtained by the case-by-case method, and indirectly the supporting argument just said that this method gives careful attention to individual judgments. In List and Pettit (2002, p. 94), the conflicting principles are called "collective rationality" and "individual responsiveness", a more explicit terminology that

Sager's. For the subsequent law literature, see Nash's (2003) critical review.

[4] When they call *case-by-case* one of the solutions to the synchronic problem, Kornhauser and Sager may still be echoing the diachronic version, for which this expression is more appropriate. List (2004) is the only logical aggregation theorist ever to have addressed that version.

we will retain from now on.

Beyond the psychological shock of the paradox, the clash between the two methods poses a *dilemma*, in the usual sense of a forced choice between two unsatisfactory options, since the argument to want one is also an argument not to want the other, and even a *theoretical* dilemma, since two basic principles clash, as was just explained. Pettit (2001), then List and Pettit (2002), definitely move the doctrinal paradox in this abstract direction. However, the novelty of the *discursive dilemma*, which they promote as an alternative concept, cannot lie just in this reinterpretation. It must also go beyond the authors' claim that the doctrinal paradox occurs outside of the legal context, because this is so immediately obvious. Pettit points in particular to the deliberative entities of democratic institutions, review panels and authorities of economic regulation, clubs or other groups whose members coopt, and even, to some degree, political parties, unions and churches. Being a matter of empirical observation, the list can go on. The only problematic item is the whole of political society, which Pettit chooses to include, relying as is on the theory of deliberative democracy, on which he has expanded elsewhere (see also Brennan, 2001). This speculative extension of the doctrinal paradox also goes beyond the normal range of the discursive dilemma, and so cannot really mean a difference between the two. Nor is the distinction clarified by Pettit's labelling of the two methods as the "*premiss-driven way*" and the "*conclusion-driven way*" (2002, p. 274). This new terminology, which List and Pettit passed on to logical aggregation theorists, only serves as a reminder that the initial problem extends beyond the legal realm.

What substantially distinguishes the discursive dilemma from the doctrinal paradox is not to be found on the interpretive side, as would be the case for the previous suggestions, but rather in the logical formalism underlying the two problems. Here is how List and Pettit (2002) reconstruct the judiciary example. They associate formulas to all the considerations that influence deliberation, and for the legal doctrine, take $d \longleftrightarrow v \wedge b$. Supposing then that the court votes on each formula, they bring to light the logical contradiction that it would face:

A	v	$\neg b$	$\neg d$	$d \longleftrightarrow v \wedge b$
B	$\neg v$	b	$\neg d$	$d \longleftrightarrow v \wedge b$
C	v	b	d	$d \longleftrightarrow v \wedge b$
$Court$	v	b	$\neg d$	$d \longleftrightarrow v \wedge b$

Whereas the doctrinal paradox was defined in terms of two methods to relate conclusions to premisses, the discursive dilemma is defined by a contradiction within the overall collective judgment, without the need to distinguish between premisses and conclusions. Presented in this way, the problem falls within the scope of ordinary propositional logic, and it thus opens the way to the formalism of logical aggregation theory. In retrospect, the many results obtained suggest that the problem was reformulated appropriately.

Still, if one is concerned to deepen the doctrinal paradox within its original legal context, there are some reasons to doubt that List and Pettit opened the right path. Indeed, in this context, their wide-ranging definition of collective judgment is questionable for two reasons. For one, the distinction between the issues and the case gave its fine structure to the problem, and by ignoring it, one simply destroys the connections with other problems in legal theory; our brief discussion of jurisprudential reasoning can flesh out this objection. For another, the legal doctrine calls for a separate analysis, but List and Pettit make it a proposition comparable in every way with the others. To unpack the critique here, they take it for granted (i) that the doctrine can be represented by a formula of ordinary logic, like the propositions describing the issues and the case, and (ii) that it falls under the scope of the same decision rule as these propositions. It transpires from Kornhauser and Sager's later writings that they have doubts about logical aggregation theory, but they have never expressed them fully.[5] As we suggest, there could be a disagreement about (i), (ii), or even more radically, the underlying claim (iii) that the doctrine is a proposition rather than a command or a rule. The paper will not take up this last objection, which would make any recourse to logic dubious, but section 7 does take the first two into account.

3 The formal framework of logical aggregation theory

The theory is developed from the specific notions of agenda, judgment sets, and the collective judgment function, as well as a small set of axioms to be put on this mapping, and various conditions to be put on the agenda and the judgment sets. This section and the following one present the theory with a minimum of logical details, only developing the formal language, and postponing until section 5 the full definitions of inference and associated logical notions.

By definition, a *language* $\mathcal{L}$ *of the theory* is a set of formulas $\varphi, \psi, \chi, \ldots$ containing logical symbols taken from a certain set $\mathcal{S}$. It is not necessary to specify the formulas beyond the minimal requirement that $\mathcal{S}$ contains the symbol for Boolean negation $\neg$ ("not") and $\mathcal{L}$ is closed for this symbol;

[5] A debate took place between Kornhauser and Sager (2004) and List and Pettit (2005), but it does not identify the disputes as clearly as one might like.

i.e., if $\varphi \in \mathcal{L}$, then $\neg\varphi \in \mathcal{L}$. If the set $\mathcal{S}$ contains other elements, they will be symbols for the remaining Boolean connectives, $\vee$ ("or"), $\wedge$ ("and"), $\rightarrow$ ("if ..., then ..."), $\leftrightarrow$ ("if and only if ..., then ...") or for non-Boolean operators representing modalities (e.g., "it is obligatory that ...", "it is desirable that ...", "it is known that ...", or "if ..., then ..." taken in a non-Boolean sense, typically with a counterfactual interpretation). For each additional element of $\mathcal{S}$, we suppose the corresponding closure rule: if $\wedge \in \mathcal{S}$, the rule says that if $\varphi \in \mathcal{L}$ and $\psi \in \mathcal{L}$ then $\varphi \wedge \psi \in \mathcal{L}$, and similarly for other symbols. We will distinguish *classical* and *non-classical* languages according to whether, respectively, $\mathcal{S}$ contains only Boolean symbols or others in addition.

In the very large class of permitted languages, the particular case of *propositional languages* $\mathcal{L}_{\mathcal{P}}$ stands out. They are defined in terms of a set $\mathcal{P}$ of elementary formulas, or *propositional variables*, which do not contain any logical symbols, and a set $\mathcal{S}$ containing the five Boolean connective symbols. Since these connectives are inter-definable, we can equivalently have $\{\neg, .\} \subseteq \mathcal{S}$, replacing the dot with any one of $\vee$, $\wedge$, $\rightarrow$ or $\leftrightarrow$. Classical propositional languages are those for which $\{\neg, .\} = \mathcal{S}$.

By a *calculus* of the theory, we mean a language $\mathcal{L}$ together with a system of axioms and rules that determine the logical links between the elements of $\mathcal{L}$. Just as with the language, there is no need to specify the system – the logic itself – in its entirety. Section 5 will show that it suffices to have an *inference relation* $B \vdash \psi$ defined for $B \subset \mathcal{L}$ and $\psi \in \mathcal{L}$, respecting some very general restrictions, but here and in the next section, we restrict ourselves to the special case of *classical propositional calculi.* Such calculi have classical propositional languages, and for their logic part, well-known systems of axioms and rules that are spelled out by introductory texts. These systems fit the ordinary mathematical intuition and need not be repeated here. They will for now fix the meaning of the inference rule $\vdash$ and of its associated notions, like logical truth, logical contradiction, consistency and inconsistency. Classical propositional calculi draw our attention only because they are so elementary. We used one of these like Jourdain used prose when we formalized the basic judiciary example. The language then was $\mathcal{L}_{\mathcal{P}}$, built from $\mathcal{P} = \{v, b, d\}$ and $S = \{\neg, \leftrightarrow\}$, and the set $\{v, b, \neg d, d \leftrightarrow v \wedge b\}$ was contradictory in the sense of any textbook system.

In $\mathcal{L}$, the theory fixes a subset X representing the propositions that are in question for the members of the group; this is the *agenda.* It can be large or small depending on the application, but in all generality it is only required to be non-empty and, as with $\mathcal{L}$, closed for negation. The agenda that List and Pettit use for the judiciary example is:

$$\overline{X} = \{v, b, d, d, \leftrightarrow v \wedge b, \neg v, \neg b, \neg(d \leftrightarrow v \wedge b)\}.$$

If we were to add to the agenda multiple negations $\neg\neg\varphi, \neg\neg\neg\varphi, \ldots$, the logic would in the end reduce them to either φ or $\neg\varphi$. It is better to anticipate that process and define agendas as sets:

$$X = \{\varphi, \chi, \psi, \ldots\}^{\pm},$$

whose elements $\varphi, \chi, \psi, \ldots$ are *positive* formulas, meaning that they do not start with $\neg$, each one being accompanied by its negation (this is what the superscript $\pm$ indicates). To simplify matters, we impose the restriction - going beyond what the theory needs - that agendas consist of *contingent* formulas, i.e., are neither logical truths, nor logical contradictions.

The theory represents individual and group judgments by subsets of X, *judgment sets*, which can be made to fulfil certain logical constraints, the most natural being consistency. They will be denoted by $B, B', \ldots$ generally, and by $A_i, A'_i, \ldots A, A', \ldots$ when they belong to, respectively, individuals i and the group they form. A formula φ from one of these sets represents a proposition, in the ordinary sense of a semantic object endowed with a truth value. If φ is used also to represent a judgment, in the sense of a cognitive operation, then it is in virtue of the natural interpretive rule:

> (R) i judges that φ iff $\varphi \in A_i$, and the group judges that φ iff $\varphi \in A$.

We treat the formula φ in this statement as if it were itself the proposition that it expresses; this terminological ease is commonplace in logic and will be taken for granted in what follows. Thanks to (R), judgments obey a distinction between two types of negation, internal and external, which has no analogue on the level of propositions or formulas, these being negated in only one way. Indeed, "judging that not" ($\neg\varphi \in B$) is different from "not judging that" ($\varphi \notin B$). Once the inference relation is defined, the logical consistency of judgment sets will relate one negation to the other as could be expected, i.e., "judging that not" entails "not judging that", without the converse always holding.

From what we have said, it can be seen that logical aggregation theory is connected to a particular philosophical conception of judgments and propositions. The language $\mathcal{L}$ represents all expressible propositions, i.e., all propositions that can become the object of a judgment, but only those in X will actually become so. It is typical of the modern concepts of proposition and judgment – since Frege and Russell – that the former has a wider range than the latter; this definitely clashes with the ancient view – that of Aristotle, which is still to be found in Kant.[6] Logical aggregation theory uses the Fregean concept of *assertion*, here rendered as $\varphi \in B$, and

[6] Kant's *Logic*, published in 1800, is a famous sample of the ancient view. The modern one is best exemplified by Frege's *Logical Investigations* (1918–1919).

as can be checked, it is faithful to the principle, also typically modern and Fregean, that assertion remains unaffected by logic operators. For it does not matter whether φ is positive or negative, conditional or unconditional, modal or non-modal; the indicator chosen for assertion – set membership – works always in the same way. Of course the modern conception does allow distinctions which are made by the logic to be lifted to judgments, but only derivatively. Thus one may speak of a "positive judgment" or "negative judgment" of φ, to mean that φ or $\neg\varphi$, respectively, belongs to the judgment set in question; however, it is only the negation that is or is not in front of φ that differentiates the two cases.

Returning to the formal framework, we index the individuals by $i = 1, \ldots, n$, assuming that $n \geq 2$, and define the *collective judgment function*, which associates a collective judgment set to each configuration, or *profile*, of judgment sets for the n individuals:

$$A = F(A_1, \ldots, A_n).$$

Like social choice theory, logical aggregation theory usually deals with finite sets of individuals.[7] As a generalization of the Arrovian social welfare function, F formalizes the decision rules that the group would apply to the formulas in the agenda. According to its standard definition, the only one considered here, F has a *universal domain*, i.e., is defined on the set of all possible profiles, given the logical constraints imposed on judgment sets. These constraints, to be explained now, may also affect the range of F.[8]

A judgment set B can be expected to satisfy some or all of the following:

- B is *deductively closed*, i.e., for all $\varphi \in X$, if $B \vdash \varphi$ then $\varphi \in B$.
- B is *consistent*, i.e., for no $\varphi \in X$ do we have $B \vdash \varphi$ and $B \vdash \neg\varphi$.
- B is *complete*, i.e., for all $\varphi \in X$, either $\varphi \in B$ or $\neg\varphi \in B$.

Various families of judgment sets result from combining these properties. The main cases are:
- the set D of consistent and complete judgment sets, which satisfy the three properties or, equivalently, the last two (the first easily follows from them),
- the set $D^* \supsetneqq D$ of consistent and deductively closed judgment sets, as defined by the first two properties.

[7] Dietrich and Mongin (2007), and then Herzberg and Eckert (2010) and Herzberg (2010), have looked at infinite sets. Their results translate those already obtained in social choice theory, in particular by Kirman and Sonderman (1972).

[8] Logical aggregation theory is only now beginning to look at restricted domains; see List (2003) and Dietrich and List (2010).

From there, one could freely combine restrictions on the domain and range of F, but the following options are the most relevant:
(i) $F : D^n \to D$,
(ii) $F : D^n \to D^*$,
(iii) $F : (D^*)^n \to D^*$,
(iv) $F : D^n \to 2^X$ or $F : (D^*)^n \to 2^X$.
In (iv), collective judgment sets are markedly different from individual sets. This case is only given to help explain the others. In the beginning, only (i) was considered. It makes proofs easier but is called into question by cognitive psychology, which would favour weaker logical hypotheses. We can also – a more elaborate argument – question (i) by calling upon the modern notion of judgment that underlies the formalism. By ruling out abstention, D destroys the possibility it offers, unlike the ancient one, of dealing with a proposition *without having to assert it or its negation.* Also, the theory loses the distinction between internal and external negation, since "not judging" becomes equivalent to "judging that not". There is therefore more than one reason to develop the options based on D^*, i.e., (ii) and (iii).

Two ways of formalizing group decision rules suggest themselves: one can either specify F so that it coincides with a determined rule, or determine F by axiomatic conditions to represent the general principles that specific rules obey. The same two possibilities occur concerning the social welfare function, and as its record shows, one gets the most by following both paths at the same time.[9] *Proposition-wise majority voting*, which is the decision rule associated with the judiciary example, will illustrate the process. This rule is defined here as the collective judgment function $F_{maj} : D^n \to 2^X$ such that, for every profile $(A_1, \ldots, A_n)$ of the domain,

$$F_{maj}(A_1, ..., A_n) = \{\varphi \in X : |\{i : \varphi \in A_i\}| \geq q\},$$
$$\text{with } q = \tfrac{n+1}{2} \text{ if } n \text{ is odd and } q = \tfrac{n}{2} + 1 \text{ if } n \text{ is even.}$$

Note that the range is not D because there can be unbroken ties between φ and $\neg\varphi$ when n is even, and it is not even D^* in view of the judiciary example, which exhibits an inconsistent collective judgment set. Having defined proposition-wise majority, we introduce its salient normative properties. This section mentions three such properties, defined abstractly for any F, that together allow for the easy proof of an impossibility theorem – the first to have occurred in the literature. The list will be extended in section 4 with more advanced results.

Systematicity. For every pair of formulas $\varphi, \psi \in X$, and for every pair of profiles $(A_1, \ldots, A_n)$, $(A'_1, \ldots, A'_n)$, if, for every $i = 1, \ldots, n$, the

[9] According to Mongin (2003), the two ways connect the axiomatic method of social choice theory to that of logic. The definition of the rules, say majority voting, plays the role of semantic models with respect to the syntax constituted by the axioms, say IIA, and characterization theorems approximate completeness theorems proved in logic.

equivalence $\varphi \in A_i \Leftrightarrow \psi \in A'_i$ holds, then so does the equivalence

$$\varphi \in F(A_1, \ldots, A_n)) \Leftrightarrow \psi \in F(A'_1, \ldots, A'_n).$$

Systematicity means that the group, when confronted with a profile of individual judgment sets, gives the same answer concerning a formula as they would give concerning a *different* formula, when faced with a *different* profile, whenever the individual judgments concerning the first formula in the first profile are the same as the individual judgments concerning the second formula in the second profile. The rule F_{maj} clearly respects systematicity, whose analogue in social choice theory is neutrality (see d'Aspremont, 1985).

We will say that a collective judgment function F is a *dictatorship* if there is one individual j such that, for every profile $(A_1, \ldots, A_n)$,

$$F(A_1, \ldots, A_n) = A_j.$$

Given the universal domain assumption, there is only one such j per dictatorship, to be called the *dictator*. Obviously F_{maj} satisfies:

Non-dictatorship. F is not a dictatorship

and even more strongly:

Anonymity. For every profile $(A_1, \ldots, A_n)$, if $(A'_1, \ldots, A'_n)$ is obtained from $(A_1, \ldots, A_n)$ by permuting the individuals, then

$$F(A_1, \ldots, A_n) = F(A'_1, \ldots, A'_n).$$

The parallel with social choice theory is again clear. Note however that Arrow's dictator imposes only his strict preference, not his indifference, which means that dictatorship for him is not a projection property, as it is here.

The theory's first result made clear the conflict between anonymity and systematicity under a minor condition being imposed on X (List and Pettit, 2002, Theorem 1). In fact, the conflict can be expressed more strongly as that between non-dictatorship and systematicity (Pauly and van Hees, 2006, Theorem 4) and we therefore present that improved version of the result.

Theorem 1 (Pauly and van Hees, 2006, generalizing List and Pettit, 2002) Let $\mathcal{L}_\mathcal{P}$ be a classical propositional language with $\mathcal{S} = \{\neg, \wedge\}$; let $a, b \in \mathcal{P}$ be two distinct propositional variables such that $a, b, a \wedge b \in X$; then there is no $F : D^n \to D$ satisfying both non-dictatorship and systematicity.

Since F_{maj} satisfies non-dictatorship and systematicity on D^n, it must, by contraposition of the theorem, have a range other than D. When n is odd, the collective judgment sets are complete, so one of $F_{maj}(A_1, \ldots, A_n)$ must be inconsistent. This is exactly what the judiciary example in List and Pettit's version has tought, but the theoretical deduction supersedes the empirical finding, which was restricted to specific $\mathcal{L}_{\mathcal{P}}$, X and n. Beside generalizing its most famous example, Theorem 1 deepens the conceptual meaning of the discursive dilemma. "Collective rationality" is reflected in the assumption that the range of F is D, "individual responsiveness" in the non-dictatorship condition, but what about the systematicity axiom, which is related to neither? Although List and Pettit emphasize the continuity between the formal and informal analyses, it rather seems that the greatest value of Theorem 1 lies in its correcting the initial impression. The problem of collective judgment occurs in fact as a *trilemma in which systematicity is the additional element*. Unlike the other two, this property has no normative standing, and can only be defended in terms of its technical advantages. Nonetheless, it is involved just as much as the others in the impossibility conclusion.[10]

Due to its abstract generality, Theorem 1 covers many more rules than F_{maj}. We single out those variants F^q_{maj} that put a *quota of qualified majority* $1 \leq q \leq n$, i.e., the collective judgment functions $D^n \to 2^X$ defined as follows: for every profile $(A_1, \ldots, A_n)$ of the domain,

$$F^q_{maj}(A_1, \ldots, A_n) = \{\varphi \in X : |\{i : \varphi \in A_i\}| \geq q\}.$$

In the limit case where $q = n$, a formula is collectively accepted if and only if all individuals accept it, a *unanimity rule* to be compared with that of social choice theory (see Sen, 1970). Clearly, the F^q_{maj} functions cannot go to D, since some collective judgment sets are inconsistent for low q values, and others are incomplete for high q values (where proposition-wise majority voting defines the cut-off betwen "low" and "high"). The F^q_{maj} respect non-dictatorship and systematicity, so Theorem 1 also covers this finding, and the axiomatic method displays the unifying power for which it is justly celebrated.[11]

In summary, against the straightforward background of a classical propositional calculus, new concepts take their shape: the agenda; individual and collective judgment sets; and the collective judgment function that connects them. The last concept permits dealing with both specific rules and general axiomatic conditions. By this means, the discursive dilemma was recast

[10] The basic weakness of the axiom is that it cancels out semantic differences between propositions (see Mongin, 2008). When premisses and conclusions are distinguished, another problem is that it makes them interchangeable, whereas the former serve as reasons for the latter and not vice-versa (see Chapman, 2002).

[11] Quota rules are defined as in Dietrich and List (2007a), who, after Nehring and Puppe (2002, 2008), study them in detail; see also Dietrich (2010).

as an impossibility theorem about collective judgment. However, this first result only brushes the surface of the possible arguments.

4 Theorems based on the independence axiom

Pursuing the study of the voting rules, we now introduce three other salient properties that they typically satisfy, i.e., unanimity preservation, independence, monotonicity. The theorems below, which extend List and Pettit's in various ways, rely on these new axiomatic conditions. The first in the list requires the collectivity to reproduce the individuals' unanimous judgments. In the present framework, unanimity may be applied either to the judgment sets themselves, or – more strongly – to their formulas considered one by one. The parallel with systematicity, and indeed with other conditions that are to follow, is made clearer if we opt for the latter variant, which is also the closest analogue of the Pareto conditions in social choice theory.

Unanimity preservation. For every formula $\varphi \in X$ and every profile $(A_1, \ldots, A_n)$, if for every $i = 1, \ldots, n$ we have $\varphi \in A_i$, then $\varphi \in F(A_1, \ldots, A_n)$.

The second condition is a weakening of systematicity, hence the F^q_{maj} functions automatically satisfy it.

Independence. For every formula $\varphi \in X$ and every pair of profiles $(A_1, \ldots, A_n)$, $(A'_1, \ldots, A'_n)$, if for every $i = 1, \ldots, n$, the equivalence $\varphi \in A_i \Leftrightarrow \varphi \in A'_i$ holds, then so does the following equivalence:

$$\varphi \in F(A_1, \ldots, A_n) \Leftrightarrow \varphi \in F(A'_1, \ldots, A'_n).$$

Independence is the same as restricting systematicity to the case where $\varphi = \psi$. It eliminates the conceptual element of *neutrality*, i.e., of indifference to the semantic content of propositions, while preserving another conceptual element which dovetailed with it, that is: the collective judgment of φ depends only on the individual judgments of φ.[12] To put it differently, the set A is defined *formula-wise* from the sets $A_1, \ldots, A_n$. The theory can only express this idea by comparing a given profile with hypothetical profiles, in which the individual judgments of those $\psi \neq \varphi$ are different while the individual judgments of φ stay the same. The axiom, including its multi-profile formulation, is closely related to Arrow's independence of irrelevant alternatives.

[12] Despite the significant weakening, some normative objections remain (see Mongin, 2008). They should be balanced against the technical advantage that independence prevents strategic manipulations of agendas (see Dietrich, 2006).

Voting rules satisfy a classic strengthening property: when a collective result reflects the judgment of a group of voters, the result still holds if more voters join the group in their judgment. Like the related condition of positive responsiveness in one version of Arrow's theory, this requires a multi-profile formulation.[13]

Monotonicity. For every formula $\varphi \in X$ and every pair of profiles $(A_1, \ldots, A_n)$, $(A'_1, \ldots, A'_n)$, if for every $i = 1, \ldots, n$, the implication $\varphi \in A_i \Rightarrow \varphi \in A'_i$ holds, with at least one j such that $\varphi \notin A_j$ and $\varphi \in A'_j$, then the following implication holds:

$$\varphi \in F(A_1, \ldots, A_n) \Rightarrow \varphi \in F(A'_1, \ldots, A'_n).$$

Independence clearly does not imply monotonicity, and as the following example shows, neither does systematicity. A collective judgment function F is an *anti-dictatorship*, if there is j such that for every $(A_1, \ldots, A_n)$ and every $\varphi \in X$,

$$\varphi \notin A_j \Leftrightarrow \varphi \in F(A_1, \ldots, A_n).$$

Under the appropriate agenda restriction,[14] F has domain D^n and range D. It is systematic, but not monotonic, as illustrated by two profiles $(A_1, \ldots, A_n)$ and $(A'_1, \ldots, A'_n)$ such that $\neg\varphi \in A_j$, $\varphi \in A'_j$ and $A_i = A'_i$ for all $i \neq j$. Henceforth, *monotonic independence* and *monotonic systematicity* refer to the conjunction of the monotonicity axiom with the independence or systematicity axiom respectively.

If the impossibility conclusion of Theorem 1 could be derived from independence instead of systematicity, this would deepen the explanation of the discursive dilemma. The theory would then shift the problem of collective judgment to one of the two conceptual elements, namely formula-wise aggregation, from the other, neutrality. Systematicity would certainly remain in the conclusion – dictatorial functions, the only ones existing from Theorem 1, satisfy this property – and so also in the assumptions, but it would be better only to have recourse to it in the proof. In that way, one would also re-establish the parallel with social choice theory, where the strongest results deal with independence of irrelevant alternatives as an assumption, neutrality serving only as an intermediary step.[15]

This programme was realized by Pauly and van Hees (2006, Theorem 4) and Dietrich (2006, Theorem 1, Corollary 2), who posit independence as

[13] The 1951 version of Arrow's theorem relied on positive responsiveness, while the 1963 and still current version uses a Pareto condition..

[14] For every consistent subset $B \subseteq X$, the negated subset $\{\neg\varphi : \varphi \in B\}$ is also consistent.

[15] Here, logical aggregation theory and social choice theory have followed opposite paths. Arrow's 1951 theorem started with independence of irrelevant alternatives, and it was only later and for special cases that some theorems proceeded from neutrality. Fleurbaey and Mongin (2005) re-examine this sequence.

their starting point, and also by Mongin (2008, Theorem 2), who, unlike them, assume both independence and unanimity preservation to hold. Both Pauly and van Hees and Dietrich *derived* the latter condition in the course of their proofs; indeed, under their respective agenda conditions, it follows from the former. The ratio of conclusions to assumptions is impressive, but one could want to make more explicit the two conceptually very different principles that are at work simultaneously. In order to have unanimity preservation as a separate assumption, Mongin weakens independence so that it does not imply it anymore. For the relevant agendas, an impossibility theorem follows, which is closer to those of social choice theory than any of the preceding ones.

The three theorems have in common that they strengthen the very weak agenda conditions of Theorem 1; this is the price to pay for replacing systematicity by independence. Given a language $\mathcal{L}_{\mathcal{P}}$, let us say that X is *closed for propositional variables* if, for every formula $\varphi \in X$ and every propositional variable $a \in \mathcal{P}$ occurring in φ, $a \in X$. For example, $\overline{X}$ verifies this closure condition, since $\varphi = (d \leftrightarrow v \wedge b) \in \overline{X}$ and $v, b, d \in \overline{X}$. A *literal* is defined as some $a \in \mathcal{P}$ or its negation $\neg a$; it is denoted by $\pm a$. Given that X is closed by negation, the present condition requires more strongly that, for every $\varphi \in X$ and every $a \in \mathcal{P}$ occurring in φ, $\pm a \in X$.

Theorem 2 (Pauly and van Hees, 2006). Let $\mathcal{L}_{\mathcal{P}}$ be a classical propositional language with $\mathcal{S} = \{\neg, \wedge\}$; let X be closed for propositional variables, with at least two distinct propositional variables, and such that, for all $\pm a, \pm b \in X$, $\pm a \wedge \pm b \in X$; then every $F : D^n \to D$ satisfying both non-dictatorship and independence is a constant function.[16]

A collective judgment function F is *constant* if there is a judgment set A such that, for every profile $(A_1, \ldots, A_n)$, $F(A_1, \ldots, A_n) = A$. The case arises when one moves from systematicity to independence; indeed, a constant collective judgment function into D or D^* satisfies the latter but not the former.[17]

Until now we have not made - and we in general will not make - any assumption concerning the number of propositional variables. However, the following theorem is best stated with $\mathcal{P}$ finite (and so, modulo logical equivalence, $\mathcal{L}_{\mathcal{P}}$ also finite). We can then define the *atoms* of $\mathcal{L}_{\mathcal{P}}$, which are the formulas $\pm a_1, \wedge \ldots \wedge \pm a_k$, in which each of the k distinct propositional

[16] While being classical in the sense of section 3, Pauly and van Hees' propositional calculus allows for any finite number of truth values. Van Hees (2007) and Duddy and Piggins (2009) also go beyond the bivalent semantics (to which the present syntactical formalism implicitly subscribes).

[17] The restriction to D or D^* is essential. Otherwise, the following F is both constant and systematic: $F(A_1, \ldots, A_n) = X$ for all $(A_1, \ldots, A_n)$.

variables of $\mathcal{P}$ occurs. The set of atoms of $\mathcal{L}_\mathcal{P}$, to be denoted by $\mathcal{AT}_\mathcal{P}$, is the finest logical partition – class of logically exclusive and logically exhaustive formulas – for this propositional language; in other words, an atom describes a possible state of affairs with maximal precision. Dietrich shows that, if the agenda contains the atoms, Pauly and van Hees' conclusion still follows, and that it even suffices for independence to apply to these formulas rather than the entire agenda.

Theorem 3 (Dietrich, 2006). Let $\mathcal{L}_\mathcal{P}$ be a classical propositional language with $\mathcal{S} = \{\neg, \wedge\}$ and $\mathcal{P}$ finite, containing at least two propositional variables; let X include the set of atoms $\mathcal{AT}_\mathcal{P}$; then every $F : D^n \to D$ satisfying non-dictatorship and independence restricted to $\mathcal{AT}_\mathcal{P}$ is a constant function.

This statement is in fact only a special case of the original theorem, and we give a flavour of this stronger result by means of an example. Take $\mathcal{P} = \{a_1, a_2\}$ and $X = \{a_1, \neg a_1 \wedge a_2, \neg a_1 \wedge \neg a_2\}^{\pm}$. This agenda does not contain all the atoms of $\mathcal{L}_\mathcal{P}$ but nonetheless satisfies a related property, i.e., for every judgment set $B \in D$, there is a formula in X that is equivalent to the conjunction of the elements of B. Indeed, D contains only three judgment sets:

$$\{a, \neg(\neg a \wedge b), \neg(\neg a \wedge \neg b)\}, \{\neg a, \neg a \wedge b, \neg(\neg a \wedge \neg b)\}, \{\neg a, \neg(\neg a \wedge b), \neg a \wedge \neg b\},$$

and each of these can be represented by a member of X:

$$a, \neg a \wedge b, \neg a \wedge \neg b.$$

Since they describe the possible states of affairs with maximal precision, given the restriction of the language $\mathcal{L}_\mathcal{P}$ to the agenda X, these three formulas may be defined as the atoms of $\mathcal{L}_\mathcal{P}$ *relative to* X. Dietrich's result in fact applies to this extended notion of atoms, which extends its scope beyond what Theorem 3 asserts.

In this theorem, independence holds only of a subset of the agenda. The next result restricts the axiom similarly, albeit to a different subset, i.e., the set VP_X of propositional variables occurring in X.

Theorem 4 (Mongin, 2008). Let $\mathcal{L}_\mathcal{P}$ be a classical propositional language; let X be closed for propositional variables, with at least two propositional variables, and moreover satisfying the agenda conditions stated in section 7. Then there is no $F : D^n \to D$ that satisfies non-dictatorship, unanimity preservation, and independence restricted to VP_X.

Theorems 2, 3 and 4 have a common ground, which is to clarify the negative role of the independence condition. The first two essentially say that a collective judgment function degenerates if it proceeds formula-wise on an agenda whose formulas are logically interconnected. The last theorem implicitly accepts this diagnosis, since it restricts independence to the only formulas that - in a classical propositional calculus - are not logically interconnected, i.e., to VP_X. The impossibility conclusion then follows from adding unanimity preservation, and this condition becomes the new target of normative criticism. In the end, the discursive dilemma comes close to the problem of *spurious unanimity* that Mongin (1995, 1997) brought to light in the context of collective bayesianism.[18]

The judiciary example can serve to illustrate the two analyses just sketched. According to the first, the court is confronted with problems because it requires the judges to vote on each proposition considered in isolation, whereas they are logically connected by legal doctrine. According the second, even if the court makes the judges vote on logically independent propositions, it must still take care not to apply unanimity preservation unreflectingly. As it happens, the judges are not in agreement about how to make use of the legal doctrine, and this undercuts the supposed normative force of their unanimity in this circumstance.

A formal example will make the two steps of this reasoning even more explicit. Let $\mathcal{L}_{\mathcal{P}}$ be a propositional language with $\mathcal{S} = \{\neg, \vee\}$ and $\mathcal{P} = \{a, b, c\}$; let the agenda $X = \{a, b, c, a \vee b \vee c\}^{\pm}$, which fits the conditions of Theorem 4; finally, let $n = 3$ and the profile $(A_1, A_2, A_3) \in D^3$ be as follows:

$$a, \neg b, \neg c \in A_1; \neg a, b, \neg c \in A_2; \neg a, \neg b, c \in A_3.$$

By deductive closure, $a \vee b \vee c \in A_i$ for all $i = 1, 2, 3$. If the collective judgment function is F_{maj}, the collective judgment set A contains $\neg a, \neg b, \neg c, a \vee b \vee c$ and is thus contradictory. This observation illustrates the difficulty of formula-wise aggregation, given the logical connection between a, b, c established by $a \vee b \vee c$, and it reflects the spirit of Theorems 2 and 3 (the spirit but not the letter, since the chosen X does not obey their agenda conditions). Now, the contradiction would still occur if the collective judgment function were redefined as being F_{maj} on a, b, c and any unanimity-preserving F on $a \vee b \vee c$. This exactly illustrates Theorem 4, as well as the link with spurious unanimity, since the three individuals have incompatible reasons to accept the same formula $a \vee b \vee c$.

[18] Individuals can make the same expected utility comparisons although they differ both in their utility and their probability assignments. Mongin (1995) thereby explains the impossibility of collective bayesianism. Generalizing on this case, Mongin (1997) talks of *spurious unanimity* when the agreement on a collective judgment is accompanied by disagreements on the reasons for arriving at it. Nehring's (2005) abstract formalism of Paretian aggregation encapsulates related ideas.

As a matter of history, a theorem of Nehring and Puppe (2002), based on monotonic independence, came before Theorems 2–4. It was not stated in the formalism of logical aggregation theory, but it is possible to translate it to there, as the authors have shown (see Nehring and Puppe, 2010). This theorem belongs to section 6, and here, we state another result by the same authors in order to illustrate their condition of monotonic independence at work. Let us say that a collective judgment function F has a *local veto power* if there is an individual j and a formula $\varphi \in X$ such that, for every profile $(A_1, \ldots, A_n)$,

$$\varphi \notin A_j \Longrightarrow \varphi \notin F(A_1, \ldots, A_n).$$

For a given F, there can be several veto holders j, each relative to a given φ. This is a weak technical variant of dictatorship, bearing some relation to Gibbard's (1969) in social choice theory.

Theorem 5. (Nehring and Puppe, 2008). Let $\mathcal{L}_{\mathcal{P}}$ be a classical propositional language; let X be closed for propositional variables, with at least one contingent formula that is not logically equivalent to a literal. Then there is no surjective F satisfying monotonic independence and having no local veto power.

Compared to Theorems 2, 3 and 4, the axioms on F are strengthened. Independence has been supplemented with monotonicity, which, in the presence of surjectivity, can be shown to entail unanimity preservation, and the absence of veto is clearly much more demanding than the absence of a dictator. At the same time, the constraints on X are definitely reduced. Thus, various trade-offs are possible between conditions placed on the agenda and on the axioms; section 6 will develop this observation to the point of specifying meta-theoretical equivalences.

Theorem 5 weakens two results of Nehring and Puppe (2008, Theorems 1 and 2) which rely on a technical notion of their own. They define F to be an *oligarchy with default* if there are two non-empty subsets $J \subset X$ and $M \subset \{1, \ldots, n\}$ such that for every $(A_1, \ldots, A_n)$ and every $\varphi \in X$,

$$\varphi \in F(A_1, \ldots, A_n) \quad \text{iff} \quad \begin{array}{l} \text{either } \varphi \in A_j \text{ for all } j \in M, \\ \text{or } \varphi \in J \text{ and } \varphi \in A_j \text{ for some } j \in M. \end{array}$$

The members $j \in M$ are called the *oligarchs*, and the set of formulas J the *default*. In essence, if the oligarchs agree about a formula φ, it goes through to the collective judgment set, and if they are divided, then the default makes the decision between φ or $\neg\varphi$. For certain agendas (we do not give the conditions here), Nehring and Puppe show that the only F satisfying monotonic independence and surjectivity are oligarchies with default.[19]

[19] Nehring and Puppe (2010) drop the condition – called *truth-functionality* – which corresponds here to the closure of X for propositional variables. As a result, their agenda conditions become compatible with non-oligarchic collective judgment functions.

In summary, with Theorems 2–5, logical aggregation theory further deepens the discursive dilemma. Systematicity has given way to independence, sometimes posited by itself, as in Pauly and van Hees or Dietrich, sometimes strengthened by unanimity preservation, as in Mongin, and sometimes strengthened by monotonicity, as in Nehring and Puppe. The current theory favours the coupling of independence and unanimity preservation. Before introducing its results, we will return to the formal framework in order to put across its extreme generality.

5 A general logic for the theory

The theorems of sections 3 and 4 were formulated in terms of classical propositional calculi, which is restrictive. The question arises of extending them to *non-classical propositional calculi*, i.e., in which the language comprises of non-Boolean connectives and the logic has a stronger inference relation than the standard one. Equally, the question arises of extending them to *predicate calculi*, whether classical or not, which improve the analysis of propositions by using symbols for predicates, variables and quantifiers. They strengthen the preceding logics in another direction, and on the application side, they are needed to obtain social choice theorems on preference relations as corollaries of the logical aggregation theorems.

Instead of working in two steps, first by proving a logical aggregation theorem for elementary calculi, and then checking that it holds for more advanced ones, it would be better to prove it once and for all *in a general logic that encompasses all the calculi one may be interested in.* This requirement stems from Dietrich (2007a), who achieved it by axiomatizing the inference relation $\vdash$ without referring to any particular logic. We pursue the same approach using the improved axiomatization of Dietrich and Mongin (2010). Henceforth, once a theorem is proved for the general logic, it will suffice, in order for it to apply to a calculus whose language is of type $\mathcal{L}$ defined in section 4, that its inference relation obeys the axioms in question. The canonical theorem and the further results in section 6 and 7 are stated in this new formal framework.

Let us fix a binary relation $S \vdash \psi$, holding between certain sets $S \subseteq \mathcal{L}$ and certain formulas $\psi \in \mathcal{L}$. We define it to be an *inference relation*, with S being then called a *set of premisses* and ψ a *conclusion*, if it satisfies the following list of six axioms. In their statement, the notation $S \nvdash \psi$ and $\varphi \vdash \psi$ means, respectively, that $S \vdash \psi$ does not hold and that $\{\varphi\} \vdash \psi$.

(E1) There is no $\psi \in \mathcal{L}$ such that $\emptyset \vdash \psi$ and $\emptyset \vdash \neg\psi$ (non-triviality).

(E2) For every $\varphi \in \mathcal{L}$, $\varphi \vdash \varphi$ (reflexivity).

(E3) For every $S \subset \mathcal{L}$ and every $\varphi, \psi \in \mathcal{L}$, if $S \cup \{\varphi\} \vdash \psi$ or $S \cup \{\neg\varphi\} \vdash \psi$, then $S \vdash \psi$ (single-step completion).

(E4) For every $S \subset S' \subset \mathcal{L}$ and every $\psi \in \mathcal{L}$, if $S \vdash \psi$ then $S' \vdash \psi$ (monotonicity).

(E5) For every $S \subset \mathcal{L}$ and every $\psi \in \mathcal{L}$, if $S \vdash \psi$ then there is a finite subset $S_0 \subseteq S$ such that $S_0 \vdash \psi$ (compactness).

(E6) For every $S \subset \mathcal{L}$, if there is $\psi \in \mathcal{L}$ such that $S \vdash \psi$ and $S \vdash \neg\psi$, then for every $\psi \in \mathcal{L}$, $S \vdash \psi$ (non-paraconsistency).

A further property follows from these:

(E7) For every $S, T, \subset \mathcal{L}$ and every $\psi \in \mathcal{L}$, if $T \vdash \psi$ and $S \vdash \varphi$ for every $\varphi \in T$, then $S \vdash \psi$ (transitivity).

From this list, (E4) is doubtless the most important condition. It expresses the monotonicity that is typical of *deductive* inferences, as opposed to the non-monotonicity typical of *inductive* inferences, which the following example illustrates. Suppose that S says that all ravens examined up to time t are black, and ψ that all ravens are black. Now, the inductive inference from S to ψ no longer holds if S is augmented with a φ saying that a raven examined at time $t+1$ is not black. Neither the ordinary, nor the philosophical concept of judgment appears to be analytically tied with the concept of deduction; rather, they both draw upon a broader idea of reasoning that can accommodate induction. One should therefore see (E4) as a substantial restriction *on the judgments the theory is concerned with*. Incidentally, this is another reason to favour the label of *logical* aggregation promoted here.

Condition (E1) is essential for non-triviality, especially in the presence of (E4). (If it were violated for some ψ, then, by (E4), ψ and $\neg\psi$ could be concluded from every set of premisses S.) (E2) states a property that one would expect any inference, whether deductive or inductive, to have. (E3) permits suppressing unnecessary premisses, which is appropriate for deductive inferences. This condition is hidden in a more familiar one, which we will explain when discussing logical inconsistency. (E5) says that sets of premisses can be taken to be finite, a property that reflects a general concern among logicians for finiteness. Most of the results below, including the canonical theorem, need (E5), but it is used only in some specific parts of the proofs. (E6) imposes another restriction on the class of permitted inferences, but unlike (E4), it appears to be unproblematic. It excludes a group of deductive calculi - the so-called *paraconsistent* ones - which have long vexed logicians and whose peculiar situation we explain now.

Let $\mathcal{I}$ denote the set of *inconsistent* sets of $\mathcal{L}$-formulas; by definition, a set will be *consistent* if and only if it belongs to the complement of $\mathcal{I}$. One way of formalizing these notions is to define them in terms of the inference relation. According to the most standard definition in logic:

(Def*) $S \in \mathcal{I}$ if and only if for all $\psi \in \mathcal{L}$, $S \vdash \psi$.

However, paraconsistent logicians choose a weaker definition:

(Def**) $S \in \mathcal{I}$ if and only if there is $\psi \in \mathcal{L}$ such that $S \vdash \psi$ and $S \vdash \neg\psi$.
Either definition can object to the other on the ground that it gives rise to the wrong number of inconsistent sets (i.e., too many in the case of Def** and too few in the case of Def*). Mathematicians have implicitly pushed this debate aside, and (E6), which makes the two definitions coincide, reflects their views. By adopting this axiom, logical aggregation theory complies with ordinary proof intuitions and only excludes a rather uncommon family of logical calculi.[20]

Under either definition, the axiomatization (E1)–(E6) implies the following properties of $\mathcal{I}$:

(I1) $\emptyset \notin \mathcal{I}$ (non-triviality).

(I2) For every $\varphi \in \mathcal{L}$, $\{\varphi, \neg\varphi\} \in \mathcal{I}$ (reflexivity).

(I3) For every $S \subset \mathcal{L}$ and every $\varphi \in \mathcal{L}$, if $S \notin \mathcal{I}$, either $S \cup \{\varphi\} \notin \mathcal{I}$ or $S \cup \{\neg\varphi\} \notin \mathcal{I}$ (single-step completion).

(I4) For every $S \subset S' \subseteq \mathcal{L}$, if $S \in \mathcal{I}$ then $S' \in \mathcal{I}$ (monotonicity).

(I5) For every $S \subseteq \mathcal{L}$, if $S \in \mathcal{I}$, then there is a finite subset $S_0 \subseteq S$ such that $S_0 \in \mathcal{I}$ (compactness).

Like its inferential counterpart, (I1) avoids trivializing the notion to be defined. (I2) can be expected to hold when paraconsistency is put aside. (I4) is simply monotonicity restated. As to (I3), it permits completing a consistent set by a formula or its negation, a property that underlies a standard extension claim of elementary logic. In the presence of compactness - here (I5) - the step can be made from the finite consistent extension property stated in (I3) to the corresponding infinite property:

($I3^+$) For every $S \subset \mathcal{L}$, if $S \notin \mathcal{I}$, there is $T \subset \mathcal{L}$ such that (i) $S \subseteq T$, (ii) $T \notin \mathcal{I}$, and (iii), for every $\varphi, \neg\varphi \in \mathcal{L}$, either $\varphi \in T$ or $\neg\varphi \in T$ (full completion).

This is the so-called *Lindenbaum extension property*, which logicians prove from other premisses. It retrospectively justifies the definition of D in section 3. There are consistent and complete judgment sets, no matter the cardinality of the language; the infinite case raised a problem that is henceforth resolved.

The general logic can be presented in the opposite order, that is starting from the set $\mathcal{I}$ axiomatized by (I1)–(I5), and treating the relation $\vdash$ as derived. It can then be checked that $\vdash$ satisfies (E1)–(E6), which become properties rather than axioms. A new connecting definition is needed if one follows that order of doing things:

(Def***) $S \vdash \varphi$ if and only if $S \cup \{\neg\varphi\} \in \mathcal{I}$.

To reduce inference to inconsistency, as in (Def***), is no less common than to reduce inconsistency to inference, as in (Def*), and the fact that our

[20] See Priest (2002) for a survey of paraconsistent calculi and their motivations.

general logic can rely on two axiomatizations instead of one makes it easier to use in proving aggregation theorems. Since the chosen axioms or properties reflect standard deductive practice, the proofs can be carried out at the intuitive level as far the logic goes. The only exception is compactness, which one would not like to assume in all and every context, and which should accordingly be mentioned any time it comes into play (as in Dietrich and Mongin, 2010).

Whichever of the two lists is taken as a criterion, classical propositional calculi fall under the general logic. Non-classical propositional logics, on the other hand, need to be examined one by one. Among them, there are many deductive logics, as opposed to inductive or non-monotonic ones, that fulfil the criterion, but some, especially with epistemic applications, turn out not to be compact.[21] Any classical predicate calculus also obeys the general logic. In this case, the formulas of $\mathcal{L}$ are the *closed* formulas of the original language (i.e., those having no free variables in them) and the combinations of them obtained with the Boolean connectives. It is then routine to check that the inference relation of the calculus satisfies (E1)–(E6) when it is restricted to $\mathcal{L}$. There is another method to handle classical predicate calculi, which dispenses with such a direct check. It consists in extracting the propositional content from the predicate calculus by using a standard isomorphism construction (see, e.g., Barwise, 1977) and then invoking the already established fact that classical propositional calculi agree with the general logic. Of course, non-classical predicate calculi call for the same kind of reservations as their propositional counterparts.

Beyond the notions of inference and inconsistency, that we will use interchangeably out of convenience, two derived notions will enter the next theorem statements. First, a set of formulas $\mathcal{S} \subset \mathcal{L}$ is called *minimally inconsistent* if it is inconsistent and all its proper subsets are consistent. With a classical propositional calculus, this is the case, e.g., for

$$\{v, b, d \leftrightarrow v \wedge d, \neg d\},$$

but not for

$$\{\neg v, \neg b, d \leftrightarrow v \wedge b, d\}.$$

Second, given $\varphi, \psi \in X$, we say that ψ *is inferred conditionally by* φ – denoted by $\varphi \vdash^* \psi$ – if there is a *set of auxiliary premisses* $Y \subset X$ such that (i) $Y \cup \{\varphi\} \vdash \psi$ and (ii) $Y \cup \{\varphi\}$ and $Y \cup \{\neg\psi\}$ are consistent. ($Y = \emptyset$ is permitted.) Conditional inference can be reformulated as a property of minimally inconsistent sets, and that is in effect how it first arose in the theory. Indeed, under compactness (I5), the conditional inference $\varphi \vdash^* \psi$

[21] Probabilistic epistemic logics are not compact (see Heifetz and Mongin, 2001), nor are most logics of common knowledge (though some are, see Lismont and Mongin, 2002).

is equivalent to resquesting that $\varphi \neq \neg\psi$ and there be some minimally inconsistent $Y' \subset X$ with $\varphi, \neg\psi \in Y'$. Conditional inference never relates a formula to its negation. Also, it satisfies contraposition, i.e.,

$$\varphi \vdash^* \psi \Leftrightarrow \neg\psi \vdash^* \neg\varphi.$$

Before moving to the canonical theorem, we specify a notational shortcut. Sometimes, it will be necessary to transform a set $Y \subset X$ into another set Y' in which every formula of some $Z \subseteq Y$ is replaced by its negation. The set thus transformed

$$Y' = (Y \backslash Z) \cup \{\neg\varphi : \varphi \in Z\}$$

will be denoted by $Y_{\neg Z}$.

6 The canonical impossibility theorem

Beyond the fact that they restrict the chosen logic, Theorems 1 to 5 suffer from a certain imprecision. As they are formulated, they only state sufficient conditions on the agenda for there to exist no collective judgment function – or at least, no non-degenerate one – that satisfies one or more specified axiomatic properties. These hypotheses can be too strong for the conclusion, and if they are not, an additional proof of their necessity should establish this. Influenced first by Nehring and Puppe (2002, 2010), and then by Dokow and Holzman (2009, 2010a and b), logical aggregation theory has taken on the task of *characterizing*, in the sense of necessary and sufficient conditions, the agendas which turn a given list of axioms into impossibility conditions. If the results of these authors deserve, as we maintain, being called *canonical*, it is not so much because of their depth or generality, since they are far from unifying the whole theory, but rather because they have established a format of results that is now usually adopted. We will follow Dokow and Holzman's analysis, which is the more general of the two. We do not reproduce it as is, but render it into general logic, which was not the formalism intended by the authors; the significant difference will be explained at the end of the section.

Dokow and Holzman raise and solve the following aggregative problem: how to characterize the agendas X such that, if we define D with respect to X, there is no $F : D^n \rightarrow D$ that satisfies at once non-dictatorship, independence, and unanimity preservation? The answer to this problem - the mentioned canonical theorem - brings to the fore the following agenda conditions:

(a) *There is a minimally inconsistent set of formulas $Y \subset X$ and a choice of $Z \subset Y$ such that $|Z| = 2$ and $Y_{\neg Z}$ is consistent.*[22]

[22] It is equivalent to take $|Z| = 2$ or $|Z|$ even (see Dokow and Holzman, 2010, and Dietrich and Mongin, 2010).

(b) *For every pair of formulas* $\varphi, \psi \in X$*, there are formulas* $\varphi_1, \dots, \varphi_k \in X$ *such that*

$$\varphi = \varphi_1 \vdash^* \varphi_2 \vdash^* \dots \vdash^* \varphi_k = \psi.$$

Theorem 6 (Dokow and Holzman, 2010a; for sufficiency, see also Dietrich and List, 2007b). Under conditions (a) and (b), there is no $F : D^n \to D$ that satisfies non-dictatorship, unanimity preservation and independence. When the number of individuals $n \geq 3$, (a) and (b) are also necessary for this conclusion.

To illustrate Theorem 1, we reexamine $F_{maj} : D^n \to 2^X$ in the case where n is odd. Then, as was pointed out, it is equivalent to say that the range of F_{maj} is not D or that there is a profile $(A_1, \dots, A_n)$ making $F_{maj}(A_1, \dots, A_n)$ inconsistent; denote this property by (Inc). Given that F_{maj} satisfies the three axioms, Theorem 6 gives the implication (a),(b) $\Rightarrow$ (Inc). We will check this for the agenda of the judiciary example in the discursive dilemma version:

$$\overline{X} = \{v, b, d, d \leftrightarrow v \wedge b\}^{\pm}.$$

We see that (a) holds by taking:

$$Y = \{\neg v, d, d \leftrightarrow v \wedge b\} \quad \text{and} \quad Z = \{\neg v, d\}, \quad \text{or}$$
$$Y = \{v, b, d, \neg(d \leftrightarrow v \wedge b)\} \quad \text{and} \quad Z = \{v, b\},$$

or yet more choices, which suggest that (a) is easy to fulfil despite being complex to specify. As for (b), it is also satisfied, as shown in Figure 1. (We write q for $d \leftrightarrow v \wedge b$, the arrows indicate conditional inferences, and the lower-case characters auxiliary premisses that can be used to make these inferences.)

The above exemplifies (a) and (b) in their role as *sufficient* conditions for the impossibility theorem, and we still have to illustrate why they are *necessary*. We do that while taking up the argument of section 2, which suggests that there are other ways than List and Pettit's discursive dilemma to reconstruct Kornhauser and Sager's doctrinal paradox. Section 7 explores one way by replacing the classical biconditional with a non-classical biconditional in the legal doctrine formula, and this will bring about that condition (b) fails. A less drastic change, to be considered now, is to keep a classical formula for the doctrine and *make it part of the inference relation instead of the agenda*. This formalizes the possibility, which Kornhauser and Sager envisage, that the court determines its decision from a common doctrine that it does not put to vote. Let, then, the new inference relation $\vdash_{d \leftrightarrow v \wedge b}$ be defined by:

$$S \vdash_{d \leftrightarrow v \wedge b} \text{ iff } S \cup \{d \leftrightarrow v \wedge b\} \vdash \varphi,$$

with the correspondingly small agenda:

$$\overline{\overline{X}} = \{v, b, d\}^{\pm}.$$

Given the new conditional entailments, $\overline{\overline{X}}$ satisfies (a) but not (b); this is shown in Figure 2. Thus, when applied to the doctrinal paradox agenda, Theorem 6 entails a possibility result, i.e., there exists an $F : D^n \to D$ that is non-dictatorial, unanimity preserving, and independent. An example is the function $\widetilde{F}^n_{maj}$ defined on D^n as follows: for every *positive* formula $\varphi \in X$, it respects unanimity if either φ or $\neg\varphi$ belongs to all individual judgment sets, and in case of a split choice, it always chooses $\neg\varphi$. That is to say, $\widetilde{F}^n_{maj}$ strengthens the unanimity rule F^n_{maj} so as to make its collective judgment sets complete, and $\overline{\overline{X}}$ is such that these sets remain consistent as in F^n_{maj}.[23]

The statement of Theorem 6 can be simplified when the focus of attention shifts from general F to specific cases. Consider again F_{maj}. We know from one part of Theorem 6 that (a), (b) $\Rightarrow$ (Inc), but can we improve the implication in this instance? It turns out that (c) $\Longleftrightarrow$ (Inc), where the new condition is:

(c) *There is a minimal inconsistent set of formulas* $Y \subset X$ *such that* $|Y| \geq 3$.

It is easy to prove the sufficiency claim by constructing a profile $(A_1, \ldots, A_n)$ such that $Y \subset F_{maj}(A_1, \ldots, A_n)$. As for necessity, it follows from the remaining part Theorem 6 and the easy proof that (b) implies (c). For the latter, take $\varphi \in X$. Then, by (b), there is a chain of conditional inferences:

$$\varphi = \varphi_1 \vdash^* \varphi_2 \vdash^* \ldots \vdash^* \varphi_k = \neg\varphi.$$

In the absence of (c), this chain would reduce to:

$$\varphi = \varphi_1 \vdash \varphi_2 \vdash \ldots \vdash \varphi_k = \neg\varphi,$$

which is impossible because the general logic secures the transitivity of logical inference.

Why has (a) disappeared and (b) been weakened so much when $F = F_{maj}$? Heuristically, this must relate to properties of the function that Theorem 6 does not mention, and two of them stand out, which are monotonicity and systematicity. The following result, specifically part (iii), supports this analysis.

[23] An inconsistent collective judgment set would have to include one of the following minimally inconsistent subsets of $\overline{\overline{X}}$: $\{v, r, \neg d\}$, $\{\neg v, d\}$ or $\{\neg r, d\}$. However, each case is ruled out by the definition of $\widetilde{F}^n_{maj}$.

Theorem 6'. **(i)** Under conditions (a) and (c), there is no $F : D^n \to D$ that satisfies non-dictatorship, unanimity preservation and systematicity. When $n \geq 3$, (a) and (c) are also necessary for this conclusion.

(ii) Under condition (b), there is no $F : D^n \to D$ that satisfies non-dictatorship, unanimity preservation and monotonic independence, and (b) is also necessary for this conclusion.

(iii) Under condition (c), there is no $F : D^n \to D$ that satisfies non-dictatorship, unanimity preservation, systematicity and monotonicity. When $n \geq 3$, (c) is also necessary for this conclusion.

Each of these statements has been proved separately, and in particular, (ii) is the version of the canonical theorem established by Nehring and Puppe (2002, 2010). Today it is better to consider (i), (ii) and (iii) as being partial results leading to Theorem 6. Comparing them permits locating what constraint on X is equivalent to a given axiom placed on F, and in this way, the trade-off that is so typical of the new theory comes out most rigourously.

All of the preceding results allow for variants based on D^* rather than D. In a nutshell, these turn F into an oligarchic rather than a dictatorial collective judgment function, a somewhat less obvious form of degeneracy. By definition, F is an *oligarchy* if there is a non-empty subset $M \subseteq \{1, \ldots, n\}$ such that, for all $(A_1, \ldots, A_n)$,

$$F(A_1, \ldots, A_n) = \bigcap_{j \in M} A_j.$$

If F is an oligarchy, M is unique and will be called the *set of oligarchs.* Dictatorship is the particular case where M is a singleton. In section 4, we encountered a stronger and less standard notion of oligarchy; the present one is directly in line with social-choice-theoretic work. It is a fact of elementary logic, also secured by general logic, that the intersection of consistent and deductively closed sets retains both these properties; as a result, if F is defined on D^n or $(D^*)^n$ and it is an oligarchy, then its range is D^*. As nothing is specified to settle disagreements between the oligarchs, F will often produce incomplete collective judgment sets. This can be seen, e.g., from the quota rule F^n_{maj}, which corresponds to the maximal set $M = N$.

Formally, the new axiomatic condition:

Non-oligarchy. F is not an oligarchy

leads to the following impossibility results:

Theorems 7 and 7'. The statements are the same as those of Theorems 6 and 6', with $F : D^n \to D$ being replaced by $F : D^n \to D^*$ or $F : (D^*)^n \to D^*$, and non-dictatorship being replaced by non-oligarchy.

These various extensions can be found in Dietrich and List (2008) and Dokow and Holzman (2010b).[24] Like the initial results, they should be compared with related ones in social choice theory. Put briefly, each logical aggregation theorem induces a social choice theorem *via a suitably selected logical calculus*. It will typically be a fragmentary classical predicate calculus, whose language has one or more binary relation symbols to represent preferences. Axioms formulated in this language will capture the properties of preferences that one is willing to assume, such as transitivity and the like. The inference relation of the chosen predicate calculus will have to be augmented with those preference axioms, in exactly the same way as the inference relation of the judiciary example was made here to include the legal doctrine formula. That is the method followed by Dietrich and List (2007b) to derive from (the sufficiency part of) Theorem 6 a version of Arrow's theorem, in which there occur only strict preferences. They introduce a classical predicate calculus, whose language $\mathcal{L}_{\succ}$ is built from basic formulas $x \succ y$ (interpreted as "x is strictly preferred to y") and whose inference relation $\vdash_{\succ}$ incorporates the three properties of asymmetry, transitivity and completeness. As an agenda $X \subset \mathcal{L}_{\succ}$, they simply take the set of basic formulas. The proof consist in showing, first, that conditions (a) and (b) hold of X, and second, that Arrow's "social welfare function", with its relevant set of axioms, can be associated with an F meeting the conditions of Theorem 6.

To obtain Arrow's theorem in its entirety – that is to say, with weak preferences – Dokow and Holzman (2010b) take a detour via Theorem 7, which did not seem apt for this goal.[25] That same theorem, however, does straightforwardly entail Gibbard's (1969) concerning oligarchies. The field of social-choice-theoretic applications has barely been opened. Up to now, they have related to unrestricted domains of options and preferences, as in Arrow and Gibbard, rather than to specialized "economic" or "political" domains (such as those described by Le Breton and Weymark, 2003). It is easy to see why: the more concrete the domain, the more problematic it is to describe by means of a logical language.

We have stated the results of this section in terms of the general logic, which gives them wide applicability, but this presentation does not accurately reflect the historical process of discovery, which went through various technical hypotheses, each of them more restrictive than ours. Dokow and Holzman, for their part, use a formalism called *abstract aggregation*, which goes back to Fishburn and Rubinstein (1986), Wilson (1975) and Guilbaud

[24] The early oligarchic result of Gärdenfors (2006) imposes unnecessarily strong conditions on the agenda for impossibility.

[25] Dietrich (2007b) also obtains the full Arrow's theorem, but in a more complex framework than that of the canonical theorem (he assumes that formulas have relevance relations in addition to logical relations). Nehring's (2003) early derivation does not formally belong to logical aggregation theory.

(1952). Starting from a finite number k of propositions that correspond to the positive formulas of our agendas, they render the individual and collective judgments concerning these propositions by the values 0 or 1 that the individuals or the collectivity attributes to them. Thus, after fixing an arbitrary order on propositions, they can reduce the aggregative problem to the study of subsets of $\{0,1\}^k$ and of functions defined from these subsets. If $\mathcal{E} \subset \{0,1\}^k$ represents the set of admissible judgment sets, then $G : \mathcal{E}^n \to \mathcal{E}$ represents a collective judgment function, the analogue of our $F : D^n \to D$. The conditions defined on F, starting from independence and unanimity preservation, are easy to formulate in terms of G.

Such a terse statement of the aggregative problem yields quick and elegant proofs, as Guilbaud had already foreshadowed, but it tends to erase the *logical* and *linguistic* properties of judgments, along with certain conceptual distinctions that flow from these properties. The stage of defining the agenda is absorbed into the - one would expect, later - stage of defining what judgment sets are allowed. Sometimes, the same $\mathcal{E}$ corresponds to different agendas. For example, with $k = 2$, take the set

$$\mathcal{E} = \{(1,1),(0,1),(0,0)\}.$$

In a classical propositional logic, there are at least two agendas that could give rise to this, i.e.,

$$X = \{a, a \vee b\}^{\pm} \text{ and } X' = \{a \wedge b, a \to b\}^{\pm},$$

and it would be a conceptual abuse to treat them as they were the same. (This example comes from List and Puppe, 2009). Another relevant distinction, that between D and D^*, cannot be stated in the abstract aggregation framework as naturally as it is in the present logical framework. As a secondary technical issue, we may perhaps add that the initial assumption of a finite number k of propositions is too sweeping. Thanks to the flexible use of compactness, the general logic here has an advantage, the cost of which is a certain unwieldiness.

The set-theoretical formalism just discussed should not be confused with those expositions that differ from ours, less drastically, *by emphasizing the semantics over the syntax.* For example, Pauly and van Hees (2006) describe individual and collective judgments in terms of Boolean valuations, rather than sets of formulas, but they also have a formal language and, implicitly, a logic. They have simply made a choice of emphasis within a dual framework.[26]

[26] There are many other ways in which the theory lends itself to the logician's work. Here are two recent examples. Pauly (2007 and 2009) reformulates the acceptance of formulas in terms of a modal operator, rather than by set-theoretic membership, as is done here. Cariani, Pauly and Snyder (2008) define on collective judgment functions a condition of language invariance that leads to a new impossibility result.

To sum up, the discursive dilemma has guided all theoretical developments thus far. Section 3 had made clear that it was a trilemma, and section 4 that the omitted branch was independence rather than systematicity. As it appears from the present section, it is really a *quadrilemma*, with the definition of the agenda as the last branch, because it can be also resolved by dropping either condition (a) or (b). It now remains to be seen if the doctrinal paradox can be submitted to a such a thorough analysis.

7 Back to the doctrinal paradox

Briefly put, the doctrinal paradox requires that premisses and conclusions be distinguished, and it takes legal doctrine to be central and specific. We will reexamine these two aspects in turn and show – as is the final aim of the article – that it is possible to do justice to this problem using the results of the preceding sections once they are appropriately generalized.

In an axiomatized theory like the present one, the distinction between premisses and conclusions can only be made clear if at least one axiom takes care of this. For suppose otherwise, that every condition on F would apply equally to both sets of formulas; then, because the theory does not have the expressive means to separate them, we would have to choose between rejecting it, as being too crude, and rejecting the distinction itself, as being pointless. The unresolved debate between Kornhauser and Sager, on one side, and List and Pettit, on the other, seems to reflect these alternatives. We escape from this choice by selecting an axiom that encapsulates the distinction that needs expressing. The canonical theorem leaves the choice between independence and unanimity preservation, since non-dictatorship is used only to state impossibility. Let P and C be subsets of X that represent the premisses and the conclusions; we assume for simplicity that they form a partition of X. It is *independence* to which we give the discriminating role, reserving it to P, while keeping unanimity preservation applied to X.

This asymmetry between axioms can be justified by legal theory itself. In Kornhauser and Sager's classic example, the decisions about the issues are taken following a majority vote on each of them, and to impose on P both independence and unanimity preservation is merely to generalize this procedure abstractly. On the other hand, if we read them well, a decision on the case can be taken differently from by a formal vote, be it simple majority or otherwise. Respecting consensus seems to be the only norm that the procedure must then guarantee, which is what our single condition placed on C translates. A further supporting argument is that, given the assumed partition of X, the legal doctrine may fall in C rather than in P - this happens in one of the reconstructions below - and Kornhauser and Sager, as opposed to List and Pettit, do not always want to subject it to a formal vote by the judges. The agreement results, again here, from a procedure that is not entirely clear and about which we can only say that it respects unanimity preservation.

Theorem 8 shows that, under agenda conditions close to those of the canonical theorem, the new set of axioms proposed for F forces it to be a dictatorship. Theorem 9 states the corresponding oligarchic result. Variants that we do not present here would reproduce the two impossibility conclusions by supplementing independence with systematicity, monotonic independence or monotonic systematicity. The analysis, due to Dietrich and Mongin (2010), improves on the canonical theorem and its variants by recovering their conclusions from weaker hypotheses, independence or the related axioms being now applied to P instead of the whole of X. On the conceptual level, the analysis puts the doctrinal paradox to the test, and it escapes consolidated. There is no need for majority voting in order for the method of premisses and the method of conclusions to conflict: it is enough to state some of the broad conditions that they satisfy. Also, the premiss-based way, which is offered as a solution to the paradox more often than its rival, the conclusion-based way, appears to be in trouble. Indeed, it is enough to assume a modicum of conclusion-based way - unanimity preservation - on top of the premiss-based way for the latter to fall into the dictatorial trap.[27]

Formally, we define $P = \{p, \neg p, q, \neg q, \ldots\}$ as any non-empty set of X that is closed by negation. The new axioms on F revise those of the canonical theorem in terms of the partition between P and $C = X \setminus P$.

Systematicity (resp. Independence) of premisses: only for each pair of formulas $p, q \in P$ (resp. every formula $p \in P$).

Unanimity preservation for premisses (resp. conclusions): only for every formula $p \in P$ (resp. every formula $p \in C$).

Non-dictatorship for premisses: There is no $i = 1, \ldots, n$ such that $F(A_1, \ldots, A_n) \cap P = A_i \cap P$ for every $(A_1, \ldots, A_n) \in D^n$.

Non-oligarchy for premisses: There is no nonempty subset $M \subseteq \{1, \ldots, n\}$ such that $F(A_1, \ldots, A_n) \cap P = (\bigcap_{j \in M} A_j) \cap P$ for every $(A_1, \ldots, A_n) \in D^n$.

New agenda conditions also take the partition of P and C into account:

(a_P) There is a minimally inconsistent set of formulas $Y \subset X$ and a choice of $Z \subseteq Y \cap P$ such that $|Z| = 2$ and $Y_{\neg Z}$ is consistent.

[27] See Nash (2003). For their part, Kornhauser and Sager choose between the issue-by-issue and the case-by-case method depending on the instance. They also recommend that the court take a "meta-vote" on the procedure first.

(b_P) For every pair of formulas $p, q \in P$, there are formulas $p_1, \ldots, p_k \in P$ such that

$$p = p_1 \vdash^* p_2 \vdash^* \ldots \vdash^* p_k = q.$$

Remember that $p \vdash^* q$ means $Y \cup \{p\} \vdash^* q$ for some choice of $Y \subset X$; thus, the formulas of P can be logically related by formulas of C.

(c_P) There is a minimally inconsistent subset of formulas $Y \subset X$ such that $|Y \cap P| \geq 3$.

Theorem 8. Under conditions (a_P) and (b_P), there is no $F : D^n \to D$ that satisfies all of the following:

- for premisses, non-dictatorship, independence and unanimity preservation,
- for conclusions, unanimity preservation.

For a number of individuals $n \geq 3$, (a_P) and (b_P) are also necessary for this conclusion.

Theorem 9. The statement is like that of Theorem 8, with $F : D^n \to D^*$ replacing $F : D^n \to D$, and non-oligarchy for premisses instead of non-dictatorship for premisses.

Since the subset P is arbitrary, Theorems 6 and 7 can be obtained by setting $P = X$, and similarly Theorems 6' and 7' are special cases of those that we do not state. The agenda characterization for F_{maj} is also covered; indeed it can be checked that (b_P) entails (c_P) and that this last condition is necessary and sufficient for there to be some inconsistent set $F_{maj}(A_1, \ldots, A_n) \cap P$.

To strengthen the definition of premisses also strengthens the results. It would fit the ordinary notion if the formulas of P had a logical connection with those of C, and more precisely, if they axiomatized every judgment set $B \in D$. Let us therefore introduce the agenda condition:

(d_P) *For every* $B \in D$,

$$B = \{\varphi \in X \mid B \cap P \vdash \varphi\}.$$

With (d_P), Theorem 8 leads to dictatorship (on the whole of X). From this corollary, Theorem 4 follows if one specializes the general logic under the form of a classical propositional calculus. In this case, the choice of $X = PV_X$ automatically satisfies (d_P). Recall that section 4 did not fully state the conditions that Theorem 4 placed on X; the missing ones are actually (a_P) and (b_P). This also establishes the generality of the analysis.

The basic conditions (a_P) and (b_P) can be illustrated by new variants of the judiciary example. Fix the agenda as $\overline{X}$, which contains the formula $q = d \leftrightarrow v \wedge b$ for the legal doctrine. On the one hand, if $P = \{v, b\}^{\pm}$, then (b_P) is violated, and legal decisions escape from the impossibility result, contrary to what the canonical theorem would predict (see Figure 3). As (c_P) is also violated, F_{maj} no longer has any drawback (at least if n is odd). On the other hand, if $P = \{v, b, q\}^{\pm}$, then (b_P) is satisfied (see Figure 4). Furthermore, (a_P) is satisfied; so, for this P, legal decisions fall into the impossibility predicted by the canonical theorem.

Now, take $\overline{\overline{X}} = \{v, b, d\}^{\pm}$, i.e., the agenda that is associated with the modified inference relation $\vdash_q$ of the last section. With $P = \{v, b\}^{\pm}$, a natural choice, (b_P) is violated; e.g., no conditional inference ever reaches a positive formula (see Figure 5). Even (c_P) is violated, and so F_{maj} becomes unobjectionable. This case reflects the exclusive use of the premiss-based way, which is the simplest way out of the doctrinal paradox.

The agendas just discussed illustrate Theorem 8 mathematically while having some legal relevance, but it is doubtful that they represent legal doctrine appropriately. Section 2 asked - the question labelled (i) - whether the doctrine is can be rendered by ordinary logic. We now argue that the Boolean biconditional $\leftrightarrow$ in the formula $d \leftrightarrow v \wedge b$ has undesirable effects that can be avoided by the non-Boolean operator $\hookleftarrow\!\hookrightarrow$ of conditional logic. The argument parallels that which Dietrich (2010) uses more generally in favour of such logics.[28]

The Boolean agenda $\overline{X}$ yields the following list of minimally inconsistent subsets:

$$\begin{array}{ll} Y_1 = \{\neg b, d, q\}, & Y_2 = \{\neg v, d, q\}, \\ Y_3 = \{v, b, \neg d, q\}, & Y_4 = \{q, \neg q\}, \\ Y_5 = \{v, b, d, \neg q\}, & Y_6 = \{\neg v, \neg d, \neg q\}, \\ Y_7 = \{\neg b, \neg d, \neg q\}. & \end{array}$$

That Y_5, Y_6, Y_7 are inconsistent is somewhat counterintuitive. Here, judges deny that d is equivalent to v and b, and this allegedly clashes with certain positions they take on v, b or d. It seems that they may *consistently* deny the equivalence and accept these positions. Specifically, suppose that they have in mind another issue s that is not mentioned here and hold that d is equivalent to $v \wedge b \wedge s$. In this case, they may deny that d is equivalent to $v \wedge b$ and nonetheless:

[28] As is well-known, the calculi of conditional logic overcome the paradoxes of "material" (Boolean) implication, another example of which is given in the next paragraph. We may only refer to the classic work by Stalnaker (1968) and Lewis (1973). For a review, see Nuete and Cross (2001).

- accept v, b, and d because they accept s (contrast with the alleged inconstency of Y_5);
- reject v and d, whatever their views on b and s (contrast with the alleged inconsistency of Y_6);
- reject b and d, whatever their views on v and s (contrast with the alleged inconsistency of Y_7).

To put it otherwise, the following theorem of classical propositional logic

$$(*)\neg(d \leftrightarrow v \wedge b) \dashv\vdash \neg d \leftrightarrow v \wedge b$$

contradicts normal intuitions of legal deliberation. Now, returning to the table, we see there is nothing intuitively wrong either with the sets Y_1 to Y_5, or with the corresponding theorems in classical propositional logic:

$$(**)\{d \leftrightarrow v \wedge b, v, b\} \vdash d, \{d \leftrightarrow v \wedge b, d\} \vdash v, \{d \leftrightarrow v \wedge b, d\} \vdash b.$$

One would indeed expect that accepting the doctrine, as opposed to refusing it, entail the consequences derived formally.

The calculi of conditional logic axiomatize the conditional, $\hookrightarrow$, and so the biconditional, $\hookleftarrow\!\hookrightarrow$, in a way that exactly fits the divided intuitions. They give rise to a list of minimally inconsistent subsets that is reduced Y_1–Y_5, or equivalently, only retain the theorems $(**)$ excluding $(*)$. That is why they seem to us capture the doctrinal paradox so much better than classical propositional calculi. It is not necessary to decide between the various different systems, since all satisfy the general logic and any of them can do for the purpose.

We would have then to replace $\overline{X}$ with $\overline{X}' = \{v, b, d, q'\}$ with $q' = d \hookleftarrow\!\hookrightarrow v \wedge b$. How does this agenda fare with respect to the canonical theorem? It still satisfies (a), but not (b), so that the negative conclusion obtained for $\overline{X}$ in the preceding section no longer holds (see Figure 6 and compare it with Figure 1). Concerning the theorems of this section, the salient fact is that (b_P) does not hold for any of the choices for P that we have envisaged. Thus, the negative conclusion is again beaten back.

Now, what about the agenda $\overline{\overline{X}} = \{v, b, d\}^{\pm}$ when the formula placed in the inference rule is non-classical? The rule defined by

$$S \vdash_{q'} \varphi \text{ iff } S \cup \{q'\} \vdash \varphi,$$

leads to the same violations of (b) and (b_P) as $\vdash_q$ did above. Indeed, only *accepting* the doctrine is a possibility, and as was said before, classical and non-classical equivalences collapse onto each other in this case; to put it another way, the minimally inconsistent subsets obtained with $\vdash_{q'}$ and those obtained with $\vdash_q$ are the same, modulo replacing q with q'.

To summarize the main results of this section, when legal doctrine is *internal* to the agenda (case $\overline{X}'$), departing from classical logic gives way to additional possibilities, but nothing is gained when legal doctrine is *external* (case $\overline{\overline{X}}$). Is there some way to decide between internal and external representations, as we have just done between classical and non-classical inferences? We do not think so. The discursive dilemma automatically imposes the internal representation, and in section 2 - see our point (ii) - we questioned this choice, but this was not to say that the opposite one was compelling. Actually, each may be justified according to the circumstances. Legal theory only suggests that judges do not *normally* vote on the doctrine, not that they never do so. The usually best model is given by $\overline{\overline{X}} = \{v, b, d\}^{\pm}$ with the rule $\vdash_q$ (or equivalently $\vdash_{q'}$), but $\overline{X}'$ will nonetheless be sometimes appropriate. The only agenda we exclude is $\overline{X}$, that of the discursive dilemma, which signals where our analysis departs from the standard one.

8 Conclusion and some open questions

With the previous analysis of the doctrinal paradox, our interpretive account of logical aggregation theory has drawn to a close. Our guiding heuristic was that this problem had been underrated, compared with its discursive dilemma variant, and that it called for its own analytical treatment. We have shown how the current work, as epitomized by the canonical theorem, could be revised so as to take notice of the paradox and deepen its explanation. This move illustrates the flexibility and expressive power of the framework collectively put in place in the 2000s. Notice however that using logic is essential to the changes we suggest, and not all current contributors approve of the logical turn taken by judgment aggregation theory.

By and large, the doctrinal paradox appears to be less of a problem for collective judgment than does the discursive dilemma. The simplest reason is that the premiss-based approach, which is not even definable in the context of the latter problem, offers a satisfactory way out in many occurrences of the former. This was illustrated by the last section: the impossibility part of our theorem applies to the toy judiciary example only for an unintuitive construal of the set of premisses. However, this unfavourable case is a warning that the premiss-based method is not immune to logical impossibilities, contrary to what is generally believed. It is enough to assume a dose of conclusion-based method, i.e., unanimity preservation on the conclusions, to bring about impossibilities for relevant agendas. Legal theorists who are keen on the premiss-based method still have to take this finding into account. However, another finding goes in their direction: to render the legal doctrine in a non-classical logic may block the threatening impossibility for the premiss-based method, even if they grant the dose of conclusion-based method just considered.

Returning to the account of the more standard work, we should emphasize that it has not yet reached its definitive stage. To begin with, the canonical theorem has fixed a format of results that is not yet applied everywhere. In particular, the early Theorems 1, 2 and 3 should be revisited. They provide only sufficient agenda conditions for the impossibility they state, and because they derive it without the help of unanimity preservation, they are not covered by the canonical theorem or our generalization. In bringing them to the format, one may hope to clarify two theoretical issues, i.e., what agenda conditions are both necessary and sufficient for independence to entail unanimity preservation, and what impossibilities, if any, surround independence when this entailment does not hold.

In social choice theory, independence of irrelevant alternatives and the Pareto conditions are logically independent conditions, and an impossibility has famously been derived by Wilson (1972) from the former condition alone. Thus, by answering the previous group of questions, one would further tighten the connection with the antecedent theory. As we mentioned in section 6, much remains to be done on this score generally. The existing proofs of Arrow's and Gibbard's theorems indicate the natural direction for further work to proceed, i.e., from logical aggregation theory to social choice theory. However, one may wonder whether the other direction is feasible. Could a suitably doctored variant of Arrow's impossibility theorem entail a corresponding result in logical aggregation theory? Many believe that this reverse programme is a non-starter, but few have actually tried their hands at it.

A no doubt more pressing task would be to complement today's negative conclusions by a richer array of positive solutions. It would be nonsense to complain that the theory is *exclusively* negative, since every theorem stated in the canonical format can be read in the positive way, as sections 6 and 7 illustrated at length. However, after so much emphasis laid on agenda conditions, more work should be done on the axioms put on the collective judgment function. Computer scientists have opened an interesting avenue when defining *merging rules on belief sets*, and several writers - starting with Pigozzi (2002) - have recommended that logical aggregation theory borrow from this technology. A belief set is essentially the same as a judgment set in the syntactical formulation developed here, and an important class of merging rules, i.e., the *distance-based* ones, are especially easy to accommodate by this formulation. They amount to minimizing the distance from the collective set to the given profile, where the notion of distance between two sets can be defined variously from the logical language; see Konieczny and Pino-Perez (2002) for the basic principles and Miller and Osherson (2008) for relevant elaborations. By this means, the impossibility of logical aggregation is circumvented by giving up independence. To relax unanimity preservation would lead to different possibilities, but regrettably, these have been hardly discussed thus far.

The most drastic resolution of all is to move from the logical to the *probabilistic* framework, as many economists brought up in the Bayesian tradition would no doubt recommend. This move is somehow comparable with the change undergone by social choice theory in Sen's (1970) hands, i.e., when the "social welfare functional", defined on profiles of individual utility functions, replaced the "social welfare function" defined on profiles of individual preference relations. Inspection of the existing results for this richer framework shows that the same selection of axioms, *mutatis mutandis*, leads to convex combinations of probability measures instead of dictatorships or oligarchies. However, these positive solutions degenerate for suitable strengthenings of the axioms, and the richer framework needs justifying anyhow. In social choice theory, the observer or planner may be unable to define the "informational basis" that would allow him to make trade-offs between the individuals' conflicting interests, and by the same token, the group's representative may be unable to quote the numerical degrees of certainty that would allow him to balance the individuals' conflicting opinions against each other.

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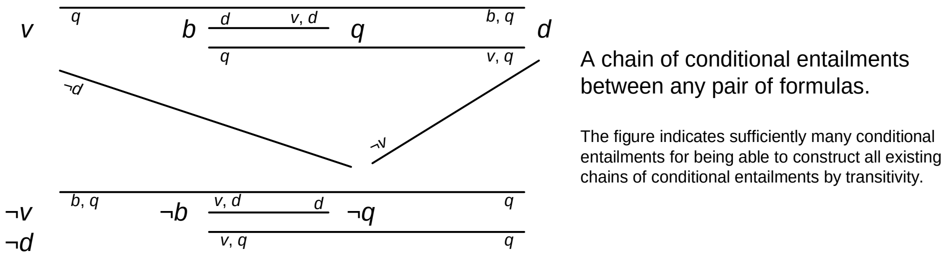

Figure 1: The agenda $\overline{X}$ satisfies (b).

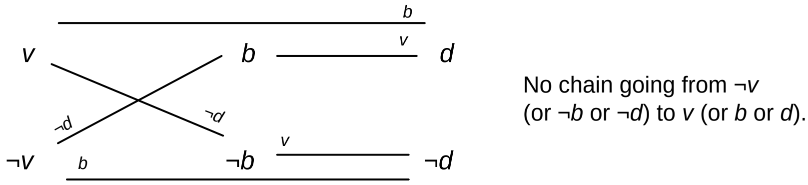

Figure 2: The agenda $\overline{\overline{X}}$ violates (b).

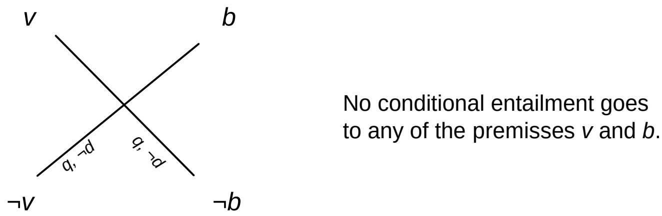

Figure 3: The agenda $\overline{X}$ with $P = \{b, v\}^{\pm}$ violates (b_P).

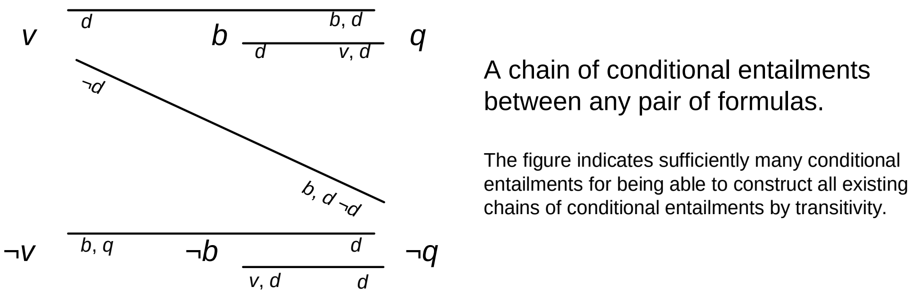

Figure 4: The agenda $\overline{X}$ with $P = \{b, v, q\}^{\pm}$ violates (b_P).

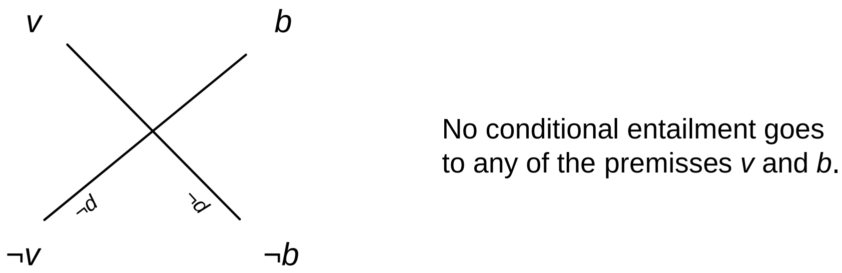

Figure 5: The agenda $\overline{\overline{X}}$ with $P = \{b, v\}^{\pm}$ violates (b_P).

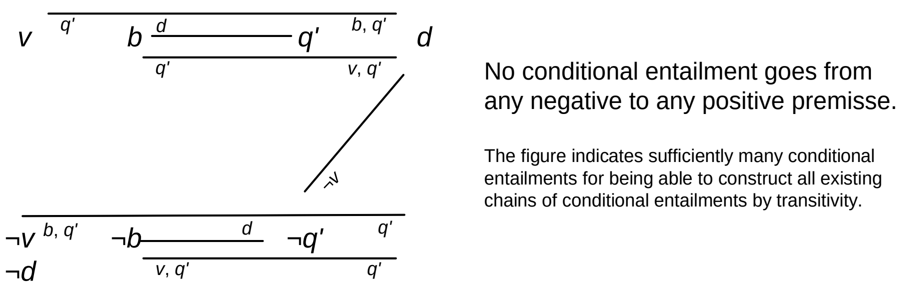

Figure 6: The agenda $\overline{X}'$ violates (b).