# AGGREGATION and RESIDUATION (non final version)

BRUNO LECLERC

*Centre d'Analyse et de Mathématique Sociales, École des Hautes Études en Sciences Sociales, 190 avenue de France, 75270 Paris Cedex France*, Bruno.Leclerc@ehess.fr

BERNARD MONJARDET

*Centre d'Économie de la Sorbonne, Université Paris I, MSE, 106-112 boulevard de l'Hôpital, 75647 Paris Cedex 13, France and CAMS, EHESS, Bernard.Monjardet@univ-paris1.fr*

**Abstract.** In this note we give a characterization of meet-projections in simple atomistic lattices which generalizes previous results on the aggregation of partitions obtained in a cluster analysis framework.

**Mathematics Subject Classifications (2010).** 06A5, 6207, 91B14.

**Key words:** Consensus theory; dependence relation; meet projection; partition; residual map; simple lattice.

## 1. INTRODUCTION

In his celebrated 1951 book *(Social Choice and Individual Values* [1]) Arrow proved that a rule to aggregate individual preferences into a collective preference, and satisfying some apparently natural conditions, must be "dictatorial". When the $n$ individual preferences are modelled by linear orders his result amounts to axiomatically characterizing *projection*s (where these projections are the $i^{\text{th}}$ projection maps associating with any n-tuple $(L_1,\ldots,L_n)$ of linear orders its $i^{\text{th}}$ coordinate $L_i$). A crucial property to obtain Arrow's result is the so-called *independence axiom* requiring that the collective preference on two alternatives must depend on the individual preferences on these only two. When applied to other types of relations like partial orders or equivalences this independence property leads to characterizations of *meet-projections*: the collective relation is a meet of some individual relations. In social choice theory, such a rule is called "oligarchic".

It is well-known that the sets of equivalences and of partial orders are lattices for the inclusion order after adjoining a greatest element to the latter. More generally one may consider the consensus (or aggregation) problem on a lattice $L$, consisting in defining "satisfactory" (i.e., satisfying some natural properties) maps associating an element of $L$ with each tuple of elements of $L$. We have shown in [18,16] that the oligarchic results obtained by Brown [4] on partial orders, by Mirkin and Leclerc [17,15] or Neumann and Norton [19] on equivalences (as well as other similar results) are particular cases of a general result on the characterization of such maps for a lattice satisfying various properties. In this note, we show that one can obtain a general result by replacing the *decisivity* property, which is a lattice form of independence, with a purely lattice property, namely the *residuation* property. On the one hand this result generalizes results obtained in the case of set partitions in [7,11] and, in the other hand it gives a characterization of meet-projections in simple atomistic lattices.

## 2 NOTATIONS AND PRELIMINARIES

### 2.1 LATTICE THEORY

Throughout this paper lattice means finite lattice. Such a lattice is denoted by $(L, \wedge, \vee, 0, 1)$ or simply by $L$. For standard notions in Lattice Theory not recalled here see for instance [8].

We denote by $J$ the set of all the *join-irreducible* elements (i.e. of the elements not the join of other elements) of $L$. A lattice is *atomistic* if all its join-irreducible elements are *atoms* (i.e., elements covering 0).

We recall some facts on Residuation Theory (equivalently, on covariant Galois connections). See [3, 5, 12] for details, the latter containing a thorough history of these notions. Let $F$ be a map from a lattice $L$ to a lattice $L' = (L', \wedge', \vee', 0', 1'))$:

$F$ is a *residual map* (respectively, a *residuated map*) if $F$ is a $\wedge$-morphism satisfying $F(1) = 1'$ (respectively, a $\vee$-morphism satisfying $F(0) = 0'$). Clearly these maps are i*sotone*. If $F$ is a residual map from $L$ to $L'$, then there exists a unique residuated map $G$ from $L'$ to $L$, such that the following relation is satisfied: for all $x \in L$, $x' \in L'$,

$$x' \leq F(x) \Leftrightarrow G(x') \leq x. \qquad \text{(R)}$$

As a consequence, $GF$ is *reductive* (i.e., $x \geq GF(x)$) and $FG$ is *extensive* (i.e., $x \leq FG(x)$). Moreover the image sets $GF(L)$ and $FG(L')$ are two isomorphic lattices.

Next we give definitions and results about two *dependence relations* defined on the set $J$ of join-irreducible elements of a lattice $L$.

Following Freese et al [13] we call dependence relation and we denote by $D$ the relation $C$ defined on $J$ by Day [9]: for $j$ and $j'$ in $J$,

$$j\,D\,j' \text{ if } j \neq j' \text{ and there exists } x \in L \text{ such that } j < j' \vee x \text{ and } j \nleq j' \vee x$$

(where $j'^{-}$ is the element covered by the join-irreducible $j'$).

A lattice is said to be *D-strong* if its dependence relation $D$ is strongly connected (i.e., if for any ordered pair $(j, j')$ of join-irreducible elements, there exists a path from $j$ to $j'$ in $D$). Recall that a lattice $L$ is called *simple* if its only congruences are the trivial one and $L^2$. The significance of the relation $D$ comes from the following result:

THEOREM 1. *A lattice is simple if and only if it is D-strong.*

REMARK. This result is a consequence of Theorem 2.35 in Freese, Jezek and Nation [13] stating that the lattice of congruences of a (finite) lattice is isomorphic to the lattice of (order) ideals of the order defined on the strongly connected classes of the relation $D$. The authors write that this theorem is a translation of Lemma 2.33 attributed by them to Jónsson and Nation in [14]. In fact this Lemma is Theorem 9.2 in the latter but it is only stated for a particular class of lattices, whereas it is given for arbitrary (finite) lattices in Day [9] (see Item 3.4).

In order to prove results in lattice consensus theory (see below) we defined in [18, 16] another dependence relation on $J$ denoted by δ: for $j$ and $j'$ in $J$,

$$j\ \delta\ j' \text{ if } j \neq j' \text{ and there exists } x \in L \text{ such that } j \not\leq x \text{ and } j < j' \vee x.$$

Then observe that $j' \not\leq x$, that $j < j'$ implies $j\ \delta\ j'$ and that $D \subseteq \delta$. It is easy to see that a lattice is distributive if and only $j < j'$ is equivalent to $j\ \delta j'$. As above a lattice is called δ-*strong* if its dependence relation δ is strongly connected. Then distributive δ-strong lattices have at most two elements. For completeness we give the proof of the following easy result (mentioned in [6]).

LEMMA 2. *A lattice L is atomistic if and only if its two dependence relations D and* δ *are equal.*

*Proof.* If $L$ is atomistic one has obviously $\delta \subseteq D$. Conversely, let $\delta \subseteq D$ and assume that there exists a join irreducible $j'$ strictly greater than an atom $j$. Then, $j\ \delta\ j'$ and $j\ D\ j'$ what is impossible (since $j\ D\ j'$ implies $j \not\leq j'$). ∎

Then the characterization of simple lattices in Theorem 1 gives immediately:

PROPOSITION 3. *An atomistic lattice is simple if and only if its dependence relation* δ *is strongly connected.*

As examples of simple atomistic lattices one finds the set partition lattice, the lattice of preorders (dual of the lattice of topologies) or the lattice of orders -after adjoining a greatest element to the latter- on a set. All such lattices are significant in "pure" as well as in "applied" mathematics. For instance, set partition lattice is a fundamental object of Combinatorial Geometry, and in applied mathematics it occurs in data analysis (its chains are the support of hierarchical classifications and its lattice operations are used to define "strong" or "weak" classes of similar objects), in game theory (Gilboa and Leher's "global games"), in information theory (measures of information on this lattice) or in the designs of experiments (relations on this lattice).

## 2.2 LATTICE CONSENSUS THEORY.

In this theory one wants to "aggregate" $n$-tuples ($n \geq 2$) of elements of a lattice $L$ into an element of this lattice representing their "consensus".

A *consensus* (or *aggregation*) *function* on $L$ is a map $F$ from $L^n$ to $L$ associating a unique element $x = F(\underline{x})$ of $L$ with each $n$-tuple $\underline{x} = (x_1,\ldots, x_n)$ of elements of $L$ (so, it is a $n$-ary operation on $L$).

In particular, a consensus function $F$ from $L^n$ to $L$ is a *meet-projection* if there exists a set $M$ (with $\emptyset \subseteq M \subseteq N = \{1,\ldots, n\}$) such that for every $\underline{x} \in L^n$, $F(\underline{x}) = \wedge_{i \in M}\, x_i$. We will denote such a *meet-projection* by $\wedge_M$. Observe that if $M = \emptyset$, then $F$ is the constant function $F^1$ which maps each $n$-tuple $\underline{x} \in L^n$ to the greatest element 1 of $L$.

For $\underline{x} = (x_1,\ldots, x_n) \in L^n$ and $x \in L$, we write $N_x(\underline{x}) = \{i \in N\colon x \leq x_i\}$. In particular, for a consensus function $F$, we define several properties based on the sets $N_j(\underline{x})$, $j \in J$.

A consensus function $F$ on $L$ is *decisive* if for every $j \in J$ and for all $\underline{x}, \underline{x}' \in L^n$,

$$[N_j(\underline{x}) = N_j(\underline{x}')] \Rightarrow [j \leq F(\underline{x}) \Leftrightarrow j \leq F(\underline{x}')].$$

A consensus function $F$ on $L$ is *neutral monotonic* if for all $j, j' \in J$ and for all $\underline{x}, \underline{x}' \in L^n$,

$$[N_j(\underline{x}) \subseteq N_{j'}(\underline{x}')] \Rightarrow [j \leq F(\underline{x}) \Rightarrow j' \leq F(\underline{x}')].$$

Observe that the latter implies the decisivity property as well as two other properties called respectively *monotonicity* (obtained when $j = j'$) and *neutrality* (obtained when $[N_j(\underline{x}) = N_{j'}(\underline{x}')]$ Moreover, if $F$ is monotonic it is isotone.

The following (easy to prove) result will be useful in the sequel:

LEMMA 4**.** *Let $F$ be a neutral monotonic consensus function on $L$, $j \in J$, $x \in L$ and $\underline{x}, \underline{x}' \in L^n$ such that $j \leq F(\underline{x})$ and $N_x(\underline{x}') \supseteq N_j(\underline{x})$. Then, $x \leq F(\underline{x}')$.*

A consensus function $F$ is *Paretian* if for every $\underline{x} \in L^n$,

$$N_j(\underline{x}) = N \Rightarrow\ j \leq F(\underline{x}).$$

Such axioms are abstract forms of "Arrowian" properties i.e., properties used by Arrow or other authors to prove impossibility or possibility results on consensus functions bearing on preferences, choices or classifications. For instance when preferences are modelled by (arbitrary) order relations the join-irreducible elements are (arbitrary) ordered pairs and the *classical independence property* is: for every $(x, y)$ and for all $\underline{x}, \underline{x}'$ $n$–tuples of orders $[N_{(x,y)}(\underline{x}) = N_{(x,y)}(\underline{x}')] \Rightarrow [(x,y) \in F(\underline{x}) \Leftrightarrow (x,y) \in F(\underline{x}')]$. Here it is replaced with the decisivity property.

We will also use classical ordinal or algebraic axioms. Since the direct product $L^n$ is a lattice (with $\underline{x}\wedge\underline{x}' = (x_1\wedge x'_1,\ldots, x_n\wedge x'_n)$, $\underline{x}\vee\underline{x}' = (x_1\vee x'_1,\ldots, x_n\vee x'_n)$), we may consider consensus functions which are $\wedge$-morphisms or $\vee$-morphisms.
Let us denote by $\underline{x}^*$ the constant $n$-tuple $(x,\ldots, x)$. Then the greatest (respectively, least) element of the lattice $L^n$ is $\underline{1}^*$ (respectively, $\underline{0}^*$), and $F$ is a residual map (respectively, a residuated map) if $F$ is a $\wedge$-morphism satisfying $F(\underline{1}^*) = 1'$ (respectively, a $\vee$-morphism satisfying $F(\underline{0}^*) = 0'$).
We say that $F$ is *meet-dominating* if

$$\wedge_N \leq F$$

One easily checks that the Paretian and the meet-dominating properties are equivalent.

## 3. THE RESULTS

In the proof of the following theorem, we adopt special notations for some $n$-tuples which will occur frequently. Let, for instance, $(A, B, C)$ be a partition of the set $N$. Then $\underline{x} = (A\text{: } x, B\text{: } y, C\text{: } z)$ is the $n$-tuple for which for every $i$ in $A$ (respectively, in $B$, $C$) $x_i = x$ (respectively, $y$, $z$). In Condition (2) below $F^0$ is the constant function which maps each $n$-tuple $\underline{x} \in L^n$ to the least element 0 of $L$.

THEOREM 5. Let L be a δ-strong atomistic lattice of size greater than 2 and $F\colon L^n \to L$ a consensus function. The following are equivalent:

(1) *F is decisive and Paretian*;

(2) *F is neutral monotonic and it is not equal to* $F^0$;

(3) *F is a* $\wedge$*-morphism and meet-dominating*;

(4) *F is a residual map and* $F(\underline{j}^*) \geq j$ *for any* $j \in J$;

(5) *F is a meet projection* $\wedge_M$ (with $\varnothing \subseteq M \subseteq N$).

*Proof.*

(1) ⇔ (2). This is proved for any δ-strong lattice in [17].

(2) ⇒ (3). By the above equivalence $F$ is Paretian, and so meet-dominating (since it has been above observed that these two properties are equivalent). It has also be observed that if $F$ is neutral monotonic it is isotone, and so $F(\underline{x}\wedge\underline{x}') \leq F(\underline{x})\wedge F\underline{x}')$ holds. Assume that $F$ is not a $\wedge$-morphism i.e., that there exists $\underline{x}, \underline{x}' \in L^n$ such that $F(\underline{x}\wedge\underline{x}') < F(\underline{x})\wedge F\underline{x}')$. Thus there exists an atom $j \in J$ such that $j \nleq F(\underline{x}\wedge\underline{x}')$ and $j \leq F(\underline{x})\wedge F(\underline{x}')$.

So, $j \leq F(\underline{x})$, $j \leq F(\underline{x}')$ and (by the Paretian property) there exists $i \in N$ such that $j \nleq x_i\wedge x'_i$.

$N_j(\underline{x}\wedge\underline{x}') = N_j(\underline{x})\cap N_j(\underline{x}') \subset N_j(\underline{x})$ and $\subset N_j(\underline{x}')$ (since, if for example $N_j(\underline{x}\wedge\underline{x}') = N_j(\underline{x})$, then decisivity implies $j \leq F(\underline{x} \wedge \underline{x}')$, a contradiction).

Let $j' \in J$ such that $j' \,\delta\, j$ i.e., such that there exists $x \in J$ with $j, j' \nleq x$ and $j' < j \vee x$.

Consider then the (well defined) following $n$-tuple $\underline{x}''$:

$[N_j(\underline{x}) \backslash N_j(\underline{x} \wedge \underline{x}'): j;\ N_j(\underline{x}') \backslash N_j(\underline{x} \wedge \underline{x}'): x;\ N_j(\underline{x} \wedge \underline{x}'): j \vee x;\ N \backslash (N_j(\underline{x}) \cup N_j(\underline{x}')): 0]$.

$N_j(\underline{x}'') = N_j(\underline{x})$ and $j \leq F(\underline{x})$ imply by decisivity $j \leq F(\underline{x}'')$

$N_x(\underline{x}'') = N_j(\underline{x}')$ and $j \leq F(\underline{x}')$ imply by neutral monotony $x \leq F(\underline{x}'')$.

Then $j' < j \vee x \leq F(\underline{x}'')$ and $N_j(\underline{x} \wedge \underline{x}') = N_{j'}(\underline{x}'')$ implies by neutrality $j \leq F(\underline{x} \wedge \underline{x}')$, a contradiction.

(3) $\Rightarrow$ (4). $F$ is residual since $F$ is a $\wedge$-morphism satisfying $F(\underline{1}^*) = 1$ (by meet-domination). And $F$ meet-dominating implies $j \leq F(\underline{j}^*)$.

(4) $\Rightarrow$ (5). Consider an atom $j \in J$ and the residuated map $G$ associated with the residual map $F$. Since $j \leq F(\underline{j}^*)$, the isotonicity of $G$ and the reductivity of $GF$ imply $G(j) \leq GF(\underline{j}^*) \leq \underline{j}^*$. So for any $i \in N$, $G_i(j) \in \{0, j\}$, where $G_i(j)$ is the $i$-th component of $G(j)$. Write $M(j) = \{i \in N\colon G_i(j) = j\}$.

Let $j, j_1, \ldots, j_r \in J$ such that $j \leq \vee_{1 \leq k \leq r}\, j_k$ and the set $\{j_1, \ldots, j_r\}$ is minimal with that inequality.

Then by isotonicity and join preservation of $G$, one has $G(j) \leq G(\vee_{1 \leq k \leq r}\, j_k) = \vee_{1 \leq k \leq r}\, G(j_k)$. By the minimality assumption $G_i(j) = j$ implies $G_i(j_k) = j_k$ for all $j = 1, \ldots, r$ and $M(j) \subseteq M(j_k)$.

Now consider $j$ and $j'$ in $J$ such that $j \,\delta\, j'$ holds. Since every element $x$ of $L$ is a join of atoms, we can apply the previous considerations to obtain $M(j) \subseteq M(j')$. Since $L$ is $\delta$-strong, we obtain $M(j) = M(j') = M$, not depending of the considered pair $j, j'$.

The characterizations of the maps $G$ and $F$ follow:

for any $x \in L$, since $x$ is a join of atoms and $G$ is join preserving, $G_i(x) = x$ if $i \in M$ and $G_i(x) = 0$ if not;

for a $n$-tuple $\underline{x} = (x_1, \ldots, x_n)$, one gets from relation (R) linking the residual map $F$ and its residuated $G$, $x \leq F(\underline{x}) \Leftrightarrow G(x) \leq \underline{x} \Leftrightarrow$ [for any $i \in M$, $x \leq x_i$] $\Leftrightarrow x \leq \wedge_{i \in M} x_i$. So, $F$ is the meet-projection $\wedge_M$ associated with the coordinates in $M$ (with, if $M = \emptyset$, $F = F^1$ mapping any $n$-tuple to the greatest element 1 of $L$).

(5) $\Rightarrow$ (1) Obvious. □

□□□□□

REMARKS. 1. One observes that the condition $F(\underline{j}^*) \geq j$ in (4) is a weakening of the meet-domination property and could replace the latter in (3). Condition (3) could also be replaced by (3′): *F is a* $\wedge$*-morphism and* $F(\underline{x}^*) \geq x$ *for any* $x \in L$. In [7] (Theorem 1) the equivalence between

conditions (3′) and (5) (both augmented with another condition excluding the map $F^1$) is proved for the lattice of partitions. In [11] (Theorem 4) the equivalence between conditions (1), (3) and (5) (also augmented with a condition excluding the map $F^1$) is proved for this same lattice.
2. Since $F$ is a residual map and $G$ the associated residuated map, the lattices $GF(L^n)$ and $FG(L)$ are isomorphic. One easily checks that $GF(L^n) = \{(x_1,\ldots, x_n)$ : for some $x \in L$, $x_i = x$ if $i \in M$ and $x_i = 0$ otherwise$\}$ and $FG(L) = L$ (if $M$ is nonempty).

The consequence of Theorem 5 and Proposition 3 is the following characterization of meet-projections in simple atomistic lattices.

COROLLARY 6. *An n-ary operation F on a simple atomistic lattice is a meet-projection if and only if it is a residual map satisfying* $F(\underline{j}^*) \geq j$ *for any* $j \in J$.

Obviously, all the above results can be dualized for simple coatomistic lattices.

## 4. CONCLUSION

The lattice of partitions of a set is a simple geometric lattice, so an atomistic and coatomistic lattice. As already said the application of the above results to this simple atomistic lattice gives again results obtained in [7, 11] for the lattice of partitions. The dual results on this lattice give a characterization of *join-projections* as a residuated map, providing a strengthening of Theorem 1 in [11] and to be compared with the characterizations given in [19] and 16]. Clearly, the interest of the abstract "axiomatic" lattice approach to aggregation theory is to give results applicable to several different problems. For instance, Theorem 5 gives a characterization of meet-projections ("oligarchic" consensus functions) for partial orders or for preorders. The abstract lattice approach has been also introduced for aggregation procedures based on distances in Barthélemy and Janowitz [2] and it has been developed by several authors. A review of these works may be found in Day and McMorris [10].

**Acknowledgment**
We thank the editor and two anonymous reviewers for their numerous suggestions and corrections on the first version of this paper.

## References

1. Arrow, K.J. : Social Choice and Individual Values. Wiley, New York (1951)

2. Barthélemy, J.P., Janowitz, M.F.: A formal theory of consensus. SIAM J. Discr. Math. **4**, 305-322 (1991)

3. Blyth, T.S., Janowitz, M.F.: Residuation theory. Pergamon Press, Oxford (1972)

4. Brown, D.J.: Aggregation of preferences. Quaterly J. of Economics **89**, 456-469 (1975)

5. Caspard, N., Leclerc, B., Monjardet, B.: Ensembles ordonnés finis : concepts, résultats, usages, Springer-Verlag, Berlin (2007)

6. Caspard, N., Monjardet, B.: On a dependence relation in finite lattices. Discrete Mathematics. **165-166**, 497-505 (1997)

7. Chambers C.P., Miller A.D. : Rules for Aggregating Information. Social Choice and Welfare. **36(1)**, 75-82, (2011)

8. Davey, B.A. , Priestley H.A. : *Introduction to lattices and order*. Cambridge University Press, Cambridge (2002)

9. Day A.: Characterizations of finite lattices that are bounded-homomorphic images or sublattices of free lattices. Canad. J. Math. **31**, 69–78 (1979)

10. Day,W.H.E., McMorris F.R.: Axiomatic Consensus Theory in Group Choice and Biomathematics. SIAM, Philadelphia (2003)

11. Dimitrov D., Marchant T., Mishra N.: Separability and Aggregation of Equivalence Relations, Economic Theory. To appear (2011)

12. Erné M. : Adjunctions and Galois connections: Origins, history and development. In: Denecke K., Erné, M., Wismath Sh. (eds.), Galois Connections and Applications, pp 1-138.Kluwer, Dordrecht (2004)

13. Freese R., Jezek K., Nation J.B.: Free lattices. American Mathematical Society, Providence (1995)

14. Jonsson B., Nation J.B.: A report on sublattices of a free lattice. In : Contributions to Universal Algebra. Coll. Math. Soc. Janos Bolyai,vol. 17, pp 233-257. North-Holland, Amsterdam (1977)

15. Leclerc, B.: Efficient and binary consensus functions on transitively valued relations. Mathematical Social Sciences. **8,** 45-61 (1984)

16. Leclerc, B. Monjardet, B.: Latticial theory of consensus. In: Barnett, V., Moulin, H., Salles M., Schofield N. (eds.) Social Choice, Welfare and Ethics. pp 145-159. Cambridge University Press, Cambridge (1995)

17. Mirkin, B.G.: On the problem of reconciling partitions. In: Quantitative Sociology, International Perspectives on Mathematical and Statistical Modelling, pp 441-449. Academic Press, New-York pp 441-449 (1975)

18. Monjardet, B.: Arrowian characterizations of latticial federation consensus functions, Mathematical Social Sciences. **20**, 51-71 (1990)

19. Neumann, D.A., Norton Jr, V.T.: Clustering and Isolation in the Consensus Problem for Partitions. J. of Classification. **3**, 281-297 (1986)