Notes

Implementation via approval mechanisms✩

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Abstract

We focus on the single-peaked domain and study the class of Generalized Approval Mechanisms (GAMs): First, players simultaneously select subsets of the outcome space and scores are assigned to each alternative; and, then, a given quantile of the induced score distribution is implemented. Our main finding is that essentially for every Nash-implementable welfare optimum – including the Condorcet winner alternative – there exists a GAM that Nash-implements it. Importantly, the GAM that Nash-implements the Condorcet winner alternative is the first simple simultaneous game with this feature in the literature.

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1. Introduction

In the single-peaked domain, the Nash-implementable welfare optima, practically, coincide with the outcomes of Generalized Median Rules (GMRs). In simple terms, the outcome of a GMR is the median of a set of points that consists of: a) the voters’ ideal policies and b) some exogenous values also known as phantoms. As proved by Moulin (1980) GMRs are the unique social choice rules that satisfy efficiency and strategy-proofness, while Berga and Moreno (2009) established that strategy-proof rules which are “not too bizarre” (in the context of Sprumont, 1995) are the only implementable ones.

However, one should note that the direct revelation game of each GMR need not lead to the same outcome as the GMR itself. In this respect, the direct revelation games of GMRs share a common feature with other strategy-proof mechanisms: They admit a large multiplicity of Nash equilibria, some of which produce different outcomes (see Saijo et al., 2007). For instance, the direct revelation game triggered by the pure median rule – whose outcome is the Condorcet winner alternative – exhibits a large set of equilibria: As long as every player announces the same alternative x, this constitutes an equilibrium with outcome x since no unilateral deviation affects the median choice. This leads to the following conclusion: The direct revelation game of a GMR does not Nash-implement the GMR (see Repullo, 1985 for similar results).

So how do we Nash-implement GMRs in a simple manner? Yamamura and Kawasaki (2013) propose the class of averaging mechanisms. Each player announces an alternative and a monotonic transformation of the average alternative is implemented. The equilibrium outcome coincides with the outcome of a GMR with an important restriction: All phantoms must be interior, which prevents, among others, the implementation of the Condorcet winner alternative. Moreover, Gershkov et al. (2016) have recently shown that sequential quota mechanisms can also implement GMRs. Indeed, being able to implement GMRs by the means of simple sequential games is very important, but ideally one would like to be able to do the same using simple simultaneous games as well.

In this paper, we design the class of Generalized Approval Mechanisms (GAMs). These mechanisms are quite easy to describe and belong to the class of simultaneous voting games. First, players select subsets of the outcome space and scores are assigned to each alternative (hence, Approval). Given a subset of alternatives, two different GAMs may assign different positive scores to the same approved alternative (hence, Generalized). Then, a given quantile of the score distribution induced by the players’ choices is implemented. Our main finding is that every generic GMR – including the Condorcet winner alternative – can be Nash-implemented by some GAM. We explain how to derive a GAM for each GMR and we explicitly design the

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1. In the present paper’s context, a welfare optimum is the outcome of social choice rule (Maskin, 1999), the set of alternatives is $A = \{0, 1\}$ and the set of possible preference relations consists of the single-peaked ones in $A$.
2. That is, restricting attention to anonymous rules that implement each of the alternatives for at least one preference profile.
3. Experimental evidence shows that strategy-proof mechanisms need not lead a large share of the agents to reveal their true type (see Atiyeh et al., 2000; Kawagoe and Mori, 2001; Kagel and Levin, 1993 and Cason et al., 2006 among others).
4. A game/mechanism Nash-implements a social choice rule if it admits a unique equilibrium outcome which coincides with the outcome of the social choice rule (see Maskin, 1999).
5. More precisely, their sequential mechanisms are obtained by modifying a sequential voting scheme suggested by Bowen (1943).
6. We consider that a GMR is generic if its interior phantoms – if any – are all non-identical.
one that implements the Condorcet winner alternative, also known as the pure median rule. To
our knowledge, this is the first simple simultaneous game that implements the Condorcet winner
alternative and arguably this finding is of interest on its own. The equilibrium strategies of most
players\(^7\) take an easy “I approve every alternative at most (least) as large as the implemented
alternative” form. In fact, every player with a preferred alternative to the left (right) of the imple-
mented one approves the implemented alternative and all the alternatives to its left (right). That
is, \(GAMs\) not only Nash-implement \(GMRs\), but also promote sincerity and agreement, in the
sense that most players include both their ideal policies and the implemented outcome in their
approval sets.

Naturally, the present analysis relates to the wider Approval voting literature. Approval vot-
ing has been studied since Weber (1995) and Brams and Fishburn (1983), and has been shown to
exhibit interesting properties in a variety of contexts: For example, it improves the quality of deci-
sions in common value problems compared to plurality rule (Bouton and Castanheira, 2012) and
leads to the sincere revelation of preferences in certain private value settings (see Laslier, 2009;
Laslier and Sanver, 2010 and Núñez, 2014). As we show, in the single-peaked domain Approval
voting can additionally help a society reach, essentially, every feasible welfare optimum.

In what follows we describe the setting (section 2) and present an example (section 3). Then
we provide our formal results and explain how to implement the Condorcet winner through a
GAM (section 4).

2. The setting

2.1. Basic concepts and definitions

Let \(A := [0, 1]\) denote the set of alternatives and \(N := \{1, \ldots, n\}\) the set of players with \(n \geq 2\).
Let \(U\) be the set of single-peaked preferences. Each player \(i\) has utility function \(u_i\) in \(U\) with
\(u_i(x)\) the utility of player \(i\) when \(x \in A\) is implemented. Each player \(i\) has a unique peak, \(t_i\),
so that \(u_i(x') < u_i(x'')\) when \(x' < x'' \leq t_i\) and when \(t_i \leq x'' < x'\).\(^8\) We let \(t = (t_1, \ldots, t_n)\) stand
for a peak profile and \(u = (u_1, \ldots, u_n) \in \mathcal{U} := \prod_{j=1}^n U\). A social choice function is a function
\(f: \mathcal{U} \to A\) that associates every \(u \in \mathcal{U}\) with a unique alternative \(f(u)\) in \(A\). For any finite
collection of points \(x_1, \ldots, x_3\) in \([0, 1]\), we let \(m(x_1, \ldots, x_3)\) denote their median: \(m(x_1, \ldots, x_3)\)
is the smallest number in \(\{x_1, \ldots, x_3\}\) which satisfies \(\frac{1}{3}\#\{x_i \mid x_i \leq m(x_1, \ldots, x_3)\} \geq \frac{1}{2}\) and
\(\frac{1}{3}\#\{x_i \mid x_i \geq m(x_1, \ldots, x_3)\} \geq \frac{1}{2}\). A social choice function is a generalized median rule (\(GMR\))
if there is some collection of points \(p_1, \ldots, p_{n-1}\) in \([0, 1]\) such that, for each \(u \in \mathcal{U}\), \(f(u) = m(t, p_1, \ldots, p_{n-1})\). We refer to \(p_1, \ldots, p_{n-1}\) as the phantoms of the \(GMR\). A \(GMR\) is consid-
ered to be generic if its interior phantoms – if any – are non-identical.

A mechanism is a function \(\theta: S \to A\) that assigns to every \(s \in S\), a unique element \(\theta(s)\) in
\(A\), where \(S := \prod_{i=1}^n S_i\) and \(S_i\) is the strategy space of player \(i\). Given a mechanism \(\theta: S \to A\),
the strategy profile \(s \in S\) is a Nash equilibrium of \(\theta\) at \(u \in \mathcal{U}\), if \(u_i(\theta(s_i, s_{-i})) \geq u_i(\theta(s'_i, s_{-i}))\)
for all \(i \in N\) and any \(s'_i \in S_i\). Let \(N^\theta(u)\) be the set of Nash equilibria of \(\theta\) at \(u\). The mechanism
\(\theta\) implements the social choice function \(f\) in Nash equilibria if for each \(u \in \mathcal{U}\), \(a\) there exists
\(s \in N^\theta(u)\) such that \(\theta(s) = f(u)\) and \(b\) for any \(s \in N^\theta(u)\), \(\theta(s) = f(u)\).

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\(^7\) If a player’s peak coincides with the equilibrium outcome, then this player may be employing a different kind of
strategy.

\(^8\) For simplicity, we assume that \(t_i \neq t_j\) for any \(i, j \in N\). Our results are not affected when relaxing this constraint.
2.2. Generalized approval mechanisms

We let $\mathcal{B}$ denote the collection of closed intervals of $A$. A $GAM$ is a mechanism $\theta : \mathcal{B}^n \to A$ which requires each player to play simultaneously a strategy in $\mathcal{B}$ and determines for each strategy profile some alternative in $A$. For each $b_i \in \mathcal{B}$, we write $\underline{b}_i = \min b_i$ and $\overline{b}_i = \max b_i$. The set $\mathcal{B}$ includes elements of different dimensions: singletons and positive length intervals. Since each $b_i$ is a convex set, its dimension is well-defined so that for each approval profile $b = (b_i, b_{-i})$, we let $\dim(b) = \max_{i \in N} \dim(b_i)$. The set of zero-dimensional and one-dimensional strategies are respectively labeled by $\mathcal{B}_0$ and $\mathcal{B}_1$ with $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1$. Similarly, $\mathcal{B}^n_0$ denotes the set of profiles in which every player plays a singleton and $\mathcal{B}^n_1$ the set of profiles such that at least one player plays a one-dimensional strategy.

In order to state a precise definition of a $GAM$, we let $\eta : \mathbb{R} \to \mathbb{R}$ be a differentiable and strictly increasing function with $\eta(0) = 0$ and $\eta(1) = 1$ and $q$ a non-negative real number. We assume that when player $i$ submits the interval $b_i$, he is endowed with a weight of $q + \eta(\overline{b}_i) - \eta(b_i)$ to be distributed over $b_i$. More precisely, if $\dim(b_i) = 1$, then the strategy $b_i$ assigns an individual score of $s_x(b_i, q, \eta)$ to each $x \in [0, 1]$ as follows:

$$s_x(b_i, q, \eta) = \frac{q}{\overline{b}_i - \underline{b}_i} + \eta'(x)$$

so that $s(b_i, q, \eta) = \int_0^1 s_x(b_i, q, \eta) \, dx$ equals $q + \eta(\overline{b}_i) - \eta(b_i)$ as defined. On the contrary, when $\dim(b_i) < \dim(b)$, $b_i$ is a singleton and some player announces a one-dimensional interval, we let $s_x(b_i, q, \eta) = 0$ for every $x \in [0, 1]$ so that his strategy is not taken into account.

Collectively, each profile $b$ in $\mathcal{B}^n_1$ assigns to each alternative $x$ a score of $s_x(b, q, \eta)$ with $s_x(b, q, \eta) = \sum_{i=1}^n s_x(b_i, q, \eta)$. Hence, the score distribution is the function $\phi_{q, \eta} : \mathcal{B}^n_1 \times [0, 1] \to [0, 1]$ such that

$$\phi_{q, \eta}(b, z) = \int_0^z \frac{s_x(b, q, \eta)}{\sum_{i=1}^n s_x(b_i, q, \eta)} \, dx.$$ 

A $GAM$ $\theta_{\alpha, q, \eta}$ associates any profile $b \in \mathcal{B}^n_0$ with $\theta_{\alpha, q, \eta}(b) = m(b_1, b_2, \ldots, b_n)$ and any profile $b \in \mathcal{B}^n_1$ with

$$\theta_{\alpha, q, \eta}(b) = \min\{x \in [0, 1] \mid \phi_{q, \eta}(b, x) = \alpha\},$$

where $\alpha \in (0, 1)$.

In other words, a $GAM$ selects the $\alpha$-quantile of the distribution endogenously generated by $b$ given $q$ and $\eta$ when at least some player announces a positive length interval; otherwise, it selects the median of the announced singletons. In what follows, we write $\theta$ rather than $\theta_{\alpha, q, \eta}$.

The initial step is to show that any $GAM$ is well-defined.

**Lemma 1.** For any admissible $q$, $\alpha$ and $\eta$, the associated $GAM$ is well-defined.

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9 This assumption can be relaxed by allowing any finite union of closed and convex subsets of $A$ to be the set of pure strategies. Relaxing it however would imply more cumbersome notation and proofs since then two strategies that differ by a zero-measure set can have equivalent consequences.

10 This assumption is made for the sake of completeness. We could use any alternative $\lambda : [0, 1]^n \to [0, 1]$ rather than using the median of the singletons without affecting the results.
**Proof.** Note first that for any \( b \in \mathcal{B}_n \), \( \frac{\sum_{x=1}^{n} s_i(b,q,\eta)}{\sum_{x=1}^{n} s(b,q,\eta)} \geq 0 \) for every \( x \in [0,1] \). It suffices to show that its integral over \([0,1]\) equals 1, which is equivalent to \( \phi_{q,\eta}(b,1) = 1 \). But this is satisfied since \( \int_0^1 s_x(b,q,\eta)dx = \sum_{i=1}^{n} s(b_i,q,\eta) \). \( \square \)

For each \( GAM \) \( \theta \), we let the points \( \kappa_1, \kappa_2, \ldots, \kappa_{n-1} \) denote the phantoms of \( \theta \), with \( \kappa_j = \max\{0, \min[1, \eta^{-1}(\gamma_j)]\} \) and \( \gamma_j = \frac{n(n+1) - n-j}{n(n+1) - 1} \), for each \( j \in \{1, \ldots, n-1\} \). Note that \( \eta^{-1}(\gamma_j) \in (0,1) \), then \( \kappa_j = \eta^{-1}(\gamma_j) \). Moreover, by the means of standard algebraic manipulations, one can show that for each \( j \in \{1, \ldots, n-1\} \), each \( q \in \mathbb{R}^+ \), and each \( \alpha \in (0,1) \), \( \gamma_j < \gamma_{j+1} \). The previous inequality, combined with \( \eta \) being differentiable and strictly increasing, implies that \( 0 \leq \kappa_1 \leq \kappa_2 \leq \ldots \leq \kappa_{n-1} \leq 1 \), and that as long as \( \kappa_i, \kappa_j \in (0,1) \), \( \kappa_i \neq \kappa_j \).

For each \( j \in \{1, \ldots, n-1\} \) and each \( x \in [0,1] \), we let \( \mathcal{B}(j,x) := \{ b \in \mathcal{B}_n \ | \ \# \{i \geq 1 \text{ s.t. } x < b_i \} = n-j \text{ and } \# \{i \geq 1 \text{ s.t. } x > b_i \} = j \} \). Any profile \( b \) in \( \mathcal{B}(j,x) \) has \( n-j \) players playing \([0,x]\) and \( j \) players playing \([x,1]\). Note that \( \theta(b) = \theta(b') \) if \( b, b' \in \mathcal{B}(j,x) \).

**Lemma 2.** If \( \theta \) is a \( GAM \), \( j \in \{1, \ldots, n-1\} \), and \( b \in \mathcal{B}(j,x) \) with \( x \in (0,1) \) then (a) \( \theta(b) = x \) if and only if \( x = \kappa_j \); (b) \( \theta(b) > x \) if and only if \( \kappa_j > x \); and (c) \( \theta(b) < x \) if and only if \( \kappa_j < x \).

**Proof.** For each \( j \in \{1, \ldots, n-1\} \), each \( x \in (0,1) \), and any \( b \in \mathcal{B}(j,x) \), the distribution \( \phi(b, x) \) satisfies \( \phi(b, x) = \frac{(n-j)x + (n-j)\alpha}{qn + (n-j)(n-\kappa_j) + (n-1)\gamma_j} \). Therefore, since \( \eta \) is differentiable and strictly increasing in \((0,1)\), \( \frac{\partial}{\partial x} \phi(b, x) > 0 \) for each \( x \in (0,1) \). This implies that, if \( \phi(b, x) = \alpha \) for some \( x \in (0,1) \), then \( x \) is unique and \( \phi(b, x) = \frac{(n-j)x + (n-j)\alpha}{qn + (n-j)(n-\kappa_j) + (n-1)\gamma_j} = \alpha \iff x = \eta^{-1}(\gamma_j) = \kappa_j \), which proves (a). Moreover, each \( b \in \mathcal{B}(j,x) \) with \( x < \kappa_j \) satisfies \( \phi(b, x) < \alpha \) since \( \phi(b, x) \) is strictly increasing in \( x \), and thus \( \theta(b) > x \). Similarly, each \( b \in \mathcal{B}(j,x) \) with \( \theta(b) > x \) is such that \( \phi(b, x) < \alpha \), which implies \( x < \kappa_j \) and proves (b). A symmetric argument proves (c) concluding the proof. \( \square \)

Notice that by the definition of a \( GAM \) – in specific by the fact that \( \alpha \in (0,1) \) – it follows that \( \theta(b) > x \) when \( b \in \mathcal{B}(n,x) \) and \( \theta(b) < x \) when \( b \in \mathcal{B}(0,x) \) for every \( x \in [0,1] \).

3. An example: the median approval mechanism

In this section we present an example that illustrates how a specific \( GAM \) works for a simple class of preference profiles. We consider a society composed of three individuals with peaks such that \( 0 = t_1 < t_2 < t_3 = 1 \). The Approval mechanisms that we consider throughout have the following common structure: a) Every player simultaneously and independently announces an interval \( b_i \in \mathcal{B} \); b) these intervals generate a score distribution, and c) the mechanism implements \( \theta(b) \) which equals some quantile of the score distribution such as the median. The Approval mechanisms differ in how this distribution is generated and in the quantile of the distribution that is implemented.

While the general structure is discussed in the rest of the paper, we stick here to the simplest interesting score assignment process: That is, we assume that when player \( i \) submits the interval \( b_i \), he assigns an individual score \( s_x(b_i) \) to each \( x \in [0,1] \) as follows:

\[
s_x(b_i) = 1 \quad \text{for any } x \in b_i \quad \text{and} \quad s_x(b_i) = 0 \quad \text{otherwise}.
\]
Collectively, each strategy profile $b$ assigns a score of $s_x(b)$ to each alternative $x$ with $s_x(b) = \sum_{i=1}^{n} s_x(b_i)$. If at least one player submits a positive length interval, the distribution is the function $\phi : \mathcal{B}_1^n \times [0, 1] \rightarrow [0, 1]$ such that

$$
\phi(b, z) = \int_{\sum_{i=1}^{n} (b_i - \bar{b}_i)}^{z} s_x(b) \, dx.
$$

The Median Approval mechanism associates any profile $b$ with the median $\theta(b)$ of the score distribution (when $\phi$ is continuous, $\phi(b, \theta(b)) = \frac{1}{2}$, while when all players announce a singleton, $\theta(b)$ corresponds to the median of these singletons).

We first notice that for any profile $b$ with $\theta(b) \neq t_i$ and $b_i \in \mathcal{B}_0$, player $i$ can effectively move the median of the score distribution closer to her peak, $t_i \in (0, 1)$, by submitting a sufficiently small – but non-degenerate – interval containing $t_i$. Hence, in equilibrium it must be the case that an individual whose peak does not coincide with the outcome uses a one-dimensional strategy and, in particular, he uses $[0, \theta(b))$ if $t_i < \theta(b)$ and $[\theta(b), 1]$ if $t_i > \theta(b)$. This is so because placing weight to alternatives to the left of the implemented one shifts the implemented alternative to the left and vice versa.

Note that for the three players example that we consider, $\theta([0, x], [0, x], [x, 1]) = \frac{1-x}{2}$ if $x \leq \frac{1}{3}$ and $\theta([0, x], [0, x], [x, 1]) = \frac{1-x}{3}$ if $x \geq \frac{1}{3}$. Similarly, $\theta([0, x], [x, 1], [x, 1]) = \frac{2x+1}{4}$ if $x \leq \frac{2}{3}$ and $\theta([0, x], [x, 1], [x, 1]) = \frac{2x-1}{3}$ if $x \geq \frac{2}{3}$. Therefore, $\theta([0, x], [0, x], [x, 1]) = x$ if and only if $x = \kappa_1 = \frac{1}{3}$ and $\theta([0, x], [x, 1], [x, 1]) = x$ if and only if $x = \kappa_2 = \frac{2}{3}$. In other words, when $n = 3$, the phantoms of the Median Approval mechanism are $\kappa_1 = \frac{1}{3}$ and $\kappa_2 = \frac{2}{3}$.

The previous arguments suggest that: a) when $t_2 < \frac{1}{3}$ the unique equilibrium is $([0, \frac{1}{3}], [0, \frac{1}{3}], [\frac{1}{3}, 1])$ with outcome $\frac{1}{2}$ and b) when $t_2 > \frac{2}{3}$ the unique equilibrium is $([0, \frac{1}{3}], [\frac{1}{3}, 1], [\frac{1}{3}, 1])$ with outcome $\frac{2}{3}$. But what happens when $t_2 \in [\frac{1}{3}, \frac{2}{3}]$? Then, in any equilibrium $b$, player 2 still uses $[0, \theta(b))$ and player 3 still uses $[\theta(b), 1]$, but player 2 can use a different kind of strategy, and have his peak implemented. Indeed, when, for example, $t_2 \in [\frac{1}{2}, \frac{1}{3}]$ an equilibrium can be such that $\theta([0, t_2], [0, 4t_2 - 1], [t_2, 1]) = t_2$. In these cases the equilibrium need not be unique, as the median player has many best responses, but the equilibrium outcome is unique and coincides with the peak of the median player. In Fig. 1 we present the unique equilibrium outcome of the Median Approval mechanism for all the preference profiles that we considered here. In Fig. 2 we present the scores assigned to each alternative, $s_x(b)$, in an equilibrium of the form $\theta([0, t_2], [0, 4t_2 - 1], [t_2, 1]) = t_2$.

4. Formal analysis

We prove first how best replies are under a $GAM$ (Lemma 3), then prove that a $GAM$ Nash-implements a $GMR$ (Proposition 1) and, finally, establish that for every generic $GMR$ there exists a $GAM$ that Nash-implements it (Theorem 1).

Next, we assert that if a player whose peak lies to the left (right) of the outcome uses a best response, then he approves of all the alternatives to the left (right) of the implemented outcome.

**Lemma 3.** If $\theta$ is a $GAM$, and $b = (b_i, b_{-i}) \in \mathcal{B}^n$ with $t_i < \theta(b)$ ($t_i > \theta(b)$), then $b_i$ is a best response to $b_{-i}$ if and only if $b_i = [0, \theta(b)]$ ($b_i = [\theta(b), 1]$).
Equilibrium outcome

\[ m(0, t_2, 1, 1/3, 2/3) \]

Fig. 1. Equilibrium outcome as a function of \( t_2 \).

An equilibrium strategy profile that implements the ideal policy of the median voter.

Proof. We only provide a proof for the case in which \( t_i < \theta(b) \) since the proof for \( t_i > \theta(b) \) is symmetric. We first consider a strategy profile \( b = (b_1, b_{-i}) \) with \( t_i < \theta(b) \) and \( b_i \neq [0, \theta(b)] \) and argue that \( b_i \) cannot be a best response of player \( i \); and then we consider a strategy profile \( b \) with \( t_i < \theta(b) \) and \( b_i = [0, \theta(b)] \) and argue that \( b_i \) is a best response of player \( i \).

When \( t_i < \theta(b) \) and \( b_i \neq [0, \theta(b)] \) there are three possibilities regarding \( b_i \): a) \( \theta(b) > \overline{b}_i \), b) \( \theta(b) < \overline{b}_i \); and c) \( \theta(b) \in b_i \). If \( \theta(b) > \overline{b}_i \) and \( \text{dim}(b) = 0 \) then \( i \) can submit a sufficiently small – but non-degenerate – interval centered at \( t_i \) and bring the implemented outcome arbitrarily closer to his peak.\(^{11} \) If \( \theta(b) > \overline{b}_i \) and \( \text{dim}(b_{-i}) = 0 \) then \( i \) can deviate to \([t_i, \theta(b)]\) and induce \( \phi(([t_i, \theta(b)], b_{-i}), b_i) > \phi(b_i, \theta(b)) \) and \( \phi(([t_i, \theta(b)], b_{-i}), t_i) \leq \phi(b_i, t_i) \); and hence bring the implemented outcome closer to her peak. If \( \theta(b) > \overline{b}_i \) and \( \text{dim}(b_i) = 1 \) then there exists \( \beta \in (0, \theta(b)) \) such that \( \theta(b) = \theta((\beta, \theta(b)), b_{-i}) \); This is so because the outcome of a GAM does not depend on the specific interval that one submits when this interval contains outcomes only to the left (right) of the implemented one, but only on the total weight assigned to policies

\(^{11} \) When \( t_i = 0 \), the player can submit an interval \([0, \varepsilon]\), with a sufficiently small \( \varepsilon > 0 \), to the described effect (the case of \( t_i = 1 \) is symmetric).
on the left (right) of the implemented outcome. We assume that \( i \) deviates to such a strategy, \([\beta, \theta(b)]\), that delivers the same outcome as \( b_i \). After this intermediate step, we simply consider marginal changes in \( \beta \). Indeed, one can show that \( \frac{\partial}{\partial \beta} \phi(\{[\beta, \theta(b)], b_{-i}\}, \theta(b)) < 0 \) which means that the implemented outcome \( \theta([\beta, \theta(b)], b_{-i}) \) continuously decreases when \( \beta \) decreases; and this clearly improves the payoff of player \( i \). That is, \( b_i \) cannot be a best response of player \( i \). Case b) admits a completely symmetric proof. Case c) is actually simpler since it is such that \( b_i \leq \theta(b) \leq \overline{b_i} \), so one can consider directly marginal changes of \( b_i \) and/or \( \overline{b_i} \) without the need for the described intermediate step.

Now consider that \( t_i < \theta(b) \) and \( b_i = [0, \theta(b)] \), and that there exists \( b'_i \) such that \( u_i(\theta(b'_i), b_{-i}) > u_i(\theta(b)) \). If \( b_i \not\subset b'_i \) then \( \phi([b'_i, b_{-i}], \theta(b)) < \phi(b, \theta(b)) \) and hence \( \theta(b'_i, b_{-i}) > \theta(b) \). If \( b_i \subset b'_i \) then \( b'_i = [0, \beta] \) for some \( \beta > \theta(b) \). One can show that \( \frac{\partial}{\partial \beta} \phi(([0, \beta], b_{-i}), \theta(b)) < 0 \) when \( \beta > \theta(b) \). That is, a transition from \( b_i \) to \( b'_i \) will induce \( \phi([b'_i, b_{-i}], \theta(b)) < \phi(b, \theta(b)) \) and hence \( \theta(b'_i, b_{-i}) > \theta(b) \). In both cases the assumption that \( b_i = [0, \theta(b)] \) is not a best response is contradicted and this concludes the argument. □

Next we establish that a GAM implements a GMR in Nash equilibria.

**Proposition 1.** If the mechanism \( \theta : \mathcal{R}^n \to A \) is a Generalized Approval Mechanism (GAM) then:

a) there is an equilibrium in pure strategies for every admissible preference profile; and

b) in every equilibrium \( b \) of \( \theta \) we have \( \theta(b) = m(t_1, t_2, \ldots, t_n, k_1, \ldots, k_{n-1}) \).

**Proof.** Take some GAM mechanism \( \theta : \mathcal{R}^n \to A \). The proof first establishes the existence of an equilibrium (Step A), and then fully characterizes the unique equilibrium outcome (Step B).

For short, we write \((t, \kappa)\) rather than \((t_1, t_2, \ldots, t_n, k_1, \ldots, k_{n-1})\).

**Step A. There is some equilibrium \( b \) of \( \theta \) with \( \theta(b) = m(t, \kappa) \).**

Step A. is divided into two cases: There is either no \( t_h \) with \( t_h = m(t, \kappa) \) (Step A.I.), or there is a \( t_h \) with \( t_h = m(t, \kappa) \) (Step A.II.).

**Step A.I. There is no \( t_h \) with \( t_h = m(t, \kappa) \).** Since there is no \( t_h \) with \( t_h = m(t, \kappa) \), there must exist \( j \in \{1, \ldots, n-1\} \) such that \( k_j = m(t, \kappa) \). Therefore, the number of elements located below and above \( k_j \) in \((t, \kappa)\) is equal to \( n-1 \), which is equivalent to:

\[
\#\{i \in N \mid t_i < k_j\} + (j - 1) = \#\{i \in N \mid t_i > k_j\} + (n - j - 1) = n - 1.
\]

The previous equalities imply that \( \#\{i \in N \mid t_i < k_j\} = n - j \) and \( \#\{i \in N \mid t_i > k_j\} = j \). Let \( b \in \mathcal{B}(j, k_j) \) be such that:

\[
b_i := \begin{cases} 
[0, k_j] & \text{if } t_i < k_j, \\
[k_j, 1] & \text{if } t_i > k_j. 
\end{cases}
\]

By Lemma 2, \( \theta(b) = k_j \) and therefore \( \theta(b) = m(t, \kappa) \). Since every player is playing a best response as defined in Lemma 3, \( b \) is an equilibrium of the game and this concludes Step A.I.

**Step A.II. There is some \( t_h \) with \( t_h = m(t, \kappa) \).** If there exists \( j \in \{1, \ldots, n-1\} \) such that \( k_j = t_h \), then either \( j = n - h \) or \( j = n - h + 1 \). Using the same line of reasoning as in A.I., one
can show that: a) when \( j = n - h \), any \( b \in \mathcal{B}(n - h, t_h) \) is an equilibrium with \( \theta(b) = t_h \) and b) when \( j = n - h + 1 \), any \( b \in \mathcal{B}(n - h + 1, t_h) \) is an equilibrium with \( \theta(b) = t_h \).

If \( t_h = m(t, \kappa) \) and \( t_h \neq \kappa_j \), there are \( n - 1 \) values strictly smaller than \( t_h \) in \((t, \kappa)\). There are essentially two cases here: a) \( t_h \in (\kappa_1, \kappa_{n-1}) \) and b) \( t_h < \kappa_1 \) (the proof for the case \( t_h > \kappa_{n-1} \) is symmetric). Below, we consider both cases in turn.

a) Choose \( j \), such that \( 1 \leq j \leq n - 2 \), with \( \kappa_j < t_h = m(t, \kappa) < \kappa_{j+1} \). Moreover \( \#(\kappa_l | \kappa_l < t_h) = j \) and \( \#(i \in N | t_i < t_h) = h - 1 \) so that: \( j + h - 1 = n - 1 \implies j = n - h \). Therefore, \( \kappa_{n-h} < t_h < \kappa_{n-h+1} \).

For each strategy \( c^* \in \mathcal{B} \), we let \( b = (c^*, b_{-h}) \) denote a strategy profile with:

\[
b_i = \begin{cases} 
[0, t_h] & \text{if } t_i < t_h, \\
c^* & \text{if } t_i = t_h, \\
[t_h, 1] & \text{if } t_i > t_h.
\end{cases}
\]

Our objective is to prove that there is some \( c^* \) such that \( \theta(b) = t_h \) is an equilibrium.

By Lemma 2, it follows that if \( \kappa_{n-h} \in (0, 1) \), \( \theta(b') = \kappa_{n-h} < t_h \) for any \( b' \in \mathcal{B}(n - h, \kappa_{n-h}) \) and, if \( \kappa_{n-h} = 0, \theta(b') < t_h \) for any \( b' \in \mathcal{B}(n - h, t_h) \). Again, due to Lemma 2, if \( \kappa_{n-h+1} \in (0, 1) \), \( \theta(b') = \kappa_{n-h+1} > t_h \) for any \( b' \in \mathcal{B}(n - h + 1, \kappa_{n-h+1}) \) and if \( \kappa_{n-h+1} = 1, \theta(b') > t_h \) for any \( b' \in \mathcal{B}(n - h + 1, t_h) \). Hence, it follows that \( \theta([0, t_h], b_{-h}) < t_h \) and \( \theta([t_h, 1], b_{-h}) > t_h \), so that there exists some \( c^* \) with \( \theta(b) = t_h \). This is so because when the rest of the players behave according to \( b_{-h}, h \) can smoothly deviate from \([0, t_h]\) to \([t_h, 1]\) – first, continuously increase his right bound of his interval up to \( t_h \) and, then, continuously increase the left bound of his interval up to his peak – and induce a continuous change of the implemented policy from \( \theta([0, t_h], b_{-h}) \) to \( \theta([t_h, 1], b_{-h}) \).

In order to prove that \( b = (c^*, b_{-h}) \) with \( \theta(b) = t_h \) is an equilibrium, suppose by contradiction that there exists some \( i \in N \) with a profitable deviation \( b_{i}' \). Yet, as proved by Lemma 3, any player with a peak different than \( t_h \) is playing a best response in \( b \). Moreover, the player with peak \( t_h \) is also playing a best response since \( \theta(b) = t_h \). Therefore, \( b \) must be an equilibrium concluding a) in Step A.

b) In this case, \( t_h = m(t, \kappa) < \kappa_1 \), and hence, \( h = n \). Therefore, in any equilibrium \( b \), the \( n - 1 \) players with peak strictly lower than \( t_h \) play \([0, t_h]\). Moreover, \( \theta([0, t_n], b_{-n}) < t_n \), since for any \( b \in \mathcal{B}(0, x) \), \( \theta(b) < x \) for every \( x \in (0, 1) \); and \( \theta([t_n, 1], b_{-n}) > t_n \), since for any \( b \in \mathcal{B}(1, x) \), \( \theta(b) > x \) if \( \kappa_1 > x \) (by Lemma 2). Hence, the existence of an interval \( A^* \) such that \( \theta(b^{A^*}) = t_n \) is ensured. This, in turn, ensures the existence of an equilibrium similar to the one described in a), which concludes the proof of step A.

**Step B. Any equilibrium \( b \) of \( \theta \) satisfies \( \theta(b) = m(t, \kappa) \).**

For each profile \((t, \kappa)\), we let \( i' = \#(i \in N | t_i \leq m(t, \kappa)) \) denote the number of players with peak lower than \( m(t, \kappa) \) and \( j' = \#(j \in \{1, \ldots, n - 1 \} | \kappa_j \leq m(t, \kappa)) \) stand for the number of phantoms lower than \( m(t, \kappa) \). Since \((t, \kappa)\) has \( 2n - 1 \) elements, it follows that \( i' + j' \geq n \) so that \( n - i' \leq j' \).

Suppose, by contradiction, that there is some \( GAM, \theta \), that admits an equilibrium \( b \) with \( 1 > \theta(b) > m(t, \kappa) \).\(^\text{13}\) The rest of the proof inspects the different cases for each value of \( n - i' \). A symmetric argument applies when \( 0 < \theta(b) < m(t, \kappa) \).

\(^{13}\) An equilibrium with \( \theta(b) = 1 \) can be trivially ruled out since it requires that all players announce singletons. Obviously, any \( i \) with \( t_i < 1 \) can gain by deviating to \([t_i, t_i + \varepsilon]\) for \( \varepsilon > 0 \) and small enough.
Step B.I. \( n - i' \in [0, n) \). Assume first that there is some equilibrium \( b \) with \( n - i' = 0 \). It follows that \( i' = n \) players have a peak lower than \( m(t, \kappa) \). Since, by assumption, \( m(t, \kappa) < \theta(b) \), Lemma 3 implies that each player \( i \) plays \( b_i = \{ 0, \theta(b) \} \). However, by definition \( \theta(b) \) is the \( \alpha \)-quantile of the sample generated by \( b \) given \( q \) and \( \eta \). Since \( \alpha \in (0, 1) \), it follows that \( \theta(b) \notin (0, \theta(b)) \) which is impossible. If there is some equilibrium \( b \) with \( n - i' = n \), then all players have a peak higher than \( m(t, \kappa) \). Hence, a similar contradiction to the case with \( n - i' = 0 \) arises, which concludes Step B.I.

Step B.II. \( n - i' \notin [0, n) \). Assume now that there is some equilibrium \( b \) with \( n - i' \notin [0, n) \) and let \( i'' = \# \{ t \in N \mid t_i < \theta(b) \} \) denote the number of players with a peak strictly lower than the outcome \( \theta(b) \). Since, by assumption, \( \theta(b) > m(t, \kappa) \) it follows that \( i' = i'' \).

Given that \( \kappa_j \leq \kappa_{j+1} \) for any \( j \in \{1, 2, \ldots, n - 2\} \), \( n - i' \leq j' \) and \( i' \neq n \), the following inequality holds:

\[
\kappa_{n-i'} \leq \kappa_{j'} \leq m(t, \kappa).
\]

If \( i'' = n \), there are \( n \) players with a peak strictly lower than \( \theta(b) \). Lemma 3 implies that each player plays \( [0, \theta(b)] \), which, in turn, implies that \( \theta(b) \) is in the interior of \( [0, \theta(b)] \), a contradiction. Therefore, \( i'' \leq n - 1 \iff n - i'' \geq 1 \). Moreover by definition we have that \( i' \leq i'' \iff n - i'' \leq n - i' \) which implies that

\[
\kappa_{n-i''} \leq \kappa_{n-i'} \leq \kappa_{j'} \leq m(t, \kappa).
\]

If there is no \( t_h \) with \( t_h = \theta(b) \) then by Lemma 3, \( i'' \) players play \( [0, \theta(b)] \) and \( n - i'' \) players play \( [\theta(b), 1] \). Therefore, \( b \) to be an equilibrium it must be the case that \( \theta(b) = \kappa_{n-i''} \) which contradicts \( \theta(b) > m(t, \kappa) \). Thus, there is no equilibrium \( b \) with \( \theta(b) \neq t_h \).

If there is some \( t_h \) with \( t_h = \theta(b) \) then \( i'' \) players play \( [0, \theta(b)] \), \( n - i'' - 1 \) players play \( [\theta(b), 1] \) and player \( h \) plays the strategy \( b_h \). If \( b_h = [\theta(b), 1] \) then \( \theta(b) = \kappa_{n-i''} \) which contradicts \( \theta(b) > m(t, \kappa) \). If \( b_h \neq [\theta(b), 1] \) then \( t_h = \theta(b) < \theta([t_h, 1], b_{-h}) \) with \( ([t_h, 1], b_{-h}) \in \mathcal{B}(n - i'', t_h) \). Hence, by Lemma 2, \( t_h < \theta([t_h, 1], b_{-h}) \iff t_h < \kappa_{n-i''} \). Thus, we have that \( t_h = \theta(b) < \kappa_{n-i''} \) which contradicts \( \theta(b) > m(t, \kappa) \). Thus, there is no equilibrium \( b \) with \( \theta(b) = t_h \), which ends the proof. \( \square \)

We now have all the tools that are necessary to state the main result of this paper.

Theorem 1. For every generic GMR there exists a GAM that Nash-implements it.

Proof. Take some generic GMR with phantom vector \( p = (p_1, \ldots, p_{n-1}) \). We want to prove that there is some GAM with phantom vector \( \kappa = (\kappa_1, \ldots, \kappa_{n-1}) \) that Nash-implements it. Given the result of Proposition 1, it is sufficient to show that there exists a GAM with \( \kappa = p \).

Assume first that every \( p_j \in (0, 1) \). In this case, it suffices to set \( \alpha \in (0, 1), q \in \mathbb{R}^+ \) and some function \( \eta \) so that, for each \( j \in \{1, \ldots, n - 1\} \):

\[
p_j = \eta^{-1}(\frac{\alpha(nq + j) - (n - j)q}{(n - j) - \alpha(n - 2j)}),
\]

leading to \( \kappa = p \) as wanted.
Assume now that there is some pair \( a, b \in \{1, \ldots, n-1\} \) such that \( p_a = 0 \) and/or \( p_b = 1 \) with \( p_i \in (0, 1) \) if \( i \in (a, b) \).\(^{14}\) As previously argued, it must be the case that \( p_1 \leq p_2 \leq \ldots \leq p_{n-1} \). Hence, for any \( s \leq a \), \( p_s = 0 \) and for any \( t \geq b \), \( p_t = 1 \).

Take now some \( q \) and \( \alpha \) such that
\[
\frac{\alpha(nq + a) - (n - a)q}{(n - a) - \alpha(n - 2a)} = 0 \quad \text{and} \quad \frac{\alpha(nq + b) - (n - b)q}{(n - b) - \alpha(n - 2b)} = 1. \tag{2}
\]
This ensures that \( \kappa_a = 0 \) and \( \kappa_b = 1 \). The previous equalities are equivalent to
\[
q = \frac{a(nq + a - (n - a))}{n(1 - a) - a}, \tag{3}
\]
while \( \alpha \) depends on the value of \( a + b \). More precisely,
\[
\alpha = \frac{1}{(n - a - b)n}((n - a)(n - b) - \sqrt{ab(n - a)(n - b)}) \quad \text{if } a + b < n \tag{4}
\]
and
\[
\alpha = \frac{1}{(n - a - b)n}((n - a)(n - b) + \sqrt{ab(n - a)(n - b)}) \quad \text{if } a + b > n. \tag{5}
\]
Moreover, since \( 0 \leq \kappa_1 \leq \kappa_2 \leq \ldots \leq \kappa_{n-1} \leq 1 \), it follows that, for any \( s \leq a \), \( \kappa_s = 0 \) and for any \( t \geq b \), \( \kappa_t = 1 \).

If \( b = a + 1 \), then we are done, since \( \kappa = p \). If \( b > a + 1 \), then by assumption, any \( p_j \) with \( j \in \{a, \ldots, b\} \cap \{1, \ldots, n-1\} \) satisfies \( p_j \in (0, 1) \). Then, given that \( q \) and \( \alpha \) are given by (1), it is enough to suitably select \( \eta \) such that for any \( j \in \{a, \ldots, b\} \cap \{1, \ldots, n-1\} \), \( p_j = \eta^{-1}(\alpha(nq + j) - (n - j)q) / (n - j - \alpha(n - 2j)) \) which ensures that \( \kappa = p \) as wanted. \( \square \)

Finally, we discuss some examples that show the usefulness of the analysis above. The first one is concerned with the implementation of the Condorcet winner. The second attempts to illustrate how to implement GMRs with interior phantoms.

**Example 1: Implementing the Condorcet winner.** Let \( N = \{1, 2, 3\} \) be the set of players with \( t_1 < t_2 < t_3 \) and set \( a = 1 \), \( \alpha = 1/2 \) and \( \eta(x) = x \). Namely, each player is endowed with a weight of \( 1 + b_i - b_i \) and the outcome selected corresponds to the median of the distribution generated by \( b \). For short, we let \( \theta(b) \) denote the mechanism outcome and \( \phi(b, z) \) the cumulative distribution associated to any profile \( b \). The unique equilibrium outcome of this game is the selection of \( t_2 \), the median of the peaks and the Condorcet winner policy.

We first prove that \( t_2 \) is an equilibrium outcome and then show that it is the unique one. Let \( b = (b_i, b_{-i}) \) be a strategy profile with \( \theta(b) = t_2 \) for any \( t_2 \in (0, 1) \). If \( b \) is an equilibrium, Lemma 3 implies that \( b_1 = [0, t_2] \) and \( b_3 = [t_2, 1] \). Thus, in order to prove that there is an equilibrium with outcome \( t_2 \), it suffices to show that there is some \( b_2 \) with \( b = (0, t_2, b_2, [t_2, 1]) \) satisfying \( \theta(b) = t_2 \). If \( b_2 = [0, t_2] \), \( \phi(b, t_2) = \int_0^{t_2} \frac{2 + x^2}{3 + 2x + 1 - t_2} dx > \frac{1}{2} \) so that \( \theta(b) < t_2 \), whereas, if \( b_2 = [t_2, 1] \),

\(^{14}\) To ensure that \( \alpha \in (0, 1) \), when \( p_1 = 0 \) and \( p_{n-1} < 1 \), we consider that \( b = n - \frac{1}{2} \) and when \( p_1 > 0 \) and \( p_{n-1} = 1 \), we consider that \( a = \frac{1}{2} \).
\( \phi(b, t_2) = \int_0^{t_2} \frac{1 + \frac{1}{2} \sqrt{1 - 4t^2}}{3 + t_2 + 2(1 - t_2)} dx < \frac{1}{2} \) which implies that \( \theta(b) > t_2 \). Therefore, player 2 can change smoothly her strategy from \([0, t_2]\) to \([t_2, 1]\) and find a strategy \( b_2 \) which leads to \( \phi(b, t_2) = 1/2 \) – that is, to the implementation of \( t_2 \). The precise strategy of player 2 depends on the value of \( t_2 \). For each \( t \in (0, 1) \), let \( w(t) = \sqrt{1 - 2t + 4t^2} + 2t - 1 \). If \( t_2 \leq 1/2 \), then \( b_2 = [0, w(t_2)] \) ensures that \( \theta(b) = t_2 \), whereas, when \( t_2 \geq 1/2 \), then \( b_2 = [1 - w(1 - t_2), 1] \) ensures that \( t_2 \) is the outcome.

Now, in order to prove that there is no other possible equilibrium outcome, assume by contradiction that there is some \( z \neq t_2 \) elected at some equilibrium \( b \). Assume that \( z < t_2 \), the case with \( z > t_2 \) being symmetric. If \( z \in (t_1, t_2) \), then \( b \) must be such that the three players play \([z, 1]\) which leads to an outcome larger than \( z \). Finally, if \( z = t_1 \), then in any equilibrium \( b \), players 2 and 3 play \( b_2 = b_3 = [t_1, 1] \). However, if player 1 plays \([0, t_1]\), then \( \phi([0, t_1], b_2, b_3, t_1) < \frac{1}{2} \) for any \( t_1 \in [0, 1] \). Thus, \( \theta([0, t_1], b_2, b_3, t_1) > t_1 \). Yet, since \([0, t_1] \) is the best response of any player with peak to the left of \( t_1 \), it follows that \( [0, t_1] \notin \arg \min \theta(b_1, b_2, b_3) \). Therefore, \( \theta(b_1, b_2, b_3) > t_1 \) for any \( b_1 \in \mathbb{B} \) so that there is no equilibrium with outcome \( t_1 \). All in all, the unique equilibrium outcome associated with \( \theta \) is \( t_2 \).

To see how the previous argument extends to any number of players, consider that the number of players is odd.\(^{15}\) We let \( n = 2k + 1 \) for some non-negative integer \( k \) and let \( t^* \) denote the median peak of the \( n \) players. It follows that there are exactly \( k \) players with a peak smaller than \( t^* \) and \( k \) players with a peak larger than \( t^* \). We now set \( q = k, \alpha = 1/2 \) and \( \eta(x) = x \). In other words, each player is now endowed with a weight of \( k + b_i - b_1 \) and the outcome selected corresponds to the median of the distribution generated by \( b \). Using the equalities (3) and (4) in the proof of Theorem 1, the previous specifications ensure that half of the phantoms are located at zero and half of them at one, which leads to the implementation of the median. In equilibrium, the \( k \) players with a peak smaller than \( t^* \) approve of \([0, t^*] \) whereas the \( k \) ones with a peak larger than \( t^* \) approve of \([t^*, 1] \). The player with peak at \( t^* \) just needs to play some strategy \( b^*_i \) that ensures that the median of the profile equals \( t^* \). As in the case with just three players, one can prove by continuity that such strategy exists since the median of the score distribution is smaller than \( t^* \) when he plays \([0, t^*] \) and larger than \( t^* \) when he plays \([t^*, 1] \).

**Example 2: A GAM with interior phantoms.** If we set \( q = 0, \alpha = \frac{1}{2} \) and \( \eta(x) = x \), we get the Median Approval mechanism discussed in Section 3. The phantoms of this Approval mechanism must satisfy, for any \( j \in \{1, 2, \ldots, n - 1\} \), \( \eta(\kappa_j) = \frac{j + q(2j - n)}{n} \iff \kappa_j = \frac{2j}{n} \) (as already shown in Section 3, when \( N = \{1, 2, 3\} \) we have \( \kappa_1 = \frac{1}{3} \) and \( \kappa_2 = \frac{2}{3} \)). The equilibria with this mechanism in the case in which \( m(t_1, t_2, t_3, \frac{1}{3}, \frac{2}{3}) \) equals one of the peaks is similar to the ones of the Approval mechanism that implements the Condorcet winner.

\(^{15}\) This ensures the existence of a unique median player (or Condorcet Winner). When \( n = 2k \) for some \( k > 1 \), there are two median players: player \( k \) and player \( k + 1 \). Setting \( a = k \) and \( b = k + 1 \) and replacing these values in equalities (3) and (6) leads to the GAM with parameters \( q = \frac{\sqrt{k^2 - 1}}{k} \) and \( a = \frac{q}{2q + 1} \) and \( \eta(x) = x \) for any \( x \in [0, 1] \). This GAM implements the peak of the leftist median. Conversely, setting \( a = k - 1 \) and \( b = k \) in the equalities (3) and (5) leads to the GAM with parameters \( q = \frac{\sqrt{k^2 - 1}}{k} \) and \( a = \frac{kq + q}{2kq + k - 1} \) and \( \eta(x) = x \) for any \( x \in [0, 1] \). This GAM implements the peak of the rightist median player.
However, in the precise case in which $m(t_1, t_2, t_3, \frac{1}{3}, \frac{2}{3}) = \frac{1}{3}$ (the case $m(t_1, t_2, t_3, \frac{1}{3}, \frac{2}{3}) = \frac{2}{3}$ being symmetric), the logic is different. Indeed, the mechanism admits a unique equilibrium $b^*$ with $b^*_1 = b^*_2 = [0, \frac{1}{2}]$ and $b^*_3 = [\frac{1}{3}, 1]$. In general, if the equilibrium outcome coincides with a phantom and not with a type, there is a unique equilibrium (all players playing either to the left or to the right of the outcome) whereas this is not the case when a player’s peak is the equilibrium outcome (this player can play in several ways, while the rest of the players play either to the left or to the right of the outcome).

References