Karlsruhe Institute of Technology (KIT)

Condorcet Admissibility

Indeterminacy and Path-Dependence under Majority Voting on Interconnected Decisions

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The Judgement Aggregation Problem Defined

- A **judgement aggregation problem** consists in the aggregation of combined yes/no decisions on a set of interrelated binary issues (List and Pettit, 2002).
- With K issues, a *judgement set (a "view")* is an element of $\{0,1\}^K$. Importantly, not all of $\{0,1\}^K$ may be feasible.
- $X \subseteq \{0,1\}^K$ *feasible views*.
- $\{1, ..., I\}$ *set of individuals*, $F : X^I \rightrightarrows X$ *aggregation correspondence*.
- **Example (Preference Aggregation):** Strict orderings over alternatives a, b, c. Issue 1: "$a \succ b$?", issue 2: "$b \succ c$?", issue 3: "$c \succ a$?" Thus, $X^{\text{pref}} = \{0,1\}^3 \setminus \{(0,0,0),(1,1,1)\}$.

Inconsistency of the Issue-wise Majority Decision

- The issue-wise majority view may be **infeasible**: E.g. one third of the population endorse $(1, 1, 0)$ ["$a \succ b \succ c$"], one third endorse $(0, 1, 1)$ ["$b \succ c \succ a$"], and another third endorse $(1, 0, 1)$ ["$c \succ a \succ b$"], then issue-wise majority view $(1, 1, 1) \notin X^{\mathrm{pref}}$.
- **Example (Resource Allocation):** Budget M to be spent on L public goods. Issues: "spend at least j dollars for good l?" with feasibility constraint that exactly M dollars spent in total.
- E.g. one third of the population endorse $(M - 2, 1, 1)$, one third $(1, M - 1, 0)$, and another third $(0, 0, M)$. Then, majority view $(1, 1, 1) \notin X^{\mathrm{alloc}}$ if $M > 3$.

Other Examples

- **Example (Committee Selection):** K candidates for election into a committee with at least I members ($I \leq K$) and at most J members ($I \leq J \leq K$). Issues: "elect candidate k?"
- Again, feasibility problem arises: E.g. one third of the population endorse each of $(1, 0, 1, 0, 0)$, $(0, 1, 0, 1, 0)$ and $(0, 0, 1, 0, 1)$, respectively. Then, majority view $(0, 0, 1, 0, 0) \notin X^{\text{com}}$ if $I = 2$.
- Large number of further applications: aggregation of weak orderings, equivalence relations, partial orders, social classification and group identification á la Kasher/Rubinstein, reason based choice in legal contexts (the "doctrinal paradox"), probability aggregation, horizontal equity, etc.

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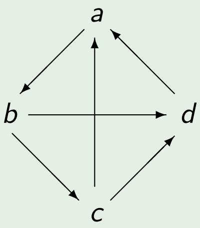
Condorcet Admissibility Defined

Definition

Given a profile $\mu \in X^I$ of feasible views, the **Condorcet admissible set** *$Cond(X, \mu) \subseteq X$ is the set of all $x \in X$ such that no feasible view coincides with the (issue-wise) majority view on a strictly larger set of issues than x.*

Example

Consider alternatives a, b, c, d and suppose that one third of the population endorses the preference orderings $a \succ_1 b \succ_1 c \succ_1 d$, $b \succ_2 c \succ_2 d \succ_2 a$ and $c \succ_3 d \succ_3 a \succ_3 b$, respectively. The Condorcet admissible set consists of the following five orderings: $a \succ b \succ c \succ d$, $b \succ c \succ d \succ a$, $c \succ d \succ a \succ b$, $d \succ a \succ b \succ c$, $c \succ a \succ b \succ d$.



Example

Consider $X^{\text{com}}_{I,J;K}$, and suppose that $M \subseteq \{1, ..., K\}$ is the set of candidates that receive majority support under the profile μ. The Condorcet admissible set is given as follows:

$$\begin{aligned} \text{If } I \leq \#M \leq J, \text{ then } & Cond(X,\mu) = \{\mathbf{1}_M\}, \\ \text{if } \#M < I, \text{ then } & Cond(X,\mu) = \{\mathbf{1}_H : M \subset H \text{ and } \#H = I\}, \\ \text{if } J < \#M, \text{ then } & Cond(X,\mu) = \{\mathbf{1}_H : H \subset M \text{ and } \#H = J\}. \end{aligned}$$

The Slater rule

- The **Slater rule** chooses the Condorcet admissible views $\mathit{Slater}(X, \mu) \subseteq X$ that maximize the *number* of issues in which there is agreement with the majority view.
- In general, $\mathit{Slater}(X, \mu) \subsetneqq \mathit{Cond}(X, \mu)$.
- E.g. $\mathit{Slater}(X^{\mathrm{pref}}, \mu) = \{b \succ c \succ d \succ a\}$ in the example above.

Proposition

For all μ, $\mathit{Cond}(X^{\mathrm{alloc}}, \mu) = \mathit{Slater}(X^{\mathrm{alloc}}, \mu)$ *and* $\mathit{Cond}(X^{\mathrm{com}}, \mu) = \mathit{Slater}(X^{\mathrm{com}}, \mu)$

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Sequential Majority Voting

Definition

For any given sequence γ of issues, **sequential majority voting** *$F^{\gamma}(\mu)$ yields the majority judgement in each issue unless consistency (feasibility) with the previous judgements in γ requires the opposite.*

Proposition

For all μ, $Cond(X, \mu) = \{F^{\gamma}(\mu)\}_{\gamma}$.

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1. Defining Majoritarianism: The Condorcet Admissible Set
   - The Judgement Aggregation Problem
   - The Condorcet Admissible Set
   - Sequential Majority Voting
2. Path-Dependence and Indeterminacy
   - Unanimity Violations
   - Quantifying Indeterminacy
3. Conclusion

When is respect for unanimity guaranteed?

- As above, suppose that one third of the population endorse each of the preference orderings $a \succ_1 b \succ_1 c \succ_1 d$, $b \succ_2 c \succ_2 d \succ_2 a$, $c \succ_3 d \succ_3 a \succ_3 b$, respectively.
- Consider any sequence of issues that decides (d, a), (a, b), (b, c) first.
- Then, sequential majority voting yields $d \succ a \succ b \succ c$, in particular $d \succ c$ by transitivity.
- But there is **unanimous** agreement that $c \succ d$!

- A *forbidden fragment* of length $k \leq K$ is a collection of judgements on a subset of k issues that cannot be extended to a feasible view on X. A forbidden fragment is called *critical* if it does not contain a strictly smaller forbidden fragment.

Theorem

$F^{\gamma}(\mu)$ never overrides a unanimous judgement if and only if all critical fragments have length ≤ 3.

Compare to the following result:

Theorem (Nehring and Puppe, 2007)

The Condorcet admissible set $Cond(X, \mu)$ is a singleton for all profiles μ (i.e. X is "majoritarian determinate") if and only if all critical fragments have length ≤ 2.

Can one **design** respect for unanimity?

- The above result states that unanimity is violated for *some* profile μ and *some* sequence γ provided that there is a critical fragment of length > 3 (which is typically the case).
- Can one *choose* a suitable path γ such that respect for unanimity is guaranteed for all profiles μ?
- In general, no! E.g. in the spaces X^{alloc} and X^{com} no path guarantees respect for unanimity.
- But in the spaces X^{pref} there **does** exist a path that guarantees respect for unanimity: number the alternatives from 1 to N and decide the (binary) issues according to the lexicographic ordering, i.e. first $(1,2),(1,3),...,(1,N)$, then $(2,3),(2,4),...$ etc.

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Issue-wise Indeterminacy and Total Indeterminacy

- A profile μ is called *issue-wise indeterminate* if sequential majority voting can give either answer in *each* issue, i.e. for each issue k there exist $x, y \in Cond(X, \mu)$ such that $x_k \neq y_k$.
- A profile μ is called *totally indeterminate* if $Cond(X, \mu) = X$.
- A space X is called *issue-wise indeterminate* if there exists an issue-wise indeterminate profile, and X is called *totally indeterminate* if there exists a totally indeterminate profile.

General Characterization of Issue-wise Indeterminacy

- Let w be a critical fragment. Say that a profile μ *activates* w if the majority view given μ coincides with w (on $\mathrm{supp}(w)$).
- Denote by $W(X, \mu)$ the set of all critical fragments activated by μ, and by $Indet(\mu)$ the indeterminate issues given μ.
- Thus, μ is issue-wise indeterminate if and only if $Indet(\mu) = \{1, ..., K\}$.

Theorem

For all X and all μ,

$$Indet(\mu) = \bigcup_{w \in W(X,\mu)} \mathrm{supp}(w).$$

- Say that $x \in \{0,1\}^K$ is *critical for X* if there exists a collection of critical fragments $w_1, \ldots, w_n$, each of length at least 3, such that (i) $w_j \subseteq x$ for all j, and (ii) $\cup_{j=1}^{n} \mathrm{supp}(w_j) = \{1, \ldots, K\}$.

Theorem

Let $X \subseteq \{0,1\}^K$.
(a) *A profile μ is issue-wise indeterminate if and only if the corresponding (issue-wise) majority view is critical for X.*
(b) *The space X is issue-wise indeterminate if and only if some (issue-wise) majority view is critical for X.*

Relation to McKelvey (1979)

Proposition

Consider the space X^{pref}, and assume that there are at least three alternatives. A majority tournament x is critical for X^{pref} if and only if its top cycle contains every alternative. Thus, a profile μ on X^{pref} is issue-wise indeterminate if and only if the top cycle of the corresponding majority tournament has "full range."

Issue-wise Indeterminacy on McGarvey spaces

- Say that X is *McGarvey* if every $x \in \{0,1\}^K$ is the (issue-wise) majority view for some profile μ on X (Nehring and Pivato, 2011a).

Corollary

If X is McGarvey, then X is issue-wise indeterminate if and only if there exists some $x \in \{0,1\}^K$ that is critical for X.

Application to Preference Aggregation

- The spaces X^{pref} are McGarvey if there are at least three alternatives (McGarvey, 1953),
- hence X^{pref} is issue-wise indeterminate if there exists x that is critical for X^{pref},
- which (as noted above) holds if and only the top cycle of x has full range;
- but such x are easily constructed,
- thus the spaces X^{pref} are issue-wise indeterminate if there are at least three alternatives.

- Are *all* (non-degenerate) McGarvey spaces issue-wise indeterminate? No! But we haven't found yet a "natural" counter example.
- Are perhaps all "natural" spaces that are not median spaces issue-wise indeterminate? No!
- For instance, if $I > 0$, then the spaces $X^{\text{com}}_{I,K;K}$ are issue-wise indeterminate if and only if $I < K/2$.
- How many individuals does an issue-wise indeterminate profile minimally involve? Often the minimal necessary number is only 3.

General Characterization of Total Indeterminacy

- A view $z \in \{0,1\}^K$ is called a *panopticon for X* if no element of X is between z and any other element of X (from z one can "see" all elements of X without the view being blocked by other elements).

Proposition

Let $X \subseteq \{0,1\}^K$.
(a) *A profile μ is totally indeterminate if and only if the corresponding (issue-wise) majority view is a panopticon for X.*
(b) *The space X is issue-wise indeterminate if and only if some (issue-wise) majority view is a panopticon for X.*

- The spaces $X^{\mathrm{com}}_{I,l;K}$ are totally indeterminate if $I > K/2$ (consider the uniform distribution).
- In McGarvey spaces, the existence of a panopticon is not only necessary but also sufficient for total indeterminacy, however:

Proposition

Suppose X contains $\mathbf{1}$ and $\mathbf{1}_k$ for all $k = 1, ...K$. Then X is McGarvey but admits no panopticon.

- Summarizing, our results suggest that many natural aggregation problems are issue-wise indeterminate, but not total indeterminate. In this respect, the spaces X^{pref} are typical.
- An exception are the spaces X^{alloc} in which a profile is issue-wise indeterminate if and only if it is totally indeterminate, which holds if and only if the corresponding majority view is $\mathbf{0}$.
- While few spaces are totally indeterminate, many are *asymptotically* totally indeterminate, i.e. the relative (logarithmic) size of the maximal Condorcet admissible set converges to 1. For instance, the relative size of the maximal Condorcet admissible set in the spaces X^{pref} converges to 1 when the number of alternatives grows without bound.

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Conclusion and Outlook

- Possible to define "pure" majoritarianism for a large class of interesting aggregation problems.
- Generally, the Condorcet admissible set coincides with sequential majority voting.
- Typically, one obtains path-dependence and indeterminacy.
- The extent of path-dependence and indeterminacy depends on the specific aggregation problem, and in particular on its *symmetries*.
- For instance, the preference aggregation problem has a particular combinatorial structure that makes it special in many respects.

Outlook

- Complement purely majoritarian principles with other considerations, e.g. majority margins (Nehring and Pivato, 2011)
- **Median rule:** Choose the feasible view that maximizes the sum of the popular support over each issue (Kemeny, 1959; Young and Levenglick, 1978). For the spaces X^{alloc}: Lindner, Nehring and Puppe, 2011.