Non-manipulable domains for the Borda count*

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Summary. We characterize the preference domains on which the Borda count satisfies Arrow's "independence of irrelevant alternatives" condition. Under a weak richness condition, these domains are obtained by fixing one preference ordering and including all its cyclic permutations ("Condorcet cycles"). We then ask on which domains the Borda count is non-manipulable. It turns out that it is non-manipulable on a broader class of domains when combined with appropriately chosen tie-breaking rules. On the other hand, we also prove that the rich domains on which the Borda count is non-manipulable for all possible tie-breaking rules are again the cyclic permutation domains.

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1 Introduction

Ever since their publication, the two most important results of social choice theory, the impossibility theorems of Arrow and Gibbard-Satterthwaite, have led to a steady search for possibility results on restricted domains (see Gaertner (2002) for a recent overview). The usual approach is to fix an appropriate set of admissible preferences, and to investigate which social welfare functions satisfy Arrow's conditions, respectively which social choice functions are non-manipulable, on that preference domain. Classic examples of this approach are Black (1958) and Moulin (1980) who consider the domain of single-peaked preferences.[1]

A somewhat different view on the question has been developed by Dasgupta and Maskin (2003) based on Maskin (1995). These authors consider specific preference aggregation rules such as majority rule, plurality rule or the Borda count, and ask on what domains these rules satisfy desirable conditions in the spirit of Arrow's conditions. The present paper follows this approach. Specifically, we restrict our attention here to the Borda count and, slightly more generally, to scoring methods (cf. Moulin (1988)). We characterize the preference domains on which scoring methods satisfy Arrow's conditions ("Arrovian domains"). In contrast to Dasgupta and Maskin (2003), we impose the original independence of irrelevant alternatives condition, not their stronger neutrality condition. By consequence, the Arrovian domains for the Borda count determined here encompass the domains that satisfy Dasgupta and Maskin's characterizing condition of "quasi-agreement." Our analysis also shows that all Arrovian domains for the Borda count that are minimally "rich" in the sense that any social alternative is on top of at least one preference ordering, are obtained by fixing one preference ordering over the alternatives and including all its cyclic permutations. Remarkably, these are precisely the configurations of preferences that give rise to the Condorcet paradox. The rich domains on which the Borda count "works well" thus turn out to be exactly the problematic domains for majority voting.[2]

We then consider the question on which domains the Borda count is strategy-proof. Since the Borda count does in general not select one single social alternative, we have to consider tie-breaking rules here. It turns out that the Borda count is strategy-proof with any given tie-breaking rule on all Arrovian domains. The converse is not true, however. We show by example that there exist rich domains on which the Borda count violates the independence of irrelevant alternatives condition but is nevertheless strategy-proof when combined with *some* suitable tie-breaking rule. On the other hand, as our main result we prove that the rich domains on which the Borda count is strategy-proof for *any* tie-breaking rule are again exactly the Arrovian domains (one fixed preference ordering together with all its cyclic permutations). While the result is quite intuitive, its proof is surprisingly complex due to the dependence on the way of how ties are broken.

Our analysis confirms the general view of the literature that the Borda count is highly vulnerable to strategic manipulation.[3] This intuition is made precise here in two

[1]See also Barberá, Gul and Stacchetti (1993) for a multi-dimensional extension, and Nehring and Puppe (2003, 2004) for a unifying approach to impossibility and possibility results based on "generalized single-peaked preferences" in the context of strategy-proofness.

[2]The apparent conflict of this conclusion with Dasgupta and Maskin's (2003) robustness result for majority voting is due to the fact that we do not impose their strong neutrality condition here.

[3]Recently, there have been different approaches to measuring the degree of "vulnerability" of voting procedures to strategic manipulation, see Saari (1990), Aleskerov and Kurbanov (1999) and Smith

ways. First, for any preference ordering, there is only *one* rich domain that contains the given preference ordering and that renders the Borda count non-manipulable. By contrast, for other choice rules there may exist many different rich and non-manipulable domains that contain a given preference ordering; for instance, there are many rich single-peaked domains that contain a given single-peaked preference ordering. Secondly, any fixed rich domain on which the Borda count is non-manipulable is as small as it could possibly be, since it contains just as many orderings as there are social states. Again, this strong restriction does not apply to single-peaked domains, for instance. Thus, the overall conclusion from our analysis is that the Borda count fares poorly in terms of strategic manipulation, in the sense that there are very few non-manipulable domains all of which are, moreover, very small.

The plan of the paper is as follows. In Section 2 we introduce our basic notation and definitions. Section 3 provides the characterization of the Arrovian domains for general scoring methods. In Section 4, we consider the most interesting special case of the Borda count. For this case, we provide an alternative global characterization which in particular yields the cyclic permutation structure of all rich Arrovian domains. In Section 5, we then investigate the structure of non-manipulable domains. While some proofs are included in the main text, the more technical ones are collected in an appendix.

2 Basic Notation and Definitions

2.1 Social Welfare Functionals and the Arrow Conditions

Let X be a finite universe of social states or social alternatives. By $\mathcal{P}_X$, we denote the set of all linear orderings (irreflexive, transitive and total binary relations) on X, and by $\mathcal{P} \subseteq \mathcal{P}_X$ a generic subdomain of the unrestricted domain $\mathcal{P}_X$. Moreover, denote by $\mathcal{R}$ the set of all weak orderings (reflexive, transitive and complete binary relations).

Definition (Social welfare functional) A mapping $F : \bigcup_{n=1}^{\infty} \mathcal{P}^n \to \mathcal{R}$ that assigns a social preference ordering $F(\succ_1, ..., \succ_n) \in \mathcal{R}$ to each n-tuple of linear orderings and all n is called a *social welfare functional (SWF).*

Thus, we do allow for non-trivial indifferences on the social level but not on the individual level. Note also that we require a SWF to be defined for societies with any finite number of agents. For some of our results this will be important. Alternatively, we could have assumed a continuum of agents as e.g. in Dasgupta and Maskin (2003).

A SWF F satisfies the *Pareto rule* on $\mathcal{P}$ if, for all $x, y \in X$, all $\succ_i \in \mathcal{P}$ and all n,

$$[x \succ_i y \text{ for all } i = 1, ..., n] \;\Rightarrow\; x \succ y,$$

where $\succ$ is the strict part of the social preference relation $\succeq\, = F(\succ_1, ..., \succ_n)$.

A SWF F is called *non-dictatorial* on $\mathcal{P}$ if, either $\#\mathcal{P} = 1$ or, for all $n \geq 2$ and all $i = 1, ..., n$, there exist $x, y \in X$ and $\succ_i \in \mathcal{P}$ such that $x \succ_i y$ and $y \succeq x$, where

(1999), among others. The most relevant study in our context is Favardin, Leppelley and Serais (2002) who characterize (for the case of three alternatives) the preference profiles at which the Borda count is manipulable. Their conclusion is that the Borda count is significantly more vulnerable than, say, the Copeland method. Note that, in contrast to this literature, our aim is not to determine the relative frequency of possible manipulation on an unrestricted domain, but to characterize the restricted domains on which manipulation can *never* occur.

$\succeq = F(\succ_1, ..., \succ_n)$.

A SWF F satisfies *independence of irrelevant alternatives (IIA)* on $\mathcal{P}$ if, for all $x, y \in X$, all n and all $\succ_i, \succ_i' \in \mathcal{P}$,

$$[\succ_i|_{\{x,y\}} = \succ_i'|_{\{x,y\}} \text{ for all } i = 1, ..., n] \Rightarrow \succeq|_{\{x,y\}} = \succeq'|_{\{x,y\}},$$

where $\succeq = F(\succ_1, ..., \succ_n)$, $\succeq' = F(\succ_1', ..., \succ_n')$, and $\succ|_{\{x,y\}}$ denotes the restriction of the binary relation $\succ$ to the pair $\{x, y\}$.

Definition (Scoring method) Let q be the cardinality of X, and let $s : \{1, ..., q\} \to \mathbf{R}$ satisfy $s(1) \geq s(2) \geq ... \geq s(q)$ and $s(1) > s(q)$. Moreover, let $rk[x, \succ]$ denote the *rank* of alternative x in the ordering $\succ$ (i.e. $rk[x, \succ] = 1$ if x is the top alternative in the ranking $\succ$, $rk[x, \succ] = 2$ if x is second-best, and so on). A SWF is a *scoring method* if for some function s, all $x, y \in X$, all n and all $\succ_i$, $i = 1, ..., n$,

$$x \succeq y \Leftrightarrow \sum_{i=1}^{n} s(rk[x, \succ_i]) \geq \sum_{i=1}^{n} s(rk[y, \succ_i]),$$

where $\succeq$ is the social preference corresponding to $(\succ_1, ..., \succ_n)$. The scoring method corresponding to the function $s : \{1, ..., q\} \to \mathbf{R}$ will be denoted by F^s. A scoring method is called *proper* if s is strictly decreasing.

Definition (Borda count) The *Borda count (or "rank-order voting rule")*, denoted by F^B, is the proper scoring method corresponding to the function $s(k) = q + 1 - k$ for $k = 1, ..., q$.

Clearly, all scoring methods are non-dictatorial; moreover, any proper scoring method satisfies the Pareto rule. On the other hand, scoring methods do not generally satisfy the IIA condition. A characterization of the domains on which scoring methods satisfy this condition will be provided in Section 3 below.

2.2 Social Choice Functions and Non-Manipulability

Definition (Social choice function) A mapping $f : \bigcup_{n=1}^{\infty} \mathcal{P}^n \to X$ that assigns a social alternative to each n-tuple of linear orderings and all n is called a *social choice function (SCF)*.

A SCF f satisfies *unanimity* on $\mathcal{P}$ if, for all $x \in X$, all $\succ_i \in \mathcal{P}$ and all n,

$$[rk[x, \succ_i] = 1 \text{ for all } i = 1, ..., n] \Rightarrow x = f(\succ_1, ..., \succ_n).$$

A SCF f is called *non-manipulable, or strategy-proof* on $\mathcal{P}$ if for all n, all $\succ_i, \succ_i' \in \mathcal{P}$ and all $\succ_{-i} \in \mathcal{P}^{n-1}$,

$$f(\succ_i, \succ_{-i}) \succeq_i f(\succ_i', \succ_{-i}).$$

Example (Borda count with tie-breaking rule) For our purposes, a *tie-breaking rule* is simply a linear ordering τ on X. Given a tie-breaking rule τ, any SWF F uniquely defines a SCF f by associating to each preference profile $(\succ_1, ..., \succ_n)$ the τ-best element of $F(\succ_1, ..., \succ_n) \subseteq X$. Below we will specifically consider the Borda count F^B together with a tie-breaking rule τ; the resulting SCF will be denoted by f_τ^B. Note that f_τ^B satisfies unanimity.

Obviously, sufficiently "small" domains can give rise to strategy-proofness in a trivial way. For instance, any SCF is vacuously strategy-proof on any domain consisting of one single preference ordering. We will therefore be often interested in domains that are "rich" in the sense that any alternative is on top of some preference ordering.[4]

Definition (Rich domain) A domain $\mathcal{P}$ is called *rich* if for any $x \in X$ there exists $\succ \in \mathcal{P}$ such that $rk[x, \succ] = 1$.

3 Arrovian Domains for Scoring Methods

It is well-known that scoring methods violate the IIA condition on the unrestricted domain $\mathcal{P}_X$. However, scoring methods may well satisfy this condition on restricted domains. We will say that $\mathcal{P}$ is an *Arrovian domain* for the SCW F if F is non-dictatorial and satisfies the Pareto rule as well as IIA on $\mathcal{P}$.

Definition (Equal score difference) A domain $\mathcal{P}$ satisfies the *equal score difference* condition with respect to s if, for all $x, y \in X$, either all orderings in $\mathcal{P}$ agree on $\{x, y\}$, or if not, then

$$s(rk[x, \succ]) - s(rk[y, \succ]) = s(rk[x, \succ']) - s(rk[y, \succ'])$$

for all $\succ, \succ' \in \mathcal{P}$ such that $\succ|_{\{x,y\}} = \succ'|_{\{x,y\}}$.

Note that for the Borda count the latter condition reduces to

$$rk[x, \succ] - rk[y, \succ] = rk[x, \succ'] - rk[y, \succ']$$

for all $\succ, \succ' \in \mathcal{P}$ such that $\succ|_{\{x,y\}} = \succ'|_{\{x,y\}}$, which we will also refer to as the *equal rank difference* condition.

Proposition 1 *A domain is Arrovian for the proper scoring method F^s if and only if it satisfies the equal score difference condition with respect to s.*

Proof Clearly, any scoring method is non-dictatorial and satisfies the Pareto rule on any domain. Let $\mathcal{P}$ satisfy the equal score difference condition. In order to verify IIA consider any $x, y \in X$ and $\succ_i \in \mathcal{P}$ for $i = 1, ..., n$. Suppose that $x \succeq y$, where $\succeq = F^s(\succ_1, ..., \succ_n)$, i.e. suppose that

$$\sum_{i=1}^{n} [s(rk[x, \succ_i]) - s(rk[y, \succ_i])] \geq 0. \tag{3.1}$$

If all orderings in $\mathcal{P}$ agree on $\{x, y\}$, we must in fact have $x \succ y$, and this relative ranking of x and y holds for the social preference corresponding to *any* profile. Thus, assume that not all orderings in $\mathcal{P}$ agree on the pair $\{x, y\}$. Then, by the equal score difference condition, the inequality (3.1) is preserved when any voter i's ordering $\succ_i \in \mathcal{P}$ is replaced by an ordering $\succ'_i \in \mathcal{P}$ that agrees with $\succ_i$ on $\{x, y\}$. This shows that F^s satisfies IIA on $\mathcal{P}$.

Conversely, suppose that the domain $\mathcal{P}$ does not satisfy the equal score difference condition. Then, there exist $x, y \in X$ and three orderings $\succ, \succ', \succ'' \in \mathcal{P}$ such that

$$l \;:=\; s(rk[y, \succ'']) - s(rk[x, \succ'']) \;>\; 0$$

[4]This condition is frequently imposed in the literature. It is much weaker than the richness condition used, for instance, in Nehring and Puppe (2004).

and

$$s(rk[x,\succ]) - s(rk[y,\succ]) \ =: \ m \ > \ m' \ := \ s(rk[x,\succ']) - s(rk[y,\succ']) \ > \ 0.$$

Choose n_1 and n_2 such that

$$\frac{l}{m'} > \frac{n_1}{n_2} > \frac{l}{m} \ ,$$

and consider the following two profiles of $n_1 + n_2$ individual preferences. In the first profile, denoted by $\Pi = (\succ, ..., \succ, \succ'', ... \succ'')$, the first n_1 voters have the preference $\succ$ and the remaining n_2 voters have the preference $\succ''$; in the second profile, denoted by $\Pi' = (\succ', ..., \succ', \succ'', ..., \succ'')$, the first n_1 voters have the preference $\succ'$, and the remaining n_2 voters have the preference $\succ''$. By construction, x is ranked strictly above y in the social ranking $F^s(\Pi)$ corresponding to the first profile, while y is strictly above x in the social ranking $F^s(\Pi')$ corresponding to the second profile. This yields the desired violation of IIA and completes the proof.

4 A Special Case: The Borda Count

The restrictiveness of the equal score difference condition depends on the scoring rule. For instance, suppose that $X = \{x, y, z\}$ and consider any scoring method s that does not coincide with the Borda count, i.e. $s(2) - s(1) \neq s(3) - s(2)$. It is easily seen that any domain that satisfies the equal score difference condition with respect to such s can consist of at most two preference orderings on X. More generally, one can show that, for arbitrary X, no scoring method different from the Borda count can satisfy the equal score difference condition on any rich domain. On the other hand, for the Borda count there are rich domains satisfying the corresponding (equal rank difference) condition. In the following, we will provide a "global" characterization of all such domains. Before we do so, we briefly want to compare our equal rank difference condition to Dasgupta and Maskin's (2003) condition of "quasi-agreement." That condition requires that any triple $\{x, y, z\}$ admit one member, say x, such that all orderings in the domain agree on either (i) x being the best element among the three, or (ii) x being the middle element, or (iii) x being the worst element among the triple. Dasgupta and Maskin (2003) show that the property of quasi-agreement characterizes the domains on which the Borda count satisfies an appropriate neutrality condition stronger than Arrow's independence of irrelevant alternatives considered here. By consequence, quasi-agreement is more restrictive than the equal rank difference condition. This can be directly verified by contraposition, as follows. Suppose that a domain violates the equal rank difference condition, i.e. there exist three orderings $\succ_1$, $\succ_2$ and $\succ_3$ such that

$$rk[y,\succ_1] - rk[x,\succ_1] > rk[y,\succ_2] - rk[x,\succ_2] > 0 \qquad (4.1)$$

and $rk[y,\succ_3] - rk[x,\succ_3] < 0$. By (4.1), there exists a third alternative z such that $x \succ_1 z \succ_1 y$ but not $(x \succ_2 z \succ_2 y)$, in which case the three orderings violate quasi-agreement on the triple $\{x, y, z\}$.

Equal rank difference as well as quasi-agreement are "local" conditions; the former imposes restrictions on any pair, the latter on any triple. It is therefore not evident how these conditions are reflected in the "global" structure of the corresponding domains. We now provide an alternative characterization of equal rank difference domains that makes this global structure explicit. An ordering $\succ'$ is called a *cyclic permutation* of $\succ$ if $\succ'$ can be obtained from $\succ$ by sequentially shifting the bottom element to the top while leaving the order between all other alternatives unchanged. Thus, for instance, the cyclic permutations of the ordering *abcd* are *dabc*, *cdab* and *bcda*. The set of all cyclic permutations of a fixed ordering $\succ$ is denoted by $\mathcal{Z}(\succ)$. Say that a domain $\mathcal{P}$ is *hierarchically cyclic* if there exists a partition $\{X_1, ..., X_r\}$ of X such that for all $\succ \in \mathcal{P}$ and all $i \in \{1, ..., r\}$,

(i) $x \succ y$ whenever $x \in X_i$, $y \in X_j$ and $j > i$, and

(ii) $\{\succ' |_{X_i} : \succ' \in \mathcal{P}\} \subseteq \mathcal{Z}(\succ |_{X_i})$ or $\#\{\succ' |_{X_i} : \succ' \in \mathcal{P}\} \leq 2$

Thus, a domain is hierarchically cyclic if the universe of alternatives can be partioned in such a way that (i) the partition elements themselves are ordered unambiguously and identically by all orderings, and (ii) within each partition element X_i, the restrictions to X_i give rise to at most two different orderings on X_i, or they are cyclic permutations of each other. The following table shows a typical domain satisfying this condition.

Table 1: A hierarchically cyclic domain

$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$	$\succ_5$
x_1	x_2	x_4	x_2	x_4
x_2	x_3	x_1	x_3	x_1
x_3	x_4	x_2	x_4	x_2
x_4	x_1	x_3	x_1	x_3
y_1	y_1	y_3	y_3	y_1
y_2	y_2	y_2	y_2	y_2
y_3	y_3	y_1	y_1	y_3
u	u	u	u	u
z_1	z_2	z_3	z_1	z_2
z_2	z_3	z_1	z_2	z_3
z_3	z_1	z_2	z_3	z_1

In the example shown in Table 1, the partition from the definition of a hierarchically cyclic domain is given by $X_1 = \{x_1, x_2, x_3, x_4\}$, $X_2 = \{y_1, y_2, y_3\}$, $X_3 = \{u\}$ and $X_4 = \{z_1, z_2, z_3\}$. Note that the preferences are cyclic permutations of one fixed ordering on X_1 and X_4. The two different restrictions on X_2 are not cyclic permutations of each other; nevertheless, the domain satisfies the defining condition since $\#X_2 \leq 2$.

We have the following result.

Theorem 1 *The following statements are equivalent.*
a) *$\mathcal{P}$ is an Arrovian domain for the Borda count.*
b) *$\mathcal{P}$ satisfies the equal rank difference condition.*
c) *$\mathcal{P}$ is hierarchically cyclic.*

The equivalence of the first two statements follows at once from Proposition 1 above. Furthermore, it is easily verified that any hierarchically cyclic domain satisfies the equal rank difference condition. The more difficult proof of the converse statement is deferred to the appendix. As an immediate corollary of Theorem 1, we obtain the following result showing that all *rich* Arrovian domains for the Borda count are obtained by fixing one preference ordering and including all its cyclic permutations; such domains will henceforth be referred to as *cyclic permutation domains.* Note that the cyclic permutation domains on three alternatives are precisely the "Condorcet cycles."

Corollary *For any linear ordering $\succ$, there is exactly one rich Arrovian domain for the Borda count that contains $\succ$, namely the cyclic permutation domain $\mathcal{Z}(\succ)$.*

5 Non-Manipulable Domains

We now want to ask on what domains the Borda count with tie-breaking rule is non-manipulable. The following result shows that the equal score/rank difference condition is sufficient for non-manipulability.

Proposition 2 *Suppose that the domain $\mathcal{P}$ satisfies the equal score difference condition. Then, any scoring method with any tie-breaking rule is strategy-proof on $\mathcal{P}$.*

Proof Take any preference profile $(\succ_1, ..., \succ_n)$, and suppose that x is the chosen alternative. Consider any alternative y that voter i prefers to x. Since y was not chosen there must exist another voter j such that $x \succ_j y$. By the equal score difference condition, any preference that favours y over x must display the same score difference between these alternatives as $\succ_i$. In particular, voter i cannot change the difference in total scores of y relative to x by reporting a preference that favours y over x. Since y is arbitrary this shows that voter i cannot successfully manipulate.

We now turn to the question of the necessary conditions for non-manipulability. This is a more difficult problem, and we will concentrate on the most interesting case of the Borda count. As already noted, if many conceivable preference orderings are excluded, strategy-proofness can result simply from the lack of misrepresentation possibilities. We will thus focus in the following on rich domains. Recall that the rich domains satisfying the equal rank difference condition are the cyclic permutation domains. First, we show by example that the Borda count may be non-manipulable also on domains that do not have the form of cyclic permutation domains, provided the tie-breaking rule is appropriately chosen.

Example (Non-manipulability without equal rank difference) Consider on the universe $X = \{a, b, c, d\}$ the domain $\{\succ_I, \succ_{II}, \succ_{III}, \succ_{IV}\}$, where $a \succ_I b \succ_I c \succ_I d$, $b \succ_{II} a \succ_{II} d \succ_{II} c$, $c \succ_{III} d \succ_{III} a \succ_{III} b$ and $d \succ_{IV} c \succ_{IV} b \succ_{IV} a$. Clearly, this domain is rich and not a cyclic permutation domain. Observe that the equal rank difference condition is only violated by the two pairs (a, d) and (b, c). Hence, manipulation is only possible between alternatives a and d, or b and c, respectively. In particular, one can easily check that a voter of type I, II, III or IV can potentially benefit only by reporting type II, I, IV or III, respectively. Note that for any manipulation of this kind, a voter can increase the total score difference only of two alternatives simultaneously over the other two alternatives; moreover, any such change in the score difference is by exactly two units. This property makes the domain rather special.

We will now show that the Borda count is non-manipulable when combined with the tie-breaking rule $a\tau b\tau c\tau d$. Suppose that a profile with n voters consists of k, l, m

and p preferences of types $\succ_I$, $\succ_{II}$, $\succ_{III}$ and $\succ_{IV}$, respectively. Then, we have

$$\sum_{i=1}^{n} rk\,[a, \succ_i] + rk\,[d, \succ_i] = \sum_{i=1}^{n} rk\,[b, \succ_i] + rk\,[c, \succ_i] = 5\,(k + l + m + p)\,. \qquad (5.1)$$

It follows from (5.1) that, if there is to be room for manipulation at all, the total scores of all four alternatives have to be close to each other. Consider the case in which a was chosen by f_τ^B; the other cases can be treated analogously. If a was chosen, then only a voter of type III might potentially benefit from manipulating (by misreporting to be of type IV). By the above observations and by the form of the tie-breaking rule, alternative d could "overtake" a only if before both received the same total score, or if a led only by one unit. In the first case, all four alternatives received the same total score by (5.1), while in the latter case b's total score was greater or equal to the total score of d, again by (5.1). Hence, misreporting type IV either does not change the outcome, or makes b the winner, which is not beneficial to a type III voter.

The example shows that on rich domains the equal rank difference condition is not necessary for non-manipulability of the Borda count together with a fixed tie-breaking rule. However, if we require non-manipulability of the Borda count when combined with *any* tie-breaking rule, the equal rank difference condition re-emerges, as shown by the following result.

Theorem 2 *Suppose that the Borda count is non-manipulable on the rich domain $\mathcal{P}$ for all tie-breaking rules τ. Then, $\mathcal{P}$ satisfies the equal rank difference condition, i.e. $\mathcal{P}$ is a cyclic permutation domain.*

The proof of Theorem 2 is provided in the appendix. The following example shows that the richness assumption in Theorem 2 is needed. Consider on $X = \{a, b, c, d\}$ the domain consisting of the three preference orderings *abcd*, *dabc* and *dacb*. This domain violates the equal rank difference condition (in fact even any equal score difference condition). But the Borda count is non-manipulable with any tie-breaking rule. Indeed, alternatives b and c can never win, while there are obviously no manipulation possibilities between alternatives a and d.

Appendix: Remaining proofs

For the proof of Theorem 1, we need the following notation. For any $1 \leq i \leq j \leq q = \#X$ let $\succ|_{[i,j]}$ be the restriction of $\succ$ ranging from the ith position to the jth position of $\succ$, i.e., $\succ|_{[i,j]} = \succ|_{\{x_i, x_{i+1}, \ldots, x_j\}}$ where $x_1 \succ \ldots \succ x_i \succ \ldots \succ x_j \succ \ldots \succ x_q$. In addition, for any $1 \leq i \leq j \leq q$, we define $\mathcal{P}_{[i,j]} := \left\{ \succ|_{[i,j]} : \succ \in \mathcal{P} \right\}$. Furthermore, for any linear ordering $\succ$ on $X' \subseteq X$ we shall denote by $T_i(\succ)$ the set of the top i alternatives of $\succ$, i.e., $T_i(\succ) = \{x \in X' : rk\,[x, \succ] \leq i\}$.

Proof of Theorem 1 "b) $\Rightarrow$ c)" We have to show that any domain satisfying the equal rank difference (henceforth: ERD) condition is hierarchically cyclic.

Step 1: We construct recursively a partition of X. Let $i_0 := 0$. To obtain the first partition element X_1, we determine the smallest integer $i \in \{i_0 + 1, \ldots, q\}$ satisfying

$$\forall x \in X, \forall \succ, \succ' \in \mathcal{P} : i_0 < rk\,[x, \succ] \leq i \Leftrightarrow i_0 < rk\,[x, \succ'] \leq i. \tag{A.1}$$

Clearly, at least q satisfies (A.1) and therefore there exists a smallest i, denoted by i_1, satisfying (A.1). Set $X_1 := \{x \in X : i_0 < rk\,[x, \succ] \leq i_1\}$ for some $\succ \in \mathcal{P}$. If $X_1 = X$, then we are finished and the partition consists only of the single set X_1. If $X_1 \neq X$, then we proceed inductively to obtain i_2 and X_2 from (A.1). Repeating this procedure, we get the desired partition $X_1, \ldots, X_r$.

In the following, we only have to consider those sets X_j for which

$$\#\mathcal{P}_{[i_{j-1}+1, i_j]} = \# \left\{ \succ'|_{X_j} : \succ' \in \mathcal{P} \right\} > 2. \tag{A.2}$$

Pick an arbitrary set X_j satisfying (A.2), and set $\mathcal{P}_j := \mathcal{P}_{[i_{j-1}+1, i_j]} = \left\{ \succ_1, \ldots, \succ_{n_j} \right\}$ and $q_j := i_j - i_{j-1}$. Clearly, $q_j \geq 3$ because of (A.2).

Step 2: First, we establish that $\mathcal{P}_j$ contains three preference relations with different top alternatives. Obviously, not all preferences can have the same top alternative, since this would be in contradiction with the construction of X_j. Thus, suppose that the preferences in $\mathcal{P}_j$ have two different top alternatives. Without loss of generality we can assume that the first $p \in \{2, \ldots, n_j - 1\}$ preferences have $a \in X_j$ as their top alternative, while the remaining preferences have another alternative $b \in X_j \setminus \{a\}$ as their top alternative. Define

$$Y := \{x \in X_j \ : \ \forall k, l \in \{1, \ldots, p\}, \ rk\,[x, \succ_k] = rk\,[x, \succ_l]\}. \tag{A.3}$$

Clearly, $a \in Y$. Moreover, we must also have $rk\,[b, \succ_k] = rk\,[b, \succ_l]$ for all $k, l \in \{1, \ldots, p\}$ by ERD, hence $b \in Y$. Let

$$\begin{aligned} J \ &:= \ \{k \in \{1, \ldots, q_j\} \ : \ \exists y \in Y, \ k = rk\,[y, \succ_1]\} \text{ and} \\ i^* \ &:= \ \max\{k \in \{1, \ldots, q_j\} \ : \ \{1, \ldots, k\} \subseteq J\}. \end{aligned}$$

Observe that i^* is well defined, since $\{1\} \subseteq J$.

Clearly, if $i^* = q_j$, then we have $\succ_1 = \ldots = \succ_p$, which cannot be the case, since the preferences $\succ_1, \ldots, \succ_p$ are distinct. Hence, we may assume that $i^* < q_j$. ERD implies that any alternative $z \in X_j \setminus Y$ must be ranked below the i^*th position by any preference relation having b on top, since z changes its rank difference to all alternatives in $T_{i^*}(\succ_1) = \ldots = T_{i^*}(\succ_p)$. Formally,

$$\forall z \in X_j \setminus Y, \ \forall l \in \{p+1, \ldots, n_j\} : rk\,[z, \succ_l] > i^*. \tag{A.4}$$

Suppose now that none of the alternatives in Y are ranked lower than i^* (i.e. suppose that $\{1, \ldots, i^*\} = J$). Then, we obtain $T_{i^*}(\succ_1) = \ldots = T_{i^*}\left(\succ_{n_j}\right)$ by (A.4), which contradicts the construction of X_j.

Thus, there exists an alternative $y \in Y$ with $rk\,[y, \succ_1] > i^*+1$ (i.e. $\{1, \ldots, i^*\} \neq J$). In this case, we will show that

$$\forall y \in Y, \forall l \in \{p+1, \ldots, n_j\} : rk\,[y, \succ_1] > i^* + 1 \Rightarrow rk\,[y, \succ_l] > i^*.^5 \tag{A.5}$$

Indeed, suppose that this is not the case, i.e. suppose that $y \in Y$ is such that $rk\,[y, \succ_1] > i^* + 1$ and $rk\,[y, \succ_l] \leq i^*$ for some $l \in \{p+1, \ldots, n_j\}$. Now pick two alternatives $u, v \in X_j \setminus Y$ and a preference relation $\succ_k$, $k \in \{2, \ldots, p\}$, such that $rk\,[u, \succ_1] = rk\,[v, \succ_k] = i^* + 1$. If $u \succ_k y$ or $v \succ_1 y$, then the pair $\{u, y\}$ or the pair $\{v, y\}$, respectively, violates ERD, since $y \succ_l u$ and $y \succ_l v$ by (A.4). Similarly, it can be verified that if $v \succ_k y \succ_k u$ and $u \succ_1 y \succ_1 v$, then at least one of the pairs $\{u, v\}$, $\{v, y\}$ or $\{u, y\}$ violate ERD by (A.4). We have thus derived a contradiction, hence (A.5) holds. Together with (A.4), this implies that all alternatives ranked below the i^*th position in the first p orderings must also be ranked below the i^*th position in all remaining orderings. But this means again that $T_{i^*}(\succ_1) = \ldots = T_{i^*}\left(\succ_{n_j}\right)$, contradicting the definition of X_j. Thus, we must have at least three different top alternatives in $\mathcal{P}_j$.

Step 3: We now show that any three top alternatives in $\mathcal{P}_j$ produce a Condorcet cycle. Pick preferences $\succ_k, \succ_m, \succ_l \in \mathcal{P}_j$ having the three different alternatives $a, b, c \in X_j$, respectively, on top. Without loss of generality we may assume that $a \succ_k b \succ_k c$. Now it can easily be verified that from the four possibilites

$$\begin{array}{l} [a \succ_k b \succ_k c,\ b \succ_m a \succ_m c,\ c \succ_l a \succ_l b]\,, \\ [a \succ_k b \succ_k c,\ b \succ_m c \succ_m a,\ c \succ_l a \succ_l b]\,, \\ [a \succ_k b \succ_k c,\ b \succ_m a \succ_m c,\ c \succ_l b \succ_l a]\,, \text{ and} \\ [a \succ_k b \succ_k c,\ b \succ_m c \succ_m a,\ c \succ_l b \succ_l a] \end{array}$$

only the Condorcet cycle satisfies ERD.

Step 4: We claim that for any three different top alternatives a, c and b, where $rk\,[a, \succ_k] = rk\,[c, \succ_l] = rk\,[b, \succ_m] = 1$, there exists $t_{k,l,m} \in \{1, \ldots, q_j\}$ such that $\{\succ_l|_{[1,t_{k,l,m}]}, \succ_m|_{[1,t_{k,l,m}]}\} \subseteq \mathcal{Z}\left(\succ_k|_{[1,t_{k,l,m}]}\right)$. By Step 3 we can assume that the top elements are ordered in the following way $a \succ_k b \succ_k c$, $c \succ_l a \succ_l b$, $b \succ_m c \succ_m a$. Take an alternative x such that $c \succ_l x \succ_l a$. Suppose that $a \succ_m x$; this implies $c \succ_k x$ by ERD. But then ERD must be violated, since x cannot maintain its rank difference to both a and b in $\succ_k$ as well as in $\succ_m$. Hence, we have $x \succ_m a$, and by ERD, the rank difference between x and a has to be the same in $\succ_l$ as in $\succ_m$. In a similar way one can establish that $rk\,[b, \succ_k] - rk\,[z, \succ_k] = rk\,[b, \succ_l] - rk\,[z, \succ_l]$ for any $a \succ_k z \succ_k b$ and that $rk\,[c, \succ_m] - rk\,[y, \succ_m] = rk\,[c, \succ_k] - rk\,[y, \succ_k]$ for any $b \succ_m y \succ_m c$.

Next we pick an alternative z satisfying $a \succ_l z \succ_l b$. Then z must be ranked below a in $\succ_m$, since by the above argument, $\{w : c \succ_l w \succ_l a\} = \{w : c \succ_m w \succ_m a\}$, $\{w : b \succ_m w \succ_m c\} = \{w : b \succ_k w \succ_k c\}$, and $\{w : a \succ_k w \succ_k b\} = \{w : a \succ_l w \succ_l b\}$. Hence, by ERD, $rk\,[c, \succ_l] - rk\,[z, \succ_l] = rk\,[c, \succ_m] - rk\,[z, \succ_m]$. Similarly, $rk\,[a, \succ_k] - rk\,[y, \succ_k] = rk\,[a, \succ_l] - rk\,[y, \succ_l]$ for any $b \succ_k y \succ_k c$, and $rk\,[b, \succ_m] - rk\,[x, \succ_m] = rk\,[b, \succ_k] - rk\,[x, \succ_k]$ for any $c \succ_m x \succ_m a$. Finally, observe that we can choose $t_{k,l,m} = rk\,[b, \succ_k] - rk\,[a, \succ_k] + rk\,[a, \succ_l] - rk\,[c, \succ_l] + rk\,[c, \succ_m] - rk\,[b, \succ_m]$.

[5] Observe that this implies $rk\,[b, \succ_1] \leq i^*$, since $b \in Y$ and $rk\,[b, \succ_{p+1}] = 1$.

Step 5: Now we can complete the proof. Assume that $a, b, c, d_4, \ldots d_{n_j}$ are the top alternatives of $\succ_1, \ldots, \succ_{n_j}$, respectively, where a, b and c are pairwise distinct. Apply Step 4 to preferences $\succ_1$, $\succ_2$ and $\succ_3$, and pick another preference relation $\succ_m \in \mathcal{P}_j$ arbitrarily.

First, if one of the three first top alternatives, say c, is also the top alternative of $\succ_m$, then Step 3 and ERD imply $rk[a, \succ_3] - rk[c, \succ_3] = rk[a, \succ_m] - rk[c, \succ_m]$ and $rk[b, \succ_3] - rk[c, \succ_3] = rk[b, \succ_m] - rk[c, \succ_m]$. Hence, by Step 4 we must have $\succ_3|_{[1,t_{1,2,3}]} = \succ_m|_{[1,t_{1,2,3}]} \in \mathcal{Z}\left(\succ_1|_{[1,t_{1,2,3}]}\right)$.

Second, suppose that $d_m \in X_j$ is distinct from a, b and c. Then it can be easily verified that $\succ_2|_{[1,t_{1,2,3}]} \in \mathcal{Z}\left(\succ_1|_{[1,t_{1,2,3}]}\right)$ and $\succ_2|_{[1,t_{1,2,m}]} \in \mathcal{Z}\left(\succ_1|_{[1,t_{1,2,m}]}\right)$ implies $t_{1,2,3} = t_{1,2,m}$.

Thus, in both cases we obtain $T_{t_{1,2,3}}(\succ_1) = \ldots = T_{t_{1,2,3}}\left(\succ_{n_j}\right)$. Therefore, we must have $t_{1,2,m} = q_j$ for all $m \in \{3, \ldots, n_j\}$ by the construction of X_j. This completes the proof of Theorem 1.

For the proof of Theorem 2, we need the following series of lemmas. Given a profile of preferences $(\succ_1, \ldots, \succ_n) \in \mathcal{P}^n$, we say that alternatives $A \subseteq X$ are *indifferent on the top* if

$$\sum_{i=1}^{n} rk[a, \succ_i] = \sum_{i=1}^{n} rk[b, \succ_i] < \sum_{i=1}^{n} rk[c, \succ_i] \tag{A.6}$$

for all $a, b \in A$ and all $c \in X \setminus A$.

Lemma A.1 *If there exists a preference profile $(\succ_1, \ldots, \succ_n) \in \mathcal{P}^n$ with alternatives $\{x, y\} \subseteq X$ being indifferent on the top and violating ERD, then there exists a tie-breaking rule such that Borda count is manipulable on $\mathcal{P}$.*

Proof of Lemma A.1 Suppose that profile $\Pi := (\succ_1, \ldots, \succ_n) \in \mathcal{P}^n$ has alternatives x and y violating ERD indifferent on the top. If according to Π we have

$$\sum_{i=1}^{n} rk[x, \succ_i] = \sum_{i=1}^{n} rk[y, \succ_i] \geq \sum_{i=1}^{n} rk[c, \succ_i] - 2(\#X - 1)$$

for some $c \in X \setminus \{x, y\}$, then we can take a 'multiple' of profile Π consisting of l preferences of type $\succ_i$ for each i such that

$$l\sum_{i=1}^{n} rk[x, \succ_i] = l\sum_{i=1}^{n} rk[y, \succ_i] < l\sum_{i=1}^{n} rk[c, \succ_i] - 2(\#X - 1) \tag{A.7}$$

for all $c \in X \setminus \{x, y\}$ and l sufficiently large by (A.6). This ensures that if only one voter reveals another preference relation, then either x or y will still be the Borda winning alternative. For notational convenience we will assume in what follows that $\Pi = (\succ_1, \ldots, \succ_n)$ already satisfies (A.7).

Since x and y are indifferent on the top, profile Π must have voters with preferences $\succ_i$ and $\succ_j$ such that $x \succ_i y$ and $y \succ_j x$. Suppose that there exists another preference $\succ' \in \mathcal{P}$ such that $x \succ' y$ and $rk[y, \succ'] - rk[x, \succ'] \neq rk[y, \succ_i] - rk[x, \succ_i]$. Now if $rk[y, \succ'] - rk[x, \succ'] > rk[y, \succ_i] - rk[x, \succ_i]$, then, taking a tie-breaking rule selecting y as the winner in case of ties between x and y, a voter having preference $\succ_i$ could manipulate by revealing preference $\succ'$. Otherwise, if $rk[y, \succ'] - rk[x, \succ'] < rk[y, \succ_i] -$

$rk[x, \succ_i]$, then we take the tie-breaking rule, which selects x as the winner in case of ties between x and y. Consider profile $(\succ_1, \dots, \succ_{i-1}, \succ', \succ_{i+1}, \dots, \succ_n)$, which has y as the Borda winner. Clearly, voter i can achieve a tie between x and y by revealing $\succ_i$ instead of $\succ'$ and therefore, enforce that x will be chosen, which he prefers to y.

Finally, if there does not exist a preference $\succ' \in \mathcal{P}$ such that $x \succ' y$ and $rk[y, \succ'] - rk[x, \succ'] \neq rk[y, \succ_i] - rk[x, \succ_i]$, then there exists a preference $\succ' \in \mathcal{P}$ such that $y \succ' x$ and $rk[x, \succ'] - rk[y, \succ'] \neq rk[x, \succ_i] - rk[y, \succ_i]$, since x and y violate ERD. Hence, to complete the proof we just have to exchange the roles of x and y while repeating the arguments of the previous paragraph.

The next lemma is a simple corollary to Lemma A.1.

Lemma A.2 *If in a rich domain $\mathcal{P}$ there exists a preference $\succ$ with its top two alternatives violating ERD, then there exists a tie-breaking rule such that Borda count is manipulable on $\mathcal{P}$.*

Proof of Lemma A.2 Let $rk[x, \succ] = 1$ and $rk[y, \succ] = 2$. Since $\mathcal{P}$ is a rich domain, we can find a preference $\succ' \in \mathcal{P}$, which has y as the top alternative. We define $d := rk[x, \succ'] - rk[y, \succ']$. Now taking one voter with $\succ'$ and d voters with $\succ$ we obtain a profile that has $\{x, y\}$ indifferent on the top, since y dominates any $z \in X \setminus \{x, y\}$. Now apply Lemma A.1.

Sometimes the set of alternatives that are indifferent on the top will contain more than two alternatives. In this case the following lemma turns out to be helpful in many cases.

Lemma A.3 *Suppose that $\mathcal{P}$ is a rich domain. If there exist two distinct preferences $\succ, \succ' \in \mathcal{P}$ and an alternative $y \in X$ satisfying*

- $rk[y, \succ] \geq 2$
- $\forall x \in X : x \succ y \Rightarrow x \succ' y,$
- $\forall x \in X : x \succ y \Rightarrow rk[y, \succ] - rk[x, \succ] \neq rk[y, \succ'] - rk[x, \succ'],$

then there exists a tie-breaking rule such that Borda count is manipulable on $\mathcal{P}$.

Proof of Lemma A.3 Let $k = rk[y, \succ]$. For all $i \in \{1, \dots, k-1\}$ we shall denote by x_i the alternative with $rk[x_i, \succ] = i$. Pick a preference $\succ'' \in \mathcal{P}$ having y as the top alternative. We define values $d_i := rk[x_i, \succ''] - rk[y, \succ'']$ for all $i \in \{1, \dots, k-1\}$. Clearly, we have $rk[y, \succ] - rk[x_i, \succ] = k - i$ for all $i \in \{1, \dots, k-1\}$. Now let $J := \arg\min_{i \in \{1,\dots,k-1\}} \frac{d_i}{k-i}$ and $A := \{x_j \in X : j \in J\}$. Pick an arbitrary $j \in J$. Then it can be verified that a profile consisting of d_j preferences of type $\succ$ and $k - j$ preferences of type $\succ''$ makes alternatives $\{y\} \cup A$ indifferent on the top. In particular, we will take a profile $(\succ_i)_{i=1}^n \in \mathcal{P}^n$ consisting of ld_j preferences of type $\succ$ and $l(k-j)$ preferences of type $\succ''$ for which

$$\sum_{i=1}^{n} rk[y, \succ_i] = \sum_{i=1}^{n} rk[a, \succ_i] < \sum_{i=1}^{n} rk[b, \succ_i] - 2(\#X - 1)$$

is satisfied for all $a \in A$ and all $b \in X \setminus (\{y\} \cup A)$, where l is a suffciently large positive integer. Thus, we can restrict our attention to alternatives in $\{y\} \cup A$.

We have to deal with two cases. First, suppose that there exists an alternative $a \in A$ such that $rk[y, \succ] - rk[a, \succ] < rk[y, \succ'] - rk[a, \succ']$. If we select a tie-breaking rule, which prefers y to all alternatives in A, then a voter having preference $\succ$ can manipulate by revealing $\succ'$, since he prefers any alternative in A to y.

Second, suppose that for all alternatives $a \in A$ we have $rk[y, \succ] - rk[a, \succ] > rk[y, \succ'] - rk[a, \succ']$. If we select a tie-breaking rule, which prefers all alternatives in A to y, and consider a profile in which one voter's preference of type $\succ$ in $(\succ_i)_{i=1}^{n}$ is replaced by $\succ'$, then this voter with preference $\succ'$ can manipulate by revealing $\succ$, since he prefers any alternative in A to y. This completes the proof of the lemma.

Proof of Theorem 2 We prove the contraposition of the statement, i.e., if a rich domain $\mathcal{P} \subseteq \mathcal{P}_X$ does not satisfy ERD, then there exists a tie-breaking rule for which the Borda count is manipulable on $\mathcal{P}$. Hence, suppose that the rich domain $\mathcal{P} \subseteq \mathcal{P}_X$ does not satisfy ERD.

Step 1: We can assume without loss of generality that the rich domain $\mathcal{P}$ violating ERD consists of exactly q preferences (recall that $q = \#X$). This can be verified as follows. Take an arbitrary rich domain $\mathcal{P}$ violating ERD with $\#\mathcal{P} > q$. Choose q preferences from $\mathcal{P}$ with different top alternatives, and denote the corresponding domain by $\mathcal{P}_0$. If $\mathcal{P}_0$ violates ERD, then we are done. On the other hand, if $\mathcal{P}_0$ does not violate ERD, then $\mathcal{P}_0 = \mathcal{Z}(\succ)$ for any $\succ \in \mathcal{P}_0$ by Theorem 1. Consider any preference ordering $\succ_0 \in \mathcal{P} \setminus \mathcal{P}_0$ and replace the preference in $\mathcal{P}_0$ with the same top alternative as $\succ_0$ by the ordering $\succ_0$. As is easily verified, the resulting domain violates ERD. Henceforth, we thus assume that $\mathcal{P} = \{\succ_1, \ldots, \succ_q\}$ is rich and violates ERD.

Step 2: We will construct a "chain" of alternatives and preferences. Start with preference $\succ_1$ and denote its top alternative by x_1 and its second ranked alternative by x_2. Without loss of generality we can assume that $\succ_2$ has x_2 on top. To describe how the procedure goes on suppose that we have already obtained a sequence of distinct alternatives $x_1, \ldots, x_k$ such that $rk[x_i, \succ_i] = 1$ for all $i \in \{1, \ldots, k\}$ and $rk[x_i, \succ_{i-1}] = 2$ for all $i \in \{2, \ldots, k\}$. Now we define x_{k+1} recursively to be the second ranked alternative of $\succ_k$. We have found a "chain" if x_{k+1} equals one of the alternatives $x_1, \ldots, x_k$. Otherwise, we can suppose without loss of generality that x_{k+1} is the top alternative of $\succ_{k+1}$. We iterate the described procedure until we obtain a "chain" of alternatives. Clearly, this procedure terminates in at most q steps. Thus, we can determine indices $m, p \in \{1, \ldots, q\}$ such that $m < p$, $x_m, \ldots, x_p$ are all distinct, $rk[x_i, \succ_i] = 1$ and $rk[x_{i+1}, \succ_i] = 2$ for all $i \in \{m, \ldots, p-1\}$, and $rk[x_p, \succ_p] = 1$ and $rk[x_m, \succ_p] = 2$. In what follows we can assume without loss of generality that $m = 1$. Nevertheless we will still denote the length of the chain by p. Furthermore, let $X' := \{x_1, \ldots, x_p\}$ and $\mathcal{P}_1 := \{\succ_1, \ldots, \succ_p\}$.

Step 3: We can manipulate by Lemma A.2 for some tie-breaking rules if there exists a preference $\succ_i \in \mathcal{P}_1$ in which the top two alternatives violate ERD. Hence, in the following analysis we can assume that the top two alternatives of all $\succ_1, \ldots, \succ_p$ satisfy ERD. But this implies that the top p alternatives of the preferences in $\mathcal{P}_1$ follow the pattern shown in Table 2. Clearly, if $p = q$, we cannot have a violation of ERD by Proposition 1, hence we must have $p < q$.

Case (i): Suppose that there exists an alternative $y \in X$ that is ranked by two distinct preferences $\succ_i$ and $\succ_j$ $(i, j \in \{1, \ldots, p\})$ at the $p{+}1$th position. Then y violates ERD with all alternatives $x_1, \ldots, x_p$, since $\mathcal{P}$ is a rich domain and all alternatives $x_1, \ldots, x_p$ are ranked differently according to $\succ_i$ and $\succ_j$ while y is ranked identically

Table 2: A full cycle on the top

$\succ_1$	$\succ_2$	$\dots$	$\succ_{p-1}$	$\succ_p$	$\dots$
x_1	x_2	$\dots$	x_{p-1}	x_p	$\dots$
x_2	x_3	$\dots$	x_p	x_1	$\dots$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	
x_{p-1}	x_p	$\dots$	x_{p-3}	x_{p-2}	$\dots$
x_p	x_1	$\dots$	x_{p-2}	x_{p-1}	$\dots$
$\vdots$	$\vdots$		$\vdots$	$\vdots$	

by these two preferences. Hence, taking $\succ_i$, $\succ_j$ and y we can apply Lemma A.3.

Case (ii): Suppose that the alternatives $y_1, \dots, y_p \in X$ are all distinct and are ranked $p+1$th by the preferences $\succ_1, \dots, \succ_p$, respectively. Let $Y := \{y_1, \dots, y_p\}$.

We claim that if there exists an alternative $y_i \in Y$ and a preference $\succ_j \in \mathcal{P}_1$ such that $rk\,[y_i, \succ_j] - rk\,[x_i, \succ_j] \neq p$, then y_i violates ERD with all alternatives in X', and manipulation is possible by Lemma A.3, taking $\succ_i$, $\succ_j$ and y_i as $\succ$, $\succ'$ and y, respectively. We check this claim without loss of generality for alternative y_p.[6] Of course, $rk\,[y_p, \succ_p] - rk\,[x_p, \succ_p] = p$ and therefore, $rk\,[y_p, \succ_j] - rk\,[x_p, \succ_j] \neq p$ implies that x_p and y_p violate ERD. Suppose that $d := rk\,[y_p, \succ_j] - rk\,[x_p, \succ_j] < p$. Note that we have $rk\,[x_p, \succ_j] = p - j + 1$ and therefore, it follows that $rk\,[y_p, \succ_p] - rk\,[x_i, \succ_p] = p - i > rk\,[y_p, \succ_j] - rk\,[x_i, \succ_j] = d - i$ for all $i \in \{1, \dots, j-1\}$. In addition, for all $i \in \{j, \dots, p-1\}$ we have $rk\,[y_p, \succ_p] - rk\,[x_i, \succ_p] = rk\,[x_p, \succ_j] - rk\,[x_i, \succ_j] < rk\,[y_p, \succ_j] - rk\,[x_i, \succ_j]$. Now suppose that $d = rk\,[y_p, \succ_j] - rk\,[x_p, \succ_j] > p$. Then clearly, $rk\,[y_p, \succ_j] - rk\,[x_i, \succ_j] > d > p > rk\,[y_p, \succ_p] - rk\,[x_i, \succ_p] = p - i$ for all $i \in \{j, \dots, p-1\}$. Furthermore, for all $i \in \{1, \dots, j-1\}$ we have $rk\,[y_p, \succ_p] - rk\,[x_i, \succ_p] = p - i < rk\,[y_p, \succ_j] - rk\,[x_i, \succ_j] = d - i$. Hence, in any case y_p and x_i violate ERD for all $i \in \{1, \dots, p\}$.

We still have to investigate the case in which for all alternatives $y_i \in Y$ and for all preferences $\succ_j \in \mathcal{P}_1$ we have $rk\,[y_i, \succ_j] - rk\,[x_i, \succ_j] = p$. For this case the first p preferences are illustrated in Table 3. If $2p < q$, then we can mimic the arguments given so far for alternatives ranked, by some preference relations in $\mathcal{P}_1$, at the $2p+1$th position. By doing so, in a similar way as in case (i), we can derive that manipulation is possible through an appropriately selected tie-breaking rule if an alternative is ranked twice at the $2p+1$th position by some preferences in $\mathcal{P}_1$. Otherwise, let z_i be the alternative for which $rk\,[z_i, \succ_i] = 2p+1$ and let $Z := \{z_1, \dots, z_p\}$. Now, in an analogous way as in the beginning part of case (ii) one can argue that we can manipulate if there exists an alternative $z_i \in Z$ such that there exists a preference $\succ_j \in \mathcal{P}_1$ such that $rk\,[z_i, \succ_j] - rk\,[y_i, \succ_j] \neq p$. The case that remains to be investigated whenever $3p \leq q$ is illustrated in Table 4.

Alternatives $y_1, \dots, y_p$ are all top alternatives of a certain preference relation since $\mathcal{P}$ is a rich domain. We shall denote the set of these preferences by $\mathcal{P}_2$. Without

[6] If we relabel the alternatives and preferences cyclically, then the claim follows for all the other alternatives $y_1, \dots, y_{p-1}$ in the same way.

Table 3: Two consecutive full cycles

$\succ_1$	$\succ_2$	$\cdots$	$\succ_{p-1}$	$\succ_p$	$\cdots$
x_1	x_2	$\cdots$	x_{p-1}	x_p	$\cdots$
x_2	x_3	$\cdots$	x_p	x_1	$\cdots$
$\vdots$	$\vdots$	$\iddots$	$\vdots$	$\vdots$	
x_{p-1}	x_p	$\cdots$	x_{p-3}	x_{p-2}	$\cdots$
x_p	x_1	$\cdots$	x_{p-2}	x_{p-1}	$\cdots$
y_1	y_2	$\cdots$	y_{p-1}	y_p	$\cdots$
y_2	y_3	$\cdots$	y_p	y_1	$\cdots$
$\vdots$	$\vdots$	$\iddots$	$\vdots$	$\vdots$	
y_{p-1}	y_p	$\cdots$	y_{p-3}	y_{p-2}	$\cdots$
y_p	y_1	$\cdots$	y_{p-2}	y_{p-1}	$\cdots$
$\vdots$	$\vdots$		$\vdots$	$\vdots$	

Table 4: Three consecutive full cycles

$\succ_1$	$\cdots$	$\succ_p$	$\cdots$
x_1	$\cdots$	x_p	$\cdots$
$\vdots$	$\iddots$	$\vdots$	
x_p	$\cdots$	x_{p-1}	$\cdots$
y_1	$\cdots$	y_p	$\cdots$
$\vdots$	$\iddots$	$\vdots$	
y_p	$\cdots$	y_{p-1}	$\cdots$
z_1	$\cdots$	z_p	$\cdots$
$\vdots$	$\iddots$	$\vdots$	
z_p	$\cdots$	z_{p-1}	$\cdots$
$\vdots$		$\vdots$	

Table 5: Two full cycles on the top

$\succ_1$	$\dots$	$\succ_p$	$\succ_{p+1}$	$\dots$	$\succ_{2p}$	$\dots$
x_1	$\dots$	x_p	y_1	$\dots$	y_p	$\dots$
$\vdots$	⋰	$\vdots$	$\vdots$	⋰	$\vdots$	
x_p	$\dots$	x_{p-1}	y_p	$\dots$	y_{p-1}	$\dots$
y_1	$\dots$	y_p	$\vdots$		$\vdots$	
$\vdots$	⋰	$\vdots$				
y_p	$\dots$	y_{p-1}				
z_1	$\dots$	z_p				
$\vdots$	⋰	$\vdots$				
z_p	$\dots$	z_{p-1}				
$\vdots$		$\vdots$				

loss of generality we can assume that $rk\,[y_i, \succ_{p+i}] = 1$ for all $i \in \{1, \dots, p\}$. Thus, $\mathcal{P}_2 = \{\succ_{p+1}, \dots, \succ_{2p}\} \subset \mathcal{P}$. In what follows we have to consider four subcases.

Subcase (a): There exists a preference $\succ_{p+i} \in \mathcal{P}_2$ that ranks an alternative $u \in X \setminus (X' \cup Y)$ second, i.e., $rk\,[u, \succ_{p+i}] = 2$. Then y_i and u violate ERD and $\mathcal{P}$ is manipulable with respect to an appropriate tie-breaking rule by Lemma A.2.

Subcase (b): The set of second ranked alternatives of all preferences in $\mathcal{P}_2$ is a subset of Y. If there exists a preference in $\mathcal{P}_2$ with top two alternatives violating ERD, then we can apply Lemma A.2. Otherwise, if the top two alternatives of all preferences in $\mathcal{P}_2$ satisfy ERD, then $\mathcal{P}_2$ must have a very special structure, since y_i is ranked just above $y_{i\oplus 1}$ for all $i \in \{1, \dots, p\}$ whenever y_i is ranked above $y_{i\oplus 1}$.[7] Thus, we must have preferences as shown in Table 5. Let

$$A := \left\{ x \in X' : \sum_{i=1}^{p} rk\,[x, \succ_{p+i}] \leq \sum_{i=1}^{p} rk\,[u, \succ_{p+i}] \text{ for all } u \in X' \right\}.$$

Now pick an alternative $x_i \in A$ and we will make $A \cup Y$ indifferent on the top. Define $d := \left(\sum_{j=1}^{p} rk\,[x_i, \succ_{p+j}]\right) - \frac{1}{2}p(p+1)$. For each $\succ \in \mathcal{P}_1$ taking d voters and for each $\succ' \in \mathcal{P}_2$ taking p^2 voters we obtain a profile in which all $x \in A$ and all $y \in Y$ are indifferent on the top. In particular, any $x \in A$ beats all alternatives in $X' \setminus A$ and any $y \in Y$ beats all alternatives in $X \setminus (X' \cup Y)$, while alternatives $x \in A$ and $y \in Y$ receive the same Borda score. Pick an alternative $x_i \in A$ and consider a voter having preference $\succ_{i\oplus 1}$. Suppose that the tie-breaking rule prefers $y_{i\oplus 1}$ to x_i and x_i to all

[7] For two integers $k, l \in \{1, \dots, p\}$, if $k + l \neq p$ and $k + l \neq 2p$, we define $k \oplus l := (k + l) \bmod p$, while if $k + l = p$ or $k + l = 2p$, we define $k \oplus l := p$.

Table 6: The final case of subcase (c)

$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$	$\succ_5$	$\succ_6$	$\dots$
x_1	x_2	x_3	y_1	y_2	y_3	$\dots$
x_2	x_3	x_1	x_1	x_2	x_3	$\dots$
x_3	x_1	x_2	x_2	x_3	x_1	$\dots$
y_1	y_2	y_3	x_3	x_1	x_2	$\dots$
y_2	y_3	y_1	$\cdot$	$\cdot$	$\cdot$	$\dots$
y_3	y_1	y_2	y_2	y_3	y_1	$\dots$
z_1	z_2	z_3	y_3	y_1	y_2	$\dots$
z_2	z_3	z_1	$\cdot$	$\cdot$	$\cdot$	
z_3	z_1	z_2	$\cdot$	$\cdot$	$\cdot$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

other alternatives. Then a voter having preference $\succ_{i\oplus 1}$ could manipulate by revealing $\succ_{i\oplus 2}$.

Subcase (c): The set of second ranked alternatives of all preferences in $\mathcal{P}_2$ is a subset of X'. If there exists a preference $\succ_{p+j}\in \mathcal{P}_2$ that ranks an alternative x_i with $i \neq j$ second, then the top two alternatives of $\succ_{p+j}$ violate ERD[8] and therefore, by Lemma A.2 we can find a tie-breaking rule making manipulation possible. Hence, in what follows we can assume that $rk\,[x_i, \succ_{p+i}] = 2$ for all $i \in \{1, \dots, p\}$. Since we know that the alternatives in X' satisfy ERD, $\mathcal{P}_2$ must have again a very special structure because x_i is ranked just above $x_{i\oplus 1}$ for all $i \in \{1, \dots, p\}$ whenever x_i is ranked above $x_{i\oplus 1}$. Thus, any $\succ\in \mathcal{P}_2$ must rank the alternatives of X' from the 2nd to the $p + 1$th position in a cyclic pattern. Therefore, in any preference $\succ_{p+i}\in \mathcal{P}_2$ we have $x_j \succ_{p+i} y_j$ for all $j \in \{1, \dots, p\} \setminus \{i\}$. Hence, if there exists a preference $\succ_{p+i}\in \mathcal{P}_2$ and a pair of alternatives x_j, y_j ($j \in \{1, \dots, p\} \setminus \{i\}$) such that $rk\,[y_j, \succ_{p+i}] - rk\,[x_j, \succ_{p+i}] \neq p$, then the top two alternatives y_j and x_j of $\succ_{p+j}$ violate ERD and we are done by applying Lemma A.2.

We still have to investigate the case in which for all preferences $\succ_{p+i}\in \mathcal{P}_2$ and for all pairs of alternatives x_j, y_j ($j \in \{1, \dots, p\} \setminus \{i\}$) we have $rk\,[y_j, \succ_{p+i}] - rk\,[x_j, \succ_{p+i}] = p$. For the case of $p = 3$ we illustrate this case in Table 6. Clearly, this case can only occur whenever $2p < q$. Observe that the $p+2$nd positions of each preference in $\mathcal{P}_2$ have to be filled with an alternative from $X \setminus (X' \cup Y)$. Suppose that we have $rk\,[u, \succ_{p+1}] = p+2$ for an alternative $u \in X \setminus (X' \cup Y)$. Then u violates ERD with all alternatives ranked by $\succ_{p+1}$ above u (i.e., with all alternatives in $X' \cup \{y_1\}$), since $\mathcal{P}$ is a rich domain. More specifically, if $u \neq z_2$, then we can apply Lemma A.3 with $\succ_{p+1}, \succ_1$ and u; while if $u = z_2$, then we can apply Lemma A.3 with $\succ_{p+1}, \succ_2$ and u.

Subcase (d): We still have to investigate the case in which the second ranked alternatives in $\mathcal{P}_2$ come from both X' and Y. First, observe that as in subcase (c), if there exists a preference $\succ_{p+j}\in \mathcal{P}_2$ that ranks an alternative x_i with $i \neq j$ 2nd, then x_i and

[8] In fact, looking at Table 3 it is easy to verify that all pairs x_i and y_j ($i \neq j$) violate ERD, while all pairs $x_i \in X'$ and $y_i \in Y$ satisfy ERD on $\mathcal{P}_1$. For instance, if $i < j$, then the sequence $(rk\,[x_i, \succ_k])_{k=1}^{p}$ decreases until $k = i$ and jumps up by $p - 1$ afterwards, whereas $(rk\,[y_i, \succ_k])_{k=1}^{p}$ still decreases after $k = i$. Hence, the rank differences between x_i and y_j differ according to preferences $\succ_i$ and $\succ_{i+1}$. One can argue analogously in case of $1 \leq j < i \leq p$.

Table 7: The final case of subcase (d)

$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$	$\succ_5$	$\succ_6$	$\dots$
x_1	x_2	x_3	y_1	y_2	y_3	$\dots$
x_2	x_3	x_1	y_2	x_2	$\cdot$	$\dots$
x_3	x_1	x_2	x_2	x_3	$\cdot$	$\dots$
y_1	y_2	y_3	x_3	x_1	$\cdot$	$\dots$
y_2	y_3	y_1	x_1	$\cdot$	$\cdot$	$\dots$
y_3	y_1	y_2	$\cdot$	$\cdot$	$\cdot$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

y_j violate ERD and we can apply Lemma A.2. Hence, in what follows we can assume that if $rk\,[u, \succ_{p+i}] = 2$ and $u \in X'$, then $u = x_i$.

Second, if there exists a preference $\succ_{p+i} \in \mathcal{P}_2$ that ranks an alternative $y \in Y \setminus \{y_{i\oplus 1}\}$ second, then y_i and y violate ERD and we are done by Lemma A.2.

Finally, we can assume that there exists $\succ_{p+i} \in \mathcal{P}_2$ such that $rk\,[y_{i\oplus 1}, \succ_{p+i}] = 2$ and $rk\left[x_{i\oplus 1}, \succ_{p+(i\oplus 1)}\right] = 2$. Now if $rk\,[x_{i\oplus 1}, \succ_{p+i}] > 3$, then the top two alternatives $y_{i\oplus 1}$ and $x_{i\oplus 1}$ of $\succ_{p+(i\oplus 1)}$ violate ERD and we are finished by Lemma A.2. Otherwise, if $rk\,[x_{i\oplus 1}, \succ_{p+i}] = 3$, then we can make $x_{i\oplus 1}$ and y_i indifferent on the top by taking for all preferences $\succ \in \mathcal{P}_1$ two voters each and p^2 voters with $\succ_{p+i}$.[9] In particular, y_i beats any other alternative in $X \setminus X'$ and $x_{i\oplus 1}$ beats any other alternative in X', while y_i and $x_{i\oplus 1}$ receive the same Borda score. Since $x_{i\oplus 1}$ and y_i violate ERD, we can apply Lemma A.1.

[9] For $p = 3$ and $i = 1$ this case is illustrated in Table 7.

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