

The Single-Peaked Domain Revisited: A Simple Global Characterization*

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Revision, December 2017

*I thank two anonymous referees and an associate editor for helpful comments and remarks. I am indebted to Tobias Dittrich and Michael Müller who provided excellent research assistance. Tobias Dittrich also created the figures based on a graphic tool developed by David McCooey. This work was presented in seminars at Higher School of Economics in Moscow, Université de Cergy-Pontoise, Université Paris Dauphine, the University of Glasgow, the University of Siena, TU Berlin, at the Workshop on Game Theory and Social Choice at Corvinus University Budapest, at the “Tagung des Theoretischen Ausschusses des Vereins für Socialpolitik” in Basel, at COMSOC conference in Toulouse, at ESEM conference in Lisbon, at the VfS conference in Vienna and at the Workshop on Social Choice in Lausanne. I am grateful to the participants for valuable discussion and comments. Special thanks to Jürgen Eichberger, Ulle Endriss, Martin Hellwig, Anke Gerber, Bettina Klaus, Gleb Koshevoy, Jérôme Lang, Jean-Francois Laslier, Bernard Monjardet, Hervé Moulin, Stefan Napel, Klaus Nehring, Rolf Niedermeier, Georg Nöldeke, Matthias Nunez, Hans Peters, Marcus Pivato, Anup Pramanik, Ernesto Savaglio, Karl Schlag, Adi Shamir, Attila Tasnádi, Jörgen Weibull and Bill Zwicker for helpful comments and/or encouragement.

Abstract *It is proved that, among all restricted preference domains that guarantee consistency (i.e. transitivity) of pairwise majority voting, the single-peaked domain is the only minimally rich and connected domain that contains two completely reversed strict preference orders. It is argued that this result explains the predominant role of single-peakedness as a domain restriction in models of political economy and elsewhere. The main result has a number of corollaries, among them a dual characterization of the single-dipped domain; it also implies that a single-crossing ('order-restricted') domain can be minimally rich only if it is a subdomain of a single-peaked domain. The conclusions are robust as the results apply both to domains of strict and of weak preference orders, respectively.*

JEL Classification D71, C72

Keywords: Social choice, restricted domains, Condorcet domains, single-peakedness, single-dippedness, majority voting, single-crossing property.

1 Introduction

A subset of preference orders on a finite set of alternatives is called *single-peaked* if there exists a left-to-right arrangement of alternatives such that all upper contour sets are connected ('convex') with respect to the given left-to-right arrangement of alternatives. The celebrated median voter theorem of Black [1948] and Arrow [1951] states that the domain of all single-peaked linear orders with respect to a fixed underlying spectrum of alternatives form a 'Condorcet domain,' i.e. pairwise majority voting with an odd number of individuals each of whom has preferences from the given domain induces a transitive relation. Moreover, the domain of all single-peaked preferences is *minimally rich* in the sense that every alternative is on top of at least one preference order; it is *connected* in the sense that every two single-peaked orders can be obtained from each other by a sequence of transpositions of neighboring alternatives such that the resulting order remains single-peaked at each step; and it contains two *completely reversed* orders (namely, the orders that rank the alternatives in descending order from left to right and from right to left, respectively).

This paper's main result (Theorem 1) shows that, conversely, every minimally rich and connected Condorcet domain which contains at least one pair of completely reversed orders must be single-peaked.¹ As is easily verified, any single-peaked domain contains at most *one* pair of completely reversed orders. We thus obtain as a corollary that, for any given pair of completely reversed orders, there is a unique *maximal* Condorcet domain that contains them and is minimally rich as well as connected: the domain of *all* orders that are single-peaked with respect to either one of the given pair of completely reversed orders (Corollary 1).

This result is remarkable in particular in view of the fact that quite a number of non-single-peaked Condorcet domains have been identified in the literature, among others the domains satisfying Sen's 'value restriction' condition (Sen [1966]) with the 'single-dipped' domain (Inada [1964]) as a special case, the domains satisfying the so-called 'intermediate-ness' property (Grandmont [1978], Demange [2012]), and the 'order-restricted' domains identified by Rothstein [1990]; the latter domains are sometimes also referred to as the domains with the *single-crossing property* (Gans and Smart [1996], Saporiti [2009], Puppe and Slinko [2017]). Our analysis shows that none of these domains can jointly satisfy the three conditions of minimal richness, connectedness and the inclusion of a pair of completely reversed orders unless it is *also* single-peaked. In particular, a single-crossing domain can be minimally rich only if it is at the same time single-peaked (Corollary 3).

The purpose of the present analysis is not to justify the assumption of single-peakedness *per se* and, in fact, the empirical evidence for single-peakedness is mixed, see the review of the literature below. The main argument put forward here is that, among all domains that guarantee consistency of pairwise majority voting, the single-peaked domain is distinguished by a remarkably simple set of additional requirements: connectedness, minimal richness and the existence of two completely reversed orders. The main conclusion to be

¹In fact, as detailed in Section 2 below, the condition of connectedness can be substantially relaxed in this result to the condition that there exist *one* path that connects a pair of completely reversed orders.

drawn from the present analysis is therefore that, *if* a modeler wishes to guarantee transitivity of the majority relation for any possible profile of agents' preferences, *then* the assumption of single-peakedness follows from a small number of natural additional conditions. In this sense, the present study may thus be interpreted as a *conditional defense* of single-peakedness.

The results presented here are robust as they generalize with some additional work, but using the same underlying logic, to the case of weak preference orders, i.e. to the case in which individual preferences may display indifferences. In this case, the domain of *all* (weakly) single-peaked weak orders does not form a Condorcet domain in our sense since, even with an odd number of voters, the indifference relation corresponding to pairwise majority voting may be intransitive. Moreover, the notion of connectedness has to be suitably adapted since two 'neighboring' weak orders may differ in the ranking of more than one pair of alternatives if one of these orders displays an indifference class with more than two elements. Nevertheless, we still obtain that any connected, minimally rich Condorcet domain that contains at least two completely reversed strict orders must be single-peaked with respect to either one of the pair of completely reversed orders (Theorem 3). However, if one allows for indifferences – in particular also on the top of individual preferences – there are different possibilities to formulate the condition of minimal richness. If one only requires that each alternative be *among* the top elements of at least one order in the domain, the stated conditions characterize a class of different non-isomorphic maximal Condorcet domains, all of which are proper subdomains of the domain of all single-peaked orders. If, on the other hand, one imposes the stronger requirement that each alternative be the *unique* top element of at least one order in the domain, the stated conditions characterize a unique maximal Condorcet domain of single-peaked orders (with respect to the left-to-right arrangement of alternatives given by either one of the completely reversed strict orders).

The above characterization results of the single-peaked domain imply 'dual' characterization results for the *single-dipped* domain in a straightforward way, both in the case of linear and in the case of weak orders. Specifically, any connected Condorcet domain containing two completely reversed orders such that every alternative is (among) the least preferred alternative(s) for some order in the domain must be single-dipped (see Theorems 2 and 4).

Relation to the literature

The literature on single-peaked preferences is abundant both in economics and political science. Their application ranges from the Hotelling-Downs model of political competition to models of local public good provision (for a modern treatment see, e.g., Austen-Smith and Banks [1998]). It is well-known that the assumption of single-peakedness enables possibility results both in the theory of preference aggregation (Black [1948], Arrow [1951]) and in the theory of strategy-proof social choice (Moulin [1980]). Moreover, it has frequently been argued that the assumption of single-peakedness is reasonable in contexts in which alternatives are naturally arranged according to an exogenous one-dimensional scale, e.g., in terms of political views on a left-to-right spectrum, or in terms of objective

distance, temperature, etc. The empirical evidence on single-peakedness is mixed. Some authors have argued that a tendency towards single-peakedness may be assumed under certain circumstances, in particular when there is repeated interaction and/or ‘deliberation’ (see, e.g., Spector [2000], DeMarzo et al. [2003], List et al. [2013]). Others have cast doubt on the applicability of single-peakedness, in particular in cases where compromises are difficult to reach and a search for them threatens to lead to a deadlock (Egan [2014]).

The paper in the literature that is closest to the present analysis is Ballester and Haerlinger [2011]. These authors also provide an axiomatic characterization of the single-peaked domain (though only in the case of linear orders).² However, the conditions employed by these authors are very different in character from the ones used here. Specifically, Ballester and Haerlinger [2011] use two families of conditions. The first is the condition that among every triple of alternatives there should be at least one that is never the worst of the three for any voter. This is one of Sen’s family of ‘value restrictions’ (Sen [1966]), and it evidently amounts to assuming single-peakedness on all triples, hence it directly implies transitivity of the majority relation. However, it is also known that single-peakedness on all triples (‘local single-peakedness’) is not sufficient to guarantee single-peakedness globally (cf. Inada [1964]). Therefore, additional conditions are needed to characterize the single-peaked domain. The important contribution of Ballester and Haerlinger [2011] is to show that the absence of a certain preference constellation on all quadruples of alternatives does the job. By contrast, the present analysis only assumes transitivity of the majority relation corresponding to every profile with an odd number of voters, and derives the single-peakedness of the domain from the three conditions of connectedness, minimal richness and the existence of two completely reversed orders (either of which then represents the underlying left-to-right spectrum). The latter three conditions are *global* properties of a domain, hence the title of this paper. By contrast, all conditions used by Ballester and Haerlinger [2011] are ‘local’ conditions as they apply simultaneously either to all triples or to all quadruples of alternatives. While the conditions of the present analysis can be suitably adapted to yield a corresponding characterization of the weakly single-peaked domain, it is not obvious how to appropriately formulate Ballester’s and Haerlinger’s local conditions in the case of weak orders.

In the case of linear orders, this paper’s main result could be deduced from the theory of ‘tiling domains’ developed in Danilov et al. [2012] by observing that the single-peaked domain is the only minimally rich Condorcet domain of ‘tiling type.’ By contrast, the present paper’s analysis is completely elementary and, unlike the proof strategy via tiling domains, the proof for the main characterization result presented here can be adapted to the case of weak orders.

The present study also informs the literature that aims at identifying ‘large’ Condorcet domains, see the excellent survey Monjardet [2009] and the more recent work on this topic by Danilov et al. [2012], Danilov and Koshevoy [2013]. Indeed, our main result suggests that Condorcet domains with the maximal *number* of elements on a given set of alternatives, the so-called *maximum* Condorcet domains, are most likely not minimally

²Strictly speaking, Ballester and Haerlinger [2011] characterize the domain of all single-peaked *profiles* which is a slightly different task.

rich. This may seem particularly surprising, as it is known that the cardinality of a maximum Condorcet domain exceeds the cardinality of the domain of all single-peaked domain considerably; for instance, with n alternatives a single-peaked domain has at most 2^{n-1} elements, while the cardinalities of the maximum Condorcet domains are 9, 20, 45, for $n = 4, 5, 6$, respectively, and these numbers are all attained by connected Condorcet domains containing a pair of completely reversed orders. The general structure and precise cardinality of the maximum Condorcet domains for larger n is unknown. However, it is known that the largest cardinality of a connected Condorcet domain on n alternatives that contains two completely reversed orders always exceeds 2^{n-1} , see Fishburn [1997]. By this paper's main result, the corresponding domains can never be minimally rich.

We finally emphasize the implications of our analysis for the literature on the single-crossing property. In the present finite framework, a domain is said to have the single-crossing property if the agents can be arranged in a fixed linear order such that, for every pair of alternatives, if two voters prefer one alternative to the other, then so do all agents that are between them in the given linear order of voters.³ It has frequently been noted that single-peakedness and the single-crossing property in this sense are logically independent conditions, see, e.g., Saporiti [2009]. However, since every single-crossing domain can be extended to a connected single-crossing domain containing two completely reversed orders, our main result yields as a corollary that all minimally rich single-crossing domains must *also* be single-peaked, i.e. a subset of the domain of all single-peaked orders (cf. Elkind et al. [2014]). Since for $n > 3$, only proper subsets of the domain of *all* single-peaked preferences can have the single-crossing property, this has the further interesting consequence that no minimally rich single-crossing domain can constitute a *maximal* Condorcet domain.

The remainder of this paper is organized as follows. In Section 2 we state and prove our main result in the case of linear ('strict') orders. We also show by means of examples that none of the conditions can be omitted in the characterization of the single-peaked domain. Section 3 generalizes the characterization to the case of weak orders. The paper concludes with a brief discussion on the relation of the present analysis to a recent literature that aims at characterizing preference domains admitting strategy-proof social choice functions (with suitable additional properties) in terms of generalized notions of single-peakedness.

2 Characterizing the single-peaked domain: The case of strict preference orders

2.1 Statement of main result

Consider a finite set of alternatives X and the set $\mathcal{P}(X)$ of all *linear (strict) orders* (i.e., complete, transitive and antisymmetric binary relations) on X . A subset $\mathcal{D} \subseteq \mathcal{P}(X)$ will be

³This condition is related to but *prima facie* different from the well-known Spence-Mirrlees 'single-crossing' condition which requires that agents' types are unambiguously ordered according to their marginal rate of substitution uniformly across the good space.

called a *domain of preferences* or simply a *domain*. A *profile* $\pi = (P_1, \dots, P_n)$ on \mathcal{D} is an element of the Cartesian product \mathcal{D}^n for some number $n \in \mathbb{N}$ of ‘voters,’ where the linear order P_i represents the preferences of the i th voter over the alternatives from X . A profile with an odd number of voters will simply be referred to as an *odd profile*. Frequently, we will denote linear orders simply by listing the alternatives in descending order, e.g. the linear order that ranks a first, b second, c third, etc., is denoted by $abc\dots$.

The *majority relation* associated with a profile π is the binary relation P_π^{maj} on X such that $xP_\pi^{\text{maj}}y$ if and only if more than half of the voters rank x above y . Note that, according to this definition, the majority relation is asymmetric and for any odd profile π and any two distinct alternatives $x, y \in X$, we have either $xP_\pi^{\text{maj}}y$ or $yP_\pi^{\text{maj}}x$. The class of domains $\mathcal{D} \subseteq \mathcal{P}(X)$ such that, for all odd n , the majority relation associated with any profile $\pi \in \mathcal{D}^n$ is transitive has received significant attention in the literature, see the excellent survey of Monjardet [2009] and the references therein. In the following, we will refer to any such domain as a *Condorcet domain*. A domain \mathcal{D} is called a *maximal Condorcet domain* if every Condorcet domain (on the same set of alternatives) that contains \mathcal{D} as a subset must in fact coincide with \mathcal{D} . It is well-known that any maximal Condorcet domain \mathcal{D} is *closed* in the sense that the majority relation of any odd profile from \mathcal{D} is again an element of \mathcal{D} (and not only of $\mathcal{P}(X)$), cf. [Puppe and Slinko, 2017, Lemma 2.1].

A domain \mathcal{D} is *single-peaked with respect to the linear order $>$* on X if, for all $P \in \mathcal{D}$ and all $w \in X$, the upper contour sets $U_P(w) := \{y \in X : yPw\}$ are connected (‘convex’) in the order $>$, i.e. $\{x, z\} \subseteq U_P(w)$ and $x < y < z$ together imply $y \in U_P(w)$. A domain \mathcal{D} is simply called *single-peaked* if there exists *some* linear order $>$ such that \mathcal{D} is single-peaked with respect to $>$. The domain of *all* orders that are single-peaked with respect to the fixed order $>$ on X is denoted by $\mathcal{SP}_>(X)$. If a domain is single-peaked with respect to $>$, we will call the linear order $>$ the *spectrum* underlying the single-peaked domain.

A *path* in $\mathcal{P}(X)$ is a subset $\{P_1, \dots, P_m\} \subseteq \mathcal{P}(X)$ with $m \geq 2$ such that for all $j = 1, \dots, m - 1$, the two consecutive orders P_j and P_{j+1} differ in the ranking of exactly one pair x, y of distinct alternatives; note in that case x and y must be adjacent alternatives in both orders P_j and P_{j+1} . A pair of orders which differ in the ranking of exactly one (adjacent) pair of alternatives will be called *neighbors*. A domain \mathcal{D} will be called *connected* if, for every pair $P, P' \in \mathcal{D}$ of distinct orders in \mathcal{D} , there exists a path $\{P_1, \dots, P_m\}$ that connects P and P' (i.e. $P_1 = P$ and $P_m = P'$) and that lies entirely in \mathcal{D} (i.e. $P_j \in \mathcal{D}$ for all $j = 1, \dots, m$).

Two orders P and P^{inv} are called *completely reversed* if P and P^{inv} rank the alternatives in X in exactly the opposite way, i.e. for all distinct x and y , $xPy \Leftrightarrow \text{not}(xP^{\text{inv}}y)$. Note that by the completeness assumption, two orders $P, P^{\text{inv}} \in \mathcal{P}(X)$ are completely reversed if and only if $xPy \Leftrightarrow yP^{\text{inv}}x$. A domain is said to have *maximal width* if it contains at least one pair of completely reversed orders.⁴ The following property may look artificial at first, but turns out to be conceptually very natural. Say that a domain $\mathcal{D} \subseteq \mathcal{P}(X)$ is *semi-connected* if it contains two completely reversed orders P and P^{inv} and an entire path connecting them (cf. [Danilov et al., 2012, p.938]). Evidently, semi-connectedness implies maximal width, and is implied by, but logically weaker than, the conjunction of

⁴Domains with that property are called ‘normal’ in Danilov and Koshevoy [2013].

connectedness and maximal width. Finally, a domain \mathcal{D} will be called *minimally rich* if, for every alternative $x \in X$, there exists an order $P \in \mathcal{D}$ such that P has x as the top alternative.

The following characterization of the single-peaked domain is this paper's first main result.

Theorem 1. **a)** *For every linear order $>$ on X , the domain $\mathcal{SP}_{>}(X)$ of all single-peaked orders with respect to $>$ is a connected and minimally rich Condorcet domain with maximal width. (In particular, $\mathcal{SP}_{>}(X)$ is semi-connected.)*

b) *Conversely, let $\mathcal{D} \subseteq \mathcal{P}(X)$ be a semi-connected and minimally rich Condorcet domain. Then, \mathcal{D} is single-peaked.*

Except perhaps for the connectedness, the properties of the domain of all single-peaked orders stated in part a) are straightforward to verify. Clearly, the two completely reversed orders are $>$ itself and its reverse. Note that a single-peaked domain can contain at most one pair of completely reversed orders,⁵ therefore such pair uniquely determines a corresponding maximal single-peaked domain, and we have the following corollary.

Corollary 1. *Let $\mathcal{D} \subseteq \mathcal{P}(X)$ be a maximal Condorcet domain that is (semi-)connected, minimally rich and contains the pair P, P^{inv} of completely reversed orders, then $\mathcal{D} = \mathcal{SP}_{>}(X)$ where the spectrum $>$ is given by either P or P^{inv} .*

Figure 1 below depicts the (unique) maximal single-peaked domain containing the pair $abcd$ and dcb of completely reversed orders on the set $X = \{a, b, c, d\}$. Neighboring orders are connected by an edge; the domain consists of the orders marked in red color, i.e. $abcd$, $bacd$, $bcad$, $bcda$, $cbad$, $cbda$, $cdab$ and $dcba$.

The proof of Theorem 1 is provided in Section 2.3 below. Note that in the sufficiency part b) it is *not* asserted that \mathcal{D} must contain *all* orders that are single-peaked with respect to some given linear order (and this does in fact not follow); on the other hand, due to the semi-connectedness, any domain satisfying the conditions of Theorem 1b) must contain at least $\#X \cdot (\#X - 1)/2 + 1$ elements (because every ordered pair of alternatives has to be switched at least once on any path connecting two completely reversed orders).

Before providing the proof, we discuss the meaning and significance of the characterizing conditions in Theorem 1. We will also show by means of concrete examples that the characterization of the single-peaked domain provided by Theorem 1 is *tight* in the sense that each condition in part b) is indeed necessary to obtain the conclusion.

2.2 Discussion

The conditions imposed on preference domains in Theorem 1 will now be discussed. In particular, we demonstrate by means of examples that each condition in part b) is necessary to obtain the single-peakedness of the domain. A secondary purpose of this subsection is to illustrate the great diversity of the class of (maximal) Condorcet domains.

⁵This follows, e.g., immediately from Fact 2.2 below.

2.2.1 Consistency of majority voting

The condition of consistency of pairwise majority voting lies at the heart of the present analysis and ‘defines’ the problem. Transitivity of the majority relation (for odd profiles) is certainly a strong requirement but, as already noted in the introduction, the goal here is not to justify it but rather to study its implications. Clearly, the condition that \mathcal{D} be a Condorcet domain is necessary for the conclusion that \mathcal{D} is single-peaked. For instance, the universal domain $\mathcal{P}(X)$ evidently satisfies all other conditions in Theorem 1b), but the universal domain is clearly not single-peaked.

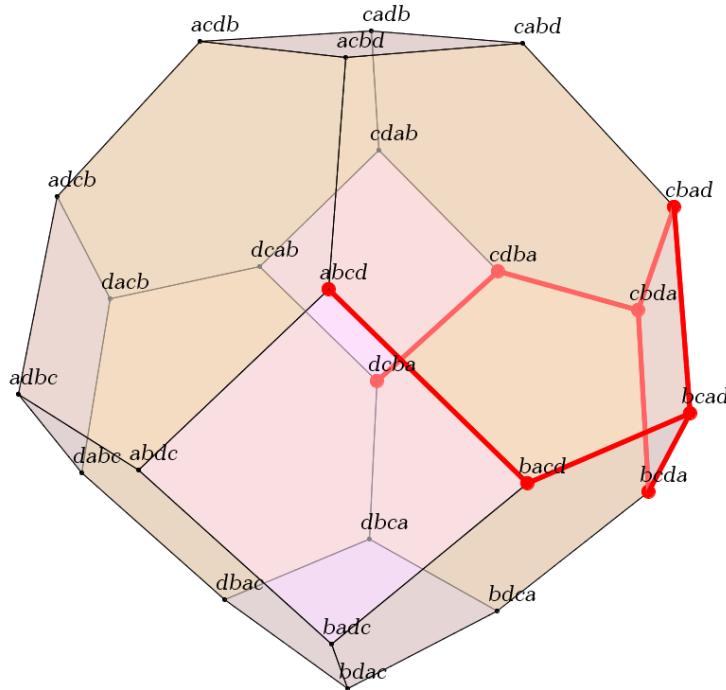


Fig. 1: A maximal single-peaked domain on $X = \{a, b, c, d\}$.

2.2.2 Maximal width: The existence of two completely reversed orders

One may think of a domain as the description of a ‘society.’ Under this interpretation, the existence of two orders in the domain that are completely reverses of each other is a condition of maximal ‘width’ of opinions: the underlying society is required to admit two maximally opposed views. As the other conditions in Theorem 1, the maximal width condition represents a substantial requirement.⁶ On the other hand, it is not evident whether there are natural classes of maximal Condorcet domains that violate the maximal width condition. Figure 2 illustrates the necessity of the maximal width condition in

⁶Also mathematically, it has significant consequences. Indeed, it is well-known that, together with the consistency of majority voting, maximal width implies that the domain can be embedded in a distributive lattice (cf. Abello [1991], Chameni-Nembua [1989], Monjardet [2009], Danilov and Koshevoy [2013], Puppe and Slinko [2017]).

Theorem 1b) by displaying a maximal Condorcet subdomain of $\mathcal{P}(X)$ on $X = \{a, b, c, d\}$ that is not single-peaked but connected as well as minimally rich. The connectedness and minimal richness of the depicted domain $\{abcd, bacd, bcad, bcd, bdca, cbad, cbda, dbca\}$ (marked in red) is evident. To verify that it is not single-peaked, note first that a and d are the only two alternatives that occur at the bottom of each marked order. Since $abcd$ is a member, this implies that if the domain is to be single-peaked with respect to $>$, we must have either $a > b > c > d$ or $d > c > b > a$. However, in either case the contained order $bdca$, for instance, would not qualify as single-peaked.

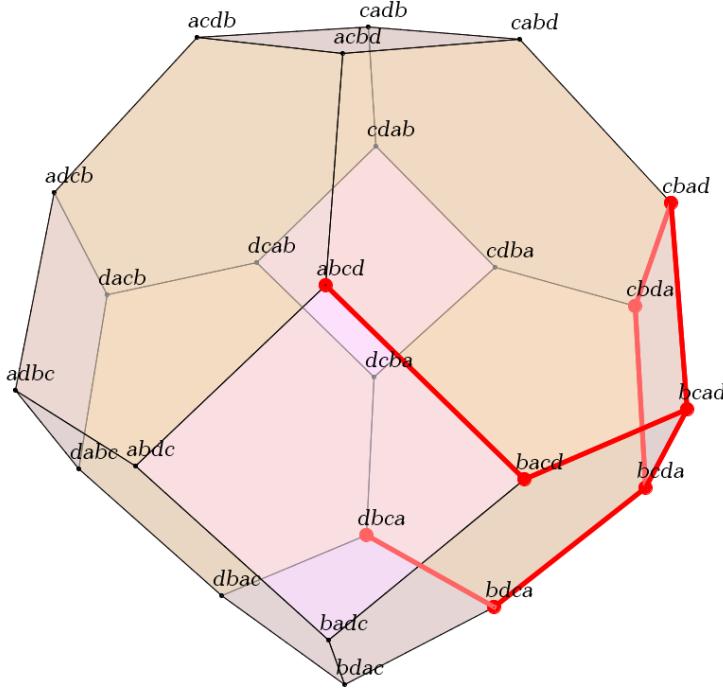


Fig. 2: A connected and minimally rich Condorcet domain without a pair of two completely reversed orders.

The fact that the domain depicted in Fig. 2 is indeed a Condorcet domain can be easily inferred from the following well-known result.⁷

Fact 2.1. *Let $\mathcal{D} \subseteq \mathcal{P}(X)$ with X finite. The following statements are equivalent.*

- i) *\mathcal{D} is a Condorcet domain, i.e. the majority relation corresponding to every odd profile on \mathcal{D} is an element of $\mathcal{P}(X)$.*
- ii) *The majority relation corresponding to every profile on \mathcal{D} is acyclic.⁸*

⁷See, e.g. [Monjardet, 2009, p. 142]). Condition iii) is Sen's [1966] 'value restriction' and condition iv) has been introduced by Ward [1965] as the 'absence of a Latin square' (in other terminology, it requires the absence of a 'Condorcet cycle'; cf. Condorcet, 1785). In light of this condition, Condorcet domains of (linear) orders are sometimes referred to as 'acyclic sets of linear orders' (e.g. by Fishburn [1997]).

⁸An asymmetric binary relation P is *acyclic* if there does not exist a subset $\{x_1, \dots, x_m\} \subseteq X$ such that $x_1Px_2, x_2Px_3, \dots, x_{m-1}Px_m$ and x_mPx_1 .

- iii) In any triple $x, y, z \in X$ of pairwise distinct alternatives, there exists one element that either never has rank 1, or never has rank 2, or never has rank 3 in the restrictions of the orders in \mathcal{D} to the set $\{x, y, z\}$.
- iv) For no triple $P_1, P_2, P_3 \in \mathcal{D}$, and no triple $x, y, z \in X$ of pairwise distinct alternatives one has xP_1yP_1z , yP_2zP_2x and zP_3xP_3y .

We finally note that the domain depicted in Fig. 2 is in fact a *maximal Condorcet domain*.⁹

2.2.3 Connectedness and semi-connectedness

Continuing with the metaphor of a domain representing a society, connectedness has a clear meaning as well: it must be possible to reach any admissible opinion from any other admissible opinion by a series of minimal changes in the corresponding rankings while staying in the domain at each step. This may be viewed as a ‘homogeneity’ condition which forbids that opinions are clustered around a few ‘representative’ opinions. Figure 3 illustrates this; it depicts a maximal Condorcet domain on $X = \{a, b, c, d\}$ that satisfies all conditions of Theorem 1b) except connectedness and semi-connectedness.

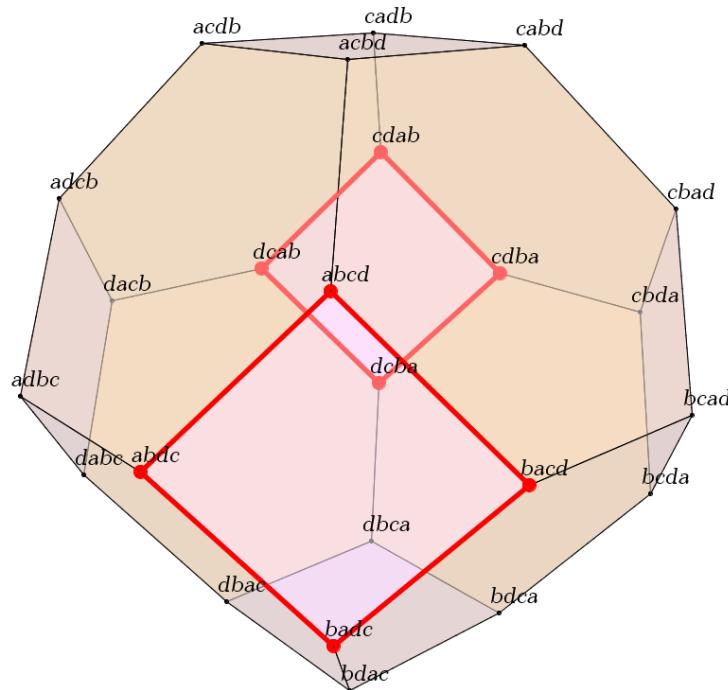


Fig. 3: A non-(semi-)connected, minimally rich Condorcet domain containing pairs of completely reversed orders.

⁹The verification of this statement is straightforward if somewhat tedious.

The domain depicted in Fig. 3 is given by the set of orders $\{abcd, abdc, bacd, badc, cdab, cdba, dcab, dcba\}$. The ‘society’ corresponding to this domain is clustered around the ‘opinion’ that each alternative in the pair $\{a, b\}$ dominates each alternative in the pair $\{c, d\}$ (the 4-cycle in the front) and the opposite ‘opinion’ that the alternatives in the pair $\{c, d\}$ dominate the alternatives in the pair $\{a, b\}$ (the 4-cycle in the back). That the depicted domain is a maximal Condorcet domain follows again easily using Fact 2.1; that it is not single-peaked follows at once from the fact that it violates the following simple (and well-known) necessary condition for single-peakedness.

Fact 2.2. *Suppose that $\mathcal{D} \subseteq \mathcal{P}(X)$ is single-peaked. Then there are at most two alternatives in X which can occur at the bottom of any order in \mathcal{D} .*¹⁰

The conditions of connectedness and also its weakening to semi-connectedness are arguably the most substantial and restrictive conditions used in Theorem 1b) (besides the consistency of majority voting). Indeed, the minimally rich Condorcet domain displayed in Fig. 3 is an instance of a general procedure that yields ‘large’ Condorcet domains that are neither connected nor even semi-connected, Fishburn’s so-called *replacement scheme* (Fishburn [1997]). The scheme takes two Condorcet domains $\mathcal{D}_1 \subseteq \mathcal{P}(X_1)$ and $\mathcal{D}_2 \subseteq \mathcal{P}(X_2)$ on two disjoint sets of alternatives and replaces one alternative, say $x \in X_1$, in each of the orders in \mathcal{D}_1 by each of the orders in \mathcal{D}_2 to obtain a new Condorcet domain $\mathcal{D}_1 * \mathcal{D}_2$ on the set $(X_1 \setminus \{x\}) \cup X_2$. It is easily verified that the domain $\mathcal{D}_1 * \mathcal{D}_2$ is not semi-connected. On the other hand, $\mathcal{D}_1 * \mathcal{D}_2$ evidently is minimally rich whenever both \mathcal{D}_1 and \mathcal{D}_2 are, and it contains two completely reversed orders whenever both \mathcal{D}_1 and \mathcal{D}_2 do. Whether the replacement scheme is important for economic applications is open to debate.

2.2.4 Minimal richness

Minimal richness has a straightforward interpretation as well: no alternative should a priori be ruled as the individually most desired choice. The condition is termed ‘*minimal*’ here because in the literature much stronger ‘richness’ conditions have been imposed.¹¹ Note that the domain of all single-peaked preferences with respect to some fixed linear order $>$ on X in fact also satisfies a stronger richness condition, namely that each alternative occurs not only sometimes as the best but also as the second-best alternative.

Despite its innocuous appearance, the minimal richness condition has quite some bite as well, as illustrated by the two domains depicted in Figure 4. The domain on the left hand side of Fig. 4 is given by the set of orders $\{abcd, acbd, acdb, adcb, dacb, dcab, dcba\}$, while the domain on the right hand side is given by the set of orders $\{abcd, abdc, bacd, badc, bdac, bdca, dbac, dbca, dcba\}$. Both domains are maximal, connected Condorcet domains and contain the pair $abcd$ and $dcba$ of completely reversed orders. Evidently, neither

¹⁰Note that this condition is clearly not sufficient for single-peakedness as the domain depicted in Fig. 2 shows.

¹¹Our terminology follows Aswal et al. [2003] and Chatterji and Sen [2011]; the latter work also discusses domains violating the minimal richness requirement. Stronger richness conditions have been used, e.g., in Chatterji et al. [2013], Nehring and Puppe [2007]).

domain is minimally rich, and by Fact 2.2 above, neither domain is single-peaked. That the depicted domains are indeed Condorcet domains follows again from Fact 2.1, and their respective maximality can be verified in a straightforward way.

Interestingly, among all maximal connected Condorcet domains on $X = \{a, b, c, d\}$ with maximal width, the domain on the left hand side of Fig. 4 has the *minimal* number of elements (in this case, $\#X \cdot (\#X - 1)/2 + 1 = 4 \cdot 3/2 + 1 = 7$) and the domain on the right hand side has the *maximal* number of elements. In fact, the domain depicted on the r.h.s. of Fig. 4 has the maximal number of elements among *all* Condorcet domains on a set of four alternatives, namely 9 (Monjardet [2009]). A Condorcet domain with the maximal number of elements is sometimes referred to as a *maximum Condorcet domain*. It is known that for $\#X \leq 6$, the maximal number of elements of a Condorcet domain is attained by connected domains with maximal width,¹² and that the maximal number of elements of such domains always exceeds the number of elements of any single-peaked domain (Monjardet [2009]). Theorem 1b) thus implies that these domains are never minimally rich, and that the maximal cardinality of a semi-connected and minimally rich Condorcet domain is $2^{\#X-1}$, the number of all single-peaked orders with respect to some fixed linear order of X .

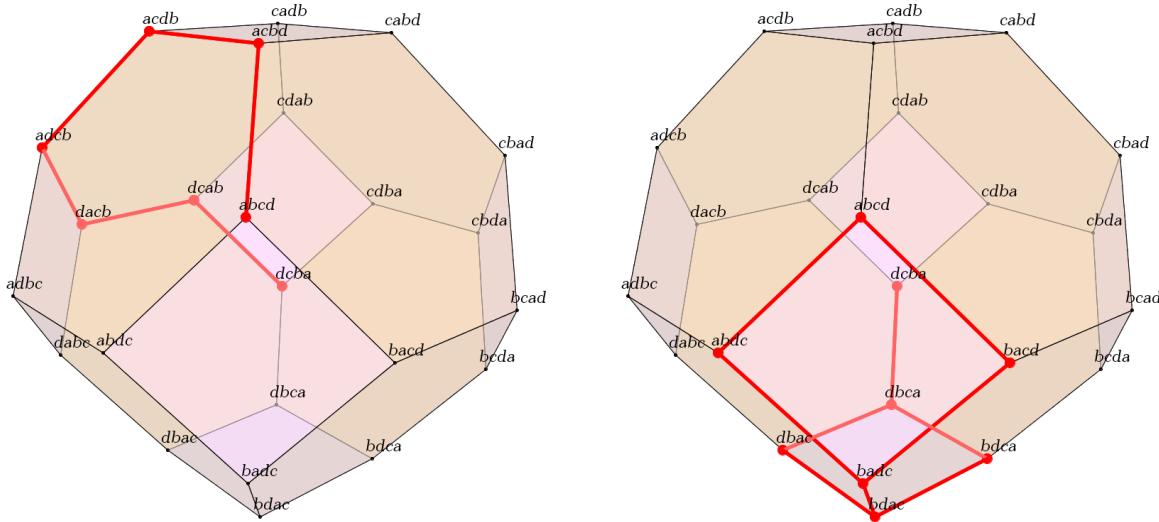


Fig. 4: Two connected but not minimally rich Condorcet domains containing a pair of completely reversed orders.

The strength of the minimal richness condition when imposed jointly with semi-connectedness can also be inferred from the following immediate corollary of Theorem 1b) and Fact 2.2.

¹²For $\#X = 3, 4, 5, 6$, the maximum Condorcet domains are connected and have 4, 9, 20, 45 elements, respectively (Fishburn [1997, 2002]). For $\#X = 7$, the maximal number of elements of a connected Condorcet domain is 100 (Galambos and Reiner [2008]), but it is not known whether this is also the maximal number of elements among *all* Condorcet domains on a set with 7 elements.

Corollary 2. Let $\#X \geq 3$. There does not exist a semi-connected and minimally rich Condorcet domain on X such that every alternative in X is worst for some order in \mathcal{D} .

Note that the domain depicted in Fig. 3 above satisfies all conditions in Corollary 2 except for the semi-connectedness.

2.3 Proof of the main result

First, we show that the domain $\mathcal{SP}_>(X)$ of all single-peaked orders with respect to any spectrum $>$ is connected, see also Obraztsova et al. [2013].¹³ The other properties of $\mathcal{SP}_>(X)$ asserted in Theorem 1a) are obvious.

Thus consider the domain $\mathcal{SP}_>(X)$ on X with $a > b > c > \dots$. We proceed by induction over $\#X$; for the case $\#X = 3$ the assertion is obvious. Thus assume that the assertion holds for all cardinalities $< \#X$. Take any two single-peaked orders P and Q in $\mathcal{SP}_>(X)$. By moving the alternative a in each of these orders stepwise down to the bottom we obtain two orders P_a and Q_a that agree with P and Q , respectively, on $X \setminus \{a\}$. Since a is an extreme point of the spectrum $>$, the resulting orders at each step as well as P_a and Q_a are all single-peaked. Thus, P and P_a , as well as Q and Q_a are connected by a path in $\mathcal{SP}_>(X)$, respectively. (Note that we allow for the possibility that $P = P_a$ and $Q = Q_a$.) Now consider the restrictions $P|_{X-a}$ and $Q|_{X-a}$ of P and Q to the set $X \setminus \{a\}$. Clearly, $P|_{X-a}$ and $Q|_{X-a}$ are single-peaked on $X \setminus \{a\}$ with respect to the restricted spectrum $b > c > \dots$. By the induction hypothesis, there exists a path $\{P^1, P^2, \dots, P^k\}$ of single-peaked orders on $X \setminus \{a\}$ that connects $P|_{X-a}$ and $Q|_{X-a}$ such that $P^1 = P|_{X-a}$ and $P^k = Q|_{X-a}$. For each j , denote by P_a^j the order on X that coincides with P^j on $X \setminus \{a\}$ and that puts alternative a at the bottom. Evidently, (i) $P_a^1 = P_a$ and $P_a^k = Q_a$, (ii) each order P_a^j is single-peaked on X with respect to $>$, and (iii) the set $\{P_a^1, P_a^2, \dots, P_a^k\}$ constitutes a path in $\mathcal{SP}_>(X)$. Combining the path connecting P and P_a with the path connecting P_a and Q_a and the path connecting Q_a and Q , we obtain a concatenated path connecting P and Q , as desired. This proves part a) of Theorem 1.

The proof of Theorem 1b) starts with the following simple observation. For every $\mathcal{D} \subseteq \mathcal{P}(X)$ and all triples $\{x, y, z\} \subseteq X$ of pairwise distinct alternatives, denote by $\mathcal{D}_{\{x,y,z\}} \subseteq \mathcal{P}(\{x, y, z\})$ the domain of the restrictions of all orders $P \in \mathcal{D}$ to the set $\{x, y, z\}$, i.e.

$$\mathcal{D}_{\{x,y,z\}} := \{P|_{\{x,y,z\}} : P \in \mathcal{D}\}.$$

Fact 2.3. Let $\mathcal{D} \subseteq \mathcal{P}(X)$ and consider any triple $\{x, y, z\} \subseteq X$ of pairwise distinct alternatives.

- a) If \mathcal{D} is a Condorcet domain on X , then $\mathcal{D}_{\{x,y,z\}}$ is a Condorcet domain on $\{x, y, z\}$.
- b) If \mathcal{D} is connected on X , then $\mathcal{D}_{\{x,y,z\}}$ is connected on $\{x, y, z\}$.
- c) If \mathcal{D} has maximal width on X , then $\mathcal{D}_{\{x,y,z\}}$ has maximal width on $\{x, y, z\}$.

¹³Theorem 4.10 in Obraztsova et al. [2013] asserts the same result; a proof (different from the one presented here) is available in the full (working paper) version of Obraztsova et al. [2013]. I am grateful to Edith Elkind for notifying me.

- d) If \mathcal{D} is semi-connected on X , then $\mathcal{D}_{\{x,y,z\}}$ is semi-connected on $\{x, y, z\}$.
- e) If \mathcal{D} is minimally rich on X , then $\mathcal{D}_{\{x,y,z\}}$ is minimally rich on $\{x, y, z\}$.

Proof. Part a) follows at once from Fact 2.1 iv) above, since a Condorcet cycle in $\mathcal{D}_{\{x,y,z\}}$ would induce a corresponding Condorcet cycle in \mathcal{D} . Part b) also follows easily. Consider any two orders \tilde{P} and \tilde{Q} in $\mathcal{D}_{\{x,y,z\}}$, and let $P, Q \in \mathcal{D}$ be such that $\tilde{P} = P|_{\{x,y,z\}}$ and $\tilde{Q} = Q|_{\{x,y,z\}}$. By assumption there exists a path $\{P^1, \dots, P^k\}$ connecting P and Q in \mathcal{D} . Evidently, then $\{P^1|_{\{x,y,z\}}, \dots, P^k|_{\{x,y,z\}}\}$ is a path in $\mathcal{D}_{\{x,y,z\}}$ that connects \tilde{P} and \tilde{Q} (observe that $P^j|_{\{x,y,z\}}$ and $P^l|_{\{x,y,z\}}$ can coincide for distinct $j, l \in \{1, \dots, k\}$). To verify part c), simply observe that if \overline{P} and \underline{P} are completely reversed in \mathcal{D} , then $\overline{P}|_{\{x,y,z\}}$ and $\underline{P}|_{\{x,y,z\}}$ are completely reversed in $\mathcal{D}_{\{x,y,z\}}$. Part d) follows at once from parts b) and c). Finally, if \mathcal{D} is minimally rich, then in particular x, y and z must be on top of at least one order in \mathcal{D} ; this proves part e). \square

The significance of the previous result in our context stems from the observation that on a set of three alternatives, there are only two types of semi-connected Condorcet domains: the single-peaked and the ‘single-dipped’ domain (as also observed by [Danilov et al., 2012, Th. 4(ii)]). A domain $\mathcal{D} \subseteq \mathcal{P}(X)$ is called *single-peaked with respect to the linear order $>$* on X if, for all $P \in \mathcal{D}$ and all $w \in X$, the lower contour sets $L_P(w) := \{y \in X : wPy\}$ are connected (‘convex’) in the order $>$, i.e. $\{x, z\} \subseteq L_P(w)$ and $x < y < z$ jointly imply $y \in L_P(w)$. A domain \mathcal{D} is called *single-dipped* if there exists *some* linear order $>$ such that \mathcal{D} is single-peaked with respect to $>$.¹⁴ The domain of *all* orders that are single-dipped with respect to the fixed order $>$ on X is denoted by $\mathcal{SD}_>(X)$.

Fact 2.4. *There are exactly two semi-connected Condorcet domains on the set $\{x, y, z\} \subseteq X$ that contain the two completely reversed orders xyz and zyx , namely*

$$\begin{aligned}\mathcal{SP}_>(\{x, y, z\}) &= \{xyz, yxz, yzx, zyx\}, \text{ and} \\ \mathcal{SD}_>(\{x, y, z\}) &= \{xyz, xzy, zxy, zyx\},\end{aligned}$$

where the spectrum $>$ is given either by the order xyz , or by the completely reversed order zyx (see Figure 5).

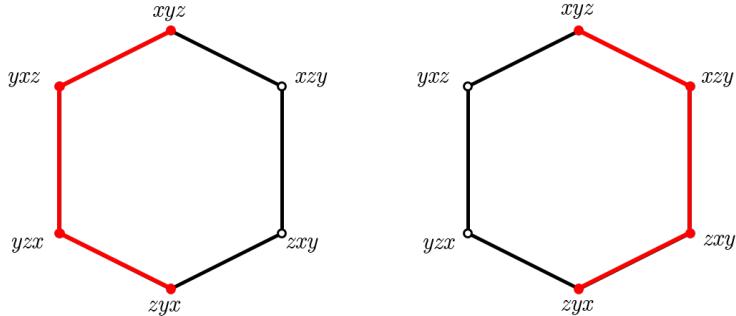


Fig. 5: The single-peaked (left) and single-dipped (right) domains on $\{x, y, z\}$.

¹⁴Inada [1964] introduced the condition of single-dippedness under the name of ‘single-cavedness.’ We follow here the terminology of Klaus et al. [1997] and others.

Proof. By the semi-connectedness of \mathcal{D} , there exists a path in \mathcal{D} connecting xyz and zyx . Thus, \mathcal{D} contains at least one neighbor of xyz , i.e. either yxz or xzy . On the other hand, \mathcal{D} cannot contain both of these neighbors because the triple yxz , xzy , zyx forms a Condorcet cycle. If \mathcal{D} contains yxz it must coincide with $\mathcal{SP}_>(\{x, y, z\})$ by the semi-connectedness; similarly, if it contains xzy it must coincide with $\mathcal{SD}_>(\{x, y, z\})$. \square

Remark. By Fact 2.4, semi-connectedness of a Condorcet domain implies its connectedness in the case of three alternatives. This does not hold in general. However, in the working paper version of this paper Puppe [2016] it is shown that every semi-connected and *closed* Condorcet domain must in fact be connected; thus, for closed (in particular, for all maximal) Condorcet domains with maximal width the notions of semi-connectedness and connectedness coincide. Indeed, every semi-connected and closed Condorcet domain satisfies the even stronger property of *connectedness without restoration* introduced by Sato [2013], see [Puppe, 2016, Prop. A.1 & Prop. A.2].

We can now complete the proof of Theorem 1b). Thus, suppose that \mathcal{D} is a semi-connected and minimally rich Condorcet domain. By the semi-connectedness there exist two completely reversed orders $\bar{P}, \underline{P} \in \mathcal{D}$ that are connected by a path in \mathcal{D} . By the minimal richness, every alternative is in top of some order in \mathcal{D} . Consider any triple $x, y, z \in X$ with $x\bar{P}y\bar{P}z$. By Fact 2.3, the domain $\mathcal{D}_{\{x,y,z\}}$ is minimally rich and semi-connected. Since the single-dipped domain $\mathcal{SD}_>(\{x, y, z\})$ is not minimally rich, we obtain by Fact 2.4 that, for all triples $x, y, z \in X$,

$$\mathcal{D}_{\{x,y,z\}} = \mathcal{SP}_>(\{x, y, z\}), \quad (2.1)$$

where $>$ is given by the restriction of \bar{P} to $\{x, y, z\}$. Thus, the restriction of \mathcal{D} to any triple is single-peaked with respect to the spectrum inherited from \bar{P} (or, equivalently, from \underline{P}).

It remains to show that this implies the single-peakedness of \mathcal{D} itself. This is the content of the following simple observation.

Fact 2.5. *Let \mathcal{D} satisfy (2.1) for all triples $x, y, z \in X$ where $>$ is given by the restriction of \bar{P} to $\{x, y, z\}$. Then \mathcal{D} is single-peaked with respect to $> = \bar{P}$.*

Proof. We have to show that all upper contour sets of all orders $P \in \mathcal{D}$ are connected in the order $> = \bar{P}$. Thus, consider any order $P \in \mathcal{D}$ and any triple x, y, z with $x > y > z$, i.e. $x\bar{P}y\bar{P}z$, and suppose that $\{x, z\} \subseteq U_P(w)$ for some $w \in X$. By (2.1) the restriction of P to $\{x, y, z\}$ takes one of the following form: $xPyPz$, $yPxPz$, $yPzPx$, or $zPyPx$ (cf. Fig. 5). Evidently, in each case $\{x, z\} \subseteq U_P(w)$ implies $y \in U_P(w)$. \square

Remark. Theorem 1 can also be deduced from the analysis in Danilov et al. [2012]. These authors characterize the class of (maximal) semi-connected Condorcet domains as the domains of ‘tiling type’ [Danilov et al., 2012, Th. 4]. Theorem 1 then follows from the observation that within this class the only minimally rich domain corresponds to the so-called ‘standard tiling’ which represents the single-peaked domain.¹⁵

¹⁵I am grateful to a referee for pointing this out to me.

2.4 A dual characterization of the single-dipped domain

Theorem 1 above entails a ‘dual’ characterization of the single-dipped domain in a straightforward way, as follows.

- Theorem 2.** **a)** *For every linear order $>$ on X , the domain $\mathcal{SD}_>(X)$ is a connected Condorcet domain with maximal width such that every alternative in X is the worst alternative for some order.*
- b)** *Conversely, let $\mathcal{D} \subseteq \mathcal{P}(X)$ be a semi-connected Condorcet domain such that every alternative in X is the worst alternative for some order in \mathcal{D} . Then, \mathcal{D} is single-dipped.*

Proof. Part a) is analogous to the proof of Theorem 1a). For part b) note that among the two admissible semi-connected domains on triples according to Fact 2.4 only the single-dipped domain has every alternative at the bottom of at least one order. Thus, if \mathcal{D} satisfies the conditions of Theorem 2b), and if $\bar{P} \in \mathcal{D}$ is one of a pair of completely reversed orders in \mathcal{D} that are connected by a path in \mathcal{D} , we obtain, for all triples $x, y, z \in X$,

$$\mathcal{D}_{\{x,y,z\}} = \mathcal{SD}_>(\{x, y, z\}), \quad (2.2)$$

where $>$ is given by the restriction of \bar{P} to $\{x, y, z\}$. By an argument completely symmetric to that given in Fact 2.5 we obtain from (2.2) that the lower contour sets of all orders in \mathcal{D} are connected in the spectrum $> = \bar{P}$, and thus that \mathcal{D} itself is single-dipped. \square

2.5 A corollary for single-crossing domains

Our analysis has an important implication for a class of domains known as ‘single-crossing’ domains. A domain $\mathcal{D} \subseteq \mathcal{P}(X)$ is a *single-crossing domain* if it can be written in the form $\mathcal{D} = \{P_1, \dots, P_m\}$ such that, for all ordered pairs $(x, y) \in X \times X$, the set $\{P \in \mathcal{D} : xPy\}$ is ‘connected’ in $\{1, \dots, m\}$, i.e., for all $x, y \in X$, xP_jy and xP_ly with $j < l$ implies xP_ky for all $k \in \{j, \dots, l\}$, and yP_jx and yP_lx with $j < l$ implies yP_kx for all $k \in \{j, \dots, l\}$. This property has been introduced in the literature by Rothstein [1990] under the name of ‘order-restriction.’ It underlies the analysis in Roberts [1977], Gans and Smart [1996], and is employed frequently under the name of ‘single-crossing property’ (see, e.g., Saporiti [2009]). It is well-known that all single-crossing domains are Condorcet domains (Rothstein [1990, 1991]).

Corollary 3. *Let $\mathcal{D} \subseteq \mathcal{P}(X)$ be a single-crossing domain. If \mathcal{D} is minimally rich, then it is single-peaked. If every alternative in X is worst in at least one order in \mathcal{D} , then \mathcal{D} is single-dipped.*

Proof. Every single-crossing domain \mathcal{D} can be extended to a semi-connected single-crossing domain $\mathcal{D}^* \supseteq \mathcal{D}$ (note that we are *not* asserting that \mathcal{D}^* is a maximal Condorcet domain). Indeed, if $\mathcal{D} = \{P_1, \dots, P_k\}$ has the single-crossing property, then so does the domain $\{P_1, \dots, P_k, P_{k+1}\}$ where $P_{k+1} = P_1^{\text{inv}}$ (we allow that $P_{k+1} = P_k$). We can now fill

possible ‘gaps’ in the sequence P_1, \dots, P_k, P_{k+1} as follows. If P_1 and P_2 differ in the ranking of more than one pair of alternatives, at least one of these pairs must be an adjacent pair in P_1 . Then, we can add the order P'_1 that switches exactly this pair and agrees with P_1 in the ranking of all other pairs, and consider the domain $\{P_1, P'_1, P_2, \dots\}$. Continuing in this fashion, we obtain a semi-connected and single-crossing domain $\mathcal{D}^* \supseteq \mathcal{D}$. Evidently, if \mathcal{D} is minimally rich, so is \mathcal{D}^* ; and if every alternative is worst in at least one order in \mathcal{D} , then the same property holds for \mathcal{D}^* . Since every single-crossing domain is a Condorcet domain, we thus obtain by Theorem 1b) that \mathcal{D}^* , and hence also \mathcal{D} , is single-peaked if \mathcal{D} is minimally rich. Similarly, if \mathcal{D} has every alternative at the bottom of some order, then \mathcal{D}^* , and hence also \mathcal{D} , is single-dipped by Theorem 2b). \square

Remark. The semi-connected domain \mathcal{D}^* constructed in the proof of Corollary 3 is in fact a *maximal* single-crossing domain in the sense that no proper superdomain of \mathcal{D}^* can be single-crossing. The maximal single-crossing domains are sometimes referred to as *maximal chains* in the literature, since they indeed correspond to the maximal chains in the so-called ‘weak Bruhat order’ (Abello [1991], Chameni-Nembua [1989], Galambos and Reiner [2008], Monjardet [2009]). Note that the maximal single-crossing domains are in general not maximal as Condorcet domains. For instance, all paths connecting the orders $abcd$ and dcb in the single-peaked domain in Fig. 1, as well as in the maximum domain on the right hand side of Fig. 4 correspond to maximal single-crossing domains; but evidently, these maximal paths do not form maximal Condorcet domains. On the other hand, the maximal single-crossing domain on the left hand side of Fig. 4 is also maximal as Condorcet domain; a simple necessary and sufficient condition for the maximality (as Condorcet domain) of a maximal single-crossing domain is given in [Monjardet, 2007, p. 79] and Puppe and Slinko [2017]. The fact that every single-crossing *profile* of linear orders in which every alternative is at the top of at least one voter must be single-peaked has also been observed by Elkind et al. [2014].

3 The case of weak preference orders

3.1 The basic concepts and definitions generalized

Consider now the set $\mathcal{R}(X)$ of all *weak orders* (i.e., complete and transitive binary relations) on X , and subdomains $\mathcal{D} \subseteq \mathcal{R}(X)$. Individual weak preferences are denoted by R, R' etc., and profiles by $\rho = (R_1, \dots, R_n) \in \mathcal{R}(X)^n$. Frequently, we will denote weak orders by listing the alternatives in descending order and putting indifferent alternatives in brackets, e.g. the weak order that ranks a first and b and c indifferently as second best is denoted by $a(bc)\dots$; similarly, the weak order that has a and b indifferently at the top and c at the following rank is denoted by $(ab)c\dots$; finally, the weak order that has a , b and c indifferently on top is denoted by $(abc)\dots$, etc.

The *majority relation* associated with a profile ρ is the binary relation R_ρ^{maj} on X such

that $xR_\rho^{\text{maj}}y$ if and only if xR_iy for more than half of the voters.¹⁶ Note that, according to this definition, the majority relation is complete for every odd profile ρ . As above, the domains $\mathcal{D} \subseteq \mathcal{R}(X)$ such that, for all odd n , the majority relation associated with any profile $\rho \in \mathcal{D}^n$ is transitive are referred to as *Condorcet domains*. Condorcet domains of weak orders have been studied much less than their counterparts with linear orders, see the monograph of Gaertner [2001] for a state-of-the-art survey. As above, a domain $\mathcal{D} \subseteq \mathcal{R}(X)$ is called a *maximal Condorcet domain (of weak orders)* if every Condorcet domain (on the same set of alternatives) that contains \mathcal{D} as a subset must in fact coincide with \mathcal{D} .

The generalization of our main result to the case of weak orders requires careful adaptation of the employed concepts since some additional complications arise. As a first indication of these, observe that the equivalences in Fact 2.1 do not carry over to the domain of weak orders. As a simple example, consider the domain on $X = \{a, b, c\}$ consisting of the three weak orders $(ab)c$, $a(bc)$ and cba . If ρ is the profile in which three voters have each one of these preferences, respectively, the majority relation is acyclic but not transitive, since a majority strictly prefers a to c while $[aR_\rho^{\text{maj}}b$ and $bR_\rho^{\text{maj}}a]$ as well as $[bR_\rho^{\text{maj}}c$ and $cR_\rho^{\text{maj}}b]$, i.e. both pairs of alternatives (a, b) and (b, c) are deemed indifferent, respectively, according to the majority relation. Thus, the domain is not a Condorcet domain in our sense.

A domain $\mathcal{D} \subseteq \mathcal{R}(X)$ is called *weakly single-peaked with respect to the linear order $>$* on X if, for all $R \in \mathcal{D}$ and all $w \in X$, the upper contour sets $U_R(w) := \{y \in X : yRw\}$ are connected ('convex') in the order $>$, i.e. $\{x, z\} \subseteq U_R(w)$ and $x < y < z$ jointly imply $y \in U_R(w)$. A domain $\mathcal{D} \subseteq \mathcal{R}(X)$ is called *weakly single-peaked* if there exists *some* linear order $>$ such that \mathcal{D} is weakly single-peaked with respect to $>$. Similarly, a domain $\mathcal{D} \subseteq \mathcal{R}(X)$ is called *weakly single-dipped* if there exists a linear order $>$ on X such that, for all $R \in \mathcal{D}$, the lower contour sets $L_R(w) := \{y \in X : wRy\}$ are connected with respect to $>$.

Remark. Several generalizations of the concept of single-peakedness in order to accommodate indifference have been discussed in the literature. [Moulin, 1988, p. 264] notes that for many purposes indifferences *across* the (unique) peak could be allowed without difficulty. On the other hand, the concept of 'single-plateaued' preferences allows for multiple best alternatives and has proven useful in some contexts ([Gaertner, 2001, p. 68]); however, this concept still assumes strict monotonicity below the optimum (Moulin [1984]). The above notion of 'weakly' single-peaked preferences has also been employed by Duggan [2016] (under this name) and is weaker than single-plateauedness as it allows for multiple 'plateaus' and, in particular, not only at the top. From an abstract perspective, it represents a natural generalization of the usual notion of single-peakedness, as it corresponds to convexity of all upper contour sets of the weak relations, just as standard single-peakedness corresponds to convexity of the upper contour sets of all strict

¹⁶It is well-known that this is not the only possible definition of majority rule with weak preferences (see, e.g., [Gaertner, 2001, Ch. 3]); for our purpose it turns out to be the appropriate one. Moreover, it represents the natural notion of majority rule inherited from the general judgement aggregation model, see List and Puppe [2009], Nehring et al. [2014, 2016].

relations. However, I do not want to argue that the adopted definitions are the only reasonable extensions of the concepts of single-peakedness and Condorcet domain to the case of weak orders. For instance, one could require only acyclicity of the majority relation for a domain to be called ‘Condorcet domain,’ or base the notion of Condorcet domain on the so-called ‘strict’ majority relation ([Gaertner, 2001, p. 27]), or require uniqueness of the top alternative in the definition of single-peakedness. However, as the subsequent analysis will show, the adopted definitions are *precisely* the appropriate notions for which the robustness of the main characterization result can be demonstrated.

A further important difference to the case of linear orders is that not all weakly single-peaked domains are Condorcet domains; in particular, the domain of *all* weak orders that are weakly single-peaked with respect to some order $>$ is not a Condorcet domain. The same example as above can be used to demonstrate this. Indeed, we have already argued above that the domain $\mathcal{D} = \{(ab)c, a(bc), cba\}$ is not a Condorcet domain. On the other hand, it is clearly weakly single-peaked with respect to the linear order $a > b > c$.

We also have to generalize the notion of ‘connectedness,’ as follows. For $R \in \mathcal{R}(X)$, denote by $\neg R$ the negation of R , i.e., for all $x, y \in X$, $x \neg R y \Leftrightarrow \text{not } x R y$. For any two orders $R, R' \in \mathcal{R}(X)$, denote by

$$[R, R'] := \{Q \in \mathcal{R}(X) : Q \supseteq R \cap R' \text{ and } \neg Q \supseteq \neg R \cap \neg R'\}. \quad (3.1)$$

the *interval* spanned by R and R' , and to its elements as the orders *between* R and R' . Thus, an order Q is between R and R' if and only if, for all pairs $x, y \in X$, (i) Q ranks x weakly above y whenever both R and R' do so, and (ii) Q ranks x strictly above y whenever both R and R' do so. Two distinct orders R and R' are called $\mathcal{R}(X)$ -*neighbors* if $[R, R'] = \{R, R'\}$. Figure 6 depicts the ‘neighborhood structure’ of $\mathcal{R}(\{a, b, c\})$, connecting all neighbors by an edge. Note that, due to transitivity of the indifference relation, the six neighbors of the weak order (abc) on the set $X = \{a, b, c\}$ each differ from (abc) in the ranking of two pairs of alternatives, respectively.¹⁷

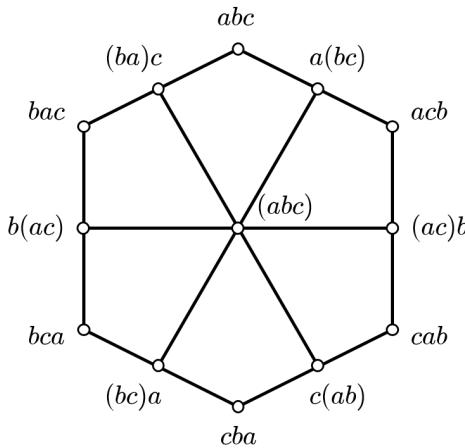


Fig. 6: The neighborhood structure of $\mathcal{R}(\{a, b, c\})$.

¹⁷The neighborhood relation can be defined analogously on the space $\mathcal{P}(X)$ of all linear orders; in this case, definition (3.1) simplifies to $[P, P'] = \{Q \in \mathcal{P}(X) : Q \supseteq P \cap P'\}$ due to the antisymmetry condition.

A *path* in $\mathcal{R}(X)$ is a subset $\{R_1, \dots, R_m\} \subseteq \mathcal{R}(X)$ with $m \geq 2$ such that for all $j = 1, \dots, m-1$, the two consecutive orders R_j and R_{j+1} are $\mathcal{R}(X)$ -neighbors. A domain $\mathcal{D} \subseteq \mathcal{R}(X)$ will be called *connected* if, for every pair $R, R' \in \mathcal{D}$ of distinct orders in \mathcal{D} , there exists a path $\{R_1, \dots, R_m\}$ that connects R and R' (i.e. $R_1 = R$ and $R_m = R'$) and that lies entirely in \mathcal{D} (i.e. $R_j \in \mathcal{D}$ for all $j = 1, \dots, m$).

As above, two orders R and R^{inv} are called *completely reversed* if, for all distinct x and y , $xRy \Leftrightarrow \text{not}(xR^{\text{inv}}y)$. Note that by the completeness assumption, two completely reversed weak orders must both in fact be linear orders, i.e. neither of the two can contain any non-trivial indifference. A domain $\mathcal{D} \subseteq \mathcal{R}(X)$ is said to have *maximal width* if \mathcal{D} contains at least one pair of completely reversed orders. Finally, a domain $\mathcal{D} \subseteq \mathcal{R}(X)$ is called *semi-connected* if \mathcal{D} contains at least two completely reversed orders and an entire path connecting them.

3.2 All semi-connected Condorcet domains on triples

As in the case of linear orders, a crucial step in our characterization of the single-peaked (and single-dipped) domain will be the complete classification of the semi-connected Condorcet domains on triples. This is provided by the next result (see Figure 7 for illustration with the domains $\mathcal{D}_1 - \mathcal{D}_6$ depicted clockwise from top left in the first row to bottom left in the second row).

Proposition 3.1. *Let $x, y, z \in X$ be pairwise distinct. The following are the semi-connected Condorcet subdomains of $\mathcal{R}(\{x, y, z\})$ that contain the completely reversed orders xyz and zyx . All of them are maximal and in fact connected.*

$$\begin{aligned}\mathcal{D}_1(x, y, z) &= \{xyz, (xy)z, yxz, y(xz), yzx, (yz)x, zyx\}, \\ \mathcal{D}_2(x, y, z) &= \{xyz, (xy)z, (xyz), z(xy), zyx\}, \\ \mathcal{D}_3(x, y, z) &= \{xyz, (xy)z, (xyz), (yz)x, zyx\}, \\ \mathcal{D}_4(x, y, z) &= \{xyz, x(yz), (xyz), z(xy), zyx\}, \\ \mathcal{D}_5(x, y, z) &= \{xyz, x(yz), (xyz), (yz)x, zyx\}, \\ \mathcal{D}_6(x, y, z) &= \{xyz, x(yz), xzy, (xz)y, zxy, z(xy), zyx\}.\end{aligned}$$

Proof. By the semi-connectedness of \mathcal{D} , there exists a path in \mathcal{D} connecting xyz and zyx . Thus, \mathcal{D} contains at least one neighbor of xyz , i.e. either $(xy)z$ or $x(yz)$. On the other hand, \mathcal{D} cannot contain both of these neighbors because the triple $\{(xy)z, x(yz), zyx\}$ is ‘forbidden’: if three voters each have one of these preference orders, respectively, we obtain a non-transitive majority relation since x and y , as well as y and z are deemed indifferent by the majority relation while x is strictly superior to z .

Case 1. Suppose first that \mathcal{D} contains $(xy)z$. By the semi-connectedness, \mathcal{D} must contain either the neighbor yxz , or the neighbor (xyz) of $(xy)z$ ‘in direction of’ zyx . Again, \mathcal{D} cannot contain both of them since the triple $\{yxz, (xyz), zyx\}$ is forbidden (in the same sense as above).

Case 1.1. Thus, suppose \mathcal{D} contains yxz in addition to $(xy)z$. By the semi-connectedness

it must contain $\mathcal{D}_1(x, y, z)$. On the other hand, $\mathcal{D}_1(x, y, z)$ is a maximal Condorcet domain. For this it remains to show that it cannot contain any order from the set $\{xzy, (xz)y, zxy, z(xy)\}$. For the two linear orders xzy and zxy this follows immediately from the existence of appropriate Condorcet cycles with two elements from $\mathcal{D}_1(x, y, z)$, respectively; for the order $z(xy)$ it follows since the triple $\{xyz, (yz)x, z(xy)\}$ is forbidden; and for $(xz)y$ it follows since the triple $\{(xy)z, yzx, (xz)y\}$ is forbidden. Thus, in Case 1.1 we obtain $\mathcal{D} = \mathcal{D}_1(x, y, z)$.

Case 1.2. Now suppose \mathcal{D} contains (xyz) in addition to $(xy)z$. Then, \mathcal{D} can neither contain $y(xz)$ nor $(xz)y$ since both triples $\{(xy)z, (xyz), y(xz)\}$ and $\{xyz, (xyz), (xz)y\}$ are forbidden.¹⁸ Thus, by the semi-connectedness \mathcal{D} must contain either $z(xy)$ or $(yz)x$, but as already argued, it cannot contain both. In the first case, \mathcal{D} thus contains $\mathcal{D}_2(x, y, z)$, and in the second case \mathcal{D} contains $\mathcal{D}_3(x, y, z)$. But by arguments completely symmetric to those given so far it follows that either of these domains is maximal.

Case 2. If \mathcal{D} contains $x(yz)$, a completely symmetric analysis yields the possible cases $\mathcal{D} = \mathcal{D}_4(x, y, z)$, $\mathcal{D} = \mathcal{D}_5(x, y, z)$, or $\mathcal{D} = \mathcal{D}_6(x, y, z)$. This completes the proof of Proposition 3.1. \square

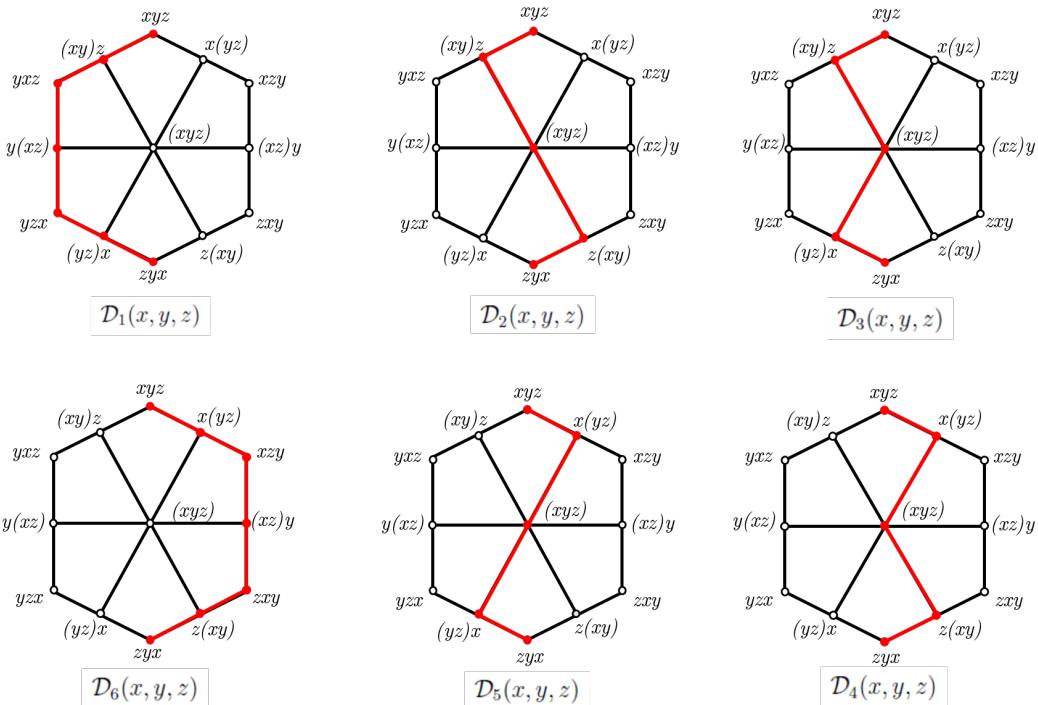


Fig. 7: The (maximal) (semi-)connected Condorcet subdomains of $\mathcal{R}(\{x, y, z\})$.

¹⁸That these two triples are forbidden hinges on the adopted definition of the majority relation. For instance, if in a profile ρ , each of the orders $(xy)z$, (xyz) , $y(xz)$ receives one third of the votes, respectively, we obtain both $xR_\rho^{\text{maj}}y$ and $yR_\rho^{\text{maj}}x$; moreover, $xR_\rho^{\text{maj}}z$ and $zR_\rho^{\text{maj}}x$ but not $zR_\rho^{\text{maj}}y$.

A crucial observation for the following is that the domains $\mathcal{D}_1 - \mathcal{D}_5$ are weakly single-peaked, while the domains $\mathcal{D}_2 - \mathcal{D}_6$ are weakly single-dipped with respect to the linear order $x > y > z$ (in particular, the five-element domains $\mathcal{D}_2 - \mathcal{D}_5$ are simultaneously weakly single-peaked and weakly single-dipped with respect to $x > y > z$).

3.3 Main result

Due to the inclusion of indifferences, there are two natural formulations of the minimal richness condition. Let us say that a domain $\mathcal{D} \subseteq \mathcal{R}(X)$ is *strongly minimally rich* if, for each alternative $x \in X$, there is at least one weak order $R \in \mathcal{D}$ that has x as the unique top alternative. Analogously, say that a domain $\mathcal{D} \subseteq \mathcal{R}(X)$ is *weakly minimally rich* if, for each alternative $x \in X$, there is at least one weak order $R \in \mathcal{D}$ such that x is among the top alternatives of R .

The following characterization extends the main result of the previous section to the case of weak orders.

Theorem 3. **a)** *For every linear order $>$ on X , there exists a unique maximal Condorcet domain $\hat{\mathcal{R}}_> \subseteq \mathcal{R}(X)$ that contains the domain $\mathcal{SP}_>(X)$ of all single-peaked linear orders with respect to $>$. The domain $\hat{\mathcal{R}}_>$ is weakly single-peaked and connected. Evidently, $\hat{\mathcal{R}}_>$ has maximal width and is strongly minimally rich.*

- b)** *Conversely, let $\mathcal{D} \subseteq \mathcal{R}(X)$ be a semi-connected and weakly minimally rich Condorcet domain. Then, \mathcal{D} is weakly single-peaked.*
- c)** *Moreover, if $\mathcal{D} \subseteq \mathcal{R}(X)$ is a semi-connected and strongly minimally rich Condorcet domain, then $\mathcal{D} \subseteq \hat{\mathcal{R}}_>$ for some linear order $>$ on X . (In particular, \mathcal{D} is weakly single-peaked.)*

The domain $\hat{\mathcal{R}}_>$ admits the following explicit characterization. First, each indifference class of every weak order in $\hat{\mathcal{R}}_>$ has at most two elements, i.e. any indifference prevails over at most two distinct alternatives. Moreover, whenever all indifferences are ‘resolved’ for a weak order in $\hat{\mathcal{R}}_>$ by transforming all indifferent pairs into strictly ranked adjacent alternatives, one obtains an order in $\mathcal{SP}_>(X)$, no matter which combination of strict rankings for the indifferent pairs is chosen. For illustration, consider Figure 8 which depicts the domain $\hat{\mathcal{R}}_>$ on $X = \{a, b, c, d\}$ for the spectrum $a > b > c > d$. As an example, consider the weak order $(bc)(ad)$ in the middle of Fig. 8, i.e. the order that declares b and c as indifferent on the top and a and d as indifferent at the bottom. There are exactly four ways to transform these two indifferent pairs into adjacent strictly ranked pairs resulting in the linear orders $bcad$, $cbad$, $bcda$, and $cbda$, all of which belong to $\mathcal{SP}_>(\{a, b, c, d\})$ for the spectrum $a > b > c > d$. By contrast, the order $a(bc)d$ does not belong to the domain $\hat{\mathcal{R}}_>$ for the spectrum $a > b > c > d$, since the linear order $acbd$ is not single-peaked. Note that the weak order $a(bc)d$ is nevertheless weakly single-peaked.

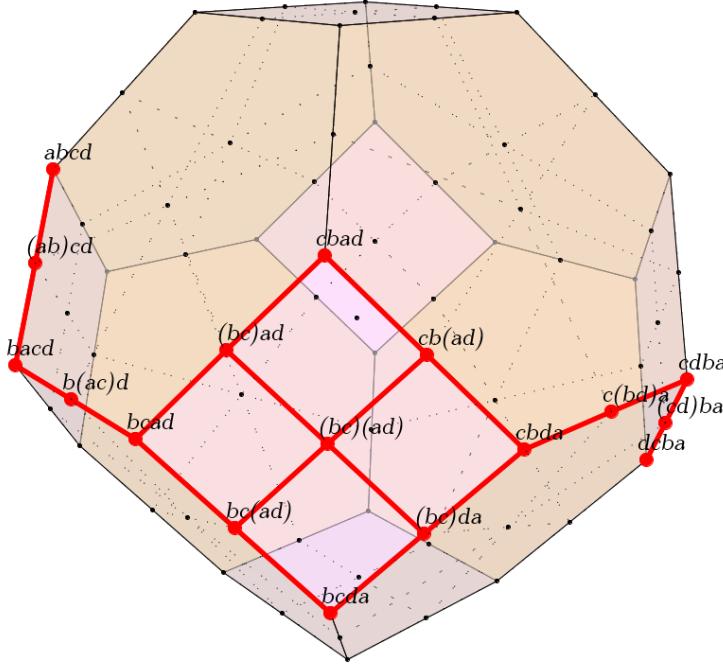


Fig. 8: The domain $\hat{\mathcal{R}}_>$ on $X = \{a, b, c, d\}$ with spectrum $a > b > c > d$.

Proof of Theorem 3. a) Let $\mathcal{D} \subseteq \mathcal{R}(X)$ be a maximal Condorcet domain with $\mathcal{D} \supseteq \mathcal{SP}_>(X)$ for a given spectrum $>$ on X . As is easily verified, for every triple of distinct alternatives $x, y, z \in X$ with $x > y > z$, the restriction $\mathcal{D}_{\{x,y,z\}}$ of \mathcal{D} to $\{x, y, z\}$ is a maximal Condorcet domain of weak orders on $\{x, y, z\}$ that contains the domain $\mathcal{SP}_>(\{x, y, z\})$ (strictly speaking, in the latter term ‘ $>$ ’ denotes the restriction of $>$ to $\{x, y, z\}$). By Fact 2.1, $\mathcal{D}_{\{x,y,z\}}$ can neither contain the linear order xzy nor the linear order zxy . Moreover, $\mathcal{D}_{\{x,y,z\}}$ can contain neither of the weak orders $x(yz)$, $(xz)y$, $z(xy)$, nor the complete indifference relation (xyz) ; indeed, the following are forbidden triples of weak orders: $\{x(yz), yxz, zyx\}$, $\{(xz)y, yxz, zyx\}$, $\{z(xy), xyz, yzx\}$, $\{(xyz), xyz, yzx\}$. This implies that $\mathcal{D}_{\{x,y,z\}} \subseteq \mathcal{D}_1(x, y, z)$, where $\mathcal{D}_1(x, y, z)$ is defined as in Proposition 3.1 above. By the maximality of both $\mathcal{D}_{\{x,y,z\}}$ and $\mathcal{D}_1(x, y, z)$, we obtain that in fact $\mathcal{D}_{\{x,y,z\}} = \mathcal{D}_1(x, y, z)$.¹⁹ This implies the desired conclusion $\mathcal{D} = \hat{\mathcal{R}}_>$ since evidently no indifference class in any order in \mathcal{D} can have more than two elements; moreover, for no weak order in \mathcal{D} , one can obtain a linear order outside $\mathcal{SP}_>(X)$ by strictly ranking all indifferent pairs and keeping the relative position of all other pairs of alternatives fixed (indeed, if this was possible there would exist a triple $x > y > z$ such that $\mathcal{D}_{\{x,y,z\}}$ would

¹⁹Note that we cannot invoke Proposition 3.1 to directly conclude this, since we do not yet know whether or not \mathcal{D} is semi-connected. In fact, there exist several maximal Condorcet domains of weak orders on a triple $\{x, y, z\}$ different from the ones listed in Proposition 3.1; these domains are either not connected or do not have maximal width (cf. Dittrich [2016]). The argument just given in the main text shows that none of them contains the single-peaked domain $\mathcal{SP}_>(\{x, y, z\})$ as a subdomain.

contain at least one of the weak orders $x(yz)$, $(xz)y$, or $z(xy)$). From the above it is immediate that the domain $\hat{\mathcal{R}}_>$ is weakly single-peaked. It is easily seen that $\hat{\mathcal{R}}_>$ is semi-connected: indeed, any two single-peaked orders in $\mathcal{SP}_>(X)$ that differ in the ranking of exactly one pair are connected to each other in $\hat{\mathcal{R}}_>$ by the weak order that declares this pair of alternatives as indifferent (keeping the position of all other alternatives fixed). This implies the full connectedness of $\hat{\mathcal{R}}_>$ recursively: each weak order R in $\hat{\mathcal{R}}_> \setminus \mathcal{SP}_>(X)$ has a neighbor in $\hat{\mathcal{R}}_>$ that displays one indifference less by resolving one binary indifference to a strict preference (in any direction). By recursion, any element of $\hat{\mathcal{R}}_> \setminus \mathcal{SP}_>(X)$ is thus connected by a path in $\hat{\mathcal{R}}_>$ to some element in $\mathcal{SP}_>(X) \subsetneq \hat{\mathcal{R}}_>$. Since every two elements in $\mathcal{SP}_>(X)$ are connected by a path in $\hat{\mathcal{R}}_>$, the domain $\hat{\mathcal{R}}_>$ is connected. This completes the proof of part a).

b) Suppose that $\mathcal{D} \subseteq \mathcal{R}(X)$ is a semi-connected and weakly minimally rich Condorcet domain. Denote by \bar{P} one of the pair of orders in \mathcal{D} that are complete inverses of each other and are connected by a path in \mathcal{D} (recall that each one of a pair of completely reversed orders in $\mathcal{R}(X)$ must be a linear order). For each triple of distinct alternatives $x, y, z \in X$ with $x\bar{P}y\bar{P}z$, the restriction $\mathcal{D}_{\{x,y,z\}}$ is a semi-connected and weakly minimally rich Condorcet domain that contains the completely reversed orders xyz and zyx . By Proposition 3.1, $\mathcal{D}_{\{x,y,z\}}$ coincides with one of the domains $\mathcal{D}_1(x, y, z) - \mathcal{D}_6(x, y, z)$, but by the weak minimal richness in fact with one of the domains $\mathcal{D}_1(x, y, z) - \mathcal{D}_5(x, y, z)$ (observe that no weak order in $\mathcal{D}_6(x, y, z)$ has y among its top alternatives). But we have already observed that the domains $\mathcal{D}_1(x, y, z) - \mathcal{D}_5(x, y, z)$ are all weakly single-peaked. As in Section 2 above, the weak single-peakedness of \mathcal{D} on all triples with respect to the same spectrum $> := \bar{P}$ implies its weak single-peakedness globally with respect to $>$.

c) Finally, suppose that $\mathcal{D} \subseteq \mathcal{R}(X)$ is a semi-connected and even strongly minimally rich Condorcet domain. Then, for all triples $x, y, z \in X$ with $x\bar{P}y\bar{P}z$ the restriction $\mathcal{D}_{\{x,y,z\}}$ coincides with $\mathcal{D}_1(x, y, z)$ (all other semi-connected Condorcet domains on $\{x, y, z\}$ are not strongly minimally rich). By the arguments given in part a) this implies that $\mathcal{D} \subseteq \hat{\mathcal{R}}_>$. This completes the proof of Theorem 3. \square

While, for a given spectrum $>$, the Condorcet domain $\hat{\mathcal{R}}_>$ is uniquely determined by the conditions in Theorem 3 c) and maximality, there are several non-isomorphic maximal single-peaked domains that satisfy the weaker richness property in part b). Figure 9 depicts four of them. The reason for the multiplicity of weakly single-peaked Condorcet domains that satisfy the weak minimal richness condition is that the restrictions to different triples of alternatives may correspond to *different* types of Condorcet domains on the triples (for each triple $x, y, z \in X$, one of the domains $\mathcal{D}_1(x, y, z) - \mathcal{D}_5(x, y, z)$). By contrast, under the strong minimal richness conditions, the restrictions to all triples x, y, z coincide uniformly with $\mathcal{D}_1(x, y, z)$.

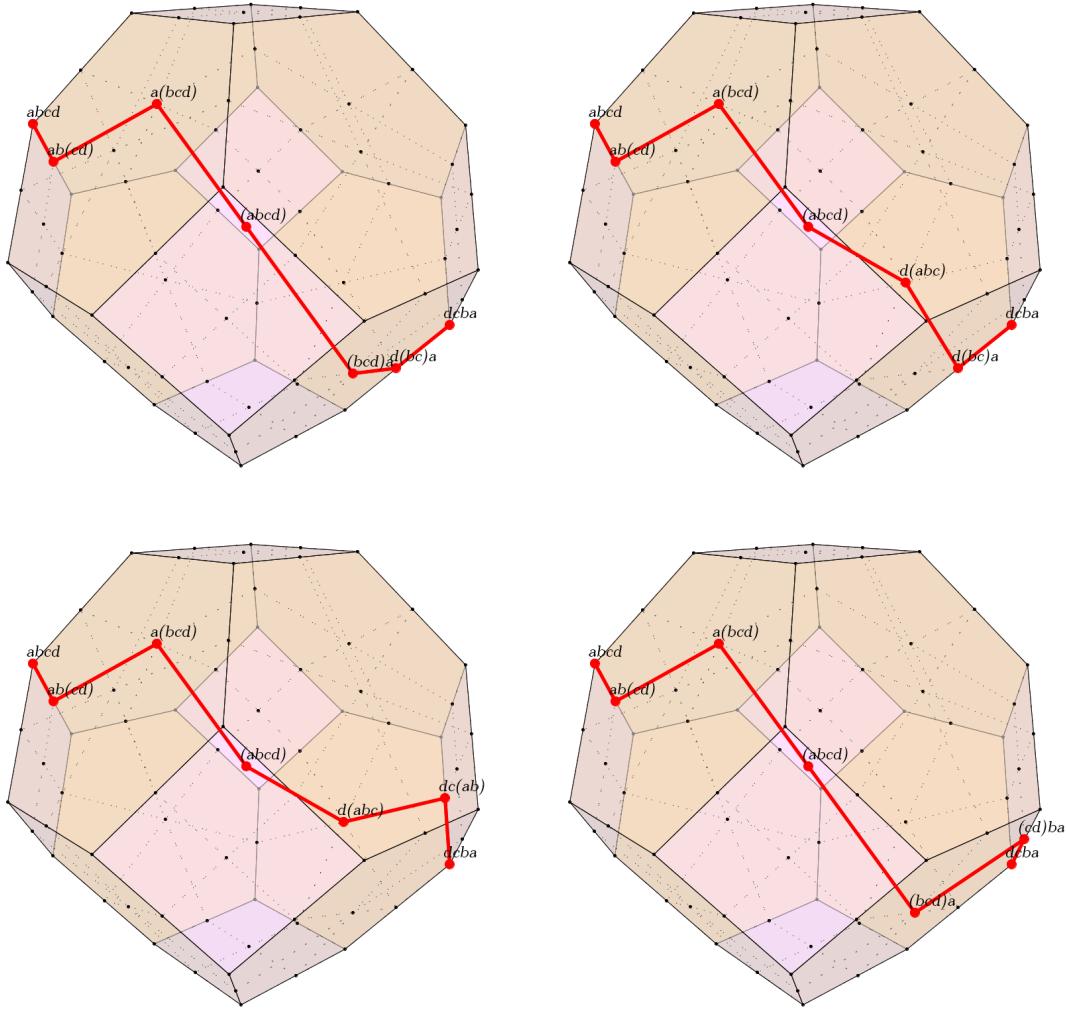


Fig. 9: Four maximal weakly single-peaked domains satisfying weak minimal richness.

3.4 A dual characterization of the weakly single-dipped domain

As in the case of linear orders, we obtain a corresponding characterization of the single-dipped domain, as follows.

- Theorem 4.** **a)** *For every linear order $>$ on X , there exists a unique maximal connected Condorcet domain $\check{\mathcal{R}}_> \subseteq \mathcal{R}(X)$ that contains the domain $\mathcal{SD}_>$ of all single-dipped linear orders with respect to $>$. The domain $\check{\mathcal{R}}_>$ is weakly single-dipped and connected. Evidently, $\check{\mathcal{R}}_>$ has maximal width and contains, for every alternative in X , at least one order that has this alternative uniquely at the bottom.*
- b)** *Conversely, let $\mathcal{D} \subseteq \mathcal{R}(X)$ be a semi-connected Condorcet domain such that every alternative is among the worst alternatives for some order in \mathcal{D} . Then, \mathcal{D} is weakly single-dipped.*

- c) Moreover, if $\mathcal{D} \subseteq \mathcal{R}(X)$ is a semi-connected Condorcet domain such that every alternative is uniquely worst for some order in \mathcal{D} , then $\mathcal{D} \subseteq \check{\mathcal{R}}_>$ for some linear order $>$ on X . (In particular, \mathcal{D} is weakly single-dipped.)

The proof of Theorem 4 is completely symmetric to the proof of Theorem 3 above; the details are left to the reader.

3.5 A corollary for weakly single-crossing domains

The following is a natural generalization of the single-crossing property for domains of weak orders. Say that a domain $\mathcal{D} \subseteq \mathcal{R}(X)$ is a *weakly single-crossing domain* if it can be written in the form $\mathcal{D} = \{R_1, \dots, R_m\}$ such that, for all ordered pairs $x, y \in X$, the sets $\{R \in \mathcal{D} : xRy\}$ and $\{R \in \mathcal{D} : x\neg R y\}$ are connected in $\{1, \dots, m\}$. As is easily verified, this condition implies that in the order $1, \dots, m$, for all distinct alternatives $x, y \in X$, either (i) all weak orders R_j that rank x strictly above y are followed by all weak orders R_j that declare x and y as indifferent, and these are followed by all weak orders R_j that rank y strictly above x or, conversely, (ii) all weak orders R_j that rank y strictly above x are followed by all weak orders R_j that declare x and y as indifferent, and these are followed by all weak orders R_j that rank x strictly above y (cf. Rothstein [1990, 1991]).

Corollary 4. *Let $\mathcal{D} \subseteq \mathcal{P}(X)$ be a weakly single-crossing domain.*

- a) *If \mathcal{D} is weakly minimally rich, then it is weakly single-peaked.*
- b) *If every alternative in X is among the worst alternatives in at least one order in \mathcal{D} , then \mathcal{D} is weakly single-dipped.*
- c) *If \mathcal{D} is strongly minimally rich, then $\mathcal{D} \subseteq \hat{\mathcal{R}}_>$ for some linear order $>$ on X . (In particular, \mathcal{D} is weakly single-peaked.)*
- d) *If every alternative in X is the uniquely worst alternative in at least one order in \mathcal{D} , then $\mathcal{D} \subseteq \check{\mathcal{R}}_>$ for some linear order $>$ on X . (In particular, \mathcal{D} is weakly single-dipped.)*

Proof. The proof follows from Theorems 3 and 4 since every weakly single-crossing domain \mathcal{D} can be extended to a semi-connected weakly single-crossing domain $\mathcal{D}^* \supseteq \mathcal{D}$ as in the proof of Corollary 3. Here is a sketch how to construct the semi-connected domain $\mathcal{D}^* \subseteq \mathcal{R}(X)$. Given the weakly single-crossing domain $\mathcal{D} = \{R_1, \dots, R_m\}$, let R_0 be any linear order satisfying, for all $x, y \in X$, (i) $xP_m y \Rightarrow yP_0 x$, and (ii) $(xI_m y \& xP_1 y) \Rightarrow xP_0 y$, where P_m, P_1, P_0 are the asymmetric parts of R_m, R_1, R_0 , respectively, and I_m is the symmetric part of R_m ; moreover, let $R_{m+1} := R_0^{\text{inv}}$ be the inverse of R_0 . Then consider the domain $\{R_0, R_1, \dots, R_m, R_{m+1}\}$, where we allow for the possibility that $R_0 = R_1$ and/or $R_m = R_{m+1}$. As is easily seen, the domain $\{R_0, R_1, \dots, R_m, R_{m+1}\}$ is again weakly single-crossing. Now fill all possible ‘gaps’ in an appropriate way, as in the proof of Corollary 3 above, whenever the interval $[R_j, R_{j+1}]$ contains weak orders distinct from both R_j and R_{j+1} for $j = 0, \dots, m$. \square

Conclusion

How restrictive is the assumption of single-peakedness as a domain restriction? In this paper it is argued that single-peakedness follows very naturally from transitivity of the majority relation for all odd profiles under a few simple and reasonable conditions of richness and connectedness. But transitivity of the majority relation for every odd profile is clearly a very demanding condition. As is well-known, its full strength is not needed in order to derive possibility results in social choice theory. For instance, the existence of non-dictatorial Arrowian aggregators and/or strategy-proof social choice functions can be demonstrated under much weaker domain restrictions (Kalai and Muller [1977]). Nevertheless, also in this context, richness and/or connectedness assumptions have frequently been imposed, and variants of the single-peakedness condition have been found to play an important role in the derivation of possibility results. The analysis in Nehring and Puppe [2007] demonstrates that the notion of *generalized* single-peakedness, which encompasses single-peakedness with respect to a tree and multi-dimensional single-peakedness as special cases, underlies a number of important possibility results in the literature on strategy-proof social choice. In a similar vein, a number of recent contributions argue that assumptions akin to (generalized) single-peakedness are in fact *necessary* to guarantee the existence of non-trivial strategy-proof social choice functions: Chatterji et al. [2013] show that certain types of strategy-proof social choice functions exist only on ‘semi-single-peaked’ domains; Chatterji and Massó [2017] derive a corresponding result for ‘semilattice single-peakedness;’ Chatterji et al. [2016] characterize domains that are single-peaked with respect to a tree using strategy-proofness (and other conditions) imposed on *random* social choice functions; finally, Achuthankutty and Roy [2017] characterize single-peaked domains as those admitting certain types of strategy-proof social choice rules, the so called ‘min-max rules.’

It seems a worthwhile task for future research to explore whether, and how, the present methodology can contribute to our understanding of the weaker domain restrictions that still enable consistent preference aggregation and/or non-dictatorial strategy-proof social choice. One important lesson that can be drawn from the present analysis in this context is that the conditions of (semi-)connectedness, minimal richness, and the presence of two completely reversed orders *do* restrict the combinatorial space of possibilities substantially. Even if these conditions are justifiable by the specific context or application at hand, they cannot be considered mere ‘technical’ requirements nor innocuous assumptions.

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