Consistent Rights on Property Spaces*

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July 14, 2021

Working Paper

forthcoming in *Journal of Economic Theory* 197 (2021)

Abstract

In many aggregation problems, subgroups of agents have the right to predetermine certain properties of the aggregate. Yet, such rights may be inconsistent. In preference aggregation, for example, the 'liberal paradox' refers to the incompatibility of minimal liberal rights with the Pareto principle (a right to society as a whole). We show that, in general, rights to properties are consistent if and only if the following simple condition holds. Whenever rights are given to a critical (i.e., minimally inconsistent) combination of properties, the respective rights holding groups must intersect to at least one common member. Rights are consistent with monotone independent aggregation (voting by properties) if and only if this condition holds under a suitable generalization of criticality. Our property formulation allows us to study a wide range of applications in social choice and judgment aggregation theory.

Keywords: expert rights, liberal rights, liberal paradox, judgment aggregation, general aggregation theory, effectivity function.

JEL Classifications: D71, D79, K0.

*I would like to thank Hans Gersbach, Tobias Rachidi and Kevin Leportier for helpful comments. I am indebted to Clemens Puppe for many insightful discussions and for his continuing encouragement.

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1 Introduction

In many aggregation problems, subgroups of agents have the right to predetermine certain *properties* of the aggregate. To respect rights, an aggregation rule must allow subgroups to enforce a property to which they hold a right if all subgroup members agree on it. In standard preference aggregation, for example, allowing individuals to fix those parts of the social ordering that falls within their private spheres corresponds to the social choice theoretic conception of *liberal rights* (Sen, 1970; Gibbard, 1974). More generally, when groups form collective beliefs or desires through aggregating judgments on a set of propositions, they may leave certain judgments to subgroups with *vested interests* or *expert knowledge* (Dietrich and List, 2008). Not least, in committees, delegations from special interest groups can often dismiss alternatives which fall short of some minimal criteria.

Yet, at least since Sen's (1970) famous 'liberal paradox' (Example 1 below), it is well understood that the rights given to different subgroups can be (jointly) inconsistent to the effect that no appealing aggregation function grants all of them at the same time. Going beyond impossibility results, in this paper, we characterize when rights to properties are (in)consistent given that properties correspond to subsets of alternatives. Such *property spaces* (Nehring and Puppe, 2007, 2010, henceforth N&P) arise for a wide range of interesting applications. As in the examples above, properties may correspond to preference statements over fixed pairs of alternatives, to judgments on propositions (and their negations) or may be naturally suggested by the structure of alternatives. Our results generalize N&P as we show that their 'intersection property' not only characterizes monotone independent aggregators (equivalent to a particular type of rights) but serves to characterize consistency of rights more generally.

Before we preview our main results and discuss the relation to the existing literature on rights, we illustrate the problem of inconsistency in three examples.

1.1 Motivating Examples

Example 1. The Sen Liberal Paradox. Ann and Bob are owners of neighboring houses which can be painted either white (w) or yellow (y). Collectively, they are faced with four possible states: (w, w), (w, y), (y, y) and (y, w) – where first entries refer to the color of Ann's house. Suppose Ann and Bob have linear preference orders[1] over these states.

Liberals subscribe to the view that there be a protected private sphere within which individuals are free from interference. Arguably, if a collective ranking $\succ$ is to be found on liberal grounds, Ann and Bob should be left alone to determine it over every pair of

[1] That is, complete, transitive and antisymmetric (preference) relations.

	$(y,y) \succ (w,y)$	$(w,y) \succ (w,w)$	$(w,w) \succ (y,y)$
Ann	$\boxed{\checkmark}$	$\times$	$\boxed{\checkmark}$
Bob	$\times$	$\boxed{\checkmark}$	$\boxed{\checkmark}$
	$\checkmark$	$\checkmark$	$\checkmark$

Table 1: The Sen liberal paradox.

states which differ only with respect to their own house color. To formulate a *minimal* requirement, we may demand that they be free to decide $\succ$ over at least one such pair of states each. However, even in such minimal form, individual rights conflict with the equally natural requirement that $\succ$ respect unanimous preference statements (the Pareto condition; a right to society as a whole).

Indeed, suppose both Ann and Bob prefer their own house to be colorful (yellow walls), the other's not (white walls). When in conflict, their preference for a neutral-colored neighborhood prevails. That is, $(y,w) \succ_{Ann} (w,w) \succ_{Ann} (y,y) \succ_{Ann} (w,y)$ and $(w,y) \succ_{Bob} (w,w) \succ_{Bob} (y,y) \succ_{Bob} (y,w)$. Thus, by unanimous agreement, $(w,w) \succ (y,y)$. By minimal liberal rights, $(y,y) \succ_{Ann} (w,y) \implies (y,y) \succ (w,y)$ and $(w,y) \succ_{Bob} (w,w) \implies (w,y) \succ (w,w)$. Consequently, every minimally liberal and Paretian $\succ$ is cyclic. The 'liberal paradox' due to Sen (1970) is the fact that such cycles occur for every minimal assignment of rights if individual preferences are unrestricted.[2] For the present case, Table 1 visualizes Ann's (Bob's) (liberal) right by a solid (dashed) box and depicts the Pareto condition as a right held jointly by Ann and Bob (dotted boxes).

Note that our presentation departs from the orthodox view of the paradox as a fundamental incompatibility of Liberalism and Welfarism (in its arguably weakest form, Paretianism). Rather, we suggest to consider it in terms of an inconsistency of individual and collective rights. ■

Example 2. The Gibbard Liberal Paradox. Reconsider Example 1. If we drop minimality, individual rights are in fact internally inconsistent (even in the absence of the Pareto condition). To see this, suppose now that Bob is conformist to the effect that he always wants to match the color of his house to that of Ann's. Ann, on the other hand, is non-conformist. By principle, she prefers to paint her house in a different color than Bob's. Under full liberal rights, $(y,w) \succ_{Ann} (w,w)$ and $(w,y) \succ_{Ann} (y,y)$ imply that $(y,w) \succ (w,w)$ and $(w,y) \succ (y,y)$. At the same time, $(w,w) \succ (w,y)$ and $(y,y) \succ (y,w)$, seeing that $(w,w) \succ_{Bob} (w,y)$ and $(y,y) \succ_{Bob} (y,w)$. Combining, we have $(y,y) \succ (y,w) \succ (w,w) \succ (w,y) \succ (y,y)$. Thus,

[2] See also Corollary 1 below.

	sales experience	techn. expertise	comm. skills	
HR	✓	✓	×	a
Sales	✓	×	✓	b
R&D	×	✓	✓	c
	✓	✓	✓	

Table 2: An inconsistent system of subcommittee rights.

no acyclic $\succ$ can respect individual liberal rights alone. The fact that, in general, such cycles cannot be avoided under full individual liberties for two agents (or more) is the 'Gibbard (liberal) paradox' (1974).[3] ■

Example 3. Departmental Rights in Hiring Committees. A company has to fill a job opening in Technical Sales by choosing one out of three candidates (a, b, c). Candidates differ with respect to their differential possession of three relevant qualifications: *sales experience* (candidates a and b), *technical expertise* (a and c) and *communication skills* (b and c). Each candidate uniquely corresponds to a set of qualifications (rows of Table 2).

Suppose the hiring committee is made up of members from Human Resources (HR), the Sales Department (Sales) and from Research and Development (R&D).[4] To each department, two qualifications are of particular interest. That is, conversely, for every qualification, there is a subcommittee of two groups with vested interests, say $\{HR \cup Sales\}$ for sales experience (solid boxes in Table 2), $\{HR \cup R\&D\}$ for technical expertise (dashed boxes) and $\{Sales \cup R\&D\}$ for communication skills (dotted boxes).

If every member is to cast a vote for one candidate, is it possible to always select an applicant while granting said subcommittees the right to insist on the respective qualifications? Suppose individual votes are homogeneous within departments and as given by the rows of Table 2. That is, HR collectively vote for candidate a, Sales for b and R&D for c. Then the rights under consideration imply that the selected candidate possess all three qualifications. As such a candidate does not exist, these rights are inconsistent. In contrast, no such inconsistent combination of qualifications is implied if some qualification can only be insisted upon by the *whole* committee. In other words, if we change the above system of rights to the effect that, for one of the qualifications, we only impose an ordinary unanimity condition, then rights are consistent. ■

[3] See also Proposition 1 below.

[4] We assume that all committee members are affiliated to one and only one department.

1.2 Overview of Results

What is the common feature that renders rights (in)consistent in the examples above? In this paper, we show that consistent rights can be characterized in terms of a simple and well-known 'intersection property' when analyzed in the framework of (abstract) aggregation on property spaces developed by N&P. On property spaces, alternatives can be distinguished by means of (binary) properties. Due to restrictions imposed by logical or physical feasibility, properties are interdependent. In the preference aggregation setting of Examples 1 and 2 – where the set of alternatives is the set of linear orders, and properties correspond to pairwise preference statements – transitivity implies logical restrictions on properties. In the context of voting on candidates in Example 3 – where properties are qualifications – the availability of applicants puts physical constraints on their joint feasibility.

We show that a system of rights to properties is consistent if and only if every collection of groups holding rights over a minimally inconsistent (i.e., *critical*) family of properties has at least one common member (Intersection Property over Critical Families, IPC). A family is inconsistent if there is no alternative that possesses all properties in it. It is minimally so if all proper subfamilies are consistent. In Example 1, the family $\{(y, y) \succ (w, y), (w, y) \succ (w, w), (w, w) \succ (y, y)\}$ is critical.[5] Rights are inconsistent as the groups $\{Ann\}$, $\{Bob\}$ and $\{Ann, Bob\}$ fail to have a common member.[6] Likewise, in Example 3, *sales experience*, *technical expertise* and *communicative skills* make for a critical combination of properties, while no committee member belongs to all of the corresponding rights holding groups simultaneously: $(HR \cup Sales) \cap (HR \cup R\&D) \cap (Sales \cup R\&D) = \emptyset$.

We study when rights allow for a particularly natural way of aggregation that is monotone within and independent across properties (voting by properties). We prove that the characterizing condition of non-empty intersection continues to hold here under a suitably generalized concept of criticality (Intersection Property over Almost Critical Collections, IPAC). We derive tractable characterizations for important classes of property spaces developed in N&P, such as totally blocked spaces (trivial rights) and median spaces (independent rights). On semi-blocked spaces (see Nehring, 2006), which include partial order aggregation and classification problems, every voter can be granted a minimal participation right if and only if every issue is decided independently via a unanimity rule.

Our work extends a fundamental result from N&P. They characterize monotone independent aggregation as voting by properties induced by some 'structure of winning coalitions'

[5] It is inconsistent by transitivity. Every proper subset can be completed to a linear order, i.e., is consistent.

[6] Of course, the singletons $\{Ann\}$ and $\{Bob\}$ have empty intersection by themselves. However, the corresponding properties do not constitute a critical family.

that satisfies IPC. Thus, expressed in terms of rights, a 'structure of winning coalitions' is a rights system which is *exhaustive* (i.e., maximally specified) to the extent that it is equivalent to a monotone independent aggregation procedure respecting it. Yet most of the rights systems one wishes to study are distinctly non-exhaustive (consider Example 1-3 above or any of the Examples below). Therefore, our results provide a crucial generalization. First, the consideration of non-exhaustive rights facilitates an analysis of rights *sui generis*. Second, unlike the case of exhaustive 'structures of winning coalitions', non-exhaustive rights engender distinct characterizations of when rights are consistent with *some* aggregation function (IPC, Theorem 1) and when they allow to be respected in monotone independent aggregation (IPAC, Theorem 2).

1.3 Relation to the Literature on Rights

Initiating the analysis of rights in economics, Sen (1970) adopted a *social choice theoretic* formulation of rights to pairwise (collective) preference statements. Drawing on this model, a large part of the early literature analyzed the robustness of the liberal paradox(es) to a weakening of rights (see, e.g., Gibbard, 1974; Blau, 1975; Kelly, 1976) and the Pareto condition (Sen, 1976; Coughlin, 1986) as well as to domain restrictions (see, e.g., Blau, 1975; Fine, 1975). More recently, Sen's paradox has been generalized to other settings. In the emerging field of judgment aggregation, Dietrich and List (2008) show that minimal group rights are incompatible with a unanimity condition when propositions are sufficiently logically connected.[7] Herzberg (2017) proves a version for probabilistic opinion pooling. It is interesting to note that, while Sen's paradox has traditionally been interpreted in terms of a fundamental incompatibility of liberalism (rights) and welfarism (the Pareto principle), our conceptualization allows to consider the Pareto principle – or, more generally, the unanimity condition – as a right held by society as a whole. Thus, Sen's paradox can also be understood as revealing a conflict between individual and collective rights.

On the other hand, the social choice theoretic formulation has met with conceptual opposition from several authors who have pointed out that the intuitive content of a (liberal) right is *not* to make individual rankings decisive for social preference. But for rights holders to have a strategy at their disposal which allows them to restrict collective choice to a subset of alternatives (see, e.g., Nozick, 1974; Bernholz, 1974; Gärdenfors, 1981; Sugden, 1985; Gaertner

[7] See Corollary 2 below.

et al., 1992).[8] This alternative view has converged to analyzing rights in game forms[9] (see, e.g., Deb, 1994, 2004; Deb et al., 1997; Peleg, 1998; Fleurbaey and Van Hees, 2000; Boros et al., 2010). An important question in this literature is whether a system of rights is representable to the effect that the effectivity function[10] induced by some game form coincides with it.

While, to our best knowledge, contributions from both the social choice theoretic literature on rights and from judgment aggregation are limited to (im)possibility results, we provide a general characterization. As compared to the game form literature on rights – where such results exist (see, e.g., Peleg, 1998) – introducing a property structure on the set of alternatives provides for an intuitive characterization in terms of (i) semantic inter-dependencies between the objects of rights and (ii) combinatorial characteristics of the corresponding rights subjects. Seeing that we (extensionally) define properties as subsets of alternatives, our model shares the basic intuition of this literature. At the same time, it differs in two respects. First, rights are *conjunctive* in our model. When individuals are part of several rights holding groups, they can exercise these rights simultaneously unless this implies enforcing an inconsistent combination of properties at the individual level. In particular, if the same group has rights to two properties, it can enforce their conjunction unless this is infeasible.[11] Second, our notion of representability (i.e., consistency) of rights differs. On the one hand, it is weaker to the effect that we study whether there is some game form that implements *at least* the considered rights (and potentially more). On the other hand, we consider representation by the restricted class of *voting* game forms. We show in Appendix A that, given our notion of weak and conjunctive representation, this is without loss of generality.

The rest of this paper is structured as follows. In Section 2, we introduce property spaces and define rights to properties. Section 3 presents our characterization of consistent rights. In Section 4, we characterize when rights are consistent with monotone independent aggregation (voting by properties). Section 5 concludes. Unless proofs are short and insightful, they are relegated to the Appendix.

[8] For example, following this line of argument, the right to one's own house color consists in the fact that one can go about painting it in every color one sees fit; thereby restricting the set of social states that can ensue. See also the continuation of Example 2 below.

[9] A game form is a game for which preferences are left unspecified.

[10] On effectivity functions, see also Moulin (1983); Peleg (2002).

[11] Undoubtedly, there are important rights which are non-conjunctive. For example, individuals have both the right to ride a bike and talk on a cell phone. However, there is no right to do both at the same time. Indeed, there is an obligation not to do it.

2 Rights on Property Spaces

Let X be some finite set of (abstract) objects, $|X| > 2$, and let $N = \{1, \ldots, n\}$ be a group of $n \geq 2$ individuals. We refer to every $x \in X$ as an *alternative* (or outcome) and to every $i \in N$ as a voter. If for every $i \in N$, $x_i \in X$, we say that $\boldsymbol{x} = (x_1, \ldots, x_n)$ is a profile (of votes). Thus, every $i \in N$ votes for exactly one alternative. An *aggregation function* is a mapping $f : X^n \to X$. It maps each profile $(x_1, \ldots, x_n) \in X^n$ to some feasible (collective) alternative $f(x_1, \ldots, x_n) \in X$.

2.1 Property Spaces

To turn the set of alternatives into a property space, we endow X with a property structure $\mathcal{P}$.

Property Space. We say that the ordered pair $(X, \mathcal{P})$ is a property space if and only if for all $P \in \mathcal{P} \subseteq 2^X$ and for all $y, z \in X$, $y \neq z$:

$$P \neq \emptyset, \tag{non-triviality}$$

$$P^c := X \backslash P \in \mathcal{P}, \tag{negation-closedness}$$

$$\exists\, Q \in \mathcal{P} : z \in Q,\ y \notin Q. \tag{separation}$$

We refer to all $P, Q \in \mathcal{P}$ as *properties*.[12]

The intuition behind the construction of $\mathcal{P}$ is the following: Every property $P \in \mathcal{P}$ is identified with the subset of alternatives (note that $P \subseteq X$) which possess it. $\mathcal{P}$ is the collection of all properties such that (i) every $P \in \mathcal{P}$ is non-empty, i.e., there is some alternative that conforms to it. (ii) Every property in $P \in \mathcal{P}$ comes with a *complement* or *negation*, $P^c = X \backslash P \in \mathcal{P}$. This ensures that property membership is binary: either $x \in X$ belongs to P or to its complement P^c. When $P \in \mathcal{P}$, we refer to $\{P, P^c\}$ as a property-negation pair or an *issue*. (iii) The property structure is exhaustive to the effect that any two alternatives are distinguishable by at least one property.

Two comments may clarify the construction. First, properties are *extensionally* defined as subsets of alternatives. Thus, the set of alternatives X is *endowed* with a property structure. This roundabout way of defining properties in terms of alternatives, instead of alternatives in terms of properties, makes it possible to consider different property structures on the same set of underlying alternatives. While a particular property structure $\mathcal{P}$ might be natural on X, it

[12]To highlight the natural connection of the property space framework with judgment aggregation theory (see Example 5 below), we depart from N&P's convention of labeling properties by $H \in \mathcal{H}$ and use $P, Q \in \mathcal{P}$ instead.

is important to keep in mind that others are possible. Second, notwithstanding what we just mentioned, the above axioms can be easily seen to imply that for all $x \in X : \{x\} = \bigcap\{P \in \mathcal{P} : x \in P\}$. That is, every alternative is uniquely identified by the set of its constituent properties. In other words, once a property structure on X is fixed, we can conveniently think of any $x \in X$ as the collection of all the properties it possesses.[13]

We call every $\mathcal{F} \subseteq \mathcal{P}$ a family (of properties) and denote the set of all non-empty families by $\mathbb{F} = 2^{\mathcal{P}} \backslash \{\emptyset\}$. There is a natural notion of consistency on $\mathbb{F}$. Consider some family $\mathcal{F} = \{P_1, \ldots, P_r\} \in \mathbb{F}$. We say that $\mathcal{F}$ is *consistent* if and only if some alternative possesses all properties in $\mathcal{F}$: $\bigcap \mathcal{F} = \bigcap_{k=1,\ldots,r} P_k \neq \emptyset$. If and only if $\bigcap \mathcal{F} = \emptyset$, $\mathcal{F}$ is *inconsistent*. Clearly, every subfamily of a consistent family is itself consistent. To study dependencies between properties on $(X, \mathcal{P})$ it is instructive to consider families which are *minimally* inconsistent to the effect that removing any property yields a consistent family. We call such families *critical* and generically denote them by $\mathcal{G} \in \mathbb{F}$.

Critical Family. Let $\mathcal{G} \in \mathbb{F}$. $\mathcal{G}$ is critical if and only if $\bigcap \mathcal{G} = \emptyset$ and for all $P \in \mathcal{G}$: $\bigcap (\mathcal{G} \backslash \{P\}) \neq \emptyset$.

All property-negation pairs $\{P, P^c\} \in \mathbb{F}$ are critical. We refer to them as the *trivial* critical families.

Example 4. Linear Orders and SWFs. Let A be a set. Define $X_{Lin(A)} = \{> \subseteq A \times A : > \text{ is a linear order}\}$. For all distinct $a, b \in A$, let $P_{a>b} = \{> \in X_{Lin(A)} : a > b\}$ and note that $P^c_{a>b} = X \backslash P_{a>b} = P_{b>a}$. Denote by $\mathcal{P}_{Lin(A)}$ the set of all such properties, i.e., $\mathcal{P}_{Lin(A)} = \{P_{a>b}\}_{a \neq b \in A}$. $\left(X_{Lin(A)}, \mathcal{P}_{Lin(A)}\right)$ defines a property space. The non-trivial critical families are those produced by a preference cycle. That is, if $r \geq 3$ and all $a_1, \ldots, a_r \in A$ are distinct, then $\{P_{a_1>a_2}, \ldots, P_{a_{r-1}>a_r}, P_{a_r>a_1}\}$ is critical. These are the only non-trivial critical families. f is an aggregation function on $\left(X_{Lin(A)}, \mathcal{P}_{Lin(A)}\right)$ if and only if it is a social welfare function (SWF). ■

Example 5. Judgment Aggregation. Let L be a set of logical propositions such that if $p \in L$ then $\neg p \in L$, where $\neg p$ means "not p" (i.e., L is closed under logical negation). An agenda $Y \subseteq L$ is a set of propositions (and their negations) on which (collective) judgments have to be made. Every $A \subseteq Y$ is referred to as a *judgment set*. A is *complete* if and only if, for all $p \in Y$, $p \in A$ or $\neg p \in A$. In standard propositional logic, *consistency* of judgment sets can be defined in the usual way. For example, for the agenda $\{u, \neg u, v, \neg v, u \to v, \neg(u \to v)\}$, the judgment set $\{u, u \to v, v\}$ is consistent, while $\{\neg u, \neg(u \to v)\}$ is inconsistent.[14]

[13] Thus, in the presence of a property structure, individual votes $x_i \in X$ can also be thought of as votes on complete and consistent combinations of properties.

[14] For a general logic, consistency can be defined in terms of an entailment relation on $2^L \times L$ (cf. Dietrich,

Let Y be an agenda and define $X_Y = \{A \subseteq Y : A \text{ complete and consistent}\}$. For every $p \in Y$, let $P_p = \{A \in X_Y : p \in A\}$. Then $P_p^c = X_Y \backslash P_p = \{A \in X_Y : p \notin A\} = \{A \in X_Y : \neg p \in A\} = P_{\neg p}$. We define $\mathcal{P}_Y = \{P_p\}_{p \in Y}$ and note that $(X_Y, \mathcal{P}_Y)$ is a property space. On $(X_Y, \mathcal{P}_Y)$, every property corresponds to the judgment on some proposition and vice versa. A family of properties is consistent if and only if the corresponding set of propositions is consistent. ■

2.2 Rights

Rights System. A rights system is a correspondence $\mathcal{R} : \mathcal{P} \rightrightarrows 2^N \backslash \{\emptyset\}$.

For every property $P \in \mathcal{P}$, $\mathcal{R}(P)$ collects all (sub)groups[15] $G \subseteq N$ that have a right to it.[16] When group $G \subseteq N$ has a right to property P, it can enforce this property on aggregate. The right is *exercised* when all members $i \in G$ vote for some alternative which conforms to P (i.e., $\forall i \in N : x_i \in P$).[17] An aggregation function $f : X^n \to X$ *respects* the rights system $\mathcal{R}$ if and only if for all $P \in \mathcal{P}$ and all profiles $\boldsymbol{x} = (x_1, \ldots, x_n) \in X^n$:

$$G \in \mathcal{R}(P) \implies [G \subseteq \{i \in N : x_i \in P\} \implies f(\boldsymbol{x}) \in P]. \tag{R}$$

That is, f respects $\mathcal{R}$ if and only if it guarantees all properties P which are unanimously endorsed by some group G with a right to P. For example, requiring that $N \in \mathcal{R}(P)$ for all $P \in \mathcal{P}$ corresponds to a standard issue-wise unanimity condition on f. By (R), every group $G \in \mathcal{R}(P)$ can exercise its right to P irrespective of how agents $i \in N \backslash G$ vote.[18] We say that the rights system $\mathcal{R}$ is *consistent* if and only if there is *some* onto aggregation function that respects it. Consequently, $\mathcal{R}$ is consistent if and only if every possible scenario of joint rights exercise is compatible with some aggregate alternative. The scope for such scenarios, however, is limited by the need to vote consistently at the individual level (for all $i \in N$: $x_i \in X$).

2007). We also assume that the agenda does not contain any tautologies and contradictions, where $p \in L$ is a contradiction if p is inconsistent and a tautology if $\neg p$ is a contradiction.

[15]Although every $G \subseteq N$ is a subgroup of N, we will also simply refer to it as a *group* for the rest of the paper.

[16]We define rights systems as mappings from properties to collections of subgroups for notational convenience. Alternatively, we could model rights by means of correspondences $\widehat{\mathcal{R}} : 2^N \backslash \{\emptyset\} \rightrightarrows \mathcal{P}$ collecting, for every group $G \subseteq N$, the set of properties to which G has a right. Seeing that, for every such $\widehat{\mathcal{R}}$, $P \mapsto \widehat{\mathcal{R}}^{-1}(P) = \{G \in 2^N \backslash \{\emptyset\} : P \in \widehat{\mathcal{R}}(G)\}$ defines a rights system, our formulation is no less general.

[17]One might want to object that, in many contexts, a right to group $G \subseteq N$ refers to its ability to enforce a property under less stringent internal support, e.g., by simple group majority. Our formulation is without loss of generality in this regard, as such demands can be reformulated as rights to subgroups of G, e.g., rights to all majority subgroups of G.

[18]Note the familiarity with the concept of (α-)effectivity in game forms. We elaborate in Appendix A.

We observe some immediate implications of (R). For rights system $\mathcal{R}$, define its (property-wise) monotone closure $\overline{\mathcal{R}}$ by $P \mapsto \overline{\mathcal{R}}(P) = \{G \subseteq N : G \supseteq G' \text{ for some } G' \in \mathcal{R}(P)\}$. Some $f : X^n \to X$ respects $\mathcal{R}$ if and only if it respects $\overline{\mathcal{R}}$. Intuitively, if some $G \subseteq N$ can force some property, then so can any group that is larger than G. Consequently, it is without loss of generality to simplify notation by restricting attention to rights systems which are minimally specified in terms of superset inclusion. Second, note that if $\mathcal{R}$ grants G a right to P, there is nothing in (R) which excludes the possibility that an aggregation rule f respecting $\mathcal{R}$ *effectively* grants this right to a proper subgroup of G. For example, declaring any $i \in G$ a *local* dictator on the issue $\{P, P^c\}$ stays true to granting G a right to P according to (R). In this sense, $\mathcal{R}$ is consistent if there exists some aggregation function that grants *at least* the rights in $\mathcal{R}$.

We say that a rights system $\mathcal{R}$ is *trivial* if $\bigcap_{G \in \mathcal{R}(P),\, P \in \mathcal{P}} G \neq \emptyset$. Indeed, if $\mathcal{R}$ is trivial, there exists some individual $j \in N$ who is part of every rights holding group. Thus, making j a dictator ($\forall \boldsymbol{x} = (x_1, \ldots, x_n) \in X^n : f(\boldsymbol{x}) = x_j$) is (trivially) consistent with $\mathcal{R}$.

3 When Are Rights Consistent?

3.1 A Characterization

We are ready to state when some given rights system $\mathcal{R}$ is (in)consistent. To gain intuition, for every profile $\boldsymbol{x} = (x_1, \ldots, x_n)$, consider the family of properties to which rights are exercised: $\mathcal{P}_{\mathcal{R}}(\boldsymbol{x}) = \{P \in \mathcal{P} : \{i \in N : x_i \in P\} \supseteq G \text{ for some } G \in \mathcal{R}(P)\}$.[19] Unless one of these families is inconsistent, we can define an onto aggregation function respecting $\mathcal{R}$ profile-wise by $\boldsymbol{x} \mapsto f(\boldsymbol{x}) \in \bigcap \mathcal{P}_{\mathcal{R}}(\boldsymbol{x})$. On the other hand, if one such family is inconsistent, there can be no $f : X^n \to X$ that respects rights.

Suppose $\boldsymbol{x}^\star = (x_1^\star, \ldots, x_n^\star) \in X^n$ is such that $\mathcal{P}_{\mathcal{R}}(\boldsymbol{x}^\star)$ is inconsistent and consider a critical subfamily $\mathcal{G}$.[20] By definition, for each $P \in \mathcal{G}$, there exists some group $G_P \in \mathcal{R}(P)$ that endorses P unanimously: $x_i^\star \in P$ for all $i \in G_P$. However, there can be no $i \in N$ who is a member of *all* these groups at once, as this would imply voting inconsistently: $x_i^\star \in \bigcap_{P \in \mathcal{G}} P = \emptyset$. On the other hand, if there exist groups with empty intersection holding rights over $\mathcal{G}$, then there is some profile $\boldsymbol{x}^\star = (x_1^\star, \ldots, x_n^\star)$ such that $\mathcal{G} \subseteq \mathcal{P}_{\mathcal{R}}(\boldsymbol{x}^\star)$, seeing that, for every $i \in N$, there is some $P \in \mathcal{G}$ such that $i \notin G_P$ (hence $\exists x_i^\star \in \bigcap \mathcal{G} \backslash \{P\} \neq \emptyset$). Thus, the following condition due to N&P characterizes consistency.

Intersection Property over Critical Families. $\mathcal{R} : \mathcal{P} \rightrightarrows 2^N \backslash \{\emptyset\}$ satisfies the *In-*

[19] Note that $\mathcal{P}_{\mathcal{R}}(\boldsymbol{x}) = \mathcal{P}_{\overline{\mathcal{R}}}(\boldsymbol{x})$.

[20] As is easily verified, every inconsistent family contains a critical subfamily.

tersection Property over Critical Families (IPC) if and only if for all critical $\mathcal{G} = \{P_1, \ldots, P_r\}$:

$$G_1 \in \mathcal{R}(P_1), \ldots, G_r \in \mathcal{R}(P_r) \implies \bigcap_{k=1}^{r} G_k \neq \emptyset. \tag{IPC}$$

Theorem 1. *A rights system $\mathcal{R} : \mathcal{P} \rightrightarrows 2^N \backslash \{\emptyset\}$ is consistent if and only if $\mathcal{R}$ satisfies* (IPC).

Theorem 1 reduces the problem of consistency to an easily interpretable condition: Whenever rights are given to a critical set of properties, the respective rights holding groups must intersect to at least one common member. (IPC) provides a characterization in terms of (i) semantic dependencies between rights objects (properties to which rights are held) and (ii) the combinatorial characteristics of the corresponding rights subjects (groups holding rights). As the set of alternatives is *endowed* with a particular property structure, the exact interpretation of (IPC) hinges on the concrete application.

Example 2 (ctd.). Using the construction in Example 4, $(X_1, \mathcal{P}_1) = \left(X_{Lin(A_1)}, \mathcal{P}_{Lin(A_1)}\right)$ with $A_1 = \{(w, w), (w, y), (y, w), (y, y)\}$ is a property space. In the social choice theoretic interpretation, if $\mathcal{R}_2$ respects full individual liberties on $(X_1, \mathcal{P}_1)$, we have, inter alia, $\{Ann\} \in \mathcal{R}_2\left(P_{(y,w)>(w,w)}\right) \cap \mathcal{R}_2\left(P_{(w,y)>(y,y)}\right)$ and $\{Bob\} \in \mathcal{R}_2\left(P_{(w,w)>(w,y)}\right) \cap \mathcal{R}_2\left(P_{(y,y)>(y,w)}\right)$.[21] As $\left\{P_{(y,w)>(w,w)}, P_{(w,w)>(w,y)}, P_{(w,y)>(y,y)}, P_{(y,y)>(y,w)}\right\}$ is critical, $\{Ann\} \cap \{Bob\} = \emptyset$ yields a violation of (IPC); $\mathcal{R}_2$ is inconsistent. Proposition 1 below contains the general statement of the Gibbard paradox.

However, as several authors have pointed out (see, e.g., Gaertner et al., 1992), the Gibbard paradox is counter-intuitive. In a more natural interpretation, individual liberal rights consist in the ability to paint one's house in the color one sees fit. Such rights are consistent. Indeed, consider the property space $(A_1, \widetilde{\mathcal{P}}_1)$, where $\widetilde{\mathcal{P}}_1 = \{P_A, P_A^c, P_B, P_B^c\}$ and $P_A = \{(w, w), (w, y)\}, P_B = \{(w, w), (y, w)\}$. Here, the properties refer directly to the color of Ann's (P_A if white) and Bob's (P_B if white) house. On $(A_1, \widetilde{\mathcal{P}}_1)$, the only critical families are the trivial ones. Thus, the rights system $\widetilde{\mathcal{R}}_2(P_A) = \widetilde{\mathcal{R}}_2(P_A^c) = \{\{Ann\}\}$ and $\widetilde{\mathcal{R}}_2(P_B) = \widetilde{\mathcal{R}}_2(P_B^c) = \{\{Bob\}\}$ satisfies (IPC) and is consistent. ■

Proposition 1. ***(cf. Gibbard, 1974, Theorem 1)*** *Let $A = A_0 \times A_1 \times \cdots \times A_n$ where A_0 is a set of public aspects and A_i are sets of aspects pertaining to the private sphere of individual $i \in N$. $|A_i| \geq 2$ for all $i \in N$. Consider $\left(X_{Lin(A)}, \mathcal{P}_{Lin(A)}\right)$. Let $\mathcal{R}$ be a rights system and*

[21] For the purpose of illustration, we denote individuals by names here instead of natural numbers.

suppose that, for all $i \in N$ and for all i-variants[22] $a, b \in A$, *we have* $\{i\} \in \mathcal{R}(P_{a>b}) \cap \mathcal{R}(P_{b>a})$. *Then $\mathcal{R}$ is inconsistent for $n \geq 2$.*

Theorem 1 generalizes a fundamental result from N&P which is stated as Fact 2 below.[23] They characterize monotone independent aggregation as *voting by properties*[24] induced by some so-called 'structure of winning coalitions' satisfying (IPC). Viewed in terms of rights, a 'structure of winning coalitions' is simply a rights system which is maximally specified or *exhaustive* to the effect that, for all groups $G \subseteq N$ and properties $P \in \mathcal{P}$, it either affords G a right to P or else $N \backslash G$ a right to P^c.[25] If consistent, such exhaustive rights in effect *define* a natural aggregation procedure that is monotone (within properties) and independent (across properties) called voting by properties. In other words, if $\mathcal{R}$ is exhaustive, the demand that some aggregation function $f : X^n \to X$ respect rights (i.e., condition (R)) is equivalent to f being voting by properties induced by $\mathcal{R}$ satisfying (IPC).

Yet the rights systems studied in the literature are generally non-exhaustive (consider any of the Examples given in this paper). Theorem 1 shows that (IPC) continues to characterize consistency for non-exhaustive rights systems. However, while a consistent exhaustive rights system *is* a monotone independent aggregation function in the aforementioned sense, Theorem 1 only ensures consistency with *some* aggregation function. Indeed, as we show in Section 3.3 below, there are interesting examples of non-exhaustive rights which are consistent yet cannot be respected in voting by properties. This raises the question of whether we can characterize when non-exhaustive rights are consistent in this stricter sense, i.e., when non-exhaustive rights can be consistently extended to an exhaustive system. Theorem 2 below provides such a characterization of *exhaustibly consistent* rights in terms of an analogous intersection property for a suitably generalized notion of criticality.

Before we develop the necessary theory in Section 4, we use Theorem 1 to relate the inconsistency of rights to structural properties of the underlying space and motivate Section 4 by discussing rights which are consistent but not exhaustibly so.

3.2 Structural Properties and a General Impossibility Result

Below, we present a general impossibility result (Proposition 2) for rights systems that grant property-wise unanimity rights (i.e., $N \in \bigcap_{P \in \mathcal{P}} \mathcal{R}(P)$). To this end, we first introduce struc-

[22] We say that $a, b \in A$ are i-variants if $\forall j \neq i : a_j = b_j$, where a_j is the projection of a on A_j.

[23] See Proposition 2.1 in Nehring and Puppe (2010); also Proposition 3.1 in Nehring and Puppe (2007).

[24] N&P refer to *voting by properties* as 'voting by issues' instead.

[25] To be precise, a 'structure of winning coalitions' as defined in N&P is an exhaustive rights system which is monotone: If group G has a right to P, then every superset of G does. However, as noted in Section 2.2 above, this distinction is immaterial as far as consistency is concerned: A rights system is consistent if and only if its (property-wise) monotone closure is.

tural characteristics of property spaces and rights systems based on entailments between properties.

Let $\mathcal{G} \in \mathbb{F}$ be critical and consider any $P \in \mathcal{G}$. The properties $\mathcal{G} \backslash \{P\}$ *entail* property P^c. Why? By criticality of $\mathcal{G}$, $\mathcal{G} \backslash \{P\}$ is consistent. That is, there exists some feasible $x \in \mathcal{G} \backslash \{P\}$. As $\bigcap \mathcal{G} = \emptyset$ and $x \in X = P \cup P^c$, we must have $x \in P^c$. In other words, every alternative consistent with $\mathcal{G} \backslash \{P\}$ must conform to P^c. Moreover, this entailment is *minimal* in the sense that no proper subset $\mathcal{G}' \subsetneq \mathcal{G} \backslash \{P\}$ entails P^c (seeing that $\mathcal{G}' \cup \{P\} \subsetneq \mathcal{G}$ is consistent by criticality of $\mathcal{G}$).

Minimal Entailment I. For $\mathcal{F} \in \mathbb{F}$, $P \in \mathcal{P}$ we define $\mathcal{F} \vdash P$ if and only if $\mathcal{F} \cup \{P^c\}$ is critical. When $\mathcal{F} \vdash P$, we say that $\mathcal{F}$ *minimally entails* P.

While $\vdash$ relates some property P to *families* of properties, at times, we are only interested in analyzing dependencies between P and some other property Q. Suppose $P \in \mathcal{F}$ and $\mathcal{F} \vdash Q$. Then *conditional* on the properties in $\mathcal{F} \backslash \{P\}$, P entails Q.

Conditional Entailment. $P \trianglerighteq Q$ if and only if there exists some $\mathcal{F} \in \mathbb{F}$ such that $P \in \mathcal{F}$ and $\mathcal{F} \vdash Q$. We denote by $\trianglerighteq^\star \subseteq \mathcal{P} \times \mathcal{P}$ the transitive closure of $\trianglerighteq$. When $P \trianglerighteq^\star Q$, we say that P *conditionally entails* Q. When we need to distinguish it from $\trianglerighteq^\star$, we refer to $\trianglerighteq$ as *direct conditional entailment.*

Note that $P \trianglerighteq Q$ if and only if there exists some critical $\mathcal{G} \in \mathbb{F}$ such that $\{P, Q^c\} \subseteq \mathcal{G}$. It follows that $P \trianglerighteq Q \iff Q^c \trianglerighteq P^c$ and $P \trianglerighteq^\star Q \iff Q^c \trianglerighteq^\star P^c$. We say that P *unconditionally entails* Q if and only if $\{P\} \vdash Q$. That is, P unconditionally entails Q if and only if $\{P, Q^c\}$ is critical. Thus, $\{P\} \vdash Q \iff P \subseteq Q$. By definition, unconditional entailment is direct. We say that $P, Q \in \mathcal{P}$ are *dependent* if and only if $P \trianglerighteq^\star Q^c$. P, Q are *directly* dependent if and only if $P \trianglerighteq Q^c$. That is, P, Q are directly dependent if and only if there exists some critical $\mathcal{G} \in \mathbb{F}$ such that $\{P, Q\} \subseteq \mathcal{G}$. We say that two *issues* $\{P, P^c\}, \{Q, Q^c\}$ are (directly) dependent if and only if we can find some (directly) dependent $\widehat{P}, \widehat{Q}$ such that $\widehat{P} \in \{P, P^c\}$ and $\widehat{Q} \in \{Q, Q^c\}$.

Median, Totally Blocked, Semi-blocked, Connected Property Spaces. Let $(X, \mathcal{P})$ be a property space. We say that $(X, \mathcal{P})$ is:

1. *median* if and only if all entailments are unconditional (cf. Nehring and Puppe, 2007, 2010),
2. *totally blocked* if and only if $\forall P, Q \in \mathcal{P} : P \trianglerighteq^\star Q$ (cf. Nehring and Puppe, 2007, 2010),
3. *semi-blocked* if and only if $(X, \mathcal{P})$ is not totally blocked and $\forall P, Q \in \mathcal{P}$: $[P \trianglerighteq^\star Q$ and $Q \trianglerighteq^\star P]$ or $[P \trianglerighteq^\star Q^c$ and $Q^c \trianglerighteq^\star P]$ (cf. Nehring, 2006),
4. *connected* if and only if all issues are directly dependent (cf. Dietrich and List, 2008).

As is easily verified, $(X, \mathcal{P})$ is median if and only if all critical families $\mathcal{G} \in \mathbb{F}$ have length two ($|\mathcal{G}| = 2$). Thus, $(A_1, \widetilde{\mathcal{P}}_1)$ from Example 2 (ctd.) above is median.[26] $(X, \mathcal{P})$ is totally blocked if and only if all properties are dependent. For example, $(X_1, \mathcal{P}_1)$ (Example 2, ctd.) is totally blocked. More generally, on every finite set of alternatives, the space of linear orders – endowed with the property structure from Example 4 – is totally blocked. When using an analogous construction for properties, important examples of semi-blocked spaces include the partial orders (Example 9 below) and the equivalence relations (Example 10 below).

Independent, Autonomous Rights. Let $\mathcal{R}$ be a rights system on a property space $(X, \mathcal{P})$.

1. $\mathcal{R}$ is *independent* (*weakly independent*) if and only if there do not exist two disjoint groups $G, G' \subseteq N$ and two dependent (directly dependent) $P, Q \in \mathcal{P}$ such that $G \in \mathcal{R}(P)$ and $G' \in \mathcal{R}(Q)$.

2. $\mathcal{R}$ is *autonomous* (*weakly autonomous*) if and only if there exist two disjoint groups $G, G' \subseteq N$ and two distinct properties $P, Q \in \mathcal{P}$ such that $G \in \mathcal{R}(P) \cap \mathcal{R}(P^c)$ and $G' \in \mathcal{R}(Q) \cap \mathcal{R}(Q^c)$ (such that $G \in \mathcal{R}(P)$ and $G' \in \mathcal{R}(Q)$).

A rights system is (weakly) independent if and only if no two disjoint groups have rights to (directly) dependent properties. It is (weakly) autonomous if and only if two disjoint groups can each autonomously decide some distinct issue (property). If $\mathcal{R}$ is autonomous and weakly independent, disjoint groups can hold rights only over issues which are not directly dependent. When $(X, \mathcal{P})$ is connected, such issues do not exist. By consequence, no rights system can be autonomous and weakly independent at the same time.

Fact 1. *Let $(X, \mathcal{P})$ be connected. There do not exist weakly independent and autonomous rights systems.*

The following proposition shows that failure of weak independence is sufficient for inconsistency when unanimity rights are granted to all properties. We use Fact 1 to derive two prominent impossibility results in the literature as corollaries. Corollary 1 establishes the 'liberal paradox' for group rights (cf. Example 1 above). Corollary 2 is its generalization to judgment aggregation: no (issue-wise) unanimous aggregation function can grant autonomous group rights on connected agendas.

Proposition 2. *Let $(X, \mathcal{P})$ be a property space and $\mathcal{R}$ be a rights system on $(X, \mathcal{P})$ such that $N \in \bigcap_{P \in \mathcal{P}} \mathcal{R}(P)$. Unless $\mathcal{R}$ is weakly independent, it is inconsistent.*

[26] Median spaces play an important role in strategy-proof social choice. They admit strong possibility results for rich single-peaked domains (Nehring and Puppe, 2007). See the same reference for more examples of median spaces.

Corollary 1. ***(cf. Sen, 1976, A3)*** *Let A be some finite set and consider $\big(X_{Lin(A)}, \mathcal{P}_{Lin(A)}\big)$. If for all $a \neq b \in A$: $N \in \mathcal{R}(P_{a>b})$ and there exist disjoint $G, G' \subseteq N$ such that, for some $c, c', d, d' \in A$, $G \in \mathcal{R}(P_{c>d}) \cap \mathcal{R}(P_{d>c})$ and $G' \in \mathcal{R}(P_{c'>d'}) \cap \mathcal{R}(P_{d'>c'})$, then $\mathcal{R}$ is inconsistent.*[27]

Corollary 2. ***(cf. Dietrich and List, 2008, Theorem 2)*** *Let Y be some agenda (of logical propositions). Suppose $(X_Y, \mathcal{P}_Y)$ is connected. If $N \in \bigcap_{p \in Y} \mathcal{R}(P_p)$ and there exist disjoint $G, G' \subseteq N$ such that, for some $p, q \in Y$, $G \in \mathcal{R}(P_p) \cap \mathcal{R}(P_{\neg p})$ and $G' \in \mathcal{R}(P_q) \cap \mathcal{R}(P_{\neg q})$, then $\mathcal{R}$ is inconsistent.*[28]

Without property-wise unanimity, however, (weak) independence is neither necessary nor sufficient for consistency. Indeed, when dropping the Pareto condition in Example 1 above, the remaining system of minimal liberal rights is not (weakly) independent but consistent. In Example 3 above, on the other hand, all rights holding groups are pairwise disjoint so that the system is trivially (weakly) independent but inconsistent. At the same time, independence completely characterizes consistency on median spaces. As an even stronger characterization holds for these spaces (see Proposition 3 below), we do not state this result here.

3.3 Consistency vs. Exhaustive Consistency

Theorem 1 shows (IPC) to be necessary and sufficient for rights to be respected by *some* aggregation function. A natural way to obtain an aggregation procedure is to decide each issue independently. In the presence of independence, monotonicity is the natural requirement that increased support for some property cannot lead to its complement being accepted if it wasn't before. Apart from its long standing tradition in social choice theory beginning with Arrow (1963), (monotone) independence is a natural requirement in many contexts (for instance, see Example 6 below).[29] Moreover, while the very nature of rights may imply that aggregation happens asymmetrically across subgroups and issues, necessitating failures of anonymity and neutrality, monotone independence seems to stand in no immediate contradiction to rights per se.

[27] The result proven in Sen (1976, A3) is slightly more general to the effect that it shows that there can be no *social decision function* (i.e., no aggregation rule producing an acyclic social relation) that satisfies minimal group rights and the Pareto criterion.

[28] In analogy to footnote 27, Theorem 2 in Dietrich and List (2008) is slightly more general as it does not require collective judgment sets to be complete. It is possible to derive the full force of both impossibility results from an analogous statement of Theorem 1 for (non-empty) aggregation correspondences $F : X^n \rightrightarrows X$. We do not do so here to keep the analysis focused on aggregation functions.

[29] From a normative point of view, Nehring and Puppe (2007) show that monotone independence is closely linked to strategy-proofness of voting rules: on rich single-peaked domains, a social choice function is strategy-proof if and only if it is monotone independent.

Consequently, it is of great interest to study whether rights are *exhaustibly* consistent in the stronger sense of being respected by a monotone independent and onto aggregator. Unlike for exhaustive rights – 'structures of winning coalitions' in N&P – there are important examples of rights which are consistent but *not* exhaustibly so. For instance, this applies to minimal Sen rights (Example 1, ctd.) or minority rights in selecting members of a committee (Example 6). Another example involving majority rights in truth-functional judgment aggregation is developed alongside the theory of Section 4 (Example 7).

Example 1 (ctd.). If we drop the Pareto condition in Example 1, minimal Sen rights are consistent. At the same time, as a corollary of Arrow's Theorem (1963, ch. VIII), the only onto and monotone independent social welfare functions are the dictatorships. Thus, minimal Sen rights are not exhaustibly consistent. More generally, suppose that there are strictly more social alternatives than individuals ($|A| > n \geq 2$). Then there exists some consistent $\mathcal{R}$ granting minimal Sen rights. However, $\mathcal{R}$ is not exhaustibly consistent.[30] ■

Example 6. Committee Selection. Suppose a committee has to be elected from a set of candidates $\{1, \ldots, K\}$ subject to the constraint that at least k' and at most k'' candidates are selected (where $0 < k' \leq k'' < K$). Thus, $X_{(K;k',k'')} = \{C \subseteq \{1, \ldots, K\} : k' \leq |C| \leq k''\}$ is the set of feasible committees. If we let, for every $k \in \{1, \ldots, K\}$, $P_k = \{C \in X_{(K;k',k'')} : k \in C\}$ and define $\mathcal{P}_{(K;k',k'')} = \{P_k, P_k^c\}_{k=1,\ldots,K}$, then $\left(X_{(K;k',k'')}, \mathcal{P}_{(K;k',k'')}\right)$ is a property space such that property k refers to whether candidate k is elected to the committee (Nehring and Puppe, 2010). Monotone independent aggregation on $\left(X_{(K;k',k'')}, \mathcal{P}_{(K;k',k'')}\right)$ amounts to the natural conception that a committee can be selected by voting on each candidate separately and that additional votes can never remove a candidate from the committee.

To safeguard their rights and interests, minority subgroups may demand representation in the committee. That is, minorities may demand the right to elect 'their' candidate to the committee.[31] If $G_1, \ldots, G_m \subseteq N$ are m such (possibly disjoint) minority groups and $k_1, \ldots, k_m$ the corresponding candidates, we consider the rights system $\mathcal{R}$ given by $\mathcal{R}(P_{k_j}) = \{G_j\}$ for $j = 1, \ldots, m$ and $\mathcal{R}(P) = \emptyset$ else. As long as $m \leq k''$, $\mathcal{R}$ is consistent.[32] Intuitively, if the number of minority candidates is less than the maximal size of the committee, each can be granted a right to membership. At the same time, if at least two minorities are disjoint –

[30]Since $(X_{Lin(A)}, \mathcal{P}_{Lin(A)})$ is totally blocked, this follows from Proposition 3 below.

[31]Such rights *are* given in reality. The Māori, a people indigenous to New Zealand, may serve as a point in case. New Zealand's national parliament reserves a number of designated 'Māori seats' for representatives elected by voters of Māori descent (see, e.g., Geddis, 2006).

[32]This can be seen by noting that the only critical families containing only non-negated properties are those containing exactly $k'' + 1$ of them; i.e., $\mathcal{G} = \{P_{k_1}, \ldots, P_{k_{k''+1}}\}$ for some pairwise distinct $k_1, \ldots, k_{k''+1} \in \{1, \ldots, K\}$. As $\mathcal{R}$ grants rights for at most $m < k'' + 1$ properties, (IPC) holds trivially.

more generally, unless $\mathcal{R}$ is trivial – this requires decisions on the other candidates to be made *depending* on whether said rights have been exercised. For example, in the simple case when $m = k''$ minorities elect 'their' candidate to the committee, all remaining candidates have to be declined irrespective of how many votes they receive. Thus, $\mathcal{R}$ is inconsistent with monotone independent aggregation, i.e., not exhaustibly consistent.[33] ■

4 Consistent Rights in Voting by Properties

On property spaces, an aggregation function $f : X^n \to X$ is independent if and only if issues are decided separately and independently of each other to the effect that the collective decision on some issue can change only if some individual changed her vote on it. Once independence is enacted, monotonicity corresponds to the natural requirement that increased support for some property cannot lead to its complement being accepted when it wasn't before. On property spaces, imposing both independence and monotonicity is equivalent to the following condition.

MONOTONE INDEPENDENCE. $f : X^n \to X$ is monotone independent if and only if for all $P \in \mathcal{P}$ and all $\boldsymbol{x} = (x_1, \ldots, x_n), \boldsymbol{y} = (y_1, \ldots, y_n) \in X^n$:

$$[\forall i \in N : x_i \in P \implies y_i \in P] \implies [f(\boldsymbol{x}) \in P \implies f(\boldsymbol{y}) \in P]\,. \tag{MI}$$

Monotone independence plays a crucial role in abstract aggregation theory as it allows for a unified characterization of all onto aggregators as *voting by properties* (Nehring and Puppe, 2010).[34] We recall this result as Fact 2 below.

4.1 Monotone Independent Aggregation as Voting by Properties

A rights system $\mathcal{R} : \mathcal{P} \rightrightarrows 2^X \backslash \{\emptyset\}$ is *exhaustive* if and only if, for all $P \in \mathcal{P}$, either $G \in \mathcal{R}(P)$ or $N \backslash G \in \mathcal{R}(P^c)$. If and only if $\mathcal{R}'$ is some rights system such that, for all $P \in \mathcal{P}$, $\mathcal{R}(P) \subseteq \mathcal{R}'(P)$, we say that $\mathcal{R}'$ *extends* $\mathcal{R}$. Thus, $\mathcal{R}'$ extends $\mathcal{R}$ if and only if it grants all rights in $\mathcal{R}$ and possibly more. $\mathcal{R}'$ is *monotone* if and only if it is equal to its (property-wise) monotone closure, $P \mapsto \overline{\mathcal{R}'}(P) = \{G \subseteq N : G \supseteq G' \text{ for some } G' \in \mathcal{R}'(P)\}$, i.e., if and only if $\mathcal{R}' = \overline{\mathcal{R}'}$.

[33] The non-trivial critical families are those described in footnote 32 as well as those containing exactly $K - k' + 1$ negated properties (i.e., $\mathcal{G} = \{P^c_{k_1}, \ldots, P^c_{k_{K-k'+1}}\}$ for some pairwise distinct $k_1, \ldots, k_{K-k'+1} \in \{1, \ldots, K\}$). As a result, $(X_{(K;k',k'')}, \mathcal{P}_{(K;k',k'')})$ is totally blocked (see also Nehring and Puppe, 2010). Inconsistency with monotone independent aggregation follows from Proposition 3 below.

[34] Note that N&P refer to *voting by properties* as 'voting by issues' instead; cf. footnote 24.

We note that every consistent exhaustive rights system $\mathcal{R}'$ is monotone.[35]

Voting by Properties. Given some exhaustive rights system $\mathcal{R}'$, we define the correspondence $F_{\mathcal{R}'} : X^n \rightrightarrows X$ by

$$(x_1, \ldots, x_n) \mapsto F_{\mathcal{R}'}(x_1, \ldots, x_n) = \bigcap \{P \in \mathcal{P} : \{i \in N : x_i \in P\} \in \mathcal{R}'(P)\}.$$

We refer to $F_{\mathcal{R}'}$ as voting by properties (induced by the exhaustive rights system $\mathcal{R}'$).

Thus, separately for each property $P \in \mathcal{P}$, $F_{\mathcal{R}'}$ accepts P iff the groups of individuals voting for P have a right to it. As shown in Nehring and Puppe (2010) – who refer to exhaustive rights systems as 'structures of winning coalitions' – $\mathcal{F}_{\mathcal{R}'}$ is a monotone independent aggregation function if and only if $\mathcal{R}'$ satisfies (IPC). Vice versa, every monotone independent aggregation function is voting by properties induced by some consistent exhaustive $\mathcal{R}'$.

Fact 2. ***(see Nehring & Puppe, 2010, Proposition 2.1)*** *An onto $f : X^n \to X$ satisfies* (MI) *if and only if it is voting by properties $F_{\mathcal{R}'}$ induced by some exhaustive $\mathcal{R}' : \mathcal{P} \rightrightarrows 2^N \backslash \{\emptyset\}$ satisfying* (IPC)*.*[36]

4.2 Rights in Voting by Properties

Before we return to the question of whether some given (possibly *non-exhaustive*) rights system $\mathcal{R}$ can be respected in monotone independent aggregation, we reappraise the notion of a right in the context of voting by properties. If and only if $\mathcal{R}$ is respected in voting by properties, there exists some consistent exhaustive $\mathcal{R}'$ such that $F_{\mathcal{R}'}$ respects $\mathcal{R}$. By (R) and the definition of $F_{\mathcal{R}'}$, this is the case if and only if, for all $P \in \mathcal{P}$, $G \in \mathcal{R}(P) \implies (\forall G' \subseteq N, G' \supseteq G : G' \in \mathcal{R}'(P))$. As $\mathcal{R}'$ is necessarily monotone, this reduces to the requirement that it extend $\mathcal{R}$.

Fact 3. *Let $\mathcal{R}$ be a rights system. There exists some onto $f : X^n \to X$ satisfying* (MI) *and* (R) *if and only if there exists some consistent exhaustive rights system $\mathcal{R}'$ such that for all $P \in \mathcal{P}$:*

$$\mathcal{R}(P) \subseteq \mathcal{R}'(P). \tag{R*}$$

Consequently, $\mathcal{R}$ is consistent with monotone independent aggregation if and only if it can be consistently *extended* to some exhaustive rights system $\mathcal{R}'$. In this case, we say that $\mathcal{R}$ is *exhaustibly consistent*. As Theorem 2 below shows, $\mathcal{R}$ is exhaustibly consistent if and

[35] Indeed, suppose that there exist some $P \in \mathcal{P}$ and some $N \supseteq G' \supsetneq G$ such that $G \in \mathcal{R}(P)$, $G' \notin \mathcal{R}(P)$. As $\mathcal{R}$ is exhaustive, $\mathcal{R}(P^c) \ni N \backslash G' \subseteq N \backslash G$, in violation of (IPC).

[36] Nehring and Puppe derive the result for monotone independent and unanimous $f : X^n \to X$. In general, unanimity implies ontoness but not vice versa. However, in the presence of monotonicity, both are equivalent.

	u	v	$u \wedge v$
voter no. 1	$\boxed{\checkmark}$	$\boxed{\checkmark}$	$\checkmark$
voter no. 2	$\boxed{\checkmark}$	$\times$	$\boxed{\times}$
voter no. 3	$\times$	$\boxed{\checkmark}$	$\boxed{\times}$
	$\checkmark$	$\checkmark$	$\times$

Table 3: A discursive dilemma.

only if it satisfies an intersection property in the spirit of (IPC) for a generalized concept of criticality. To gain intuition for why (IPC) alone is insufficient to guarantee existence of a consistent exhaustive extension, we note the following fact about consistent exhaustive rights.

Fact 4. *Let $\mathcal{R}'$ be an exhaustive rights system satisfying* (IPC)*. If $P, Q_1, \ldots, Q_r \in \mathcal{P}$ are such that $\{Q_1, \ldots, Q_r\} \vdash P$, then*

$$G_1 \in \mathcal{R}'(Q_1), \ldots, G_r \in \mathcal{R}'(Q_r) \implies \bigcap_{k=1,\ldots,r} G_k \in \mathcal{R}'(P).$$

In words, every consistent exhaustive rights system $\mathcal{R}'$ is intersection-closed under minimal entailment. When we ask whether rights system $\mathcal{R}$ can be consistently extended to some exhaustive $\mathcal{R}'$, we have to keep in mind the restrictions which are implicitly put on such extensions by way of Fact 4. We illustrate this point for majority rights in judgment aggregation.

Example 7. Discursive Dilemma. Suppose that some committee (of odd size) has to reach collective judgments on a conjunctive agenda, where the conclusion $c \leftrightarrow u \wedge v$ is endorsed if and only if the premises u and v are. Here, the *discursive dilemma* (Pettit, 2001) consists in the fact that majority voting on the premises is inconsistent with the majority judgment on the conclusion in general. And, vice versa, direct voting on the conclusion is incompatible with majority judgments on the premises. In other words, granting majority rights both on the premises and the conclusion is inconsistent. See Table 3 for an example with three voters. Boxes depict rights held by the different majorities od voters: $\{1, 2\}$ (solid), $\{1, 3\}$ (dashed) and $\{2, 3\}$ (dotted).[37]

How about majority rights on the premises alone? This *is* a consistent assignment of rights seeing that the conclusion can simply be made depending on majority judgments on

[37]The original example known as the *doctrinal paradox* (Kornhauser and Sager, 1986, 1993) refers to a court of three judges assessing individually whether a defendant owes damages to a plaintiff (the conclusion) by evaluating whether the contract was valid and whether the defendant was in breach of it (the two premises).

the premises (*premise based procedure*). Yet, it is not exhaustibly consistent: majority rights on the premises are inconsistent with *any* monotone independent method of voting on the conclusion.[38] Indeed, consider $(X_{Y_7}, \mathcal{P}_{Y_7})$ for $Y_7 = \{u, \neg u, v, \neg v, u \wedge v, \neg(u \wedge v)\}$ (cf. Example 5). Let $\mathcal{R}_7(P) = \{G \subseteq N : |G| > n/2\}$ for $P = P_u, P_{\neg u}, P_v, P_{\neg v}$, and $\mathcal{R}(P) = \emptyset$ for $P = P_{u\wedge v}, P_{\neg(u\wedge v)}$. Suppose there exists some consistent exhaustive extension $\mathcal{R}'_7$. If $i \in N$, there exist non-empty and disjoint $G, G' \subseteq N \backslash \{i\}$ such that $|G \cup \{i\}|, |G' \cup \{i\}| > n/2$. By way of Fact 4, $\{P_u, P_v\} \vdash P_{u\wedge v}$ implies that $(G \cup \{i\}) \cap (G' \cup \{i\}) = \{i\} \in \mathcal{R}'_7(P_{u\wedge v})$. Thus, $\mathcal{R}'_7$ violates (IPC) over the critical family $\{P_{\neg u}, P_{u\wedge v}\}$ (e.g., $(G \cup G') \cap \{i\} = \emptyset$), a contradiction.[39] ■

4.3 Minimal Entailment and Almost Critical Collections

As in Example 7 above, a violation of (IPC) for every consistent exhaustive extension can be *implicit* in $\mathcal{R}$ when the intersection property fails to hold over a collection of properties which are *almost* critical to the effect that they imply some critical family by minimal entailment. However, such violations might appear only after taking into account entailments at higher orders. Also, some properties may be simultaneously involved in different entailments. Consequently, violations of (IPC) can be jointly implied by *multisets* over $\mathcal{P}$.

A multiset is a generalization of a set which allows for members to appear any finite number of times. For example, if $P, Q \in \mathcal{P}$ then $\{P, Q\}$, $\{P, P, Q\}$ and $\{Q, Q, Q\}$ are multisets over $\mathcal{P}$.[40] We refer to any non-empty multiset over $\mathcal{P}$ as a *collection* (of properties).

Collection. We define $\mathbb{C} = \{\mathcal{C} : \mathcal{C} \text{ is a non-empty multiset over } \mathcal{P}\}$ and call every $\mathcal{C} \in \mathbb{C}$ a *collection* (of properties) from $\mathcal{P}$.

As $\mathcal{P}$ is finite, every $\mathcal{C} \in \mathbb{C}$ is finite.[41] Thus, we can write $\mathcal{C} = \{P_1, \ldots, P_r\}$ for some $P_1, \ldots, P_r \in \mathcal{P}$, where, possibly, $P_k = P_l$ for $k \neq l$. The concept of a collection generalizes that of a family: $\mathbb{F} \subseteq \mathbb{C}$. We define a union-operator $\sqcup$ on $\mathbb{C}$ as follows. Let $\mathcal{C} = \{P_1, \ldots, P_r\}$,

[38] Note that, given majority rights on the premises, an aggregation function is fully monotone independent if and only if it is monotone independent on the conclusion.

[39] Too see that such G, G' exist, we simply split up $N \backslash \{i\}$ in two groups of equal size. For n even, every two-member set can be obtained as the intersection of two strict majorities. A parallel argument, combined with the fact that $\{P_{u\wedge v}\} \vdash P_u, P_v$ yields a violation of (IPC) over the critical family $\{P_u, P_v, P_{\neg(u\wedge v)}\}$ provided that $n \geq 4$. If $n = 2$, majority rights are simply unanimity rights and thus respected by any *unanimity rule* (see Section 4.5.2 for a definition).

[40] Formally, a multiset over some universe U is a mapping $M : U \to \mathbb{N}_0$ which assigns to every member of $u \in U$ a *multiplicity* $M(u)$. A multiset is non-empty if $supp(M) = \{u \in U : M(u) > 0\} \neq \emptyset$. To facilitate the exposition we will stick to the informal notation introduced above. For example, if $U = \{u, u', u''\}$ we will write $M = \{u, u', u'\}$ instead of $M : U \to \mathbb{N}_0$, $M(u) = 1, M(u') = 2, M(u'') = 0$. Moreover, if $M, M' : U \to \mathbb{N}_0$ are such that for all $u \in U : M(u) \leq M'(u)$, we write $M \subseteq M'$.

[41] A multiset M is finite if $supp(M)$ is finite. In this case, we can define $|M| = \sum_{u \in supp(M)} M(u)$. Over a finite universe, every multiset is finite.

$\mathcal{C}' = \{Q_1, \ldots, Q_{r'}\} \in \mathbb{C}$. Then $\mathcal{C} \sqcup \mathcal{C}' = \{P_1, \ldots, P_r, Q_1, \ldots, Q_{r'}\}$. For $s \geq 3$ and $\mathcal{C}_1, \ldots, \mathcal{C}_s \in \mathbb{C}$, we define $\bigsqcup_{l=1,\ldots,s} \mathcal{C}_l$ inductively based on the binary case. We say that $\{\mathcal{C}_1, \ldots, \mathcal{C}_s\}$ is a partition of $\mathcal{C}$ if and only if $\mathcal{C} = \bigsqcup_{l=1,\ldots,s} \mathcal{C}_l$. To formalize our arguments from above, we introduce a generalized minimal entailment relation on $\mathbb{C}$.

Minimal Entailment II. Let $\mathcal{C}, \mathcal{C}' = \{P_1, \ldots, P_r\} \in \mathbb{C}$. We define $\mathcal{C} \Vdash \mathcal{C}'$ if and only if, for all $k = 1, \ldots, r$, there exists some $\mathcal{F}_k \in \mathbb{F}$ such that $\mathcal{F}_k \vdash P_k$ and $\mathcal{C} = \bigsqcup_{k=1,\ldots,r} \mathcal{F}_k$. We denote by $\Vdash^\star$ the transitive closure of $\Vdash$.

$\mathcal{C} \in \mathbb{C}$ minimally entails $\mathcal{C}' \in \mathbb{C}$ if and only if (i) every property in $\mathcal{C}'$ is minimally entailed (in the sense of $\vdash$) by some family and (ii) these families form a partition of $\mathcal{C}$. Note that $\Vdash$ generalizes $\vdash$. Indeed, for every $\mathcal{F} \in \mathbb{F}$ and $Q \in \mathcal{P}$, $\mathcal{F} \vdash Q \iff \mathcal{F} \Vdash \{Q\}$. For $j \geq 2$, we write $\mathcal{C} \Vdash^j \mathcal{C}'$ if and only if there exist $\mathcal{C}_1, \ldots, \mathcal{C}_{j-1}$ such that $\mathcal{C} \Vdash \mathcal{C}_{j-1} \Vdash \ldots \Vdash \mathcal{C}_1 \Vdash \mathcal{C}'$ and let $\Vdash^1 = \Vdash$. Note that $\mathcal{C} \Vdash^\star \mathcal{C}' \iff (\exists j \in \mathbb{N} : \mathcal{C} \Vdash^j \mathcal{C}')$.

Almost Critical Collections. Let $j \in \mathbb{N}$. We say that $\mathcal{C} \in \mathbb{C}$ is *almost critical* (*j-critical*) if and only if $\mathcal{C} \Vdash^\star \mathcal{G}$ ($\mathcal{C} \Vdash^j \mathcal{G}$) for some critical $\mathcal{G} \in \mathcal{F}$.

A collection is almost critical if and only if a critical family of properties can be deduced by repeated minimal entailments. As $\Vdash$ is reflexive, every critical family $\mathcal{F} \in \mathbb{F} \subseteq \mathbb{C}$ is almost critical. Moreover, j-criticality implies j'-criticality for any $j' > j$ (see Lemma 1 in Appendix C).

4.4 A Characterization

Reconsider Example 7 from above. We have $\{P_u, P_v\} \vdash P_{u\wedge v}$, thus $\{P_{\neg u}, P_u, P_v\} \Vdash \{P_{\neg u}, P_{u\wedge v}\}$. As the latter is critical, $\{P_{\neg u}, P_u, P_v\}$ is 1-critical; a fortiori, almost critical. The implicit failure of (IPC) over $\{P_{\neg u}, P_{u\wedge v}\}$ (for consistent exhaustive extensions) surfaces as a violation of a corresponding intersection property over the 1-critical family $\{P_{\neg u}, P_u, P_v\}$ for $\mathcal{R}_7$. For example, if $n = 3$, we have $\{1,3\} \cap \{1,2\} \cap \{2,3\} = \emptyset$. More generally, if for some $i \in N$, G, G' are chosen as in Example 7 above, we have: $G \cup \{i\} \in \mathcal{R}_7(P_{\neg u})$, $G' \cup \{i\} \in \mathcal{R}_7(P_u)$ and $G \cup G' \in \mathcal{R}_7(P_v)$ but $(G \cup \{i\}) \cap (G' \cup \{i\}) \cap (G \cup G') = \emptyset$.

The following condition excludes such implied inconsistencies at any order $j \in \mathbb{N}$ of j-criticality. As Theorem 2 below shows, it is necessary and sufficient in order for a consistent exhaustive extension to exist.

Intersection Property Over Almost Critical Collections. A rights system $\mathcal{R}$ satisfies the *Intersection Property over Almost Critical Collections* (IPAC) on $(X, \mathcal{P})$ if and only if for every almost critical collection $\mathcal{C} = \{P_1, \ldots, P_r\} \in \mathbb{C}$:

$$G_1 \in \mathcal{R}(P_1) \cup \{N\}, \ldots, G_r \in \mathcal{R}(P_r) \cup \{N\} \implies \bigcap_{k=1}^{r} G_k \neq \emptyset. \tag{IPAC}$$

In analogy to (IPC), (IPAC) demands that every collection of groups holding rights to an *almost* critical collection of properties must intersect to at least one common member. Note how (IPAC) takes account of the fact that every onto and monotone independent aggregation function is property-wise unanimous (i.e., $N \in \bigcap_{P\in\mathcal{P}} \mathcal{R}'(P)$ for every consistent exhaustive extension $\mathcal{R}'$). As every critical family is almost critical, (IPAC) is stronger than (IPC).

To vindicate the intuition that (IPAC) exactly excludes those violations of (IPC) that are implicit by minimal entailment for consistent exhaustive extensions, we introduce the minimal-entailment closure of some rights system. For $\mathcal{R} : \mathcal{P} \rightrightarrows 2^N\backslash\{\emptyset\}$, let $P \mapsto C^1(\mathcal{R})(P) = \{G \subseteq N : G = \bigcap_{k=1,\ldots,r} G_k$ for some $G_1 \in \mathcal{R}(Q_1), \ldots, Q_r \in \mathcal{R}(P_r)$ such that $\{Q_1, \ldots, Q_r\} \vdash P\}$ and, inductively, for $j \geq 2$, define $P \mapsto C^j(\mathcal{R})(P) = C^1(C^{j-1}(\mathcal{R}))(P)$. $C^1(\mathcal{R})$ extends $\mathcal{R}$ so as to include groups which are implied to have rights in consistent exhaustive extensions by Fact 4. In other words, $C^1(\mathcal{R})$ closes $\mathcal{R}$ with respect to $\vdash$. In the same fashion $C^j(\mathcal{R})$ closes $C^{j-1}(\mathcal{R})$ inductively for all $j \geq 2$. Lastly, for every $P \in \mathcal{P}$, let $C^\star(\mathcal{R})(P) = \bigcup_{j\in\mathbb{N}} C^j(\mathcal{R})(P)$. Then $C^\star(\mathcal{R})$ is the closure of $\mathcal{R}$ with respect to chains of minimal entailments of any length.[42] As Theorem 2 below shows, requiring $\mathcal{R}$ to satisfy (IPAC) is equivalent to imposing (IPC) on $C^\star(\mathcal{R})$.

Theorem 2. *Let $\mathcal{R} : \mathcal{P} \rightrightarrows 2^N\backslash\{\emptyset\}$. The following are equivalent:*

1. *There exists some onto $f : X^n \to X$ satisfying* (MI) *and* (R).
2. *There exists some exhaustive $\mathcal{R}' : \mathcal{P} \rightrightarrows 2^N\backslash\{\emptyset\}$ satisfying* (IPC) *and* (R*).
3. *$\mathcal{R}$ satisfies* (IPAC).
4. *$C^\star(\mathcal{R})$ satisfies* (IPC).

As Theorem 2 shows (IPAC) is what characterizes consistency with monotone independent aggregation in general. Unlike for the case of exhaustive rights – for which Fact 4 implies that $C^\star(\mathcal{R}) = \mathcal{R}$; hence (IPC) and (IPAC) coincide – the intersection property needs to be extended to hold over all almost critical families. Generalizing N&P, our results not only allow

[42]Note that for all $j \in \mathbb{N}$, $C^j(\mathcal{R})(P) \subseteq C^{j+1}(\mathcal{R})(P) \subseteq 2^N$; hence $\lim_{j\to\infty} C^j(\mathcal{R})(P) = \bigcup_{j\in\mathbb{N}} C^j(\mathcal{R})(P) = C^\star(\mathcal{R})(P)$.

to consider non-exhaustive rights but also to differentiate between simple and exhaustive consistency. While (IPC) is equivalent to (simple) consistency (Theorem 1), (IPAC) characterizes when rights can be respected in voting by properties.

4.5 Possibilities and Impossibilities on Special Domains

To check (IPAC) for some given rights system $\mathcal{R}$, we need to investigate rights holding groups over every almost critical collection. To this end, it is of considerable interest to understand the size and structure of the class of almost critical collections. Not surprisingly, it depends on the very structure of the property space $(X, \mathcal{P})$ under consideration. We focus on some special domains that are of particular interest.

4.5.1 Totally Blocked and Median Spaces

Generally speaking, the size of the class of almost critical collections tends to increase with the complexity of the agenda, i.e., with the degree of inter-dependencies between properties. At the one extreme, when $(X, \mathcal{P})$ is totally blocked, for every collection of properties, there exists some almost critical collection that contains it. Consequently, the only rights systems consistent with voting by properties are the *trivial* ones (satisfying $\bigcap_{G \in \mathcal{R}(P),\, P \in \mathcal{P}} G \neq \emptyset$). On median spaces, at the other extreme, only the critical families are almost critical. Thus, (IPAC) reduces to (IPC). Here, rights are (exhaustibly) consistent if and only if they are independent (or weakly independent, seeing that both notions of independence coincide on median spaces). We summarize in the following proposition.

Proposition 3. *Let $(X, \mathcal{P})$ be a property space.*

1. *If $(X, \mathcal{P})$ is totally blocked, every collection of properties is contained in some almost critical collection. A rights system $\mathcal{R}$ is exhaustibly consistent if and only if it is trivial.*
2. *If $(X, \mathcal{P})$ is median, the almost critical collections are exactly the critical families. If $\mathcal{R}$ is a rights system, the following are equivalent:*
   - *(a) $\mathcal{R}$ is consistent.*
   - *(b) $\mathcal{R}$ is exhaustibly consistent.*
   - *(c) $\mathcal{R}$ is independent.*
   - *(d) $\mathcal{R}$ is weakly independent.*

Example 8. Points on the Real Line. Let $a_1, \ldots, a_r \in \mathbb{R}$, $r \geq 2$ be such that $a_1 < a_2 < \cdots < a_r$. Define $X_8 = \{a_1, \ldots, a_r\}$, and, for each $k = 1, \ldots, r-1$, define $P_k = \{a \in X_8 : a \leq$

$a_k\}$ as well as $P_k^c = X_8 \backslash P_k = \{a \in X_8 : a > a_k\}$. For $\mathcal{P}_8 = \{P_k, P_k^c\}_{k=1,\dots,r-1}$, $(X_8, \mathcal{P}_8)$ is a property space (cf. Nehring and Puppe, 2007, Example 1). As $\mathcal{G} \in \mathbb{F}$ is critical if and only if $\mathcal{G} = \{P_k, P_{k'}^c\}$ for some $k \leq k'$, $(X_8, \mathcal{P}_8)$ is median.

Consider the rights system $\mathcal{R}_8$ such that, for all $k = 1, \dots, r-1$, $\mathcal{R}(P_k) = \mathcal{R}(P_k^c) = \{G \subseteq N : |G| > n/2\}$. As all rights holding groups intersect, $\mathcal{R}_8$ is (weakly) independent. Thus, by Proposition 3, it is consistent with monotone independent aggregation. Indeed, selecting the median vote (respectively, the lower/greater of the two median votes when n is even) defines an onto, monotone independent aggregation function that respects $\mathcal{R}_8$. ■

4.5.2 Semi-Blocked Spaces

On semi-blocked spaces, the conditional entailment relation, $\trianglerighteq^\star$, induces a distinctive structure for the set of properties. We can partition $\mathcal{P}$ into two sets of properties all mutually conditionally entailing each other, $\mathcal{P}^+$ and $\mathcal{P}^-$, such that all $P \in \mathcal{P}^+$ conditionally entail every $Q \in \mathcal{P}^-$ but not vice versa. This first part of Fact 5 below was shown in Nehring (2006) (to keep the exposition self-contained, we give a proof in the Appendix). Figure 1 depicts the resulting dependence structure in a graph such that $P \trianglerighteq^\star Q$ if and only if vertex Q can be reached from vertex P via some directed path. For the second part of Fact 5, we show that the almost critical collections relate to this structure as follows. Every almost critical collection contains at most one element from $\mathcal{P}^-$. Conversely, every property $P \in \mathcal{P}^-$ and every collection from $\mathcal{P}^+$ can be jointly embedded in some almost critical collection.

Fact 5. *Suppose $(X, \mathcal{P})$ is semi-blocked. There exist disjoint $\mathcal{P}^+, \mathcal{P}^- \subsetneq \mathcal{P}$ such that $\mathcal{P}^+ \cup \mathcal{P}^- = \mathcal{P}$ and, for all $P \in \mathcal{P}$:*

1. $P \in \mathcal{P}^+ \iff P^c \in \mathcal{P}^-$
2. $P \in \mathcal{P}^- \implies \forall Q \in \mathcal{P}^- : P \trianglerighteq^\star Q$
3. $P \in \mathcal{P}^+ \implies \forall Q \in \mathcal{P} : P \trianglerighteq^\star Q$.

If $\widetilde{\mathcal{C}} \in \mathbb{C}$ is almost critical on $(X, \mathcal{P})$, then it contains at most one element from $\mathcal{P}^-$. Moreover, for every multiset $\mathcal{C}$ over $\mathcal{P}^+$ and every $P \in \mathcal{P}^-$, there exists some almost critical $\widetilde{\mathcal{C}} \supseteq \mathcal{C} \sqcup \{P\}$.

As an immediate consequence, a rights system $\mathcal{R}$ is consistent with voting by properties on a semi-blocked space $(X, \mathcal{P})$ if and only if for all $G' \in \bigcup_{P \in \mathcal{P}^-} \mathcal{R}(P)$:

$$\left(\bigcap_{G \in \mathcal{R}(P),\, P \in \mathcal{P}^+} G \right) \cap G' \neq \emptyset.^{43}$$

[43] Necessity is obvious. To show sufficiency, we verify (IPAC). Let $\mathcal{C} = \{P_1, \dots, P_r\} \in \mathbb{C}$ be almost critical

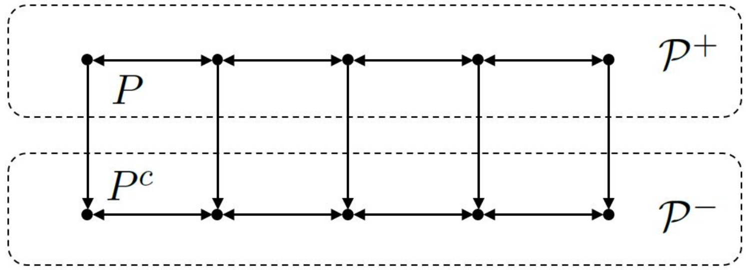

Figure 1: Conditional entailment structure on a semi-blocked space.

A fortiori, $\mathcal{R}$ is consistent with voting by properties only if $\bigcap_{G\in\mathcal{R}(P),\,P\in\mathcal{P}^+} G \neq \emptyset$. If an exhaustibly consistent $\mathcal{R}$ affords rights on $\mathcal{P}^+$, it must be *locally* trivial to the effect that there be some non-empty subgroup of individuals who belong to all rights holding groups on $\mathcal{P}^+$. It follows that exhaustive consistency is impossible if rights are autonomous or if majority rights are granted on some $P \in \mathcal{P}^+$ (see Example 7, ctd.).

Example 7 (ctd.). Note that we have $P_{u\wedge v} \trianglerighteq^\star P_u, P_v \trianglerighteq^\star P_{u\wedge v}$ and $P_{\neg(u\wedge v)} \trianglerighteq^\star P_{\neg u}, P_{\neg v} \trianglerighteq^\star P_{\neg(u\wedge v)}$. Seeing that $P_u \trianglerighteq^\star P_{\neg v}$, $(X_7, \mathcal{P}_7)$ is semi-blocked with $\mathcal{P}_7{}^- = \{P_{\neg u}, P_{\neg v}, P_{\neg(u\wedge v)}\}$. If $\mathcal{R}(\widetilde{P}) \supseteq \{G \subseteq N : |G| > n/2\}$ for some $\widetilde{P} \in \{P_u, P_v, P_{u\wedge v}\}$, then $\bigcap_{G\in\mathcal{R}(P),\,P\in\mathcal{P}^+} G \subseteq \bigcap\{G \subseteq N : |G| > n/2\} = \emptyset$ (provided that $n > 2$, cf. footnote 39). As a result, as soon as majority rights are granted on the conclusion or on either of the premises, rights are not exhaustibly consistent. In light of this, in a more fundamental sense, the discursive dilemma is the fact that majority rights of such kind are inconsistent with monotone independent aggregation. ■

Thus, the room for non-trivial, exhaustibly consistent rights on semi-blocked spaces is limited. Indeed, if voters are additionally assumed to be minimally relevant in the sense that $\mathcal{R}$ guarantees that every voter is pivotal for some issue and profile of votes, then $\mathcal{R}$ is consistent with voting by properties if only if it is the unanimity rule with default $\mathcal{P}^-$; that is, if and only if, for all $\boldsymbol{x} = (x_1, \dots, x_n) \in X^n$ and all $P \in \mathcal{P}$,

$$f(\boldsymbol{x}) \in P \iff \begin{cases} \exists i \in N : x_i \in P & \text{for } P \in \mathcal{P}^- \\ \forall i \in N : x_i \in P & \text{for } P \in \mathcal{P}^+ \end{cases}.$$

Minimal Relevance for (voter) $i \in N$. $\mathcal{R}$ satisfies minimal relevance for (voter)

and $G_1 \in \mathcal{R}(P_1), \dots, G_r \in \mathcal{R}(P_r)$. If $\mathcal{C}$ is a multiset over $\mathcal{P}^+$, we have $\bigcap_{k=1,\dots,r} G_k \supseteq \bigcap_{G\in\mathcal{R}(P),\,P\in\mathcal{P}^+} G \neq \emptyset$. If $\mathcal{C}$ contains exactly one element from $\mathcal{P}^-$, let $P_{k'}$, $k' \in \{1, \dots, r\}$, be that element. We have $\bigcap_{k=1,\dots,r} G_k = \left(\bigcap_{k=\in\{1,\dots,r\},k\neq k'} G_k\right) \cap G_{k'} \supseteq \left(\bigcap_{G\in\mathcal{R}(P),\,P\in\mathcal{P}^+} G\right) \cap G_{k'} \neq \emptyset$. Thus, (IPAC) holds and $\mathcal{R}$ is exaustibly consistent.

$i \in N$ if and only if

$$\exists P_i \in \mathcal{P}\ \exists G_i \subseteq N : i \in G_i \in \mathcal{R}(P_i) \text{ and } N \backslash G_i \cup \{i\} \in \mathcal{R}(P_i^c). \qquad \text{(MR-}i\text{)}$$

Proposition 4. *Let $(X, \mathcal{P})$ be semi-blocked and suppose $\mathcal{R}$ satisfies* (MR-i) *for all $i \in N$. If some onto $f : X^n \to X$ satisfies* (MI) *and* (R)*, then f is the unanimity rule with default $\mathcal{P}^-$; where $\mathcal{P}^-$ is as defined in Fact 5.*

Two interesting examples of semi-blocked spaces are the partial orders (Example 9 below) and the equivalence relations equipped with the natural property structure of pairwise equivalence (Example 10 below). For partial order aggregation, where it can be shown that $\mathcal{P}^-_{PO(A)} = \{P^c_{a \geq b}\}_{a \neq b \in A}$, minimal relevance rights engender widespread incomparability of (social) alternatives (seeing that the default partial order $\bigcap \mathcal{P}^-_{PO(A)}$ is the empty relation). In the context of classification problems (i.e., of aggregation of equivalence relations), Proposition 4 shows that the unique monotone independent operator which guarantees that each characteristic classification is relevant is the *meet operator*.

Example 9. Partial Orders. For a finite set A, let $X_{PO(A)} = \{\geq \subseteq A \times A : \geq \text{ is a partial order}$[44]$\}$. For all $a \neq b \in A$, define $P_{a \geq b} = \{\geq \in X_{PO(A)} : a \geq b\}$. Unlike for the linear orders in Example 4, we do not have $P^c_{a \geq b} = P_{b \geq a}$ in general, seeing that a partial order is not necessarily complete. Let $\mathcal{P}_{PO(A)} = \{P_{a \geq b}, P^c_{a \geq b}\}_{a \neq b \in A}$. $\big(X_{PO(A)}, \mathcal{P}_{PO(A)}\big)$ is a semi-blocked property space with $\mathcal{P}^-_{PO(A)} = \{P^c_{a \geq b}\}_{a \neq b \in A}$. ■

Example 10. Classification Problems. Let A be some set. $X_{Equiv(A)} = \{\sim \subseteq A \times A : \sim \text{ is an equivalence relation}$[45]$\}$. For each $a \neq b \in A$, define $P_{a \sim b} = \{\sim \in X_{Equiv(A)} : a \sim b\}$. When $\mathcal{P}_{Equiv(A)} = \{P_{a \sim b}, P^c_{a \sim b}\}_{a \neq b \in A}$, $\big(X_{Equiv(A)}, \mathcal{P}_{Equiv(A)}\big)$ is a property space. Moreover, it is semi-blocked with $\mathcal{P}^+_{Equiv(A)} = \{P_{a \sim b}\}_{a \neq b \in A}$ and $\mathcal{P}^-_{Equiv(A)} = \{P^c_{a \sim b}\}_{a \neq b \in A}$.

In classification problems, every $i \in \mathbb{N}$ is best understood not as a voter but as an attribute or characteristic (respectively, a conceptual perspective) that classifies a set of objects (cf. Fishburn and Rubinstein, 1986). For example, dogs might be classified according to sex, breed, size etc. The problem of aggregation thus consists in merging these characteristic classifications. By Proposition 4, the only monotone independent aggregator for which each attribute classification is relevant (as defined by (MR-i)) is the meet operator. ■

[44] That is, $\geq$ is reflexive, transitive and antisymmetric.
[45] That is, $\sim$ is reflexive, transitive and symmetric.

5 Conclusion

In this paper, we have provided a novel characterization of consistent rights in terms of semantic interdependencies between properties as rights objects and combinatorial characteristics of the corresponding rights subjects. We have shown that consistent rights can be characterized by means of a simple condition when alternatives differ in terms of properties: whenever rights are given to a combination of properties that is critical (minimally inconsistent), the corresponding rights holding groups must have at least one common member (Intersection Property over Critical Families, IPC). Under property-wise unanimity, rights are consistent only if they are weakly independent (no rights to directly dependent properties for disjoint subgroups).

We have demonstrated that the condition of non-empty intersection must be extended to hold over almost critical (i.e., minimally entailing some critical family) multisets (Intersection Property over Almost Critical Collections, IPAC) to characterize when rights are exhaustibly consistent in the sense of being respected by some onto and monotone independent aggregation function (voting by properties). On totally blocked spaces (where all properties are mutually dependent), rights are exhaustibly consistent if and only if they are trivial (i.e., can be respected by some dictatorship rule); on median spaces (where conditional entailment coincides with subsethood), if and only if they are (weakly) independent. On semi-blocked spaces, minimal relevance rights for all voters pin down monotone independent aggregation functions to a unanimity rule with fixed default.

Our results generalize Nehring and Puppe (2007, 2010) who characterize monotone independent aggregators as voting by properties induced by 'structures of winning coalitions' satisfying IPC. In the interpretation put forth in this paper, a 'structure of winning coalitions' is a rights system which is exhaustive (maximally specified) to the effect that it defines a (monotone independent) aggregation procedure. Thus, our work provides an analysis of rights sui generis by allowing for non-exhaustive rights and by deriving distinct characterizations for rights being respected by some aggregation function (consistent rights: IPC) and those being respected in onto and monotone independent aggregation (exhaustibly consistent rights: IPAC).

Appendix

A Relation to Effectivity Functions and Game Forms

An effectivity function is a mapping $E : 2^N \setminus \{\emptyset\} \rightrightarrows 2^X \setminus \{\emptyset\}$ such that (i) $E(N) = 2^X \setminus \{\emptyset\}$ and (ii) $X \in E(G)$ for all $G \subseteq N$. For every group G, $E(G)$ lists all subsets of outcomes for which G is *effective.* Given some game form $\Gamma = (N, (S_i)_{i\in N}, g)$ – where N is the set of players, S_i is player i's set of strategies and onto $g : \bigtimes_{i\in N} S_i \to X$ maps profiles of strategies to outcomes – we say that $\emptyset \neq G \subseteq N$ is α-effective for $Y \subseteq X$ if there exists some $s_G \in \bigtimes_{i\in G} S_i$ such that for all $s_{N\setminus G} \in \bigtimes_{i\in N\setminus G} S_i$: $g(s_G, s_{N\setminus G}) \in Y$. A game form gives rise to an (α)-effectivity function $E_\Gamma : 2^N \setminus \{\emptyset\} \rightrightarrows 2^X \setminus \{\emptyset\}$ defined by $2^N \setminus \{\emptyset\} \ni G \mapsto E_\Gamma(G) = \{Y \subseteq X : G \text{ is } \alpha\text{-effective for } Y\}$. An important question in the study of effectivity functions is when an effectivity function E can be represented by some game form Γ in the sense that $E_\Gamma = E$. We state a basic result from Peleg (1998).

Fact 6. *Let $E : 2^N \setminus \{\emptyset\} \rightrightarrows 2^X \setminus \{\emptyset\}$ be an effectivity function. Then E can be represented by some game form Γ (i.e., $E_\Gamma = E$) if and only if for all $G, G' \in 2^N \setminus \{\emptyset\}$:*

(super-additive) $(Y \in E(G), Y' \in E(G') \text{ and } G \cap G' = \emptyset) \implies Y \cap Y' \in E(G \cup G')$,

(monotone) $(Y \in E(G), X \supseteq Y' \supseteq Y) \implies Y' \in E(G)$.

As any representable effectivity function E is monotone by Fact 6, it is often insightful to consider its *basis*, the ($\subseteq$-)smallest effectivity function such that its monotone closure equals E. That is, for every $G \in 2^N \setminus \{\emptyset\}$, let $basis(E)(G) = \{Y \subseteq X : Y \in E(G) \text{ and } Y' \in E(G) \text{ for no } Y' \subsetneq Y\}$. $basis(E)$ contains the essential information about a monotone E in the sense that E is representable if and only if $basis(E)$ is super-additive.

There is a close connection between effectivity functions and rights systems as we defined them in this paper. For every $\mathcal{R} : \mathcal{P} \rightrightarrows 2^N \setminus \emptyset$, the inverse correspondence defined by $2^N \setminus \{\emptyset\} \ni G \mapsto \mathcal{R}^{-1}(G) = \{P \in \mathcal{P} : G \in \mathcal{R}(P)\}$ is a mapping $\mathcal{R}^{-1} : 2^N \setminus \{\emptyset\} \rightrightarrows \mathcal{P} \subseteq 2^X \setminus \{\emptyset\}$. Indeed, recall that properties are extensionally defined as subsets of the underlying set X. If $P \in \mathcal{R}^{-1}(G)$, the intuition that G can restrict the eventual choice to come from $P \subseteq X$ parallels that of the effectivity function approach. Due to our distinct setup, however, $\mathcal{R}^{-1}$ is not itself an effectivity function.

There are further important differences. First, conceptionally, our model is *semantic.* That is, we conceptualize rights in terms of subsets which have (respectively, are given) a meaning as and through properties. Second, our definition of respect for rights implies a *conjunctive* notion of rights. When individuals are part of several rights holding groups they can generally exercise these rights simultaneously unless this implies forcing an inconsistent combination of subsets (i.e., properties) at the individual level. In particular, if the same group G has rights to two properties P and Q, then G is effective for $P \cap Q$ under any aggregation function that respects these rights unless the conjunction of these properties is infeasible (i.e, $P \cap Q = \emptyset$).[46] This is in line with the intuition that groups and individuals have several rights which can be exercised at the same time. By contrast, for general effectivity functions, an effectivity set should be thought of as arising from a comprehensive exercise of rights by individuals or groups respectively.

Lastly, we analyze rights under a concept of representation that deviates from the general case in two respects. (i) We consider representation by the restricted class of *voting* game forms. That is, game forms for

[46] In our model of rights, for group G to exercise a right to $P \cap Q$, all of its members $i \in G$ must submit a feasible view (vote) $x_i \in P \cap Q$. If $P \cap Q = \emptyset$ such a right does not exist in the sense that it can never be exercised in an individually feasible way.

which each player's set of strategies is equal to the set of outcomes (alternatives). Indeed, $\mathcal{R}$ is consistent if and only if there exists some onto $f : X^n \to X$ satisfying (R) such that $(N, (S_i = X)_{i\in N}, f)$ defines a (voting) game form (cf. Proposition 5 below). (ii) Our concept of respect for rights is weaker than the game form notion of representability in the sense that if f respects $\mathcal{R}$ then f generally respects rights systems $\mathcal{R}'$ which extend $\mathcal{R}$ (i.e., such that $\mathcal{R}'(P) \supseteq \mathcal{R}(P)$ for all $P \in \mathcal{P}$).[47]

To formally explore the connection to property space rights, we define an analogous notion for effectivity functions. We say that Γ weakly represents E if and only if, for all $G \in 2^N\backslash\{\emptyset\}$, $E(G) \subseteq E_\Gamma(G)$. If some Γ weakly represents E, we say that E is weakly representable. The following *Intersection Property for Effectivity Functions* (IPE) is necessary and sufficient for weak representation by some game form.

Intersection Property for Effectivity Functions. We say that an effectivity function $E : 2^N\backslash\{\emptyset\} \rightrightarrows 2^X\backslash\{\emptyset\}$ satisfies the Intersection Property for Effectivity Functions (IPE) if and only if for all pairwise disjoint $G_1, \ldots, G_r \in 2^N\backslash\{\emptyset\}$:

$$Y_1 \in E(G_1), \ldots, Y_r \in E(G_r) \implies \bigcap_{k=1}^{r} Y_k \neq \emptyset. \tag{IPE}$$

Fact 7. *Let $E : 2^N\backslash\{\emptyset\} \rightrightarrows 2^X\backslash\{\emptyset\}$ be an effectivity function. The following are equivalent:*

1. *E is weakly representable.*
2. *There exists some monotone and super-additive $\overline{E}$ that extends $basis(E)$ (such that, for all $G \in 2^N\backslash\{\emptyset\}$: $basis(E)(G) \subseteq E(G) \subseteq \overline{E}(G)$).*
3. *E satisfies* (IPE).

Proof. Equivalence of 1. and 2. follows immediately from Fact 6. We show equivalence of 2. and 3.

First, suppose E satisfies (IPE). We show that (IPE) guarantees the existence of a smallest monotone and super-additive extension of E which we define, for all $G \in 2^N\backslash\{\emptyset\}$, by $G \mapsto \overline{E}(G) = \{Y \subseteq X : Y \supseteq \bigcap_{k=1,\ldots,r} Y_k$ where $Y_1 \in E(G_1), \ldots, Y_r \in E(G_r)$ for some partition $G_1, \ldots, G_r$ of $G\}$. Clearly, for all $G \in 2^N\backslash\{\emptyset\}$, $basis(E)(G) \subseteq E(G) \subseteq \overline{E}(G)$ and, by (IPE), $\emptyset \notin \overline{E}(G)$. Moreover, for all $G \in 2^N\backslash\{\emptyset\}$, $\overline{E}(G) \supseteq E(G)$ implies $X \in \overline{E}(G)$. In particular, $E(N) = 2^N\backslash\{\emptyset\}$. Thus, $\overline{E}$ is an effectivity function. Monotonicity of $\overline{E}$ is obvious. We verify that it is also super-additive. Consider two disjoint and non-empty $G, G' \subseteq N$ and suppose $Y \in \overline{E}(G)$, $Y' \in \overline{E}(G')$. Then there exist partitions $G_1, \ldots, G_r$ of G, $G'_1, \ldots, G'_{r'}$ of G' and $Y_1 \in E(G_1), \ldots, Y_r \in E(G_r)$, $Y'_1 \in E(G'_1), \ldots, Y'_{r'} \in E(G'_{r'})$ such that $Y \supseteq \bigcap_{k=1,\ldots,r} Y_k$ and $Y' \supseteq \bigcap_{k=1,\ldots,r'} Y'_k$. Clearly, $X \supseteq Y \cap Y' \supseteq \left(\bigcap_{k=1,\ldots,r} Y_k\right) \cap \left(\bigcap_{k=1,\ldots,r'} Y'_k\right)$. As $G \cap G' = \emptyset$, $G_1, \ldots, G_r, G'_1, \ldots, G'_{r'}$ partition $G \cup G'$. Consequently, we have $Y \cap Y' \in \overline{E}(G \cup G')$.

To prove the reverse implication, it suffices to note that the super-additivity property defined above for pairs of disjoint subsets generalizes to countable collections of pairwise disjoint subsets by induction. Thus, if pairwise disjoint $G_1, \ldots, G_r$ give rise to a violation of (IPE), then $\emptyset \in E(G)$ for $G = \bigcup_{k=1,\ldots r} G_k \subseteq N$ by super-additivity, contradicting the definition of an effectivity function. $\square$

When rights are rights to (combinations of) properties (i.e., when $basis(E)$ only contains subsets that are combinations of properties on $(X, \mathcal{P})$), the following proposition shows that the restriction to voting game forms is without loss of generality as long as we consider weak and conjunctive representation of rights. In this

[47] This is particularly obvious in Section 4 where we show that a rights system $\mathcal{R}$ is consistent with voting by properties if and only if there exists some consistent exhaustive rights system that *extends* $\mathcal{R}$.

case, checking whether some effectivity function E is representable by some voting game form is equivalent to analyzing consistency of $\mathcal{R}_E$, the rights system induced by it on $(X, \mathcal{P})$. For all $G \in 2^N \backslash \{\emptyset\}$, let

$$\mathcal{R}_E^{-1}(G) = \bigcup \{\widehat{\mathcal{P}} \subseteq \mathcal{P} : \bigcap \widehat{\mathcal{P}} \in basis(E)(G)\}$$

and define, for each $P \in \mathcal{P}$, $\mathcal{R}_E(P) = \{G \in 2^N \backslash \{\emptyset\} : P \in \mathcal{R}_E^{-1}(G)\}$.[48]

Conjunctive Extension. For two effectivity functions $E, \overline{E}$, we say that $\overline{E}$ *conjunctively extends* E if and only if $\overline{E}$ extends E and we have for all $G_1, \ldots, G_r \in 2^N \backslash \{\emptyset\}$ and all $Y_1 \in E(G_1), \ldots, Y_r \in E(G_r)$:

$$\left(\forall i \in \bigcup_{k=1,\ldots r} G_k : \bigcap_{k:i \in G_k} Y_k \neq \emptyset \right) \implies \bigcap_{k=1,\ldots,r} Y_k \in \overline{E} \left(\bigcup_{k=1,\ldots r} G_k \right).$$

We say that some $Y \subseteq X$ is ($\mathcal{P}$-)convex if and only if there is some $\mathcal{P}_Y \subseteq \mathcal{P}$ such that $Y = \bigcap \mathcal{P}_Y$. Note that, given the convention $\bigcap \emptyset = X$, the comprehensive set X is convex. $E : 2^N \backslash \{\emptyset\} \rightrightarrows 2^X$ is ($\mathcal{P}$-)convex valued if and only if for every $G \in 2^N \backslash \{\emptyset\}$: $Y \in E(G)$ implies that G is ($\mathcal{P}$-)convex.

Proposition 5. *Let $(X, \mathcal{P})$ be a property space and let $E : 2^N \backslash \{\emptyset\} \rightrightarrows 2^X \backslash \{\emptyset\}$ be an effectivity function with $(\mathcal{P}-)$convex valued basis. The following are equivalent:*

1. *E is weakly represented by some voting game form $\Gamma = (N, (S_i = X)_{i \in N}, f)$, where $f : X^n \to X$ respects $\mathcal{R}_E$.*
2. *There exists some monotone and super-additive effectivity function $\overline{E}$ that extends $basis(E)$ conjunctively.*
3. *$\mathcal{R}_E$ is consistent.*

Proof. We prove 1. $\implies$ 2. $\implies$ 3. $\implies$ 1.

1. $\implies$ 2. Let $\overline{E} = E_\Gamma$. By Fact 6, $\overline{E}$ is monotone and super-additive. Clearly, $\overline{E}$ extends E; a fortiori, it extends $basis(E)$. We verify that it does so conjunctively. Let $G_1, \ldots, G_r \in 2^N \backslash \{\emptyset\}$ and $Y_1 \in basis(E)(G_1), \ldots, Y_r \in basis(E)(G_r)$. We have, for some $\mathcal{P}_{Y_1}, \ldots, \mathcal{P}_{Y_r} \subseteq \mathcal{P}$, $Y_1 = \bigcap \mathcal{P}_{Y_1}, \ldots, Y_r = \mathcal{P}_{Y_r}$. Thus, for all $k = 1, \ldots, r$, $\mathcal{P}_{Y_k} \subseteq \mathcal{R}_E^{-1}(G_k) \iff (\forall P \in \mathcal{P}_{Y_k} : G_k \in \mathcal{R}_E(P))$. Suppose that, for all $i \in \bigcup_{k=1,\ldots,r} G_k$, $\bigcap_{k:i \in G_k} Y_k \neq \emptyset$, i.e., there exist $x_i^\star \in \bigcap_{k:i \in G_k} \bigcap \mathcal{P}_{Y_k}$. Then, for all $(x_i^\star)_{i \in N \backslash G} \in X^{(n-|G|)}$, all $k = 1, \ldots, r$ and all $P \in \mathcal{P}_{Y_k}$, $\{i \in N : x_i^\star \in P\} \supseteq G_k$. As f respects $\mathcal{R}_E$, we have $f(x_1^\star, \ldots, x_r^\star) \in \bigcap_{k=1,\ldots,r} \bigcap \mathcal{P}_{Y_k} = \bigcap_{k=1,\ldots,r} Y_k$. Thus, $\bigcap_{k=1,\ldots,r} Y_k \in E_\Gamma \left(\bigcup_{k=1,\ldots,r} G_k \right) = \overline{E} \left(\bigcup_{k=1,\ldots,r} G_k \right)$.

2. $\implies$ 3. We prove the contraposition. Suppose $\mathcal{R}_E$ is inconsistent. By Theorem 1, (IPC) is violated. That is, there exist some critical $\{P_1, \ldots, P_r\} \subseteq \mathcal{P}$ and $G_1 \in \mathcal{R}_E(P_1), \ldots, G_r \in \mathcal{R}_E(P_r)$ such that $\bigcap_{k=1,\ldots,r} G_k = \emptyset$. Thus, for all $i \in \bigcup_{k=1,\ldots,r} G_k$, $\bigcap_{k:i \in G_k} P_k \neq \emptyset$. Moreover, for $k = 1, \ldots, r$, there exist $\mathcal{P}_k \subseteq \mathcal{P}$ such that $P_k \in \mathcal{P}_k$ and $\bigcap \mathcal{P}_k \in basis(E)(G_k)$. Seeing that, for every effectivity function $\overline{E}$, $\bigcap_{k=1,\ldots,r} \bigcap \mathcal{P}_k \subseteq \bigcap_{k=1,\ldots,r} P_k = \emptyset \notin \overline{E} \left(\bigcup_{k=1,\ldots,r} G_k \right)$, a conjunctive extension does not exist.

3. $\implies$ 1. There exists some onto $f : X^n \to X$ that respects $\mathcal{R}_E$. Clearly, $\Gamma = (N, (S_i = X)_{i \in N}, f)$ defines a game form. We show that it weakly represents E. Let $G \in 2^N \backslash \{\emptyset\}$ and $Y \in E(G)$. There exists some $Y' \subseteq Y$ with $Y' \in basis(E)(G)$. Thus, $Y' = \bigcap \mathcal{P}_{Y'}$ for some $\mathcal{P}_{Y'} \subseteq \mathcal{P}$. By definition, we have $\mathcal{P}_{Y'} \subseteq \mathcal{R}_E^{-1}(G)$; i.e., for all $P \in \mathcal{P}_{Y'}$, $G \in \mathcal{R}_E(P)$. For all $i \in G$, let $x_i^\star \in \bigcap \mathcal{P}_{Y'}$. We have for all $(x_i^\star)_{i \in N \backslash G} \in X^{(n-|G|)}$ and all $P \in \mathcal{P}_{Y'}$: $\{i \in N : x_i^\star \in P\} \supseteq G$. Thus, as f respects $\mathcal{R}_E$, $f(x_1^\star, \ldots, x_r^\star) \in P$. This is, for all

[48] Note that $(\mathcal{R}_E)^{-1} = \mathcal{R}_E^{-1}$. That is, $\mathcal{R}_E^{-1}$ is indeed the inverse of R_E thus defined.

$(x_i^\star)_{i\in N\setminus G} \in X^{(n-|G|)}$, $f(x_1^\star,\ldots,x_r^\star) \in \bigcap \mathcal{P}_{Y'} = Y'$. Hence $Y' \in E_\Gamma(G)$. As E_Γ is monotone (cf. Fact 6), we have $Y \supseteq Y' \in E_\Gamma(G)$. □

Finally, we note that for every effectivity function on X, there is some property structure $\mathcal{P}$ such that $basis(E)$ is convex-valued on $(X,\mathcal{P})$. Indeed, for every $x \in X$, let $P_x = \{x\}$. Define $\mathcal{P}_X = \{P_x, P_x^c\}_{x\in X}$ and call $(X,\mathcal{P}_X)$ the *discrete* property space. On $(X,\mathcal{P}_X)$, every subset $Y \subseteq X$ is convex, seeing that $Y = \bigcap_{x\notin Y} P_x$.

B Proofs for Section 3

B.1 Proof of Theorem 1

$\Longleftarrow$: Let (IPC) hold. For each $\boldsymbol{x}(x_1,\ldots,x_n) \in X^n$, define $\mathcal{P}_\mathcal{R}(\boldsymbol{x}) = \{P \in \mathcal{P} : \{i \in N : x_i \in P\} \supseteq G \text{ for some } G \in \mathcal{R}(P)\}$. If, for every $\boldsymbol{x} \in X^n$, we can define $f(\boldsymbol{x}) = y_x$ for some $y_x \in \bigcap \mathcal{P}_\mathcal{R}(\boldsymbol{x})$, f is an aggregation function and respects $\mathcal{R}$ by construction. (By convention, we let for $\mathcal{P}_\mathcal{R}(\boldsymbol{x}) = \emptyset$, $\bigcap \mathcal{P}_\mathcal{R}(\boldsymbol{x}) = X$.) Since, for all $\tilde{x} \in X$, $\tilde{x} \in \bigcap \mathcal{P}_\mathcal{R}((\tilde{x},\ldots,\tilde{x}))$, we can also find a unanimous (a fortiori, onto) f that respects rights in this case.

Assume the aforementioned is not possible, i.e., suppose there exists some $\boldsymbol{x} = (x_1,\ldots,x_n) \in X^n$ s.t. $\mathcal{P}_\mathcal{R}(\boldsymbol{x})$ is inconsistent ($\bigcap \mathcal{P}_\mathcal{R}(\boldsymbol{x}) = \emptyset$ and $\mathcal{P}_\mathcal{R}(\boldsymbol{x}) \neq \emptyset$). Then there exists some critical $\mathcal{G} \subseteq \mathcal{P}_\mathcal{R}(\boldsymbol{x})$. Let $P_1,\ldots,P_r \in \mathcal{P}$ be such that $\mathcal{G} = \{P_1,\ldots,P_r\}$. By definition of $\mathcal{P}_\mathcal{R}(\boldsymbol{x})$, for each $k = 1,\ldots,r$, there exists some $G_k \in \mathcal{R}(P_k)$ s.t. $G_k \subseteq \{i \in N : x_i \in P_k\}$. By (IPC), $\bigcap_{k=1,\ldots,r} G_k \neq \emptyset$. Hence there exists some $i \in N$ such that, for all $k = 1,\ldots,r$, $x_i \in P_k$. Consequently, $x_i \in \cap\mathcal{G} = \emptyset$, a contradiction.

$\Longrightarrow$: Suppose (IPC) does not hold. That is, suppose there exist some critical family $\mathcal{G} = \{P_1,\ldots,P_k\}$ and groups $G_1 \in \mathcal{R}(P_1),\ldots,G_k \in \mathcal{R}(P_k)$ such that $\bigcap_{k=1,\ldots,r} G_k = \emptyset$. For all $i \in \bigcup_{k=1,\ldots,r} G_k$, consider $\mathcal{P}_i^\mathcal{G} = \{P_k \in \mathcal{G} : i \in G_k\}$, the family of all properties P_k in $\mathcal{G}$ for which i is part of group G_k. Evidently, $\mathcal{P}_i^\mathcal{G} \subseteq \mathcal{G}$. As $\bigcap_{k=1,\ldots,r} G_k = \emptyset$, for each $i \in \bigcup_{k=1,\ldots,r} G_k$, there exists some $k_i \in \{1,\ldots,r\}$ such that $P_{k_i} \notin \mathcal{P}_i^\mathcal{G}$. Thus, by criticality of $\mathcal{G}$, all $\mathcal{P}_i^\mathcal{G}$ are consistent. That is, for all $i \in \bigcup_{k=1,\ldots,r} G_k$, there exist $x_i^\star \in \bigcap \mathcal{P}_i^\mathcal{G}$. For all $i \in N\setminus(\bigcup_{k=1,\ldots,r} G_k)$, let $x_i^\star \in X$ be arbitrary. Now suppose some $f : X^n \to X$ respects $\mathcal{R}$. By construction, for all $k = 1,\ldots,r$, $\{i \in N : x_i^\star \in P_k\} \supseteq G_k$. By (R), $f(\boldsymbol{x}^\star) \in \bigcap_{k=1,\ldots,r} P_k = \emptyset$, a contradiction.

B.2 Proof of Proposition 1

There exist distinct $a,b,c,d \in A$ such that a and b, c and d are 1-variants; b and c, a and d are 2−variants. To see this, note that $n \geq 2$ and, for all $i = 1,\ldots,n$, $|A_i| \geq 2$. Consequently, there exist distinct $a_1, a_1' \in A_1$ and distinct $a_2, a_2' \in A_2$. For all $i \in \{0,1,\ldots,n\}\setminus\{1,2\}$, fix $a_i \in A_i$. Then $a = (a_0,a_1,a_2,\ldots,a_n)$, $b = (a_0,a_1',a_2,\ldots,a_n)$, $c = (a_0,a_1',a_2',\ldots,a_n)$ and $d = (a_0,a_1,a_2',\ldots,a_n)$ is a possible choice.

By assumption, $\{1\} \in \mathcal{R}(P_{a>b}) \cap \mathcal{R}(P_{c>d})$ and $\{2\} \in \mathcal{R}(P_{b>c}) \cap \mathcal{R}(P_{d>a})$. As $\{P_{a>b}, P_{b>c}, P_{c>d}, P_{d>a}\}$ is critical, $\{1\} \cap \{2\} = \emptyset$ implies a violation of (IPC). By Theorem 1, $\mathcal{R}$ is inconsistent.

B.3 Proof of Fact 1

Assume that $\mathcal{R}$ is weakly independent. Suppose there exist distinct $P,Q \in \mathcal{P}$ and disjoint $G,G' \subseteq N$ such that $G \in \mathcal{R}(P) \cap \mathcal{R}(P^c)$ and $G' \in \mathcal{R}(Q) \cap \mathcal{R}(Q^c)$. As $(X,\mathcal{P})$ is connected, the issues $\{P,P^c\}$ and $\{Q,Q^c\}$ are directly dependent. Thus, there exist directly dependent $\widehat{P} \in \{P,P^c\}$, $\widehat{Q} \in \{Q,Q^c\}$, contradicting weak independence. Consequently, $\mathcal{R}$ is not autonomous.

B.4 Proof of Proposition 2, Corollaries 1 and 2

B.4.1 Proposition 2

If $\mathcal{R}$ is not weakly independent, there exist disjoint $G, G' \subseteq N$ and directly dependent $P, Q \in \mathcal{P}$ such that $G \in \mathcal{R}(P)$ and $G' \in \mathcal{R}(Q)$. By direct dependence, $\{P, Q\} \subseteq \mathcal{G}$ for some critical $\mathcal{G} \subseteq \mathcal{P}$. Letting, for all $\widehat{P} \in \mathcal{G} \backslash \{P, Q\}$, $G_{\widehat{P}} = N \in \mathcal{R}(\widehat{P})$, we have $G \cap G' \cap \left(\bigcap_{\widehat{P} \in \mathcal{G} \backslash \{P,Q\}} G_{\widehat{P}}\right) = G \cap G' \cap N = \emptyset$, in violation of (IPC). Thus, $\mathcal{R}$ is not consistent.

B.4.2 Corollary 1

We have $\{P_{c>d}, P_{d>c}\} = \{P_{c>d}, P^c_{c>d}\}$ and $\{P_{c'>d'}, P_{d'>c'}\} = \{P_{c'>d'}, P^c_{c'>d'}\}$. Thus, $\mathcal{R}$ is autonomous. If $(X_{Lin(A)}, \mathcal{P}_{Lin(A)})$ is connected, we can use Fact 1 and Proposition 2 to prove the claim.

We show that $(X_{Lin(A)}, \mathcal{P}_{Lin(A)})$ is connected. Let $\{P_{a>b}, P_{b>a}\}, \{P_{c>d}, P_{d>c}\} \subseteq \mathcal{P}$ be any pair of issues. We have $a \neq b$ and $c \neq d$. If the issues are identical, they are trivially directly dependent. Suppose the issues are distinct. Then there are two possible cases: either all a, b, c, d are distinct or exactly one pair of elements are equal (if two pairs of elements are equal we have identical issues again).

Case 1: a, b, c, d are distinct. The family $\{P_{a>b}, P_{b>c}, P_{c>d}, P_{d>a}\}$ is critical; hence $\{P_{a>b}, P_{b>a}\}$ and $\{P_{c>d}, P_{d>c}\}$ are directly dependent.

Case 2: One pair of elements is equal. W.l.o.g., assume that $b = c$ (the proof for $b = d$, $a = c$ and $a = d$ is analogous). Then the family $\{P_{a>b}, P_{b>d}, P_{d>a}\} = \{P_{a>b}, P_{c>d}, P_{d>a}\}$ is critical, and thus, $\{P_{a>b}, P_{b>a}\}$ and $\{P_{c>d}, P_{d>c}\}$ are directly dependent.

B.4.3 Corollary 2

Seeing that $\{P_p, P_{\neg p}\} = \{P_p, P^c_p\}$ and $\{P_q, P_{\neg q}\} = \{P_q, P^c_q\}$, $\mathcal{R}$ is autonomous. The claim follows by Fact 1 and Proposition 2.

C Some Lemmas

We establish some basic properties of $\Vdash^j$ and $\Vdash^\star$.

Lemma 1. *Let $\mathcal{C}, \mathcal{C}', \mathcal{C}'' \in \mathbb{C}$ and $j, j' \in \mathbb{N}$.*

1. *$\Vdash^j$ and $\Vdash^\star$ are reflexive.*
2. *$(\mathcal{C} \Vdash^j \mathcal{C}', \mathcal{C}' \Vdash^{j'} \mathcal{C}'') \implies \mathcal{C} \Vdash^{(j+j')} \mathcal{C}''$.*
3. *If $j' > j$, then $\mathcal{C} \Vdash^j \mathcal{C}' \implies \mathcal{C} \Vdash^{j'} \mathcal{C}'$. Thus, if $\mathcal{C}$ is j-critical, it is j'-critical.*
4. *$\mathcal{C} \Vdash^j \mathcal{C}' = \{P_1, \ldots, P_r\} \iff (\forall k = 1, \ldots, r\, \exists \mathcal{C}_k \in \mathbb{C} : \mathcal{C}_k \Vdash^j \{P_k\}$ and $\mathcal{C} = \bigsqcup_{k=1,\ldots,r} \mathcal{C}_k)$,*
   $\mathcal{C} \Vdash^\star \mathcal{C}' = \{P_1, \ldots, P_r\} \iff (\forall k = 1, \ldots, r\, \exists \mathcal{C}_k \in \mathbb{C} : \mathcal{C}_k \Vdash^\star \{P_k\}$ and $\mathcal{C} = \bigsqcup_{k=1,\ldots,r} \mathcal{C}_k)$

Proof. 1. Let $\mathcal{C} = \{P_1, \ldots, P_r\} \in \mathbb{C}$. As $\{P, P^c\}$ is critical for all $P \in \mathcal{P}$, for $k = 1, \ldots, r$, $\mathbb{F} \ni \{P_k\} \vdash P_k$ and $\sqcup_{k=1,\ldots,r}\{P_k\} = \mathcal{C}$. Consequently, $\mathcal{C} \Vdash \mathcal{C}$. A fortiori, $\mathcal{C} \Vdash^\star \mathcal{C}$. Consider $j \geq 2$. For $l = 1, \ldots, j-1$, set $\mathcal{C}_l = \mathcal{C}$. We have: $\mathcal{C} \Vdash \mathcal{C}_{j-1} \Vdash \ldots \Vdash \mathcal{C}_1 \Vdash \mathcal{C}$, i.e., $\mathcal{C} \Vdash^j \mathcal{C}$. As $\mathcal{C}$ was chosen arbitrarily, the assertions follow.

2. Suppose $j, j' \geq 2$. There exist $\mathcal{C}_1, \ldots, \mathcal{C}_{j-1} \in \mathbb{C}$ such that $\mathcal{C} \Vdash \mathcal{C}_{j-1} \Vdash \ldots \Vdash \mathcal{C}_1 \Vdash \mathcal{C}'$ and $\mathcal{C}'_1, \ldots, \mathcal{C}'_{j'-1} \in \mathbb{C}$ such that $\mathcal{C}' \Vdash \mathcal{C}'_{j'-1} \Vdash \ldots \Vdash \mathcal{C}'_1 \Vdash \mathcal{C}''$. Letting $\mathcal{C}'_{j'} = \mathcal{C}', \mathcal{C}'_{j'+1} = \mathcal{C}_1, \ldots, \mathcal{C}'_{j'+j-1} = \mathcal{C}_{j-1}$ we have $\mathcal{C} \Vdash \mathcal{C}'_{j+j'-1} \Vdash \ldots \Vdash \mathcal{C}'_1 \Vdash \mathcal{C}''$. Hence $\mathcal{C} \Vdash^{(j+j')} \mathcal{C}''$. The proof is analogous when $j = 1$ or $j' = 1$.

3. Note that $j' - j \in \mathbb{N}$. By part 1, $\mathcal{C} \Vdash^{(j'-j)} \mathcal{C}$. By part 2, $\mathcal{C} \Vdash^{(j'-j)} \mathcal{C}, \mathcal{C} \Vdash^{j} \implies \mathcal{C} \Vdash^{j'} \mathcal{C}$.

4. For $\Vdash^j$, we prove the claim by induction over $j \in \mathbb{N}$.

   Induction basis: $j = 1$. The claim holds by definition of $\Vdash^1 = \Vdash$.

   Induction step: $j \rightsquigarrow j+1$. Assume the assertion holds for $\Vdash^j$ and consider $\mathcal{C}, \mathcal{C}' \in \mathbb{C}$ such that $\mathcal{C} \Vdash^{(j+1)} \mathcal{C}'$. There exist $\mathcal{C}_1, \ldots, \mathcal{C}_j \in \mathbb{C}$ such that $\mathcal{C} \Vdash \mathcal{C}_j \Vdash \ldots \Vdash \mathcal{C}_1 \Vdash \mathcal{C}'$. That is, $\mathcal{C} \Vdash^j \mathcal{C}_1 \Vdash \mathcal{C}' = \{P_1, \ldots, P_r\}$. By definition of $\Vdash$, there exist $\mathcal{F}_1, \ldots, \mathcal{F}_r \in \mathbb{F}$ such that, for all $k = 1, \ldots, r$, $\mathcal{F}_k \Vdash \{P_k\}$ and $\bigsqcup_{k=1,\ldots,r} \mathcal{F}_k = \mathcal{C}_1$. Let $s_1, \ldots, s_r \in \mathbb{N}$ and $P^1_1, \ldots, P^1_{s_1}, \ldots, P^r_1, \ldots, P^r_{s_r} \in \mathcal{P}$ be such that $\mathcal{F}_1 = \{P^1_1, \ldots, P^1_{s_1}\}, \ldots, \mathcal{F}_r = \{P^r_1, \ldots, P^r_{s_r}\}$. By (the inductive) assumption, there are $\mathcal{C}^1_1, \ldots, \mathcal{C}^1_{s_1}, \ldots, \mathcal{C}^r_1, \ldots, \mathcal{C}^r_{s_r} \in \mathbb{C}$ such that, for $k = 1, \ldots, r$, $l = 1, \ldots, s_k$, $\mathcal{C}^k_l \Vdash^j \{P^k_l\}$ and $\mathcal{C} = \bigsqcup_{k=1,\ldots,r, l=1,\ldots,s_k} \mathcal{C}^k_l$. Letting $\mathcal{C}^1 = \bigsqcup_{l=1,\ldots,s_1} \mathcal{C}^1_l, \ldots, \mathcal{C}^r = \bigsqcup_{l=1,\ldots,s_r} \mathcal{C}^r_l$, we have $\mathcal{C} = \bigsqcup_{k=1,\ldots,r,l=1,\ldots,s_k} \mathcal{C}^k_l = \bigsqcup_{k=1,\ldots,r} \bigsqcup_{l=1,\ldots,s_k} \mathcal{C}^k_l = \bigsqcup_{k=1,\ldots,r} \mathcal{C}^k$. Additionally, for $k = 1, \ldots, r$, $\mathcal{C}^k \Vdash^j \mathcal{F}_k \Vdash \{P_k\}$ (by Lemma 2, part 3 below); hence $\mathcal{C}^k \Vdash^{(j+1)} \{P_k\}$ (by part 2).

   If $\mathcal{C} \Vdash^\star \mathcal{C}'$ we have $\mathcal{C} \Vdash^{j^\star} \mathcal{C}'$ for some $j^\star \in \mathbb{N}$. There exist $\mathcal{C}_1, \ldots, \mathcal{C}_r \in \mathbb{C}$ such that $\mathcal{C} = \sqcup_{k=1,\ldots,r} \mathcal{C}_k$ and, for all $k = 1, \ldots, r$, $\mathcal{C}_k \Vdash^{j^\star} \{P_k\}$; hence $\mathcal{C}_k \Vdash^\star \{P_k\}$.

□

Lemma 2. *Let $j, r \in \mathbb{N}, r \geq 2$ and $\mathcal{C}_1, \ldots, \mathcal{C}_r, \mathcal{C}'_1, \ldots, \mathcal{C}'_r \in \mathbb{C}$.*

1. *If $\mathcal{C}_1 \Vdash^j \mathcal{C}'_1$ and $\mathcal{C}_2 \Vdash^j \mathcal{C}'_2$, then $(\mathcal{C}_1 \sqcup \mathcal{C}_2) \Vdash^j (\mathcal{C}'_1 \sqcup \mathcal{C}'_2)$.*
2. *If $\mathcal{C}_1 \Vdash^\star \mathcal{C}'_1$ and $\mathcal{C}_2 \Vdash^\star \mathcal{C}'_2$, then $(\mathcal{C}_1 \sqcup \mathcal{C}_2) \Vdash^\star (\mathcal{C}'_1 \sqcup \mathcal{C}'_2)$.*
3. *If $\mathcal{C}_1 \Vdash^j \mathcal{C}'_1, \ldots, \mathcal{C}_r \Vdash^j \mathcal{C}'_r$, then $\bigsqcup_{k=1,\ldots,r} \mathcal{C}_k \Vdash^j \bigsqcup_{k=1,\ldots,r} \mathcal{C}'_k$.*
4. *If $\mathcal{C}_1 \Vdash^\star \mathcal{C}'_1, \ldots, \mathcal{C}_r \Vdash^\star \mathcal{C}'_r$, then $\bigsqcup_{k=1,\ldots,r} \mathcal{C}_k \Vdash^\star \bigsqcup_{k=1,\ldots,r} \mathcal{C}'_k$.*

Proof. 1. We show the claim by induction over $j \in N$.

   Induction basis: $j = 1$. Let $\mathcal{C}'_1 = \{P^1_1, \ldots, P^1_{r_1}\}, \mathcal{C}'_2 = \{P^2_1, \ldots, P^2_{r_2}\}$. As $\mathcal{C}_1 \Vdash \mathcal{C}'_1, \mathcal{C}_2 \Vdash \mathcal{C}'_2$, there are $\mathcal{C}^1_1, \ldots, \mathcal{C}^1_{r_1}, \mathcal{C}^2_1, \ldots, \mathcal{C}^2_{r_2} \in \mathbb{C}$ such that: for $k = 1, \ldots, r_1$, $\mathcal{C}^1_k \Vdash \{P^1_k\}$ and $\mathcal{C}_1 = \bigsqcup_{k=1,\ldots,r_1} \mathcal{C}^1_k$; for $k = 1, \ldots, r_2$, $\mathcal{C}^2_k \Vdash \{P^2_k\}$ and $\mathcal{C}_2 = \bigsqcup_{k=1,\ldots,r_2} \mathcal{C}^2_k$. As $(\bigsqcup_{k=1,\ldots,r_1} \mathcal{C}^1_k) \sqcup (\bigsqcup_{k=1,\ldots,r_2} \mathcal{C}^2_k) = \mathcal{C}_1 \sqcup \mathcal{C}_2$, we have $(\mathcal{C}_1 \sqcup \mathcal{C}_2) \Vdash (\mathcal{C}'_1 \sqcup \mathcal{C}'_2)$.

   Induction step: $j \rightsquigarrow j+1$. Assume the claim holds for $\Vdash^j$. As $\mathcal{C}_1 \Vdash^{(j+1)} \mathcal{C}'_1$, $\mathcal{C}_2 \Vdash^{(j+1)} \mathcal{C}'_2$, there are $\mathcal{C}^1_1, \ldots, \mathcal{C}^1_j, \mathcal{C}^2_1, \ldots, \mathcal{C}^2_j \in \mathbb{C}$ such that $\mathcal{C}_1 \Vdash \mathcal{C}^1_j \Vdash \ldots \Vdash \mathcal{C}^1_1 \Vdash \mathcal{C}'_1$ and $\mathcal{C}_2 \Vdash \mathcal{C}^2_j \Vdash \ldots \Vdash \mathcal{C}^2_1 \Vdash \mathcal{C}'_2$. We have $\mathcal{C}^1_j \Vdash^j \mathcal{C}'_1$ and $\mathcal{C}^2_j \Vdash^j \mathcal{C}'_2$. By (the inductive) assumption and the induction basis: $\mathcal{C}_1 \sqcup \mathcal{C}_2 \Vdash \mathcal{C}^1_j \sqcup \mathcal{C}^2_j \Vdash^j \mathcal{C}'_1 \sqcup \mathcal{C}'_2$. Thus, by Lemma 1, part 2, $\mathcal{C}_1 \sqcup \mathcal{C}_2 \Vdash^{(j+1)} \mathcal{C}'_1 \sqcup \mathcal{C}'_2$.

2. There exist $j, j' \in \mathbb{N}$ such that $\mathcal{C}_1 \Vdash^j \mathcal{C}'_1, \mathcal{C}_2 \Vdash^{j'} \mathcal{C}'_2$. W.l.o.g., let $j \geq j'$. If $j = j'$, $\mathcal{C}_2 \Vdash^j \mathcal{C}'_2$. By Lemma 1, part 3, the same holds when $j > j'$. By part 1, we have $(\mathcal{C}_1 \sqcup \mathcal{C}_2) \Vdash^j (\mathcal{C}'_1 \sqcup \mathcal{C}'_2)$. A fortiori, $(\mathcal{C}_1 \sqcup \mathcal{C}_2) \Vdash^\star (\mathcal{C}'_1 \sqcup \mathcal{C}'_2)$.

3. By induction over $r \in \mathbb{N}$.

   Induction basis: $r = 2$. See part 1.

   Induction step: $r \rightsquigarrow r+1$. Assume the the claim holds $r \in \mathbb{N}, r \geq 2$. We have $\bigsqcup_{k=1,\ldots,r+1} \mathcal{C}_k = (\bigsqcup_{k=1,\ldots,r} \mathcal{C}_k) \sqcup \mathcal{C}_{r+1}$. By (the inductive) assumption, $\bigsqcup_{k=1,\ldots,r} \mathcal{C}_k \Vdash^j \bigsqcup_{k=1,\ldots,r} \mathcal{C}'_k$. Thus, by part 1, $\bigsqcup_{k=1,\ldots,r+1} \mathcal{C}_k = (\bigsqcup_{k=1,\ldots,r} \mathcal{C}_k) \sqcup \mathcal{C}_{r+1} \Vdash^j (\bigsqcup_{k=1,\ldots,r} \mathcal{C}'_k) \sqcup \mathcal{C}'_{r+1} = \bigsqcup_{k=1,\ldots,r+1} \mathcal{C}'_k$.

4. By induction over $r \in \mathbb{N}$.

   Induction basis: $r = 2$. See part 2.

   Induction step: $r \rightsquigarrow r+1$. Assume the claim holds for $r \in \mathbb{N}, r \geq 2$. We have $\bigsqcup_{k=1,\ldots,r+1} \mathcal{C}_k = (\bigsqcup_{k=1,\ldots,r} \mathcal{C}_k) \sqcup \mathcal{C}_{r+1}$. By (the inductive) assumption, $\bigsqcup_{k=1,\ldots,r} \mathcal{C}_k \Vdash^\star \bigsqcup_{k=1,\ldots,r} \mathcal{C}'_k$. Thus, by part 2, $\bigsqcup_{k=1,\ldots,r+1} \mathcal{C}_k = (\bigsqcup_{k=1,\ldots,r} \mathcal{C}_k) \sqcup \mathcal{C}_{r+1} \Vdash^\star (\bigsqcup_{k=1,\ldots,r} \mathcal{C}'_k) \sqcup \mathcal{C}'_{r+1} = \bigsqcup_{k=1,\ldots,r+1} \mathcal{C}'_k$.

□

If $\emptyset \neq \mathcal{C} \subseteq \mathcal{C}' \in \mathbb{C}$, w.l.o.g., there exist $P_1, \ldots, P_r, \ldots, P_s \in \mathcal{P}$ (with $1 \leq r \leq s$) such that $\mathcal{C} = \{P_1, \ldots, P_r\}$ and $\mathcal{C}' = \{P_1, \ldots, P_r, \ldots, P_s\}$. We define $\mathcal{C}' \backslash \mathcal{C} = \{P_{r+1}, \ldots, P_s\}$ (hence $\mathcal{C} \backslash \mathcal{C} = \emptyset$). Now for arbitrary $\mathcal{C}, \mathcal{C}' \in \mathbb{C}$, we define $\mathcal{C}' \backslash \emptyset = \mathcal{C}'$ and $\mathcal{C}' \backslash \mathcal{C} = \mathcal{C}' \backslash (\mathcal{C}' \sqcap \mathcal{C})$, where $\sqcap$ is the natural intersection operator on $\mathcal{C}$ (such that, e.g., $\{u, u', u'\} \sqcap \{u', u', u''\} = \{u', u'\}$ on the universe $U = \{u, u', u''\}$).

Lemma 3. *Let $j \in \mathbb{N}$, and $\mathcal{C} = \{P_1, \ldots, P_r\} \in \mathbb{C}$ be j-critical. Then, for $k = 1, \ldots, r$, $\mathcal{C} \backslash \{P_k\} = \{P_1, \ldots, P_{k-1}, P_{k+1}, \ldots, P_r\} \Vdash^{(2j+1)} \{P_k^c\}$. Thus, if $\mathcal{C}$ is almost critical, $\mathcal{C} \backslash \{P_k\} \Vdash^\star \{P_k^c\}$.*

Proof. We prove the assertion by induction over $j \in \mathbb{N}$.

Induction basis: $j = 1$. Note that, if $\mathcal{G} = \{Q_1, \ldots, Q_s\} \in \mathbb{F}$ is critical, we have, for $l = 1, \ldots, s$, $\mathcal{G} \backslash \{Q_l\} \vdash Q_l^c$ or, equivalently, $\mathcal{G} \backslash \{Q_l\} \Vdash \{Q_l^c\}$. Now, as $\mathcal{C}$ is 1-critical, $\mathcal{C} \Vdash \mathcal{G}$ for some critical $\mathcal{G} \in \mathbb{F}$. Let $Q_1, \ldots, Q_s \in \mathcal{P}$ be such that $\mathcal{G} = \{Q_1, \ldots, Q_s\}$. There exist $\mathcal{F}_1, \ldots, \mathcal{F}_s \in \mathbb{F}$ such that: (i) for $l = 1, \ldots, s$, $\mathcal{F}_l \vdash Q_l$ and (ii) $\bigsqcup_{l=1,\ldots,s} \mathcal{F}_l = \mathcal{C}$. By (ii), for every $k = 1, \ldots, r$, there exists some $1 \leq l_k \leq s$ such that $P_k \in \mathcal{F}_{l_k}$. By Lemma 2, part 3, we have $\mathcal{C} \backslash \mathcal{F}_{l_k} = \bigsqcup_{l \in \{1,\ldots,s\} \backslash \{l_k\}} \mathcal{F}_l \Vdash \mathcal{G} \backslash \{Q_{l_k}\}$. By reflexivity (Lemma 1, part 1), $\mathcal{F}_{l_k} \backslash \{P_k\} \Vdash \mathcal{F}_{l_k} \backslash \{P_k\}$. $\mathcal{F}_{l_k} \vdash Q_{l_k}$ implies that $\mathcal{F}_{l_k} \cup \{Q_{l_k}^c\}$ is critical. Thus, using Lemma 2, part 1, we obtain $\mathcal{C} \backslash \{P_k\} = (\mathcal{C} \backslash \mathcal{F}_{l_k}) \sqcup (\mathcal{F}_{l_k} \backslash \{P_k\}) \Vdash (\mathcal{G} \backslash \{Q_{l_k}\}) \sqcup (\mathcal{F}_{l_k} \backslash \{P_k\}) \Vdash \{Q_{l_k}^c\} \sqcup (\mathcal{F}_{l_k} \backslash \{P_k\}) = (\mathcal{F}_{l_k} \cup \{Q_{l_k}^c\}) \backslash \{P_k\} \Vdash \{P_k^c\}$. By Lemma 1, part 2, $\mathcal{C} \backslash \{P_k\} \Vdash^3 \{P_k^c\}$.

Induction step: $j \rightsquigarrow j+1$. Assume the claim holds for $j \in \mathbb{N}$. If $\mathcal{C}$ is $(j+1)$-critical, there exist $\mathcal{C}_1, \ldots, \mathcal{C}_j \in \mathbb{C}$ and some critical $\mathcal{G} \in \mathbb{F}$ such that $\mathcal{C} \Vdash \mathcal{C}_j \Vdash \ldots \Vdash \mathcal{C}_1 \Vdash \mathcal{G}$. Clearly, $\mathcal{C}_j$ is j-critical. Let $Q_1, \ldots, Q_s \in \mathcal{P}$ be such that $\mathcal{C}_j = \{Q_1, \ldots, Q_s\}$. As $\mathcal{C} \Vdash \mathcal{C}_j$, there exist $\mathcal{F}_1, \ldots, \mathcal{F}_s \in \mathbb{F}$ such that: (i) for all $l = 1, \ldots, s$, $\mathcal{F}_l \vdash Q_l$ and (ii) $\bigsqcup_{l=1,\ldots,s} \mathcal{F}_l = \mathcal{C}$. By (ii), there exists some $1 \leq l_k \leq s$ such that $P_k \in \mathcal{F}_{l_k}$. By Lemma 2, part 3, we have $\mathcal{C} \backslash \mathcal{F}_{l_k} = \bigsqcup_{l \in \{1,\ldots,s\} \backslash \{l_k\}} \mathcal{F}_l \Vdash \mathcal{C}_j \backslash \{Q_{l_k}\}$. By (the inductive) assumption $\mathcal{C}_j \backslash \{Q_{l_k}\} \Vdash^{2j+1} \{Q_{l_k}^c\}$. By reflexivity (Lemma 1, part 1), $\mathcal{F}_{l_k} \backslash \{P_k\} \Vdash \mathcal{F}_{l_k} \backslash \{P_k\}$ and $\mathcal{F}_{l_k} \backslash \{P_k\} \Vdash^j \mathcal{F}_{l_k} \backslash \{P_k\}$. As $\mathcal{F}_{l_k} \vdash Q_{l_k}$, $\mathcal{F}_{l_k} \cup \{Q_{l_k}^c\}$ is critical. Thus, using Lemma 2, part 1, we obtain $\mathcal{C} \backslash \{P_k\} = (\mathcal{C} \backslash \mathcal{F}_{l_k}) \sqcup (\mathcal{F}_{l_k} \backslash \{P_k\}) \Vdash (\mathcal{C}_j \backslash \{Q_{l_k}\}) \sqcup (\mathcal{F}_{l_k} \backslash \{P_k\}) \Vdash^{2j+1} \{Q_{l_k}^c\} \sqcup (\mathcal{F}_{l_k} \backslash \{P_k\}) = (\mathcal{F}_{l_k} \cup \{Q_{l_k}^c\}) \backslash \{P_k\} \Vdash \{P_k^c\}$. By Lemma 1, part 2, $\mathcal{C} \backslash \{P_k\} \Vdash^{2j+1+2} \{P_k^c\} = \mathcal{C} \backslash \{P_k\} \Vdash^{2(j+1)+1} \{P_k^c\}$.

Lastly, $\mathcal{C}$ is almost critical if and only if it is j'-critical for some $j' \in \mathbb{N}$. Thus, $\mathcal{C} \backslash \{P_k\} \Vdash^{(2j'+1)} \{P_k^c\}$. A fortiori, $\mathcal{C} \backslash \{P_k\} \Vdash^\star \{P_k^c\}$. □

We recall the definition of minimal entailment closure. For $\mathcal{R} : \mathcal{P} \rightrightarrows 2^N \backslash \{\emptyset\}$, define $P \mapsto C^1(\mathcal{R})(P) = \{G \subseteq N : G = \bigcap_{k=1,\ldots,r} G_k$ for some $G_1 \in \mathcal{R}(Q_1), \ldots, G_r \in \mathcal{R}(Q_r)$ such that $\{Q_1, \ldots, Q_r\} \vdash P\}$. For $j \geq 2$, we define $C^j(\mathcal{R})$ inductively by $P \mapsto C^j(\mathcal{R})(P) = C^1(C^{j-1}(\mathcal{R}))(P)$. Lastly, for every $P \in \mathcal{P}$, $C^\star(\mathcal{R})(P) = \bigcup_{j \in \mathbb{N}} C^j(\mathcal{R})(P)$.

Lemma 4. *Let $\mathcal{R} : \mathcal{P} \rightrightarrows 2^N \backslash \{\emptyset\}$, $P \in \mathcal{P}$ and $j \in \mathbb{N}$.*

1. *$G \in C^j(\mathcal{R})(P)$ if and only if there exist $r \in \mathbb{N}$, $Q_1, \ldots, Q_r \in \mathcal{P}$ and $G_1 \in \mathcal{R}(Q_1), \ldots, G_r \in \mathcal{R}(Q_r)$ such that $\{Q_1, \ldots, Q_r\} \Vdash^j \{P\}$ and $\bigcap_{k=1,\ldots,r} G_k = G$.*

2. *$G \in C^\star(\mathcal{R})(P)$ if and only if there exist $r \in \mathbb{N}$, $Q_1, \ldots, Q_r \in \mathcal{P}$ and $G_1 \in \mathcal{R}(Q_1), \ldots, G_r \in \mathcal{R}(Q_r)$ such that $\{Q_1, \ldots, Q_r\} \Vdash^\star \{P\}$ and $\bigcap_{k=1,\ldots,r} G_k = G$.*

Proof. 1. We show the claim by induction over $j \in \mathbb{N}$.

Induction basis: $j = 1$. The claim holds by definition of $C^1(\mathcal{R})$.

Induction step: $j \rightsquigarrow j+1$. Assume the claim holds for $j \in \mathbb{N}$. We have

$$\begin{aligned}
G \in \mathcal{C}^{(j+1)}(\mathcal{R})(P) &\iff \Big(G \in C^1(C^j(\mathcal{R}))(P)\Big) \\
&\overset{IB}{\iff} (\exists r \in \mathbb{N} \exists Q_1, \ldots, Q_r \in \mathcal{P} \exists G_1 \in \mathcal{C}^j(\mathcal{R})(Q_1), \ldots, G_r \in \mathcal{C}^j(\mathcal{R})(Q_r) : \\
&\qquad G = \bigcap_{k=1,\ldots,r} G_k \text{ and } \{Q_1, \ldots, Q_r\} \Vdash \{P\}) \\
&\overset{IA}{\iff} (\exists r \in \mathbb{N} \exists Q_1, \ldots, Q_r \in \mathcal{P} \exists G_1, \ldots, G_r \in 2^N \backslash \{\emptyset\} : \\
&\qquad G = \bigcap_{k=1,\ldots,r} G_k,\ \{Q_1, \ldots, Q_r\} \Vdash \{P\} \text{ and } \forall k = 1, \ldots, r \\
&\qquad \exists s_1, \ldots, s_k \in \mathbb{N} \exists Q_1^k, \ldots, Q_{s_k}^k \in \mathcal{P} \exists G_1^k \in \mathcal{R}(Q_1^k), \ldots, G_{s_k}^k \in \mathcal{R}(Q_{s_k}^k) : \\
&\qquad G_k = \bigcap_{l=1,\ldots,s_k} G_l^k \text{ and } \{Q_1^k, \ldots, Q_{s_k}^k\} \Vdash^j \{Q_k\}) \\
&\iff (\exists r \in \mathbb{N} \exists s_1, \ldots, s_r \in \mathbb{N} \exists Q_1^1, \ldots, Q_{s_1}^1, \ldots, Q_1^r, \ldots, Q_{s_r}^r \in \mathcal{P} \\
&\qquad \exists G_1^1 \in \mathcal{R}(Q_1^1), \ldots, G_{s_1}^1 \in \mathcal{R}(Q_{s_1}^1), \ldots, G_1^r \in \mathcal{R}(Q_1^r), \ldots, G_{s_r}^r \in \mathcal{R}(Q_{s_r}^r) : \\
&\qquad G = \bigcap_{k=1,\ldots,r} \bigcap_{l=1,\ldots,s_k} G_l^k \text{ and } \{Q_1^1, \ldots, Q_{s_r}^r\} \Vdash^{j+1} \{P\});
\end{aligned}$$

where the last equivalence uses Lemma 2, part 3.

2. We have, for all $P \in \mathcal{P}$:

$$\begin{aligned}
G \in C^\star(\mathcal{R})(P) &\iff \Big(\exists j^\star \in \mathbb{N} : G \in C^{j^\star}(\mathcal{R})(P)\Big) \\
&\overset{1.}{\iff} (\exists j^\star \in \mathbb{N}\, \exists r \in \mathbb{N}\, \exists Q_1, \ldots, Q_r \in \mathcal{P}\, \exists G_1 \in \mathcal{R}(Q_1), \ldots, G_k \in \mathcal{R}(Q_k) : \\
&\qquad \{Q_1, \ldots, Q_r\} \Vdash^{j^\star} \{P\} \text{ and } \bigcap_{k=1,\ldots,r} G_k = G) \\
&\iff (\exists r \in \mathbb{N}\, \exists Q_1, \ldots, Q_r \in \mathcal{P}\, \exists G_1 \in \mathcal{R}(Q_1), \ldots, G_k \in \mathcal{R}(Q_k) : \\
&\qquad \{Q_1, \ldots, Q_r\} \Vdash^\star \{P\} \text{ and } \bigcap_{k=1,\ldots,r} G_k = G).
\end{aligned}$$

□

Lemma 5. *Let $\mathcal{R} : \mathcal{P} \rightrightarrows 2^N \backslash \{\emptyset\}$ be exhaustive and satisfy* (IPC) *(i.e., be a consistent exhaustive rights system). For all $j \in \mathbb{N}$, $Q_1, \ldots, Q_r, P \in \mathcal{P}$ and $G_1 \in \mathcal{R}(Q_1), \ldots, G_r \in \mathcal{R}(Q_r)$:*

1. *$\{Q_1, \ldots, Q_r\} \Vdash^j \{P\} \implies \bigcap_{k=1,\ldots,r} G_k \in \mathcal{R}(P)$,*
2. *$\{Q_1, \ldots, Q_r\} \Vdash^\star \{P\} \implies \bigcap_{k=1,\ldots,r} G_k \in \mathcal{R}(P)$.*

Proof. 1. By induction over $j \in \mathbb{N}$.

Induction basis: $j = 1$. We have $\{Q_1, \ldots, Q_r\} \Vdash \{P\} \iff \{Q_1, \ldots, Q_r\} \vdash P \iff (\{Q_1, \ldots, Q_r, P^c\}$ is critical). If $\bigcap_{k=1,\ldots,r} G_k = N$, $N \backslash \left(\bigcap_{k=1,\ldots,r} G_k\right) = \emptyset \notin \mathcal{R}(P^c) \implies \bigcap_{k=1,\ldots,r} G_k = N \in \mathcal{R}(P)$ as $\mathcal{R}$ is exhaustive. If $\bigcap_{k=1,\ldots,r} G_k \neq N$, suppose for a contradiction that $\bigcap_{k=1,\ldots,r} G_k \notin \mathcal{R}(P)$. As $\mathcal{R}$ is exhaustive, $\bigcap_{k=1,\ldots,r} G_k \notin \mathcal{R}(P) \implies N \backslash \left(\bigcap_{k=1,\ldots,r} G_k\right) \in \mathcal{R}(P^c)$ yielding a violation of (IPC) over critical $\{Q_1, \ldots, Q_r, P^c\}$.

Induction step: $j \rightsquigarrow j+1$. Assume the claim holds for $j \in \mathbb{N}$. If $\{Q_1, \ldots, Q_r\} \Vdash^{(j+1)} \{P\}$, there exist $\mathcal{C}_1, \ldots, \mathcal{C}_j \in \mathbb{C}$ such that $\{Q_1, \ldots, Q_r\} \Vdash \mathcal{C}_j \Vdash \ldots \Vdash \mathcal{C}_1 \Vdash \{P\}$. Hence $\{Q_1, \ldots, Q_r\} \Vdash^j \mathcal{C}_1 \Vdash \{P\}$. Let $P_1, \ldots, P_s \in \mathcal{P}$ be such that $\mathcal{C}_1 = \{P_1, \ldots, P_s\}$. By Lemma 1, part 4, there exist $\mathcal{C}^1, \ldots, \mathcal{C}^s \in \mathbb{C}$ such that $\bigsqcup_{l=1,\ldots,s} \mathcal{C}^l = \{Q_1, \ldots, Q_r\}$ and, for $l = 1, \ldots, s$, $\mathcal{C}^l \Vdash^j \{P_l\}$. W.l.o.g., for $l = 1, \ldots, s$, $\mathcal{C}^l = \{Q_{r_{l-1}+1}, \ldots, Q_{r_l}\}$ for some $r_0 = 0 < r_1 < r_2 < \cdots < r_{s-1} < r = r_s$. By (the inductive) assumption, for all $l = 1, \ldots, s$, $\bigcap_{k=r_{l-1}+1,\ldots,r_l} G_k \in \mathcal{R}(P_l)$. Thus, by the induction basis, $\bigcap_{k=1,\ldots,r} G_k = \bigcap_{l=1,\ldots,s} \left(\bigcap_{k=r_{l-1}+1,\ldots,r_l} G_k\right) \in \mathcal{R}(P)$.

2. $\{Q_1, \ldots, Q_r\} \Vdash^\star \{P\} \iff \left(\{Q_1, \ldots, Q_r\} \Vdash^{j^\star} \{P\} \text{ for some } j^\star \in \mathbb{N}\right)$. Hence the claim follows from part 1.

$\square$

Lemma 6. *Let $\mathcal{R} : \mathcal{P} \rightrightarrows 2^N \backslash \{\emptyset\}$ satisfy* (IPAC)*. For all $P \in \mathcal{P}$, $G \in 2^N \backslash \{\emptyset\}$, define*

$$\mathcal{P} \ni \widehat{P} \mapsto \mathcal{R}_{(G,P)}(\widehat{P}) = \begin{cases} \mathcal{R}(\widehat{P}) & \text{if } \widehat{P} \neq P \\ \mathcal{R}(\widehat{P}) \cup \{G\} & \text{if } \widehat{P} = P \end{cases}.$$

If $G \in 2^N \backslash \{\emptyset\}$, $P \in \mathcal{P}$ are such that $G \notin \mathcal{R}(P)$ and $N \backslash G \notin \mathcal{R}(P^c)$ then $\mathcal{R}_{(G,P)}$ or $\mathcal{R}_{(N \backslash G, P^c)}$ satisfies (IPAC).

Proof. Suppose that both $\mathcal{R}_{(G,P)}$ and $\mathcal{R}_{(N \backslash G, P^c)}$ violate (IPAC). That is, there exist almost critical $\mathcal{C}_1 = \{Q_1^1, \ldots, Q_r^1\} \in \mathbb{C}$ and $\mathcal{C}_2 = \{Q_1^2, \ldots, Q_s^2\} \in \mathbb{C}$ as well as $G_1^1 \in \mathcal{R}_{(G,P)}(Q_1^1), \ldots, G_r^1 \in \mathcal{R}_{(G,P)}(Q_r^1)$ and $G_1^2 \in \mathcal{R}_{(N \backslash G, P^c)}(Q_1^2), \ldots, G_s^2 \in \mathcal{R}_{(N \backslash G, P^c)}(Q_s^2)$ such that (i) $\bigcap_{k=1,\ldots,r} G_k^1 = \emptyset$ and (ii) $\bigcap_{l=1,\ldots,s} G_l^2 = \emptyset$. Unless, for some $1 \leq k' \leq r$, $Q_{k'}^1 = P$ and $G_{k'}^1 = G$, $\mathcal{R}$ violates (IPAC), in contradiction with our assumption. Analogously, for some $1 \leq l' \leq s$, we must have $Q_{l'}^2 = P^c$ and $G_{l'}^2 = N \backslash G$. For all $k \in \{1, \ldots, r\} \backslash \{k'\}$ and all $l \in \{1, \ldots, s\} \backslash \{l'\}$, define:

$$\widehat{G}_k^1 = \begin{cases} G_k^1 & \text{if } G_k^1 \in \mathcal{W}(Q_k^1) \\ N & \text{if } G_k^1 = G \text{ and } Q_k^1 = P \end{cases} \quad \text{and } \widehat{G}_l^2 = \begin{cases} G_l^2 & \text{if } G_l^2 \in \mathcal{W}(Q_l^2) \\ N & \text{if } G_l^2 = N \backslash G \text{ and } Q_l^2 = P^c \end{cases}.$$

Thus, for all $k \neq k'$, $\widehat{G}_k^1 \in \mathcal{R}(Q_k^1) \cup \{N\}$ and, for all $l \neq l'$, $\widehat{G}_l^2 \in \mathcal{R}(Q_l^2) \cup \{N\}$. By (i), $\bigcap_{k=1,\ldots,r, k \neq k'} \widehat{G}_k^1 \subseteq N \backslash G$. By (ii), $\bigcap_{l=1,\ldots,s, l \neq l'} \widehat{G}_l^2 \subseteq G$. If we show that $\mathcal{C}_1 \backslash \{Q_{k'}^1\} \sqcup \mathcal{C}_2 \backslash \{Q_{l'}^2\}$ is almost critical, we deduce that $\mathcal{R}$ violates (IPAC), a contradiction.

To complete the proof, we show that $\mathcal{C}_1 \backslash \{Q_{k'}^1\} \sqcup \mathcal{C}_2 \backslash \{Q_{l'}^2\}$ is indeed almost critical. As $\mathcal{C}_1, \mathcal{C}_2$ are almost critical, we can use Lemma 3 to conclude that $\mathcal{C}_1 \backslash \{Q_{k'}^1\} \Vdash^\star \{(Q_{k'}^1)^c\} = \{P^c\}$ and $\mathcal{C}_2 \backslash \{Q_{l'}^2\} \Vdash^\star \{(Q_{l'}^2)^c\} = \{(P^c)^c\} = \{P\}$. By Lemma 2, part 2, $(\mathcal{C}_1 \backslash \{Q_{k'}^1\}) \sqcup (\mathcal{C}_2 \backslash \{Q_{l'}^2\}) \Vdash^\star \{P^c\} \sqcup \{P\} = \{P, P^c\}$. As $\{P, P^c\}$ is critical, $\mathcal{C}_1 \backslash \{Q_{k'}^1\} \sqcup \mathcal{C}_2 \backslash \{Q_{l'}^2\}$ is almost critical. $\square$

Lemma 7. *Let $P, Q \in \mathcal{P}$.*

1. *$P \trianglerighteq^\star Q \iff (\exists \mathcal{C} \in \mathbb{C} : P \in \mathcal{C} \Vdash^\star \{Q\})$. Moreover, $\{P\} \Vdash^\star \{Q\}$ only if $P \subseteq Q$.*

2. *If $P \trianglerighteq^\star Q$, $Q \trianglerighteq^\star P$ and $P \neq Q$, then there exists some collection $\mathcal{C}' \in \mathbb{C}$ such that $|\mathcal{C}'| \geq 2$ and $P \in \mathcal{C}' \Vdash^\star \{Q\}$.*

Proof. 1. We have

$$\begin{aligned}
P \trianglerighteq^\star Q &\iff \exists r \in \mathbb{N} \exists Q_1, \dots, Q_r \in \mathcal{P} : P \trianglerighteq Q_1 \trianglerighteq \cdots \trianglerighteq Q_r \trianglerighteq Q \\
&\iff \exists r \in \mathbb{N} \exists Q_1, \dots, Q_r \in \mathcal{P} \exists \mathcal{F}, \mathcal{F}_1, \dots, \mathcal{F}_r \in \mathbb{F} : \\
&\qquad P \in \mathcal{F} \vdash Q_1, Q_1 \in \mathcal{F}_1 \vdash Q_2, \dots, Q_r \in \mathcal{F}_r \vdash Q \\
&\iff \exists r \in \mathbb{N} \exists Q_1, \dots, Q_r \in \mathcal{P} \exists \mathcal{F}, \mathcal{F}_1, \dots, \mathcal{F}_r \in \mathbb{F} : \\
&\qquad P \in \mathcal{F} \sqcup \left(\bigsqcup_{k=1,\dots r} \mathcal{F}_k \backslash \{Q_k\} \right) \Vdash^\star \{Q\} \\
&\iff \exists \mathcal{C} \in \mathbb{C} : P \in \mathcal{C} \Vdash^\star \{Q\}.
\end{aligned}$$

By non-triviality of $\mathcal{P}$, all properties are individually consistent. Thus, for every $\widehat{P}, \widehat{Q} \in \mathcal{P}$, it holds $\widehat{P} \subseteq \widehat{Q} \iff \left(\left\{ \widehat{P}, \widehat{Q}^c \right\} \text{ is critical} \right) \iff \left\{ \widehat{P} \right\} \vdash \widehat{Q} \iff \left\{ \widehat{P} \right\} \Vdash \left\{ \widehat{Q} \right\}$. Now suppose that $\{P\} \Vdash^\star \{Q\}$. We must have $\{P\} \Vdash \{Q_1\} \Vdash \dots \Vdash \{Q_r\} \Vdash \{Q\}$ for some $Q_1, \dots, Q_r \in \mathcal{P}$. Thus, $P \subseteq Q_1 \subseteq \cdots \subseteq Q_r \subseteq Q$.

2. By part 1, there exist $\mathcal{C}_1, \mathcal{C}_2 \in \mathbb{C}$ such that $Q \in \mathcal{C}_1 \Vdash^\star P$ and $P \in \mathcal{C}_2 \Vdash^\star Q$. As $P \neq Q$, we have $|\mathcal{C}_1| \geq 2$ or $|\mathcal{C}_2| \geq 2$. If $|\mathcal{C}_2| \geq 2$, take $\mathcal{C}' = \mathcal{C}_2$. Otherwise, $\{P\} \Vdash^\star Q$. Then, by Lemma 2, part 4, $\{P\} \sqcup (\mathcal{C}_1 \backslash \{Q\}) \Vdash^\star \mathcal{C}_1 \Vdash^\star \{P\} \Vdash^\star \{Q\}$. Thus, take $\mathcal{C}' = \{P\} \sqcup (\mathcal{C}_1 \backslash \{Q\})$.

□

D Proofs for Section 4

D.1 Proof of Fact 3

The proof is contained in the main text.

D.2 Proof of Fact 4

We have $\{Q_1, \dots, Q_r\} \vdash P \iff \{Q_1, \dots, Q_r\} \Vdash \{P\}$. The claim follows by Lemma 5.

D.3 Proof of Theorem 2

Fact 3 and Theorem 1 imply equivalence of 1 and 2. We complete the proof by showing $3 \implies 2 \implies 4 \implies 3$.

D.3.1 Proof of $3 \implies 2$

If $\mathcal{R}$ is exhaustive, take $\mathcal{R}' = \mathcal{R}$. Suppose $\mathcal{R}$ is not exhaustive. That is, there exist some $G \in 2^N \backslash \{\emptyset\}$ and some $P \in \mathcal{P}$ such that $G \notin \mathcal{R}(P)$ and $N \backslash G \notin \mathcal{R}(P^c)$. By Lemma 6, there exists some rights system $\mathcal{R}^{(1)}$ satisfying (IPAC) which extends $\mathcal{R}$ such that $G \in \mathcal{R}^{(1)}(P)$ or $N \backslash G \in \mathcal{R}^{(1)}(P^c)$. If $\mathcal{R}^{(1)}$ is exhaustive, take $\mathcal{R}' = \mathcal{R}^{(1)}$. Otherwise use Lemma 6 again to extend $\mathcal{R}^{(1)}$ to some $\mathcal{R}^{(2)}$ satisfying (IPAC) such that $G' \in \mathcal{R}^{(2)}(Q)$ or $N \backslash G' \in \mathcal{R}^{(2)}(Q^c)$ for some $G' \in 2^N \backslash \{\emptyset\}$ and $Q \in \mathcal{P}$ such that $G' \notin \mathcal{R}^{(1)}(Q)$ and $N \backslash G \notin \mathcal{R}^{(1)}(Q^c)$. When continued, this procedure will produce some exhaustive $\mathcal{R}'$ satisfying (IPAC) after a finite amount of steps.

This is true because there are only finitely many pairs $(G,H) \in 2^N \backslash \{\emptyset\} \times \mathcal{P}$; seeing that both N and $\mathcal{P} \subseteq 2^X$ are finite.

D.3.2 Proof of 2 $\Longrightarrow$ 4

Let $\mathcal{R}'$ be some exhaustive rights system satisfying (IPC) and (R*). Assume for a contradiction that $\mathcal{C}^\star(\mathcal{R})$ does not satisfy (IPC). That is, there exist some critical $\{P_1,\ldots,P_r\} \in \mathbb{F}$ and $G_1 \in \mathcal{C}^\star(\mathcal{R})(P_1),\ldots,P_r \in \mathcal{C}^\star(\mathcal{R})(P_r)$ such that $\bigcap_{k=1,\ldots,r} G_k = \emptyset$. By Lemma 4, part 2, for $k=1,\ldots,r$, there exist $s_k \in \mathbb{N}$, $\{Q_1^k,\ldots,Q_{s_k}^k\} \in \mathbb{C}$ and $G_1^k \in \mathcal{R}(Q_1^k) \subseteq \mathcal{R}'(Q_1^k),\ldots,G_{s_k}^k \in \mathcal{R}(Q_{s_k}^k) \subseteq \mathcal{R}'(Q_{s_k}^k)$ such that $\{Q_1^k,\ldots,Q_{s_k}^k\} \Vdash^\star P_k$ and $G_k = \bigcap_{l=1,\ldots,s_k} G_l^k$. By Lemma 5, part 2, for all $k=1,\ldots,r$, $G_k = \bigcap_{l=1,\ldots,s_k} G_l^k \in \mathcal{R}'(P_k)$. Thus, $\mathcal{R}'$ violates (IPC), a contradiction.

D.3.3 Proof of 4 $\Longrightarrow$ 3

Let $\mathcal{C}^\star(\mathcal{R})$ satisfy (IPC). Assume for a contradiction that $\mathcal{R}$ does not satisfy (IPAC). There exist some almost critical $\{P_1,\ldots,P_r\} \in \mathbb{C}$ and $G_1 \in \mathcal{R}(P_1),\ldots,G_r \in \mathcal{R}(P_r)$ such that $\bigcap_{k=1,\ldots,r} G_k = \emptyset$. As $\{P_1,\ldots,P_r\}$ is almost critical, there exists some critical $\mathcal{G} = \{Q_1,\ldots,Q_s\} \in \mathbb{F}$ such that $\{P_1,\ldots,P_r\} \Vdash^\star \mathcal{G}$. By Lemma 1, part 4, there exist $\mathcal{C}_1 \Vdash^\star Q_1,\ldots,\mathcal{C}_s \Vdash^\star Q_s$ with $\{P_1,\ldots,P_r\} = \sqcup_{l=1,\ldots,s}\mathcal{C}_l$. W.l.o.g., for $l=1,\ldots,s$, $\mathcal{C}_l = \{P_{r_{l-1}+1},\ldots,P_{r_l}\}$ for some $r_0 = 0 < r_1 < r_2 < \cdots < r_{s-1} < r = r_s \in \mathbb{N}$. For $l=1,\ldots,s$, let $G^l = \bigcap_{k=r_{l-1}+1,\ldots,r_l} G_k$. By Lemma 4, part 2, we have $G^l = \bigcap_{k=r_{l-1}+1,\ldots,r_l} G_k \in C^\star(\mathcal{R})(Q_l)$. Seeing that $\bigcap_{l=1,\ldots,s} G^l = \bigcap_{l=1,\ldots,s}\left(\bigcap_{k=r_{l-1}+1,\ldots,r_l} G_k\right) = \bigcap_{k=1,\ldots,r} G_k = \emptyset$, $C^\star(\mathcal{R})$ violates (IPC), a contradiction.

D.4 Proof of Proposition 3

D.4.1 Proof of part 1

Let $\widetilde{\mathcal{C}} \in \mathbb{C}$. We show that there exists some almost critical $\mathcal{C}_{\widetilde{\mathcal{C}}} \in \mathbb{C}$ such that $\widetilde{\mathcal{C}} \subseteq \mathcal{C}_{\widetilde{\mathcal{C}}}$. Let $\mathcal{C} \in \mathbb{F}$ be some arbitrary almost critical collection. Let $P \in \widetilde{\mathcal{C}}, P \notin \mathcal{C}$ (if no such P exists, we have $\widetilde{\mathcal{C}} \subseteq \mathcal{C}$; thus, letting $\mathcal{C}_{\widetilde{\mathcal{C}}} = \mathcal{C}$ completes the proof) and $Q \in \mathcal{C}$. By Lemma 7, part 2, there exists some $\mathcal{C}' \in \mathbb{C}$ such that $|\mathcal{C}'| \geq 2$ and $P \in \mathcal{C}' \Vdash^\star \{Q\}$. Let $Q' \in \mathcal{C}' \backslash \{Q\} \neq \emptyset$. By Lemma 7, part 1, there exists some $\mathcal{C}'' \in \mathbb{C}$ such that $Q \in \mathcal{C}'' \Vdash^\star \{Q'\}$. Note that $\mathcal{C} \sqcup \{P\} \subseteq \mathcal{C}'' \sqcup (\mathcal{C}' \backslash \{Q'\}) \sqcup (\mathcal{C} \backslash \{Q\})$. Using Lemma 2, part 4, $\mathcal{C}'' \sqcup (\mathcal{C}' \backslash \{Q'\}) \sqcup (\mathcal{C} \backslash \{Q\}) \Vdash^\star \mathcal{C}' \sqcup (\mathcal{C} \backslash \{Q\}) \Vdash^\star \mathcal{C}$. As $\mathcal{C}$ is almost critical, so is $\mathcal{C}'' \sqcup (\mathcal{C}' \backslash \{Q'\}) \sqcup (\mathcal{C} \backslash \{Q\})$. Thus, there exists an almost critical collection containing $\mathcal{C} \sqcup \{Q\}$. In the same fashion, if $P' \in \widetilde{\mathcal{C}}, P' \notin \mathcal{C} \sqcup \{P\}$, we can find an almost critical collection containing $\mathcal{C} \sqcup \{P\} \sqcup \{P'\}$. Consequently, in finitely many steps, we can construct some almost critical collection $\mathcal{C}_{\widetilde{\mathcal{C}}}$ such that $\widetilde{\mathcal{C}} \subseteq \mathcal{C}_{\widetilde{\mathcal{C}}}$.

Let $\mathcal{P}_{2^n}$ be the collection that contains every property from $\mathcal{P}$ exactly 2^n times. As a consequence of what we just showed, there is some almost critical $\mathcal{C}_{\mathcal{P}_{2^n}} \supseteq \mathcal{P}_{2^n}$. If $\mathcal{R}$ is trivial, let $X^n \ni \boldsymbol{x} = (x_1,\ldots,x_n) \mapsto f_{i^\star}(\boldsymbol{x}) = x_{i^\star}$ for some $i^\star \in \bigcap_{G \in \mathcal{R}(P),\, P \in \mathcal{P}} G$. Then $f_{i^\star}$ respects rights and is onto and monotone independent. Conversely, suppose $\mathcal{R}$ is consistent with voting by properties. Note that, for every $P \in \mathcal{P}$, $\mathcal{R}(P) \subseteq 2^N$. Thus, $|\mathcal{R}(P)| \leq 2^n$. By Theorem 2, $\mathcal{R}$ satisfies (IPAC). When evaluated over $\mathcal{C}_{\mathcal{P}_{2^n}}$, this yields that $\bigcap_{G \in \mathcal{R}(P),\, P \in \mathcal{P}} G \neq \emptyset$.

D.4.2 Proof of part 2

As every critical family is almost critical, we only need to show the converse. We establish an auxiliary result first. By the definition of $\vdash$, on a median space $(X,\mathcal{P})$, we have that for all $\mathcal{F} \in \mathbb{F}, P \in \mathcal{P}$: $\mathcal{F} \vdash P \iff (\exists Q \in$

$\mathcal{P} : \mathcal{F} = \{Q\} \vdash P) \iff (\exists Q \in \mathcal{P} : \mathcal{F} = \{Q\}$ and $\{Q, P^c\}$ is critical$) \iff (\exists Q \in \mathcal{P} : \mathcal{F} = \{Q\}$ and $Q \subseteq P)$. Consequently, for all $\mathcal{C} \in \mathbb{C}, P \in \mathcal{P}$: $\mathcal{C} \Vdash \{P\} \iff (\exists Q \in \mathcal{P} : \mathcal{C} = \{Q\} \Vdash P) \iff (\exists Q \in \mathcal{P} : \mathcal{C} = \{Q\}$ and $Q \subseteq P)$. By transitivity of $\subseteq$, the same statements carry over to $\Vdash^\star$.

Let $\mathcal{C} \in \mathbb{C}$ be almost critical. That is, there exists some critical $\mathcal{G} \in \mathbb{F}$ such that $\mathcal{C} \Vdash^\star \mathcal{G}$. As $(X, \mathcal{P})$ is median, $\mathcal{G} = \{P_1, P_2\}$ for some $P_1, P_2 \in \mathcal{P}$. By Lemma 1, part 4, there exist $\mathcal{C}_1, \mathcal{C}_2 \in \mathbb{C}, \mathcal{C} = \mathcal{C}_1 \sqcup \mathcal{C}_2$ such that $\mathcal{C}_1 \Vdash^\star \{P_1\}$ and $\mathcal{C}_2 \Vdash^\star \{P_2\}$. By what we just showed, there exist $Q_1, Q_2 \in \mathcal{P}$ such that $\mathcal{C}_1 = \{Q_1\}, \mathcal{C}_2 = \{Q_2\}$ and $Q_1 \subseteq P_1, Q_2 \subseteq P_2$. We have $Q_1 \subseteq P_1 \subseteq P_2^c \subseteq Q_2^c$. Thus, $\{Q_1, (Q_2^c)^c\} = \{Q_1, Q_2\} = \mathcal{C}$ is critical.

As the almost critical collections are exactly the critical families (all of length two), it is without loss of generality to choose $\mathcal{R}(P) \ni G_P \neq N$ for both properties $P \in \mathcal{G}$ when checking (IPAC) over critical $\mathcal{G}$ (otherwise, the intersection is trivially non-empty). Thus, (IPAC) is equivalent to (IPC). Equivalence of (a) and (b) follows (from Theorems 1 and 2). To prove equivalence of (c) and (d), note that on median spaces all entailments are unconditional (hence direct). Thus, two properties are dependent if and only if they are directly dependent. It follows that weak independence and independence are equivalent concepts on median spaces. Lastly, to show equivalence of (a) and (d), we note that, as all critical fragments have length two on median spaces, $\mathcal{R}$ is weakly independent if and only if it satisfies (IPC) (i.e., is consistent).

D.5 Proof of Fact 5

For all $P, Q \in \mathcal{P}$ define $P \equiv Q \iff (P \trianglerighteq^\star Q$ and $Q \trianglerighteq^\star P)$. Note that $\equiv$ is an equivalence relation on $\mathcal{P}$.

We show that $\equiv$ induces exactly two distinct and non-empty equivalence classes. As $(X, \mathcal{P})$ is not totally blocked, there must be at least two equivalence classes. Let $[\equiv]_1, [\equiv]_2$ be two such classes and consider some $P \in [\equiv]_1, Q \in [\equiv]_2$. We have $P \not\equiv Q$. Thus, by semi-blockedness of $(X, \mathcal{P})$, $Q \equiv P^c$; i.e., $P^c \in [\equiv]_2$. Now, for every $\widehat{P} \in \mathcal{P} \backslash \{P, P^c\}$, $\widehat{P} \equiv P$ or $\widehat{P} \equiv P^c$. Thus, there are at most two distinct and non-empty equivalence classes.

As $[\equiv]_1 \neq [\equiv]_2$, we must have $[\equiv]_1 \not\trianglerighteq^\star [\equiv]_2$ or $[\equiv]_2 \not\trianglerighteq^\star [\equiv]_1$. W.l.o.g., assume the former. Define $\mathcal{P}^- = [\equiv]_1$ and $\mathcal{P}^+ = [\equiv]_2$. Then 2. holds by construction. To verify 1., suppose for a contradiction that $Q \equiv Q^c$ for some $Q \in \mathcal{P}$. Then, for all $P, P' \in \mathcal{P}$, $P \equiv Q$ or $P \equiv Q^c$ and $P' \equiv Q$ or $P' \equiv Q^c$. As $Q \equiv Q^c$, $P \equiv P'$ and $(X, \mathcal{P})$ is totally blocked.

To verify 3., we still need to show that $\mathcal{P}^+ \trianglerighteq^\star \mathcal{P}^-$. We first show that, when $|X| > 2$, there exists some critical family $\mathcal{G}$ such that $|\mathcal{G}| \geq 3$. Indeed, suppose for a contradiction that $|\mathcal{G}| = 2$ for all critical $\mathcal{G} \subseteq \mathcal{P}$. Then we have for all $\widehat{P}, \widehat{Q} \in \mathcal{P}$: $\widehat{P} \trianglerighteq^\star \widehat{Q} \iff \widehat{P} \subseteq \widehat{Q}$; i.e., $\widehat{P} \equiv \widehat{Q} \iff \widehat{P} = \widehat{Q}$. Consequently, $|\mathcal{P}| = 2$. It follows that, when $|X| > 2$, there are $x \neq y \in X$ such that, for all $P \in \mathcal{P}$, $x \in P \iff y \in P$, in contradiction to $(X, \mathcal{P})$ being a property space. Thus, there must exist some critical $\mathcal{G} \in \mathbb{F}$ with $|\mathcal{G}| \geq 3$. Let $\mathcal{G} \subseteq \mathcal{P}$ be critical, $|\mathcal{G}| \geq 3$. From part 1 and the fact that $\mathcal{P}^- \not\trianglerighteq^\star \mathcal{P}^+$, we have $|\mathcal{G} \cap \mathcal{P}^-| \leq 1$. Thus, $|\mathcal{G} \cap \mathcal{P}^+| \geq 2$ and there exist $\{P, Q\} \subseteq \mathcal{G} \cap \mathcal{P}^+$. We have $P \trianglerighteq^\star Q^c$. By part 1, $Q^c \in \mathcal{P}^-$. Hence for all $\widehat{P} \in \mathcal{P}^+, \widehat{Q} \in \mathcal{P}^-$: $\widehat{P} \trianglerighteq^\star P \trianglerighteq^\star Q^c \trianglerighteq^\star \widehat{Q}$.

Let $\widetilde{\mathcal{C}} = \{P_1, \dots, P_r\}$ be almost critical and suppose there exist $k', k'' \in \{1, \dots, r\}, k' \neq k''$ such that $P_{k'}, P_{k''} \in \mathcal{P}^-$. By Lemma 3, $P_{k''} \in \widetilde{\mathcal{C}} \backslash \{P_{k'}\} \Vdash^\star \{P_{k'}^c\}$. Thus, by Lemma 7, part 1 and part 1 from above, $\mathcal{P}^- \ni P_{k''} \trianglerighteq^\star P_{k'}^c \in \mathcal{P}^+$ contradicting $\mathcal{P}^- \not\trianglerighteq^\star \mathcal{P}^+$. Thus, every almost critical $\widetilde{\mathcal{C}}$ can contain at most one element from $\mathcal{P}^-$.

Let $\mathcal{C}$ be some multiset over $\mathcal{P}^+$ and $P \in \mathcal{P}^-$. We show that there exists some almost critical collection that contains $\mathcal{C} \sqcup \{P\}$. By part 1, $P^c \in \mathcal{P}^+$. Additionally, $\{P, P^c\}$ is critical; a fortiori, almost critical. Let $Q \in \mathcal{C}$. By Lemma 7, part 1 and part 3 above, there exists some $\mathcal{C}' \in \mathbb{C}, |\mathcal{C}'| \geq 2$ such that $Q \in \mathcal{C}' \Vdash^\star \{P^c\}$. Thus, $\{P, Q\} \subseteq \{P\} \sqcup \mathcal{C}'$ and $\{P\} \sqcup \mathcal{C}'$ is almost critical (seeing that $\{P\} \sqcup \mathcal{C}' \Vdash^\star \{P, P^c\}$). As $|(\{P\} \sqcup \mathcal{C}') \backslash \{P, Q\}| \geq 1$

and $(\{P\} \sqcup \mathcal{C}')\backslash\{P, Q\} \subseteq \mathcal{P}^+$, we can use Lemma 7, part 2 repeatedly until we have constructed an almost critical family containing $\mathcal{C} \sqcup \{P\}$ after finitely many steps.

D.6 Proof of Proposition 4

Suppose some monotone independent and onto $f : X^n \to X$ respects $\mathcal{R}$. By Fact 2 and the discussion in the main text, $f = F_{\mathcal{R}'}$ for some consistent exhaustive $\mathcal{R}'$ that extends $\mathcal{R}$. Thus, for all $i \in N$, $\mathcal{R}'$ satisfies (MR-i). That is, for all $i \in N$, there exist $P_i \in \mathcal{P}, G_i \subseteq N$ such that $i \in G_i \in \mathcal{R}'(P_i)$ and $(N\backslash G_i) \cup \{i\} \in \mathcal{R}'(P_i^c)$.

Consider any $P \in \mathcal{P}^-$. For all $Q \in \mathcal{P}^-$, Fact 5 establishes that there exists some almost critical $\mathcal{C} \in \mathbb{C}$ such that $\{P^c, Q^c, Q\} \subseteq \mathcal{C}$. Thus, by Lemma 3, $\{Q^c, Q\} \subseteq \mathcal{C}\backslash\{P^c\} \Vdash^\star \{P\}$. Using this with $Q = P_i$ as well as Lemma 5, part 2, we have for all $i \in N$: $\{i\} = G_i \cap ((N\backslash G_i) \cup \{i\}) \in \mathcal{R}'(P)$ (note that $N \in \bigcap_{P \in \mathcal{P}} \mathcal{R}(P)$). As $\mathcal{R}'$ is monotone, we have $\mathcal{R}'(P) = 2^N\backslash\{\emptyset\}$ and, consequently, $\mathcal{R}'(P^c) = \{N\}$. As $P \in \mathcal{P}^-$ was arbitrary, $f = F_{\mathcal{R}'}$ is a unanimity rule with default $\bigcap \mathcal{P}^-$.

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