Towards a “Borda count” for judgment aggregation

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Union College Mathematics Department

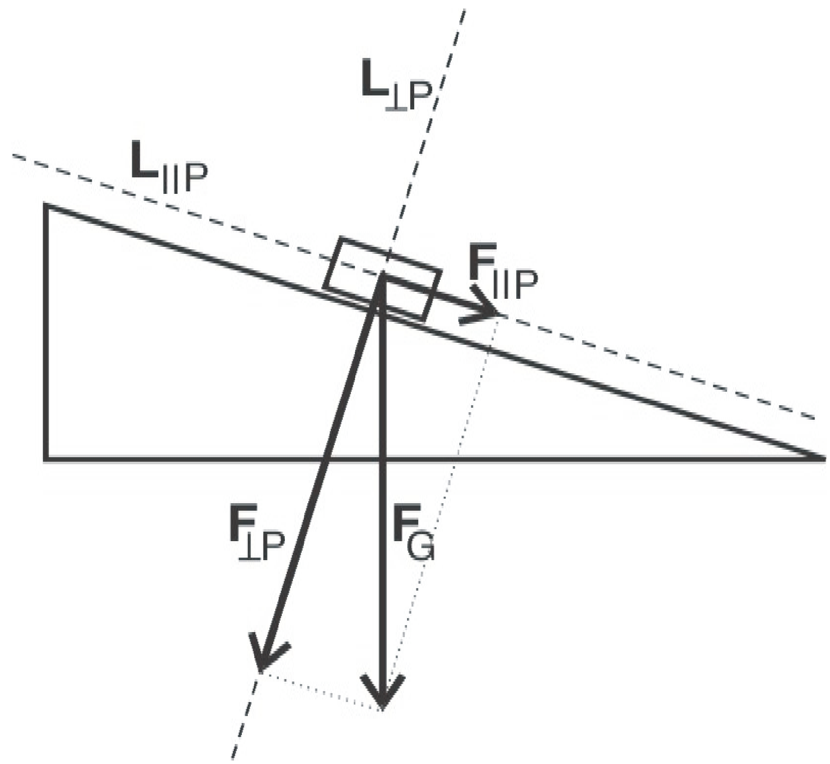
Workshop: New Developments in

Judgment Aggregation and Voting Theory

September, 2011

Orthogonal Decomposition I
(Physics analogy)

- $\mathbf{F}_G = \mathbf{F}_{||P} + \mathbf{F}_{\perp P}$
- $\mathbf{F}_{||P}$: tendency to slide
- $\mathbf{F}_{\perp P}$: resistence to slide
- Opposing implications . . . the outcome depends on which tendency dominates (relative size)



Orthogonal Decomposition I
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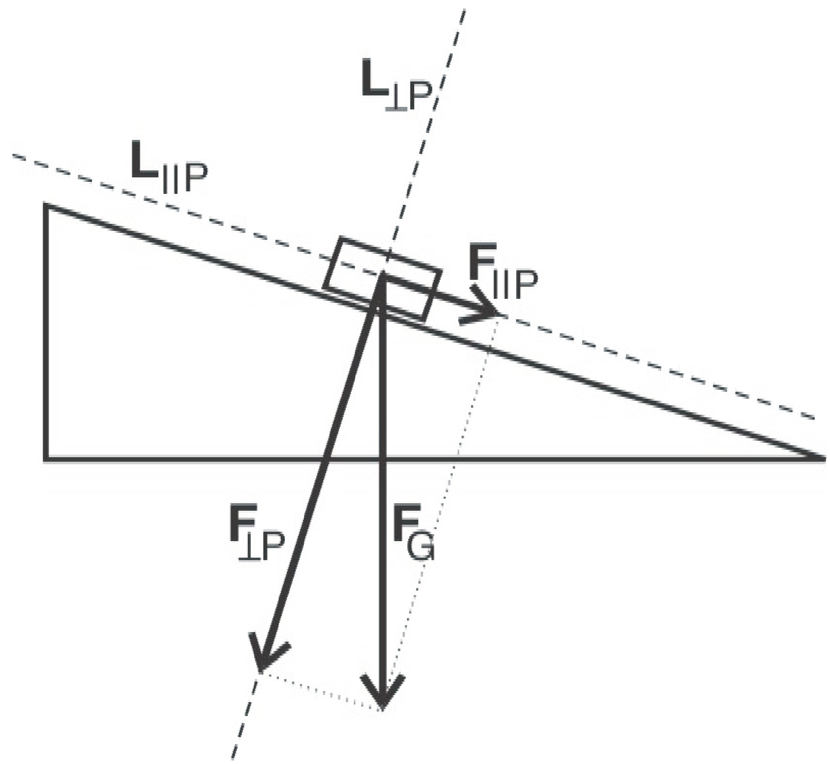
- $\mathbf{F}_G = \mathbf{F}_{||P} + \mathbf{F}_{\perp P}$
- $\mathbf{F}_{||P}$ = the orthogonal projection of **F** onto $\mathbf{L}_{||P}$
- $\mathbf{F}_{\perp P}$ = the orthogonal projection of **F** onto $\mathbf{L}_{\perp P}$
- $\mathbf{L}_{||P}$, $\mathbf{L}_{\perp P}$ are <u>orthogonal complements</u> in $\mathbf{R}^2$



Orthogonal Decomposition II
(in general)

$\mathbf{L}_1$, $\mathbf{L}_2$ are <u>orthogonal</u> <u>complements</u> in $\mathbf{R}^k$:

1) $v \in \mathbf{L}_1 \Leftrightarrow v \perp w$ for each $w \in \mathbf{L}_2 \Leftrightarrow v \bullet w = 0$ for each $w \in \mathbf{L}_2$ (and vice-versa)

equivalently

2) Each $v \in \mathbf{R}^k$ can be written uniquely as a sum $v = v_1 + v_2$ such that $v_1 \in \mathbf{L}_1$ and $v_2 \in \mathbf{L}_2$; moreover v_1, v_2 are obtained as orthogonal projections onto $\mathbf{L}_1$, $\mathbf{L}_2$

$$\mathbf{R}^k = \mathbf{L}_1 \oplus_\perp \mathbf{L}_2$$

Orthogonal Decomposition III
(Preference aggregation)

- Sample profile P for 4 alternatives a, b, c, d:

10: a > b > c > d

10: b > c > d > a

6: b > a > c > d

4: a > b > c > d

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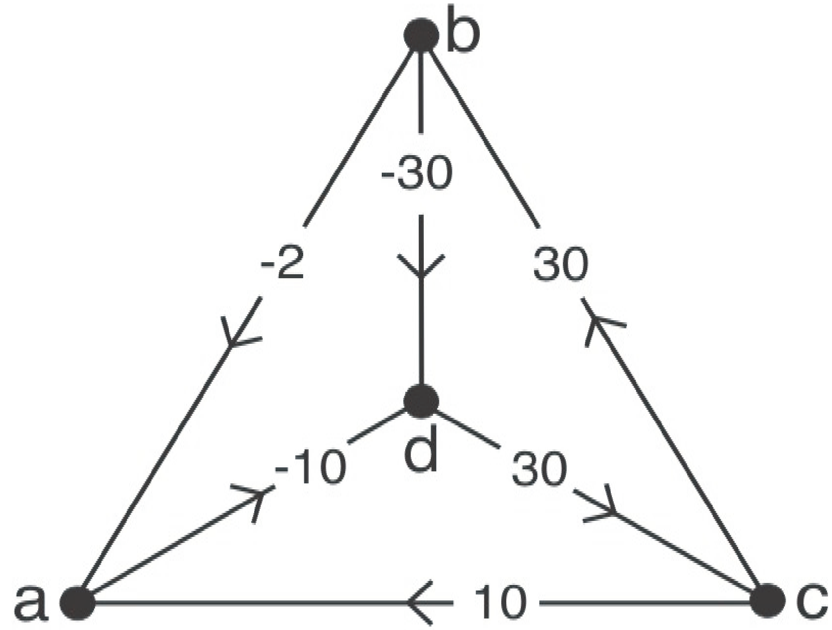
<u>Flow of net preference on digraph</u>



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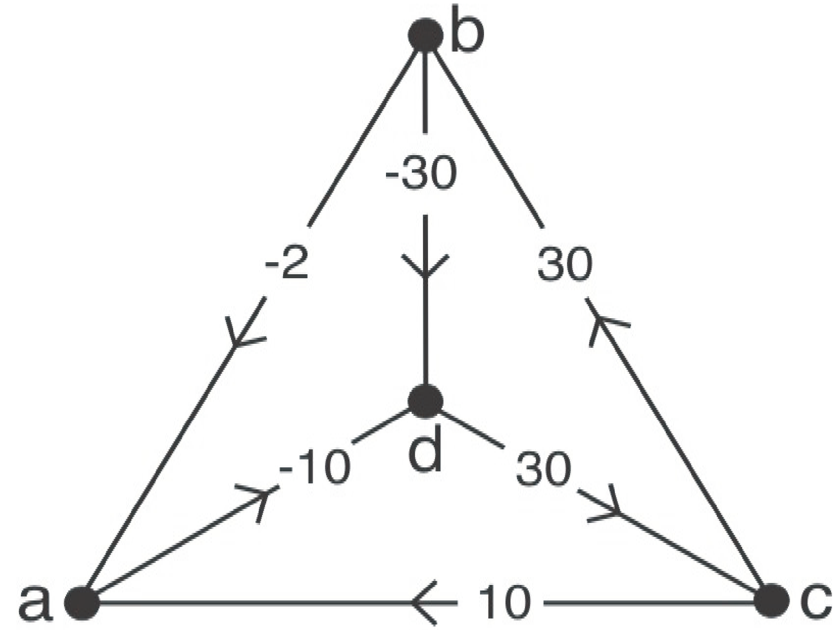
<u>Flow of net preference on digraph</u>

- $\mathrm{Net}_P(a > d) = 20 - 10 = 10$
- But edge is (arbitrarily) directed a → d, so its label is -10
- Source → Target is labeled with net preference for T over S



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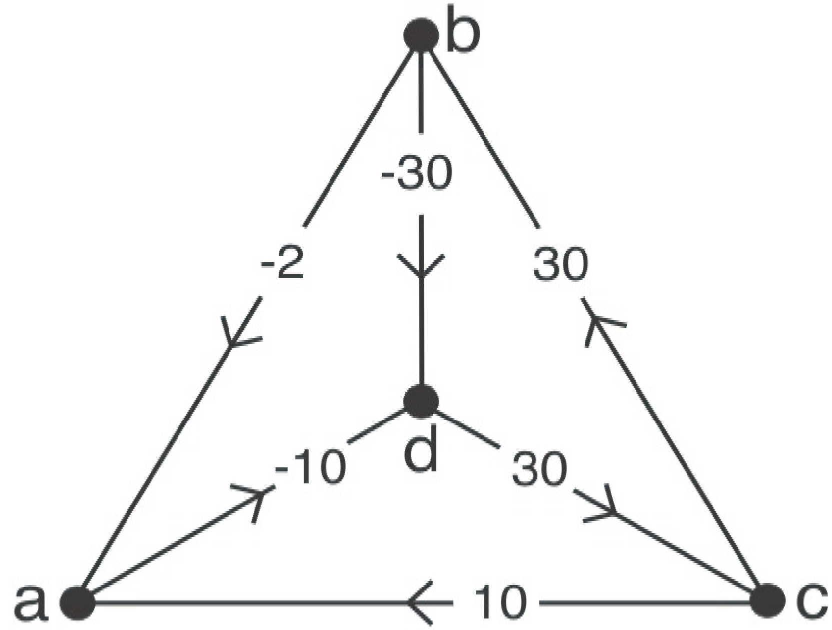
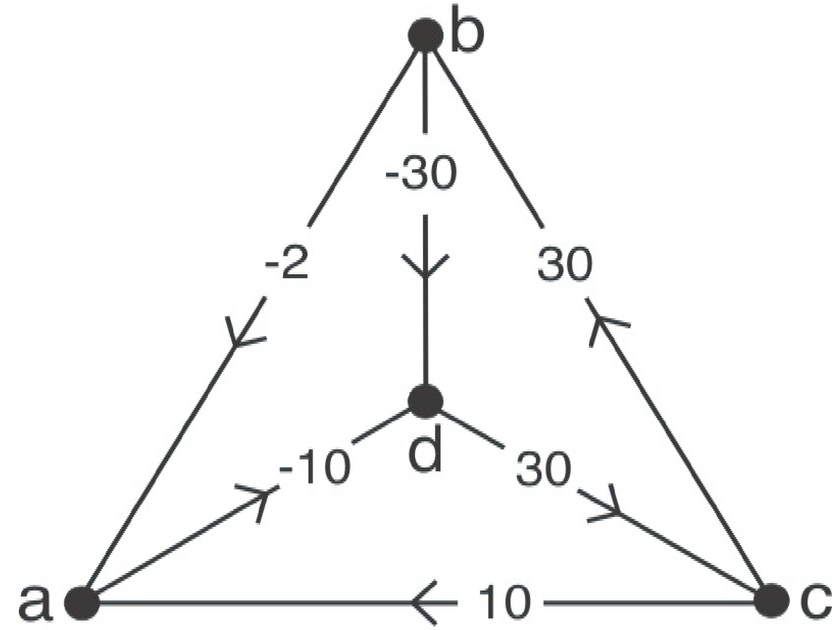
Signs of labels ➔ Condorcet ranking

- Here b > a > c > d
- (transitive for this P)



Orthogonal Decomposition III
(Preference aggregation)

- Sample profile P for a, b, c, d
- $v_P = \langle -2, -30, 30, -10, 30, 10 \rangle$, a vector in $\mathbf{R}^6$ (6 pairwise comps)



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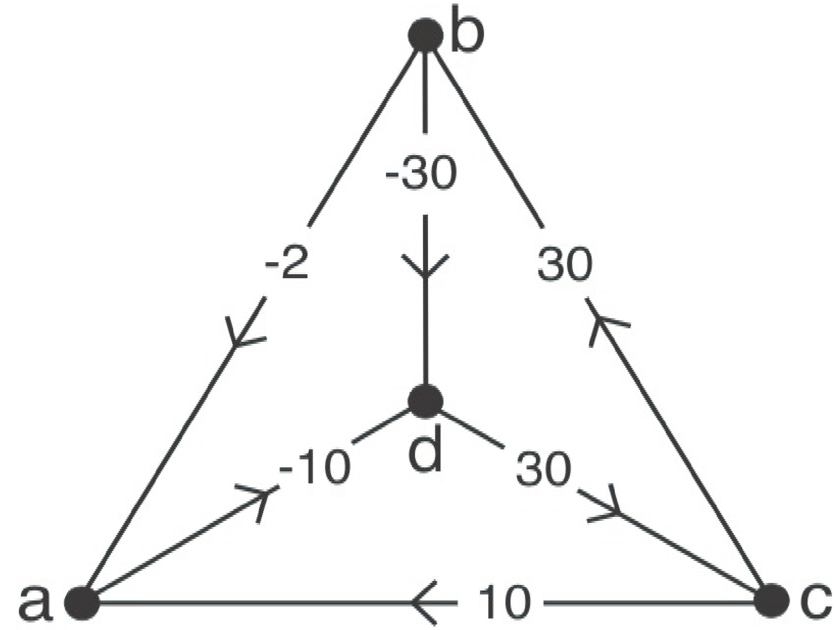

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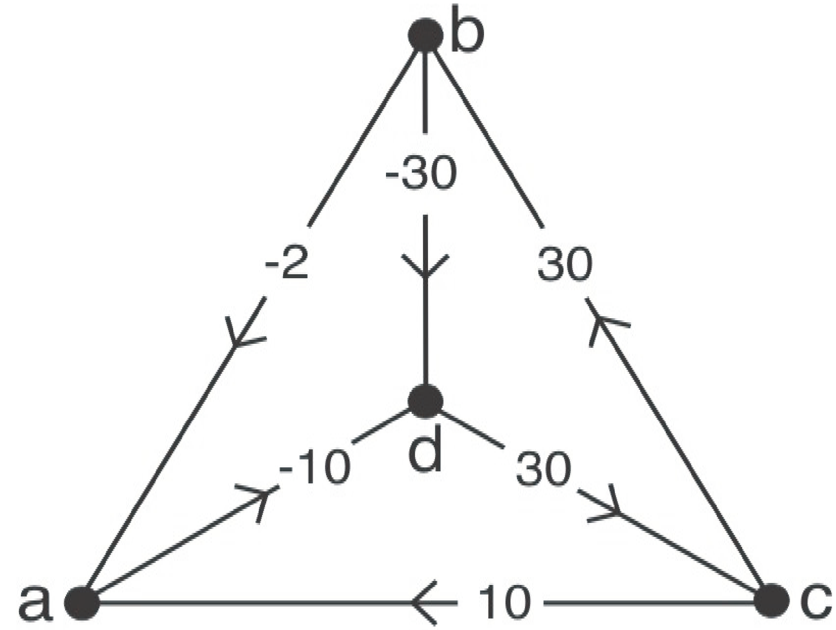
INTERPRETATION?



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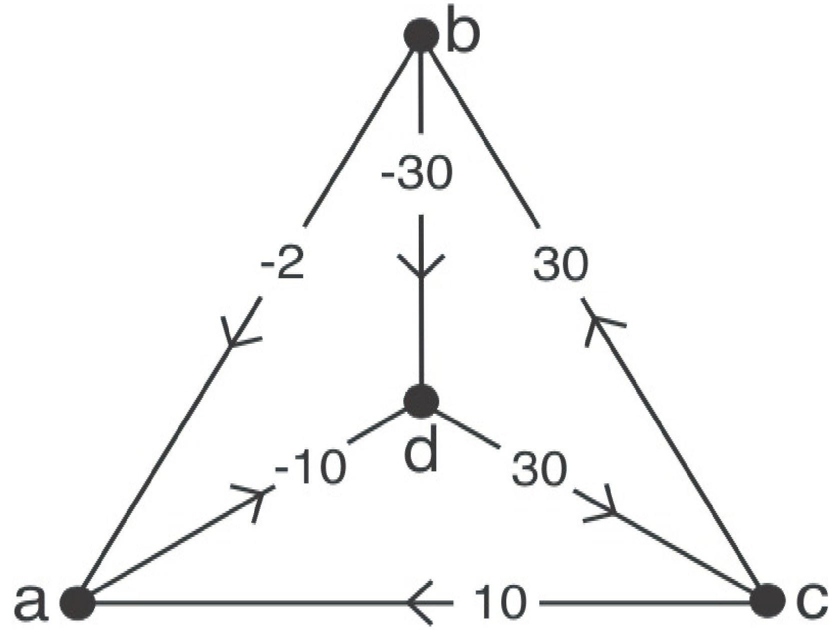
$$v_P = v_{\text{cycle}} + v_{\text{cocycle}}$$

INTERPRETATION?

- v_{cycle} = innate tendency to cycle
- v_{cocycle} = resistence to cycle

$\cong$ vector of Borda scores

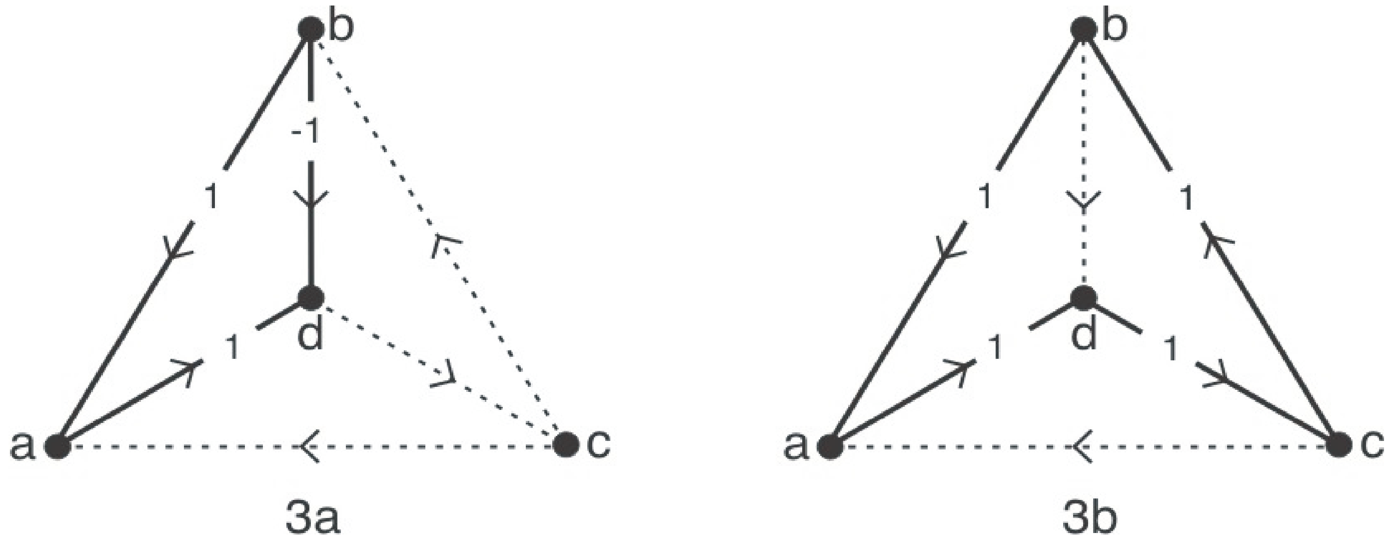


L_{cycle}, the cycle subspace

- A <u>basic</u> <u>cycle</u> is a unit loop flow (taking account of orientation)
- Two basic cycles;

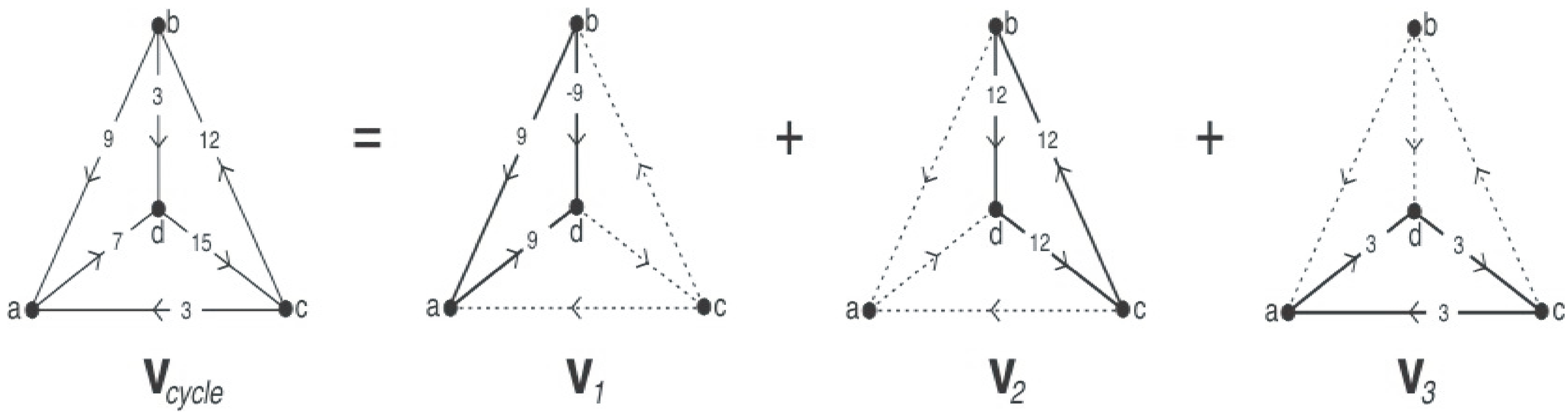
  $v_a = \langle 1, -1, 0, 1, 0, 0 \rangle$

  $v_b = \langle 1, 0, 1, 1, 1, 0 \rangle$



L_{cycle}, the cycle subspace

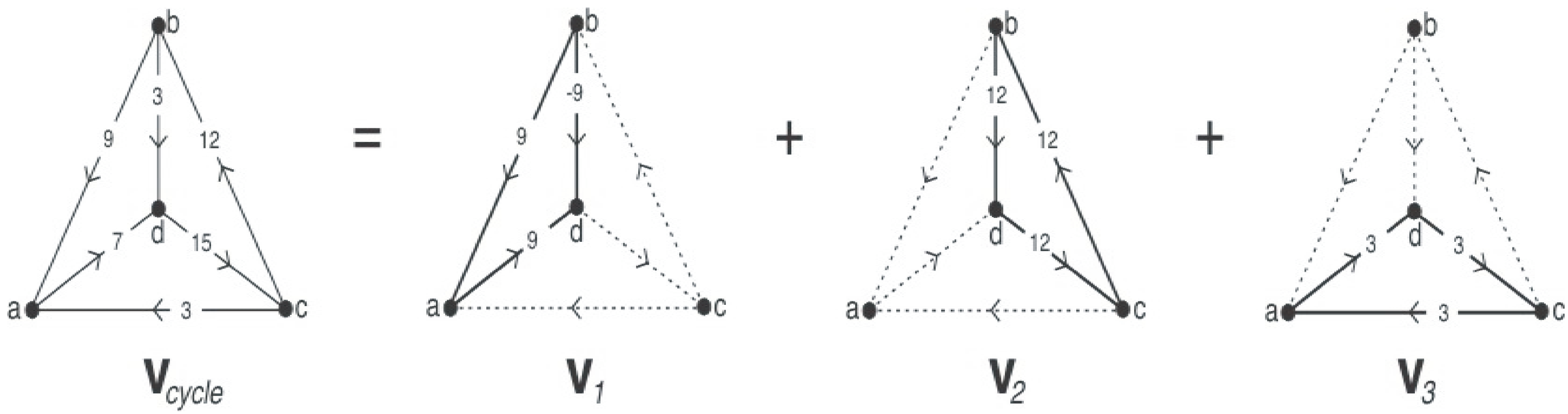
- A <u>cycle</u> is a sum of scalar multiples of basic cycles, and L_{cycle} is the space of all cycles



L_{cycle}, the cycle subspace

- A cycle is a sum of scalar multiples of basic cycles, and L_{cycle} is the space of all cycles

- Equivalently, L_{cycle} is the linear span of the basic cycles

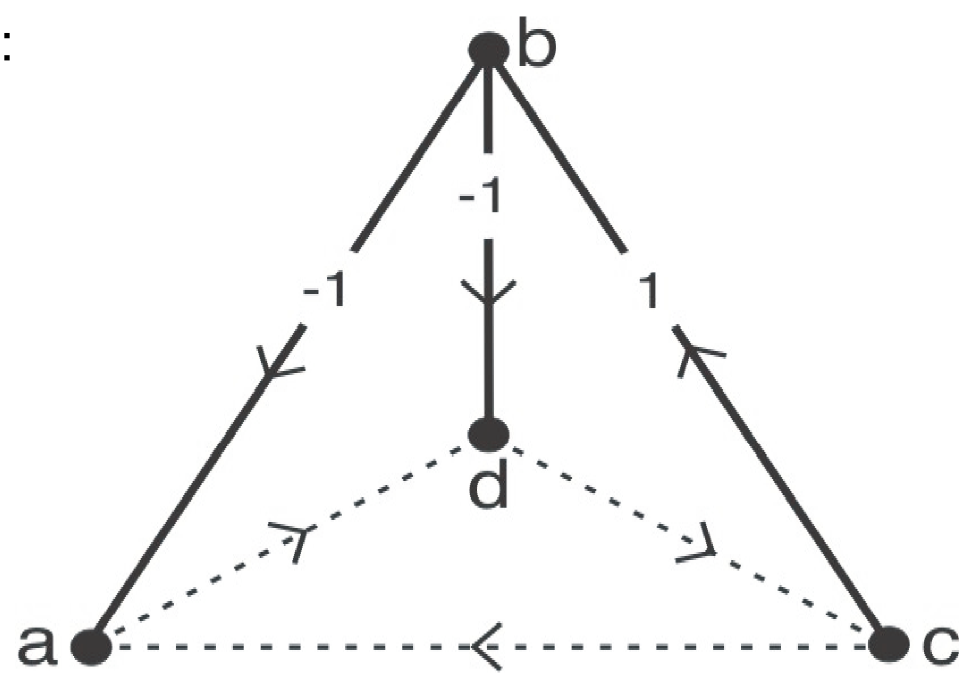


$L_{cocycle}$, the cocycle subspace

The basic cocycle for alternative x is a flow that labels:

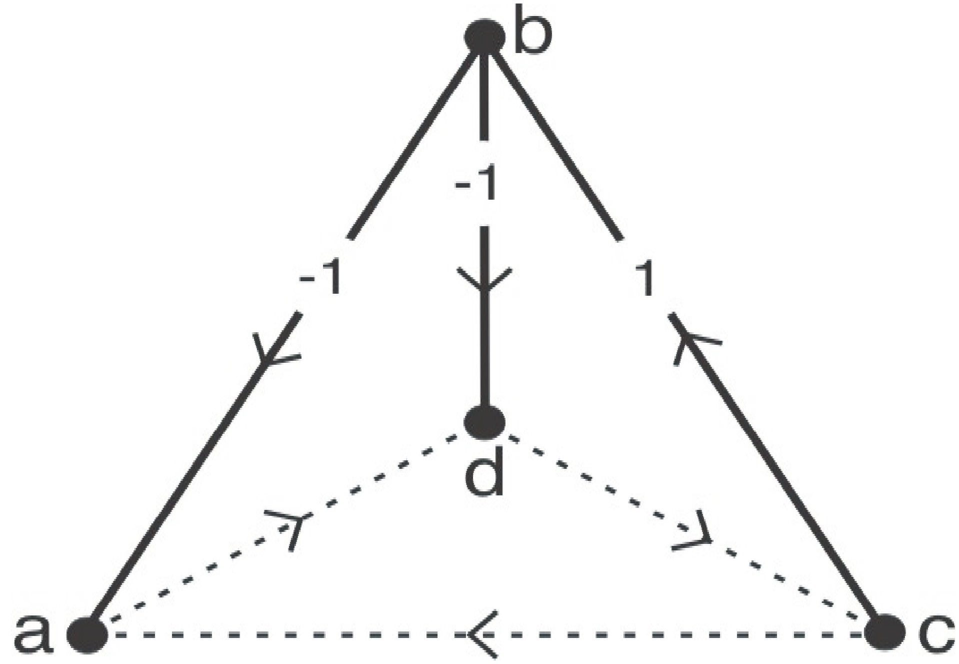
- each x ← y edge with +1
- each x → y edge with -1
- (and each edge not incident to x with 0)

Here is the basic cocyle for b:



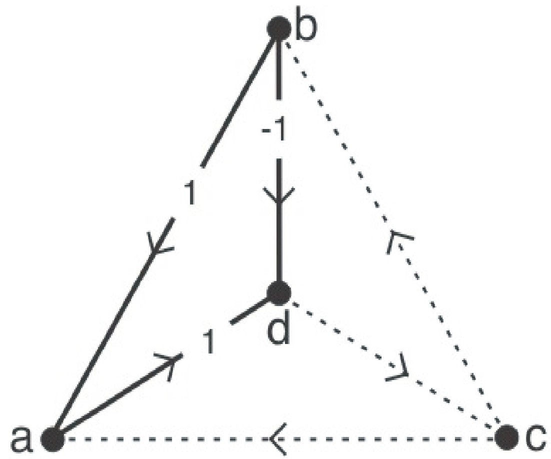
$L_{cocycle}$, the cocycle subspace

- A cocycle is a sum of scalar multiples of basic cocycles, and $L_{cocycle}$ is the space of all cocycles
- Equivalently, $L_{cocycle}$ is the linear span of the cocycles

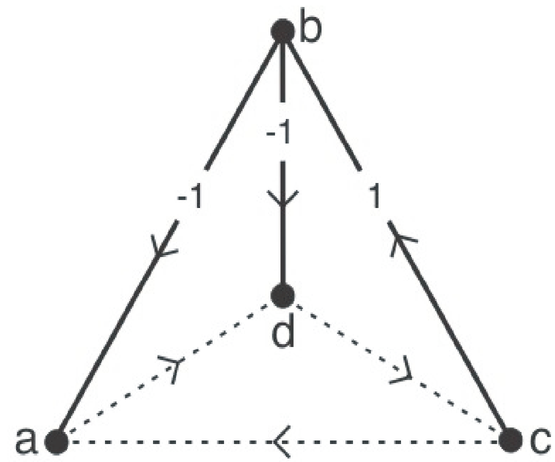


$L_{cycle} \perp L_{cocycle}$

- $v_{Left} = \langle 1, -1, 0, 1, 0, 0 \rangle$, $v_{Right} = \langle -1, -1, 1, 0, 0, 0 \rangle$
- $v_{Left} \bullet v_{Right} = (1)(-1) + (-1)(-1) + (0)(1) + (1)(0) + \ldots$
  $= -1 + 1 + 0 + 0 + 0 + 0 = 0$



Basic Cycle



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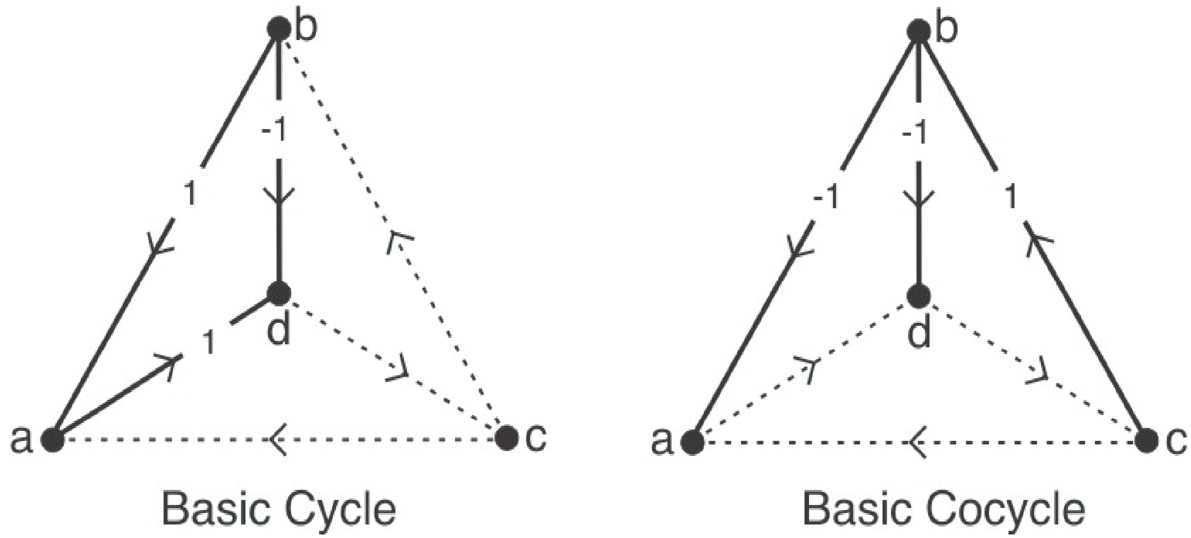
- So, any basic cycle is perpendicular to any basic cocycle
- So, $\mathbf{L}_{cycle} \perp \mathbf{L}_{cocycle}$



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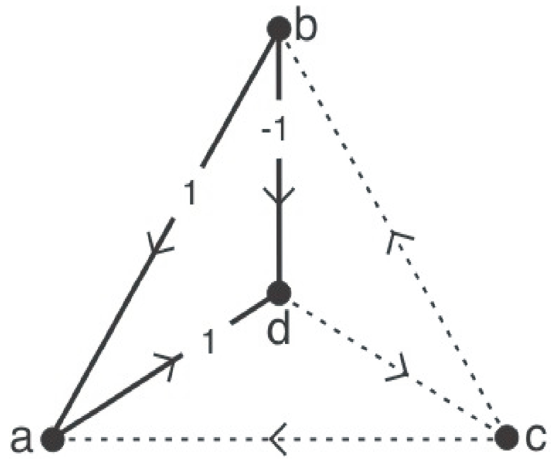
- So, any basic cycle is perpendicular to any basic cocycle
- So, $\mathbf{L}_{\text{cycle}} \perp \mathbf{L}_{\text{cocycle}}$
- In fact, $\mathbf{R}^6 = \mathbf{L}_{\text{cycle}} \oplus_{\perp} \mathbf{L}_{\text{cocycle}}$ (6 = 3 + 3)



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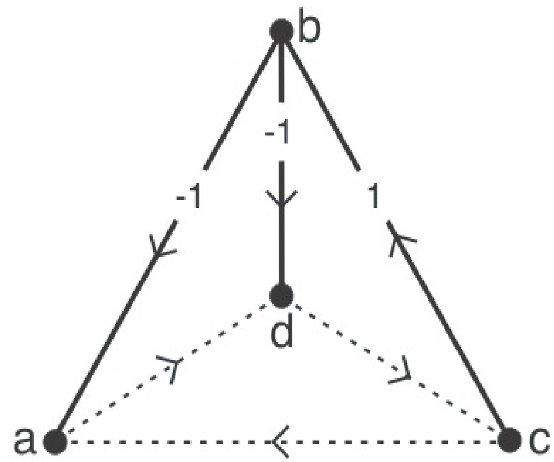

Basic Cocycle

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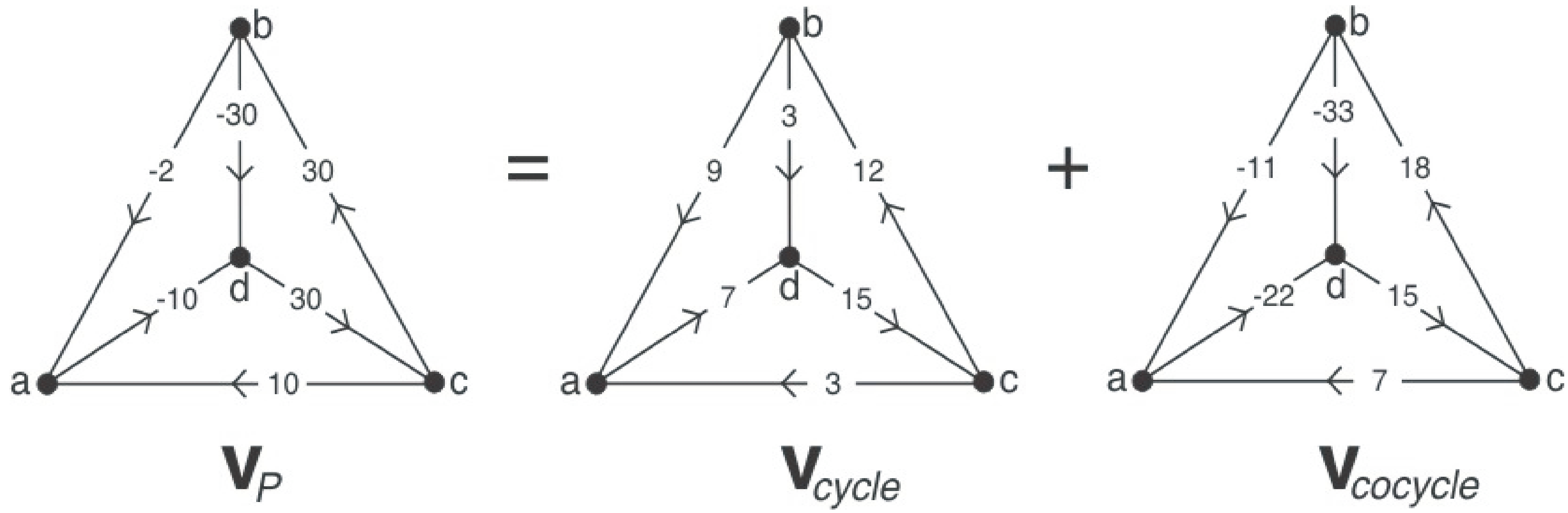
- $v_P = v_{cycle} + v_{cocycle}$
- $\langle -2, -30, 30, -10, 30, 10 \rangle =$
  $\langle 9, 3, 12, 7, 15, 3 \rangle \quad + \quad \underline{\langle -11, -33, 18, -22, 15, 7 \rangle}$



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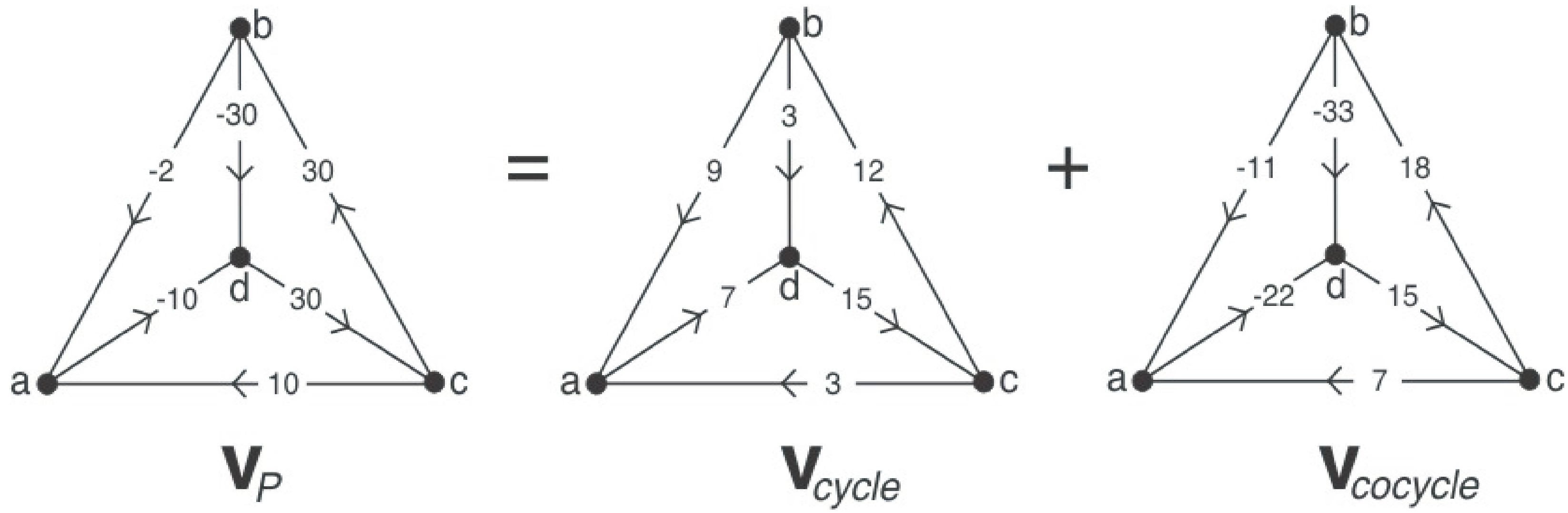
- Signs of labels on $v_{cocycle}$ ➔ Borda ranking
- These are the same as v_P: - - + - + +
- So Borda = Condorcet = b > a > c > d



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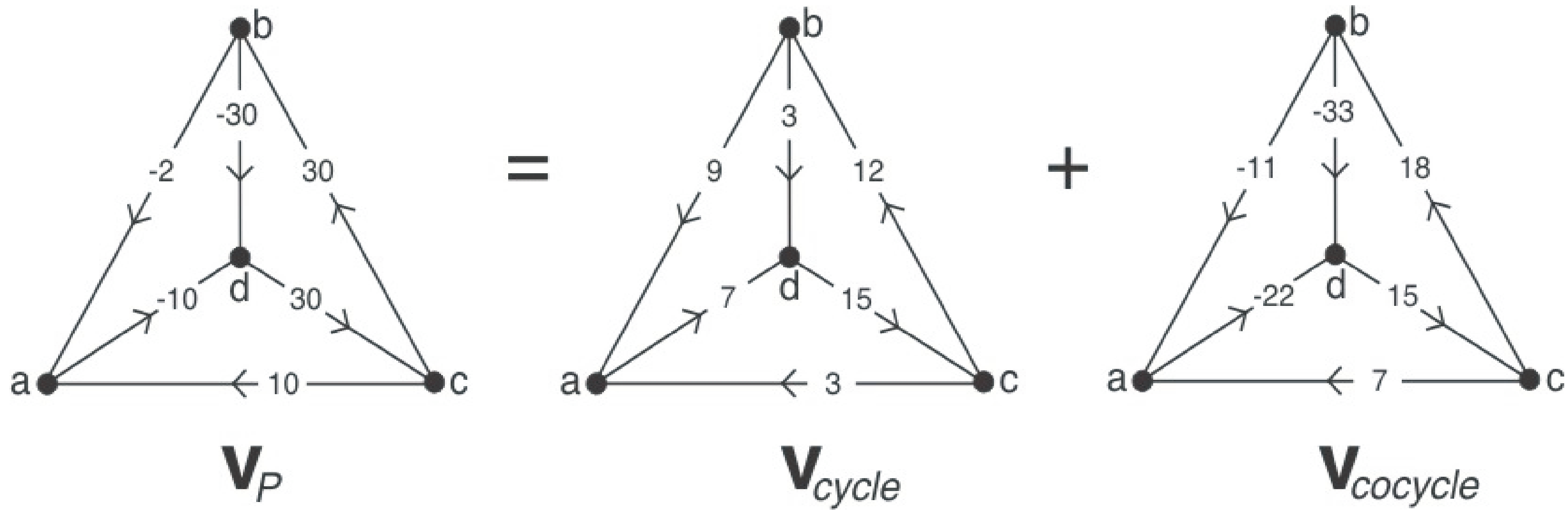

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1	$d > a > b > c$

The Profile Q

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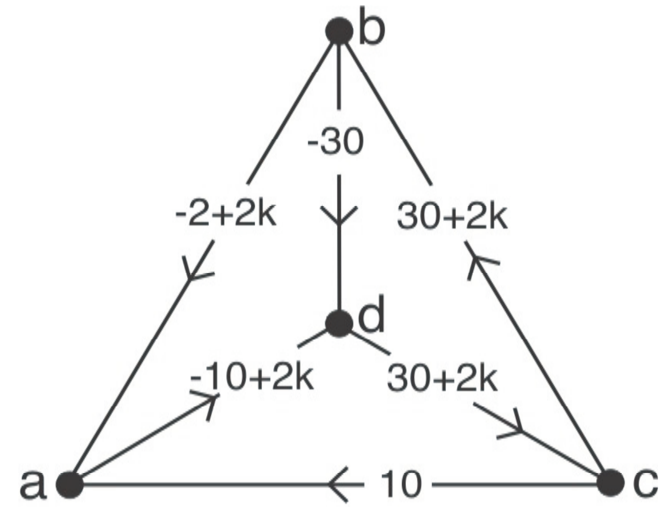

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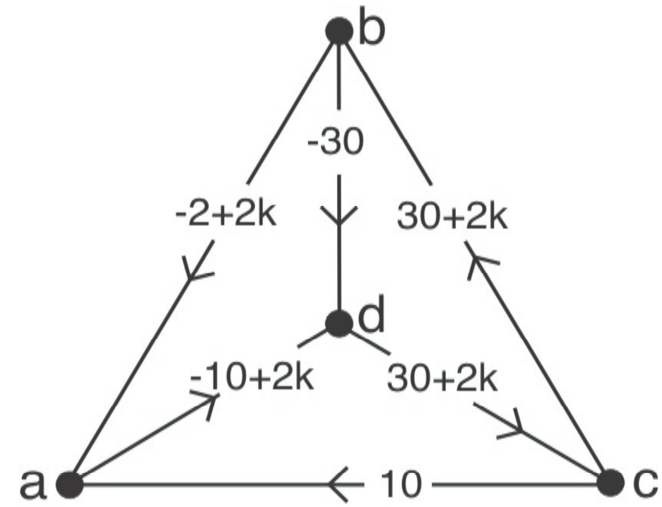

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- v_{Borda} is unchanged and Borda outcome remains b > a > c > d
- For k = 3, Condorcet outcome becomes a > b > c > d
- For k ≥ 6, Condorcet outcome becomes intransitive a>b>c>d>a

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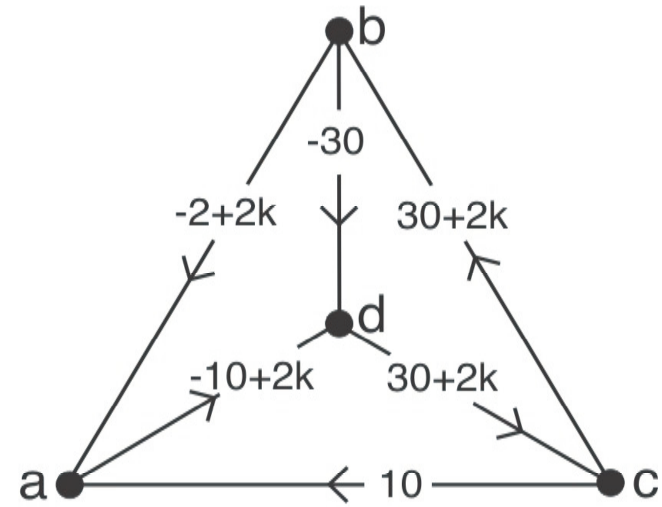

Orthogonal Decomposition IV
(Judgment aggregation)

- r binary issues
- A ballot = r-tuple of +1s, -1s
- Add ballots as vectors to obtain $\mathbf{v}_{\text{issue-wise}}$
- Signs of $\mathbf{v}_{\text{i-w}}$ components ➔ issue-wise outcome $\mathcal{IW}$

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- Orthogonal Decomposition: $\mathbf{v}_{\text{issue-wise}} = \mathbf{v}_{\text{infeas}} + \mathbf{v}_{\perp\text{infeas}}$
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- Signs of $\mathbf{v}_{\perp\text{infeas}}$ components $\Rightarrow$ "Borda JA" outcome $\mathcal{B}$
- $\mathbf{v}_{\text{infeas}}$ and $\mathbf{v}_{\perp\text{infeas}}$ have opposing tendencies
- $\mathbf{v}_{\text{infeas}} \gg \mathbf{v}_{\perp\text{infeas}}$ $\Rightarrow$ $\mathcal{IW}$ is infeasible
- $\mathbf{v}_{\text{infeas}} \ll \mathbf{v}_{\perp\text{infeas}}$ $\Rightarrow$ $\mathcal{IW}$ is feasible and = $\mathcal{B}$
- Intermediate situation?

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- $\mathbf{v}_{\text{infeas}} \ll \mathbf{v}_{\perp\text{infeas}}$ ➔ $\mathcal{IW}$ is feasible and = $\mathcal{B}$
- Intermediate situation? Similar to earlier k = 3 example: $\mathcal{IW}$ is feasible but $\neq \mathcal{B}$

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- <u>Question 1</u>: How are $\mathbf{L}_{\text{infeas}}$, $\mathbf{L}_{\perp\text{infeas}}$ defined?
- <u>Question 2</u>: Does Borda JA output <u>all</u> the feasible outcomes?
- <u>Question 3</u>: What is the effect/meaning of using Borda JA?

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- <u>Question 3</u>: What is the effect/meaning of using Borda JA?

- <u>ANS 1</u>: Code each infeasible outcome x as a vector $\mathbf{v}_x$ of +1s, -1s
  Then, $\mathbf{L}_{\text{infeas}}$ is the subspace (of $\mathbf{R}^r$) spanned by all $\mathbf{v}_x$ . . .
  and $\mathbf{L}_{\perp\text{infeas}}$ is whatever is left (the orthog complement of $\mathbf{L}_{\text{infeas}}$)
- <u>ANS 2</u>: Depends on context – is ***enough*** left in $\mathbf{L}_{\perp\text{infeas}}$?
- <u>ANS 3</u>: Depends on context

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- Orthogonal Decomposition: $\mathbf{v}_{\text{issue-wise}} = \mathbf{v}_{\text{infeas}} + \mathbf{v}_{\perp\text{infeas}}$
- $\mathbf{v}_{\text{infeas}}$, $\mathbf{v}_{\perp\text{infeas}}$ = orthogonal projections onto $\mathbf{L}_{\text{infeas}}$, $\mathbf{L}_{\perp\text{infeas}}$
- <u>Question 1</u>: How are $\mathbf{L}_{\text{infeas}}$, $\mathbf{L}_{\perp\text{infeas}}$ defined?
- <u>Question 2</u>: Does Borda JA output <u>all</u> the feasible outcomes?
- <u>Question 3</u>: What is the effect/meaning of using Borda JA?

- <u>ANS 1</u>: Code each infeasible outcome x as a vector $\mathbf{v}_x$ of +1s, -1s
  Then, $\mathbf{L}_{\text{infeas}}$ is the subspace (of $\mathbf{R}^r$) spanned by all $\mathbf{v}_x$ . . .
  and $\mathbf{L}_{\perp\text{infeas}}$ is whatever is left (the orthog complement of $\mathbf{L}_{\text{infeas}}$)
- <u>ANS 2</u>: Depends on context – is ***enough*** left in $\mathbf{L}_{\perp\text{infeas}}$?
- <u>ANS 3</u>: Depends on context

- ***Caution*** – all I have to go on so far is a few simple-minded examples

Borda JA: “3” examples

<u>1A</u> 3 projects with a budget limit

- Town swimming pool, Senior center, Repave roads
- One infeasible outcome: $\langle +1, +1, +1 \rangle$
- $\mathbf{L}_{\text{infeas}} = \{\langle x, x, x \rangle \mid x \in \mathbf{R}\}$ (a line in 3-space)
- $\mathbf{L}_{\perp\text{infeas}} = \{\langle u, v, w \rangle \in \mathbf{R}^3 \mid u + v + w = 0\}$ (a plane in 3-space)

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<u>1B</u> 3 propositional wffs, each T or F

- $\underline{p \wedge (\neg q \vee \neg r)}$, $\underline{q}$, $\underline{r \vee (\neg q \wedge \neg p)}$
- Two infeasible outcomes: $\langle +1, +1, +1 \rangle$, $\langle -1, -1, -1 \rangle$
- Same $\mathbf{L}_{infeas}$, $\mathbf{L}_{\perp infeas}$ as above

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1C 3 pairwise comparisons (Preference aggregation)

- $a > b$? $b > c$? $c > a$?
- Two infeasible outcomes: $\langle +1, +1, +1 \rangle$, $\langle -1, -1, -1 \rangle$
- Same $\mathbf{L}_{\text{infeas}}$, $\mathbf{L}_{\perp\text{infeas}}$ as above

Borda JA: “3” examples

$\mathbf{v}_{\text{i-w}} = \langle \mathcal{IW}_1, \mathcal{IW}_2, \mathcal{IW}_3 \rangle$, and

$\mathcal{IW}$ outcome = $\langle \text{sgn}(\mathcal{IW}_1), \text{sgn}(\mathcal{IW}_2), \text{sgn}(\mathcal{IW}_3) \rangle$

$\mathbf{v}_{\perp\text{infeas}} = \langle \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3 \rangle$, and

$\mathcal{B}$ outcome = $\langle \text{sgn}(\mathcal{B}_1), \text{sgn}(\mathcal{B}_2), \text{sgn}(\mathcal{B}_3) \rangle$

<u>Question 3a</u> How does $\mathcal{B}$ outcome differ from $\mathcal{IW}$ outcome?

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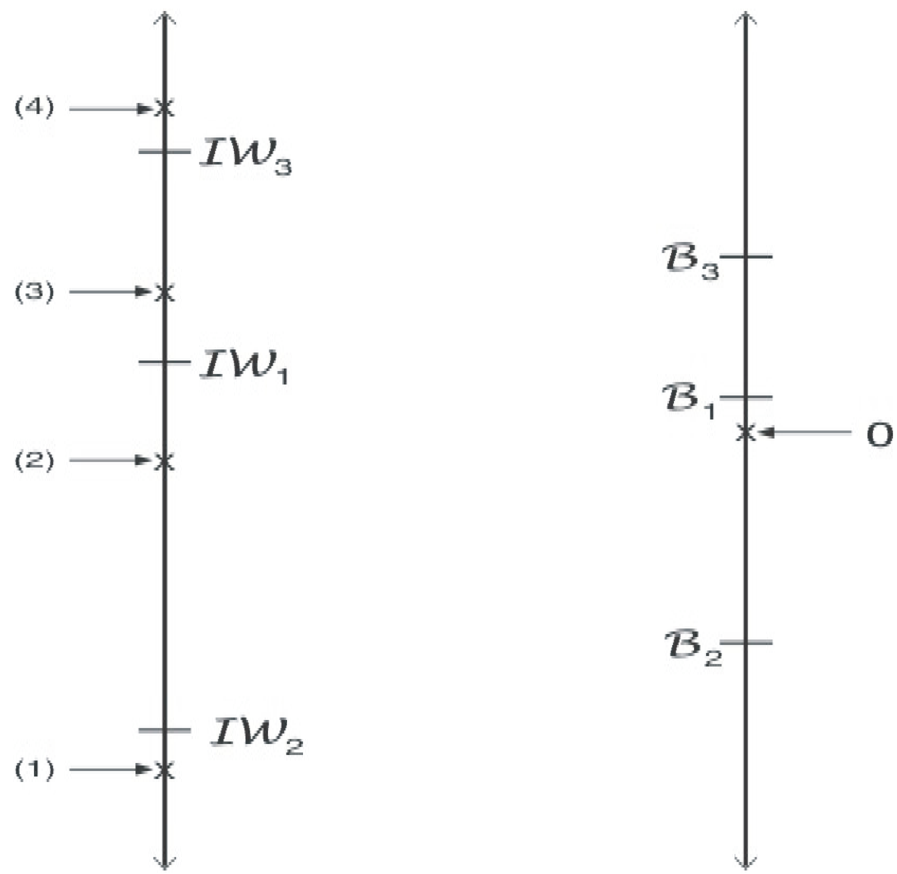
<u>Question 3a</u> How does $\mathcal{B}$ outcome differ from $\mathcal{IW}$ outcome?

<u>Question 3b</u> How does $\langle \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3 \rangle$ differ from $\langle \mathcal{IW}_1, \mathcal{IW}_2, \mathcal{IW}_3 \rangle$?

Question 3b $\langle \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3 \rangle$ VS $\langle \mathcal{IW}_1, \mathcal{IW}_2, \mathcal{IW}_3 \rangle$?

- In this (random) example, $\mathcal{IW}_3$ is greatest, $\mathcal{IW}_2$ is least, & middle # $\mathcal{IW}_1$ is twice as close to the top # as to the bottom #
- $\mathcal{B}_3$, $\mathcal{B}_2$, $\mathcal{B}_1$ will do the same (positive affine transform)



<u>Question 3b</u> $\langle \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3 \rangle$ VS $\langle \mathcal{IW}_1, \mathcal{IW}_2, \mathcal{IW}_3 \rangle$?

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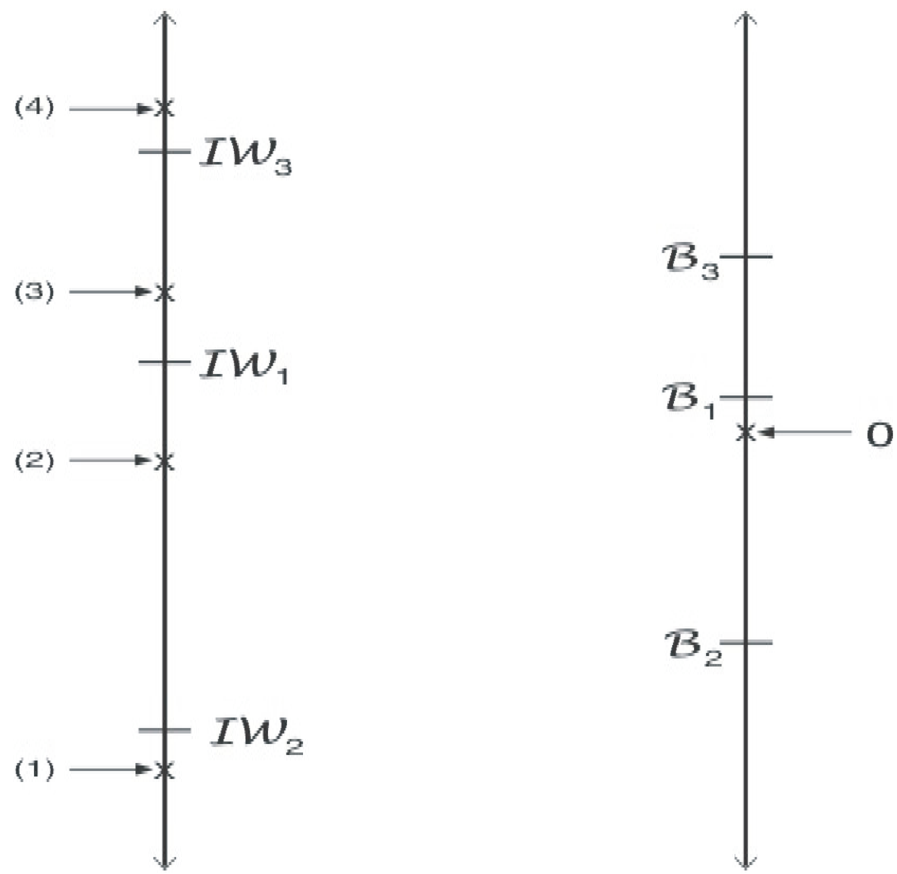
- $\mathcal{B}_3$, $\mathcal{B}_2$, $\mathcal{B}_1$ will do the same (positive affine transform)
- The "zero" can be *anywhere* among $\mathcal{IW}_3$, $\mathcal{IW}_2$, $\mathcal{IW}_1$ (so . . . possibly infeasible)



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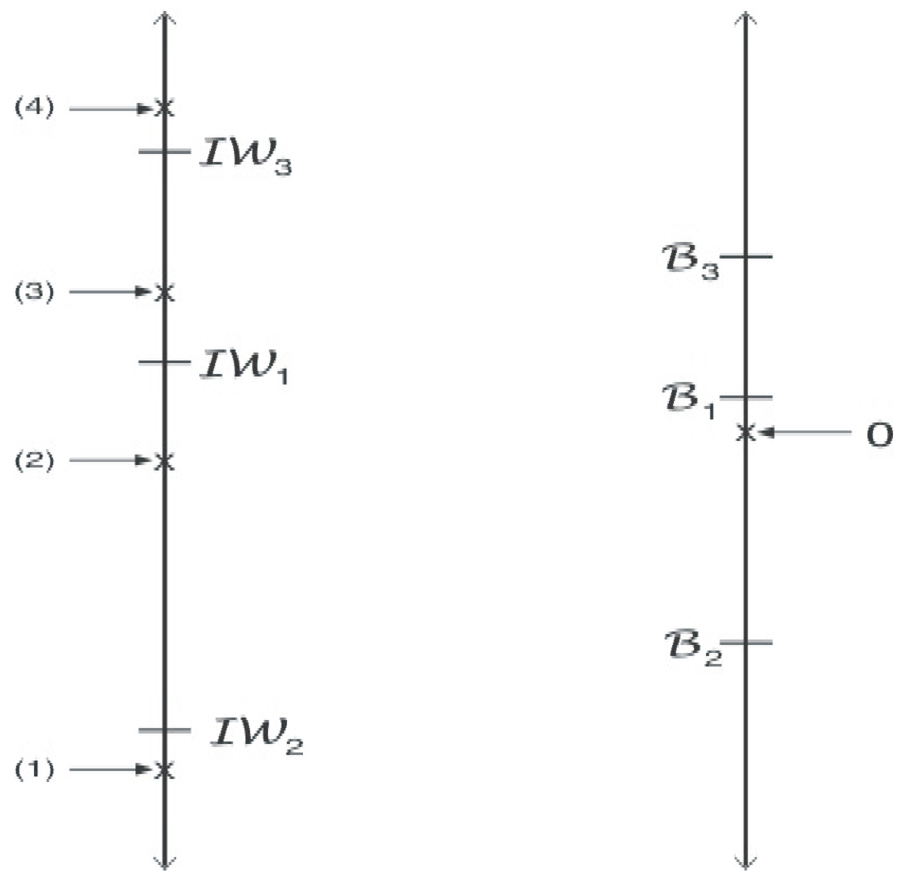
- The “zero” can be *anywhere* among $\mathcal{IW}_3$, $\mathcal{IW}_2$, $\mathcal{IW}_1$ (so . . . possibly infeasible)
- For $\mathcal{B}_3$, $\mathcal{B}_2$, $\mathcal{B}_1$, zero is at the mean: top # is above 0, bottom # is below 0, & middle # is above 0 ⇔ it’s closer to top than to bottom



0 is *anywhere* among $\mathcal{IW}_3$, $\mathcal{IW}_2$, $\mathcal{IW}_1$.
For $\mathcal{B}_3$, $\mathcal{B}_2$, $\mathcal{B}_1$, top # is above 0, bottom # is below 0,
& middle # is above 0 $\Leftrightarrow$ it's "above average"

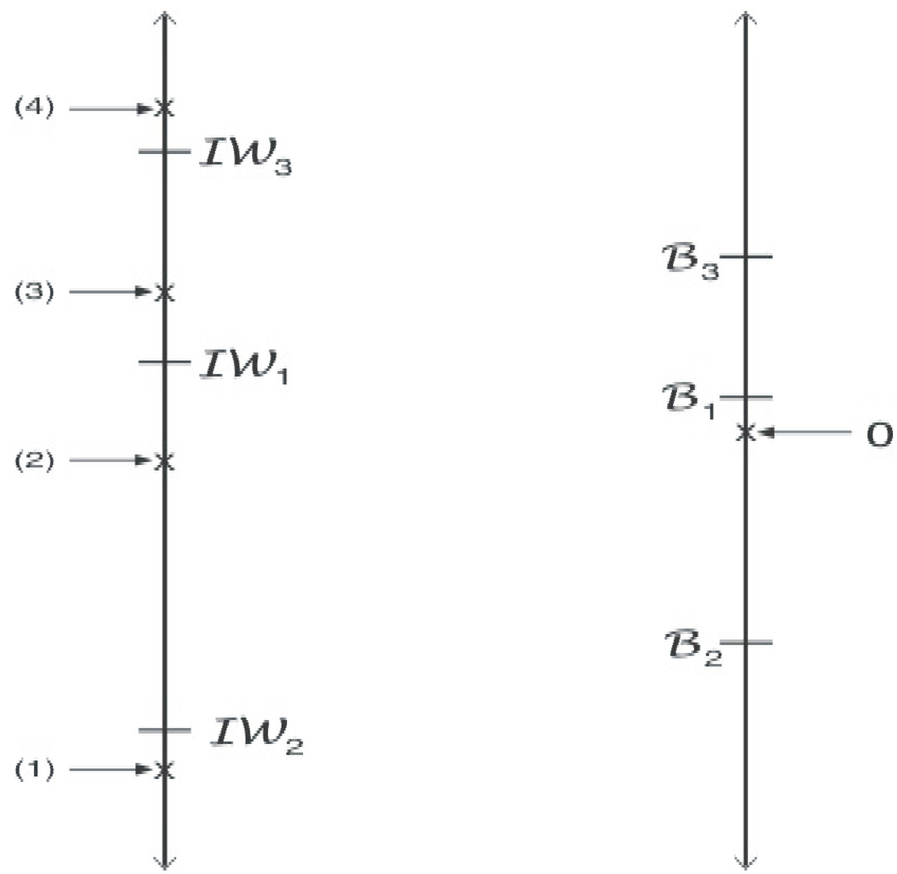
What is the $\mathcal{IW} \rightarrow \mathcal{B}$ effect?

Enforces feasibility + . . .

<u>1C</u> (Pref Aggreg)

Sign of $\mathcal{B}_1$ determines a > b VS b > a

So, the effect is large!



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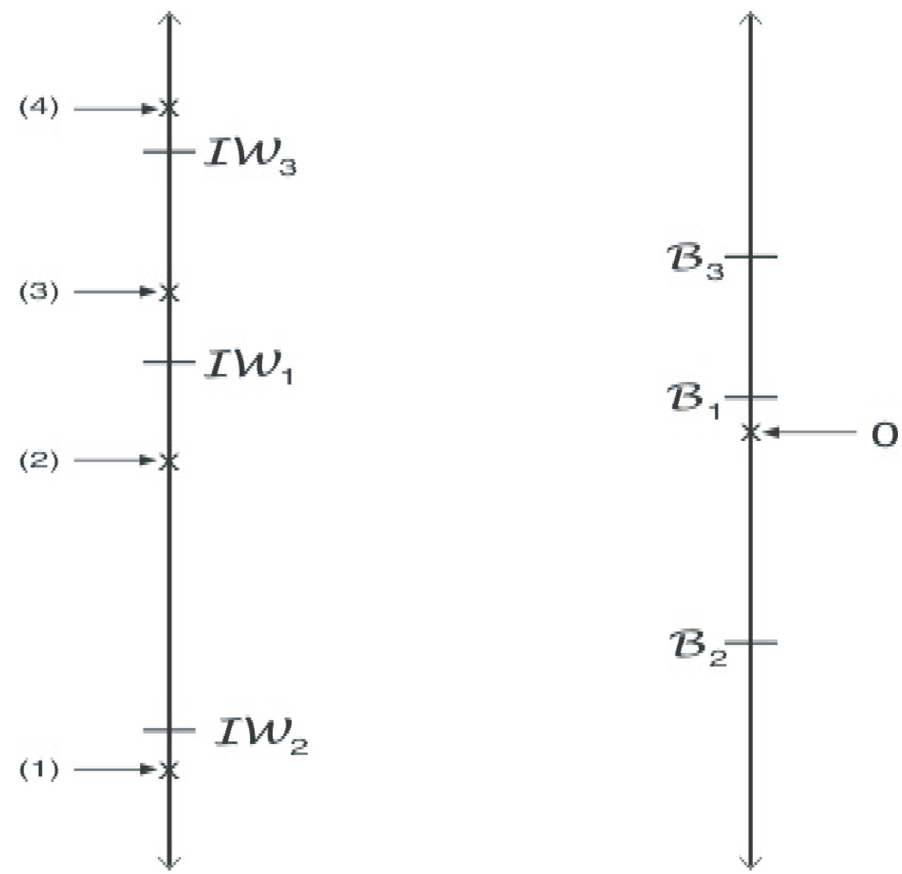
What is the $\mathcal{IW} \rightarrow \mathcal{B}$ effect?
Enforces feasibility + . . .

<u>1B</u> (Truth of three propositions)
Proposition #3 is true, and #2 is false. Sign of $\mathcal{B}_1$ determines truth of the "middle" proposition #1, according to whether the experts' certainty level for #1 better matches that of #3 or #2.

Is that reasonable? Depends on the probability distribution. If $\mathcal{IW}$ outcome is +++ or - - -, maybe yes. (Unknown bias towards + or - ?)



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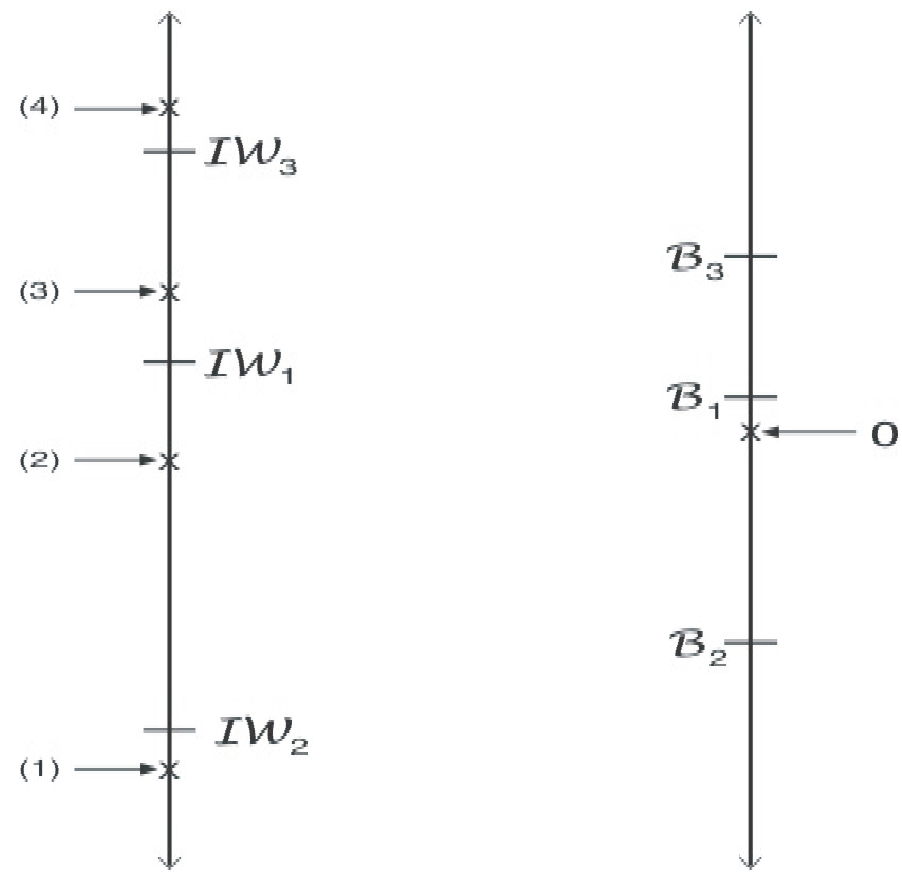
What is the $\mathcal{IW} \rightarrow \mathcal{B}$ effect?

Enforces feasibility + . . .

<u>1A</u> (Three projects w constraint)

$\mathcal{B}$ outcome will never be - - -, even though this outcome is feasible.

Borda JA seems inappropriate.



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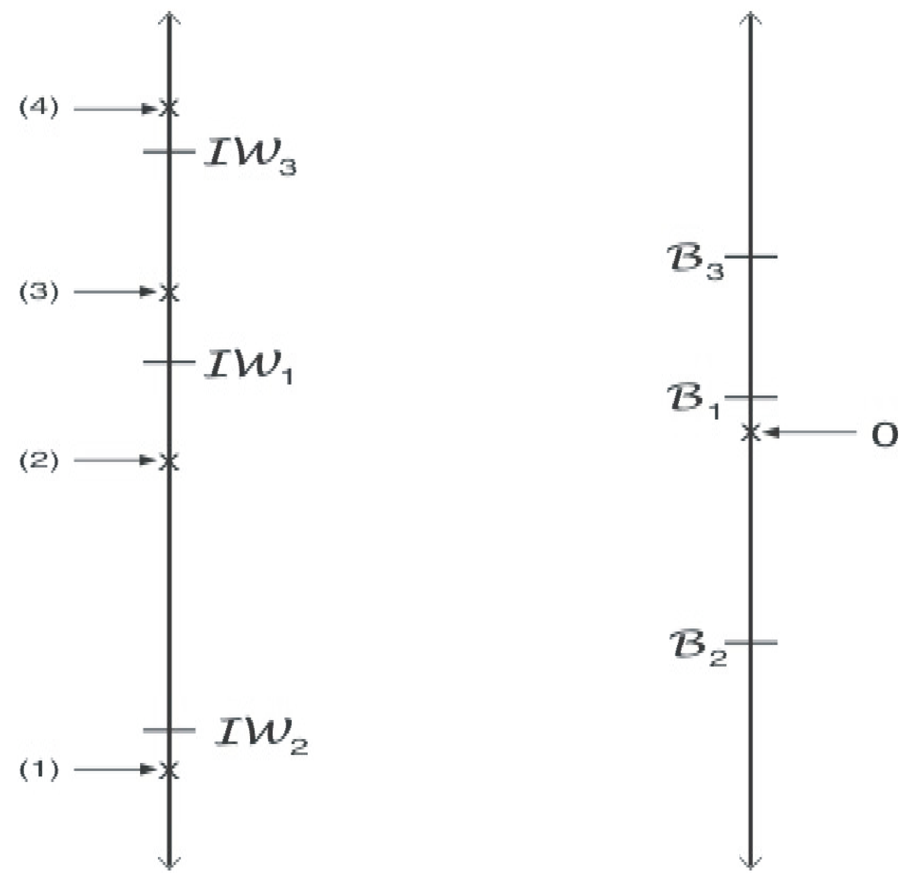
What is the $\mathcal{IW} \rightarrow \mathcal{B}$ effect?

Enforces feasibility + . . .

. . . (all three examples) . . .

An infeasible ballot can affect the $\mathcal{IW}$ outcome, and may make sense from the voter's p.o.v.

But such a ballot is wasted in Borda JA, as $\mathbf{v}_{\perp \text{infeas}} = 0$.



Tentative Conclusions

- It will be interesting to see what happens for other examples. (The picture to the right will not apply.)
- Expect: for some applications $\mathcal{B}$ will be inappropriate
- But it will have interesting implications for others
- And the normative implications of applying $\mathcal{B}$ will depend on more than just the mathematical structure – context matters