Allocating Public Goods via the Midpoint Rule

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Abstract

We study the properties of the following *midpoint rule* for determining discrete quantities of different public goods under a budget constraint. Each individual submits a proposal and the feasible allocations are ranked according to the sum of their distances (in the natural metric) to the individual proposals. One of the allocations with minimal aggregate distance is chosen. We prove that the midpoint rule is strategy-proof if all individuals have symmetric single-peaked preferences.

1 Introduction

to be written

2 Basic Notation and Definitions

A society can spend an amount L on the provision of k different public goods in discrete non-negative quantities. Throughout, we will assume that individuals have monotone preferences; moreover, for simplicity, we assume that the public goods are measured in money terms. Together, these assumptions allow us to model the allocation problem as the choice of an element of the $k-1$-dimensional simplex

$$X := \{x \in \mathbf{N}_0^k : \sum_{i=1}^{k} x_i = L\},$$

where x_i is the amount of public good i.

By $d(x,y) := \frac{1}{2}\sum_i \mid x_i - y_i \mid$ we denote the *distance* between x and y in the simplex. The set X is naturally endowed with a graph structure such that two points x and y are connected by an edge if and only if $d(x,y) = 1$ (see Figure 1 for the case $k = 3$). The metric d is then the natural graph distance given by the minimal number of edges needed to connect two points by a path. A path connecting two points with a minimal number of edges is called a *shortest path*; note that shortest paths need not be unique. We write xNy if x and y are neighbors in the graph, i.e. $xNy \Leftrightarrow d(x,y) = 1$. Furthermore, we write $xN_{ij}y$ if $x_i = y_i + 1$, $x_j = y_j - 1$ and $x_l = y_l$ for all $l \neq i, j$, i.e. if x and y are neighbors in

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the "ij-direction;" note that while the binary relation N is symmetric, the relations N_{ij} are asymmetric and satisfy $xN_{ij}y \Leftrightarrow yN_{ji}x$.

The set of all points that lie on some shortest path between x and y is also called the set of points *between x and y* and denoted by $[x,y]$. As is easily verified, we have

$$[x,y] = \{z : z_i \in [x_i, y_i] \text{ for all } i\}$$

(see Figure 1 where $[x,y] = \{x,y,z,z'\}$). The set of all points which are closer to x than to y, i.e. the set of all z such that $d(z,x) < d(z,y)$ will be denoted by $\rangle x,y\langle$. Note that $\rangle x,y\langle = \{z : x \in [z,y]\}$, i.e. a point is closer to x than to y if and only if x lies on a shortest path between the given point and y. For neighbors, the set $\rangle x,y\langle$ has the following particularly simple description (see Figure 1 in which $xN_{31}z$),

$$xN_{ij}z \Rightarrow \rangle x,y\langle = \{w : w_i \geq x_i \text{ and } w_j \leq x_j\}. \tag{1}$$

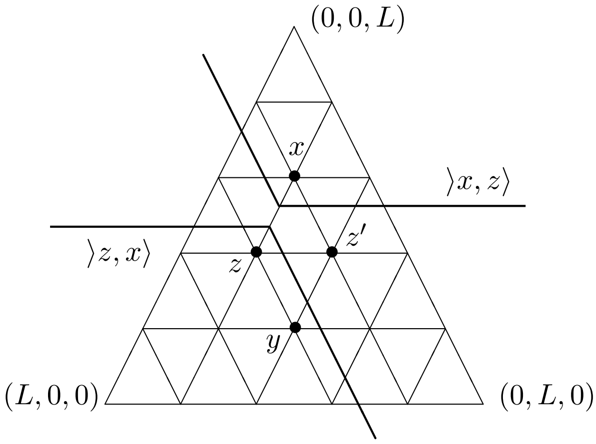

Figure 1: Illustrating the sets $[x,y]$, $\rangle x,z\langle$ and $\rangle z,x\langle$

The *midpoint rule* is defined as follows: Each individual submits a proposal $w \in X$ and allocations are ranked according to the sum of their respective distances to the individual proposals.[1] Formally, denote by p the distribution of the proposals on X, i.e. for each $w \in X$, p_w is the number of individuals who proposed w. Furthermore, for all $x \in X$, denote by $R(x) := \sum p_w d(x,w)$ the *remoteness* of x given p. An allocation $m(p) \in X$ is a *midpoint* if

$$m(p) = argmin_{x \in X} \sum_{w \in X} p_w d(x,w) = argmin_{x \in X} R(x),$$

i.e. a midpoint is a point with minimal remoteness. Let $M(p)$ denote the set of midpoints. Evidently, $M(p)$ need not be a singleton but it is always non-empty.

[1]The midpoint rule is related to other well-known aggregation procedures. In median spaces, the midpoint rules coincides with issue-by-issue majority voting (see Nehring and Puppe, 2007). In particular, in the classical case of single-peaked preferences on a line the midpoint rule chooses the median of the individual peaks. Our main theorem below thus entails the classical median voter theorem as a special case. Also outside median spaces versions of the midpoint rule have been considered in the literature, for instance by Kemeny (1959) in the context of the aggregation of preference orderings employing a specific metric on the space of preference relations.

3 The Structure of the Set of Midpoints

In this section, we prove two important properties of the set of midpoints. First, the set of midpoints is convex, i.e. all points on a shortest path between two midpoints must also be midpoints (Proposition 1). Secondly, the set of midpoints is "locally determined," i.e. whether an allocation is among the set of midpoints can be decided by comparing it only to the set of its neighbors (Proposition 2).

In order to prove these results, we need two auxiliary lemmas. For two neighbors x and y, we write xMy if $p(\rangle x, y\rangle) > p(\rangle y, x\rangle)$, i.e. if the set of allocations which are closer to x than to y has more mass than the set of allocations which are closer to y than to x. Moreover, we write xIy if neither xMy nor yMx. The first lemma shows that the ranking among neighbors induced by the midpoint rule simply corresponds to majority voting, where an individual is construed as "voting" for x in a binary comparison with y if the individual's proposal is closer to x than to y.

Lemma 1 *For any distribution p and any two neighbors x and y, $p(\rangle x, y\rangle) - p(\rangle y, x\rangle) = R(y) - R(x)$. In particular, xMy if and only if $R(y) > R(x)$.*

Proof. The assertions are immediate from the following observations: for all $w \in \rangle x, y\rangle$, $d(x, w) - d(y, w) = -1$, for all $w \in \rangle y, x\rangle$, $d(x, w) - d(y, w) = 1$, and for all other $w \in X$, $d(x, w) - d(y, w) = 0$. $\square$

The second lemma establishes a specific relationship in the respective outcomes of the majority vote among neighbors "in the same direction" (see Figure 2).

Lemma 2 *Let x, y be such that $x_i > y_i$ and $y_j > x_j$. If $xN_{ij}x'$ and $y'N_{ij}y$, then $\rangle y', y\rangle \supseteq \rangle x, x'\rangle$ and $\rangle y, y'\rangle \subseteq \rangle x', x\rangle$. Moreover, $xMx' \Rightarrow y'My$ and $yMy' \Rightarrow x'Mx$.*

Proof. By (1) we have

$$\begin{aligned}
\rangle y', y\rangle &= \{w : w_i \geq y_i + 1, w_j \leq y_j - 1\}, \\
\rangle x, x'\rangle &= \{w : w_i \geq x_i, w_j \leq x_j\}, \\
\rangle y, y'\rangle &= \{w : w_j \geq y_j, w_i \leq y_i\}, \\
\rangle x', x\rangle &= \{w : w_j \geq x_j + 1, w_i \leq x_i - 1\},
\end{aligned}$$

which gives the first assertion. From this, we obtain $p(\rangle x, x'\rangle) \leq p(\rangle y', y\rangle)$ and $p(\rangle x', x\rangle) \geq p(\rangle y, y'\rangle)$. Thus, if $p(\rangle x, x'\rangle) > p(\rangle x', x\rangle)$, then $p(\rangle y', y\rangle) > p(\rangle y, y'\rangle)$, and if $p(\rangle y, y'\rangle) > p(\rangle y', y\rangle)$, then $p(\rangle x', x\rangle) > p(\rangle x, x'\rangle)$, which proves the second claim. $\square$

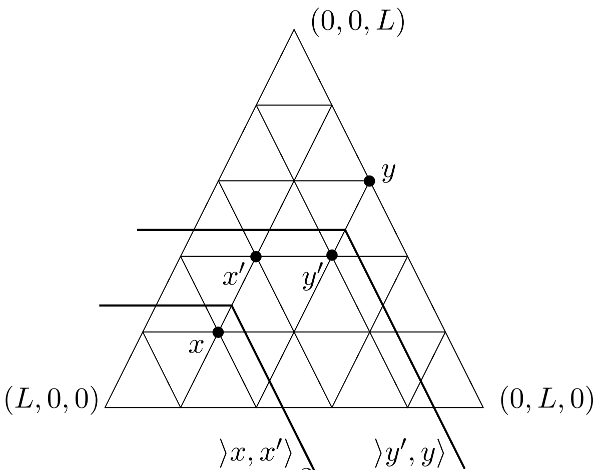

Figure 2: Illustrating Lemma 2

Proposition 1 *The set $M(p)$ of midpoints is convex, i.e. all points on a shortest path between two midpoints belong to the set of midpoints as well.*

Proof. Let x, y be two distinct midpoints and consider a shortest path between them. Suppose that there exists a point on this shortest path that is not a midpoint, i.e. that has a strictly larger remoteness than either x and y. Then, there must exist neighbors z, z' along this path such that $R(z) < R(z')$, hence by Lemma 1, zMz'. Say that z and z' are neighbors in ij-direction, i.e. $zN_{ij}z'$. Since $z, z' \in [x, y]$, this implies that either ($x_i > y_i$ and $x_j < y_j$) or ($y_i > x_i$ and $y_j < x_j$). Without loss of generality assume the former and choose y' such that $y'N_{ij}y$. By Lemma 2 we obtain $y'My$ contradicting the assumption that $y \in M(p)$. □

The next result shows that in the context of our public goods allocation problem the set of midpoints can be determined by "local" majority voting, i.e. by a pairwise comparison of any given point with all its neighbors.

Proposition 2 *An allocation x is among the midpoints if and only if x does not lose in pairwise comparison against any of its neighbors, i.e. $x \in M(p) \Leftrightarrow [\neg(yMx)$ for all y such that $yNx]$.*

Proof. By definition, a midpoint cannot lose in pairwise comparison against any of its neighbors. Conversely, let $x \in X$ be such that xMx' or xIx' for all x' with xNx'. We will show that $x \in M(p)$ by contradiction. Thus, suppose that there is some other point $z \in X$ with a strictly lower remoteness. Choose two neighbors y, y' on a shortest path from x to z such that yMy' and $y' \in [x, y]$, and assume without loss of generality that $y'N_{ij}y$. Since y' is between x and y we must have $x_i > y_i$ and $x_j < y_j$. Now consider x' with $xN_{ij}x'$. By Lemma 2, yMy' implies $x'Mx$ which contradicts the assumption that x does not lose against any of its neighbors. □

We conclude this section with the observation that a neighbor of a midpoint is itself a midpoint if and only if it does not lose against it in pairwise comparison.

Fact 1 *Let $x \in M(p)$ and yNx. Then, $y \in M(p) \Leftrightarrow xIy$.*

Proof. By Lemma 1, y has the same remoteness as x if and only if xIy. This proves the claim. □

4 Limited Manipulability of the Midpoint rule

Can an individual by submitting an appropriate non-truthful proposal influence the outcome of the midpoint rule to his or her advantage? Whether this will be so depends on the assumptions on the individuals' preferences; in general, the answer is, yes, i.e. in general the midpoint rule is not strategy-proof. However, in this section we prove the following remarkable property of the midpoint rule in the present context: Suppose that each individual has a unique most preferred allocation, his or her "peak." Then, while individuals can change the shape of the set of midpoints by unilaterally submitting different proposals, it is not possible to move this set "closer" to one's true peak by submitting a non-truthful proposal, as follows.

Fix an individual, say individual h with peak x, and denote by $M(p)$ the set of midpoints given the distribution p of proposals in which individual h submits x. Furthermore, denote by $\tilde{p}$ the distribution of proposals that differs from p only in the proposal of individual h who proposes, say $\tilde{x} \neq x$. Then, the element of $M(p)$ that is closest to x is at least as close to x than the closest element in $M(\tilde{p})$; similarly, the element of $M(p)$ that is farthest away from x is at least as close to x than the farthest element in $M(\tilde{p})$.

Theorem 1 *For all $\tilde{x} \neq x$, $\min_{w \in M(p)} d(x,w) \leq \min_{w \in M(\tilde{p})} d(x,w)$, i.e. non-truthful proposals cannot move the closest midpoint closer to one's true peak, and $\max_{w \in M(p)} d(x,w) \leq \max_{w \in M(\tilde{p})} d(x,w)$, i.e. non-truthful proposals cannot move the farthest midpoint closer to one's true peak.*

Proof. Let z be a midpoint with minimal distance to x, say $d(x,z) = r$. If $r = 0$ there is nothing to show, thus assume $r \geq 1$. For any i, j such that $z_i > x_i$ and $z_j < x_j$, let z_{ij} be the neighbor of z in direction of x, i.e. $zN_{ij}z_{ij}$. Moreover, denote by Z_x^- the set of all such neighbors, i.e.

$$Z_x^- \quad = \quad \{y \in X : yNz \text{ and } d(x,y) = r - 1\}.$$

Since z is a closest midpoint to x, we have zMy for all $y \in Z_x^-$ under the distribution p, i.e. provided that the true peak x is reported. Since x already supports any $y \in Z_x^-$ in the pairwise comparison with z, i.e. since $x \in \rangle y, z \langle$, we must have $z\tilde{M}y$ also under the distribution $\tilde{p}$, i.e. when $\tilde{x}$ is reported instead of x. By Lemma 2, $z\tilde{M}z_{ij}$ implies $y'\tilde{M}y$ for all $y \in \rangle z_{ij}, z \langle$, where $y'N_{ij}y$. This shows that, for any z_{ij}, no element of $\rangle z_{ij}, z \langle$ can be midpoint under $\tilde{p}$. The proof of the first assertion is completed by noting that

$$\{y \in X : d(x,y) < r\} \subseteq \bigcup_{z_{ij} \in Z_x^-} \rangle z_{ij}, z \langle \, .$$

Let now z be a midpoint with maximal distance to x, say $d(x,z) = r$. If the maximal distance of a midpoint to x is to be reduced by non-truthfully reporting $\tilde{x}$, z has to lose in pairwise comparison to some of its neighbors under $\tilde{p}$ by Proposition 2. Partition the set of neighbors of z as follows,

$$\begin{aligned} Z_x^- &= \{y \in X : yNz \text{ and } d(x,y) = r - 1\}, \\ Z_x^0 &= \{y \in X : yNz \text{ and } d(x,y) = r\}, \\ Z_x^+ &= \{y \in X : yNz \text{ and } d(x,y) = r + 1\}. \end{aligned}$$

Since the truthful report x already supports any element in Z_x^- in pairwise comparison against z under p, z cannot lose against such element under $\tilde{p}$. Thus, assume that, by reporting $\tilde{x}$, z loses against some $z' \in Z_x^+$, say $z'\tilde{M}z$ and $z'N_{ij}z$. Then, by Lemma 2, any element $w \in \rangle z, z' \langle$ loses in pairwise comparison against its neighbor in ij-direction, i.e. $w'\tilde{M}w$ if $w'N_{ij}w$. This shows that no element of $\rangle z, z' \langle$ can be midpoint under $\tilde{p}$. Since

$$\{y \in X : d(x,y) < r\} \subseteq \rangle z, z' \langle \, ,$$

this shows that any midpoint under $\tilde{p}$ is at least as far away from x than z. Finally, assume that by reporting $\tilde{x}$, z loses against some $z' \in Z_x^0$. Since x neither supports z nor z' in pairwise comparison, $z'\tilde{M}z$ is only possible if $z'Iz$. By Fact 1 above, this implies that z' was already a midpoint under p. There are now two possible cases. Either z' remains a midpoint under $\tilde{p}$ in which case the claim is proved, or z' loses against one of its neighbors. In the latter case, repeated application of the arguments just given shows that eventually a midpoint with distance at least r to x is obtained. $\square$

5 Strategy-Proofness

The result of the previous section has the immediate implication that the midpoint rule is strategy-proof provided that all individuals have "symmetric" single-peaked preferences of the form $y \succeq z \Leftrightarrow d(x,y) \leq d(x,z)$ for some $x \in X$, and provided that strategy-proofness is defined with respect to any extended preference relation $\succeq^*$ over subsets of X satisfying

$$M \succeq^* \tilde{M} \text{ whenever } [\min M \succeq \min \tilde{M} \text{ and } \max M \succeq \max \tilde{M}].$$

The midpoint rule is not strategy-proof for larger domains of single-peaked preferences, as shown by the following examples (see Figures 3, 4 and 5).

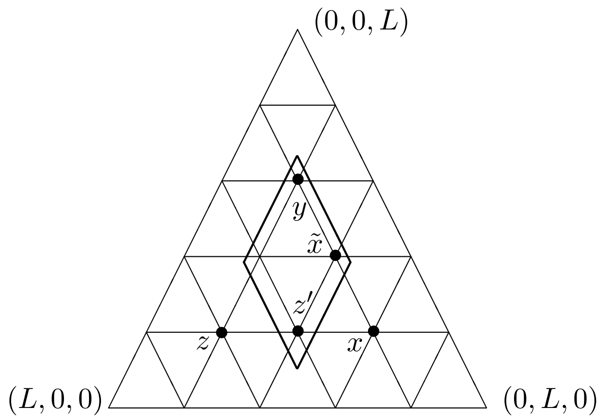

Figure 3: Manipulating the set of midpoints

Suppose that in Figure 3, two individuals have their peak at y, one individual at z, one individual at z' and another individual, say h, at x. The resulting set of midpoints the diamond-shaped set $[y, z']$. By non-truthfully reporting $\tilde{x}$ instead of x individual h removes z' from the set of midpoints, changing it to the points in the dashed triangle shown in Figure 3 (consisting of y, $\tilde{x}$ and their joint neighbor in direction of z). Whether this will be to the advantage of h depends on h's preferences *over sets* of allocations. For instance, it could be that h has (generalized) single-peaked preferences over allocations (i.e. prefers allocations closer to her peak x, see Nehring and Puppe, 2007) and furthermore prefers $\tilde{x}$ to the equally distant z'. In that case, h might prefer excluding z' from the set of midpoints (even if this does neither change the best nor the worst allocation among the midpoints).

A similar example of a possible violation of strategy-proofness of the midpoint rule is shown in Figure 4. Again suppose that two individuals have their peak at y, one individual at z, one individual at z' and another individual, say h, at x. The resulting set of midpoints is the quasi-diamond-shaped set indicated in Figure 4. By non-truthfully reporting $\tilde{x}$ instead of x individual h changes the set of midpoints to the dashed figure. Again, it is easily verified that this yields a possible violation of strategy-proofness for appropriately specified preferences over sets of outcomes. Note that, in contrast, to the example shown in Figure 3, the manipulated set of midpoints (i.e. the dashed set) is not a proper subset of the set of truthful midpoints.

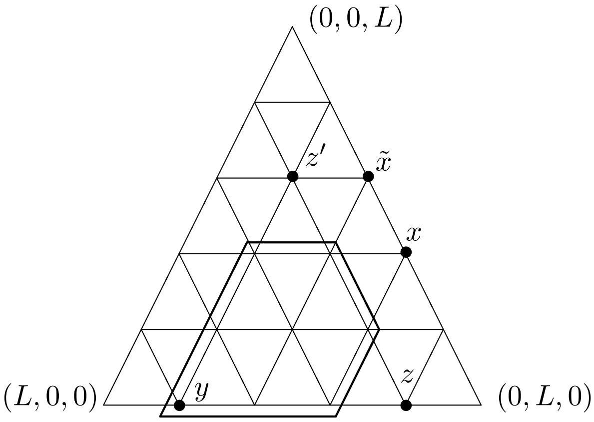

Figure 4: Another manipulation of the set of midpoints

The possible violations of strategy-proofness shown in Figures 3 and 4 hinge on an appropriate specification of preferences over *sets* of outcomes, and in particular on the impact of potential outcomes that are intermediate in preference between the best and worst elements. The following example represents a more robust violation of strategy-proofness. Consider the situation depicted in Figure 5. If two voters have their peak at y, and one voter at x and $\tilde{x}$, respectively, the unique truthful midpoint is y. However, if the voter with peak x, say individual h, strictly prefers $\tilde{x}$ to y, it seems safe to assume that h would strictly prefer to report $\tilde{x}$ instead of x, because this would change the set of midpoints from $\{y\}$ to $\{\tilde{x}, y\}$.

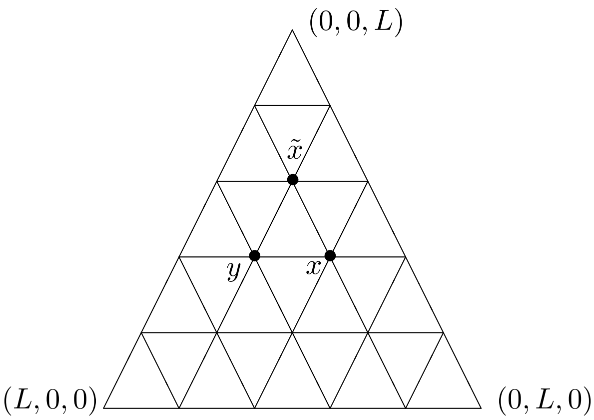

Figure 5: A more robust violation of strategy-proofness

6 More on the Structure of the Set of Midpoints

The set of midpoints has further interesting properties summarized in this section. First, we show that allocations strictly between two midpoints can have no mass, i.e. no individual can have proposed them.

Fact 2 *Suppose that $x, z \in M(p)$, $y \in [x, z]$ and $y \notin \{x, y\}$, then $p_y = 0$.*

Proof. Without loss of generality assume that $xNyNz$, $x_i > z_i$, $z_j > x_j$ and $xN_{ij}y$. Let z' be the element in X such that $z'N_{ij}z$ (z' may be equal to y). By Proposition 1, $z' \in M(p)$.

By Lemma 2, $\rangle x, z'\rangle \subset \rangle y, z\rangle$ and $\rangle z', x\rangle \supset \rangle z, y\rangle$. Suppose, by contradiction, that $p_y > 0$. Then $p(\rangle y, z\rangle) > p(\rangle x, z'\rangle)$ and $p(\rangle z, y\rangle) < p(\rangle z', x\rangle)$ which implies yMz, contradicting the fact that z is a midpoint. $\square$

Next, we give a sufficient condition for the set of midpoints to be "small" in the sense that all midpoints have distance of at most 1 to each other. Let $diamM(p) := \max_{x,y\in M(p)} d(x, y)$ denote the *diameter* of the set of midpoints, and say that a subset $Y \subseteq X$ is *connected* if, for each pair $x, y \in Y$, there is a path connecting x and y that lies entirely in Y.

Fact 3 *If supp(p) is connected, then* $diamM(p) \leq 1$.

Proof. Suppose $diamM(p) \geq 2$. Then, there exist $x, y, z \in M(p)$ with xNy, yNz, $x_i > z_i$, $z_j > x_j$ and $xN_{ij}y$. Denote by z' the point in X with $z'N_{ij}z$. By Proposition 1 $z' \in M(p)$. By Lemma 2 we have $\rangle x, y\rangle \subset \rangle z', z\rangle$. Since all these points are midpoints one obtains xIy, $z'Iz$. Because a midpoint strictly wins against neighbors which are not in the set of midpoints and (by Fact 2) there cannot be any mass on allocations which are on a shortest path between midpoints, one can easily show that $p(\rangle x, y\rangle) > 0$ and $p(\rangle z, z'\rangle) > 0$. Since *supp*$(p)$ is connected there exists a path from $\rangle x, y\rangle$ to $\rangle z, z'\rangle$ in $X\setminus \{\rangle x, y\rangle \cup \rangle z, z'\rangle\}$ which is in *supp*(p). As in Fact 2, one obtains $p(\rangle z', z\rangle) > p(\rangle z, z'\rangle)$ which contradicts the assumption. $\square$