Condorcet Domains:
The Mathematics of Coherent Collective Decision-Making

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Foreword by Hervé Moulin

At a critical moment in the history of democracy, Condorcet argued that pairwise majority comparisons is the correct interpretation of the 'volonté générale', and alerted us that this may result in 'irrational' cycles. Two centuries later, Arrow's [1951] famous impossibility theorem generalises this observation: any voting rule that decides to prefer a over b solely from the list of individual preferences over just this pair, must also produce some irrational cycles.

This difficulty disappears if all individual preferences are single-peaked with respect to a common alignment of the outcomes [Black, 1948]. Clemens Puppe and Arkadii Slinko's review of the vast body of research poised to discover other domains of preferences where majority comparisons do not cycle is masterful, comprehensive, and the first of its kind.

The theory of Condorcet domains straddles the social sciences and discrete mathematics. Social scientists will learn, already in Chapter 1, why discrete convexity, median graphs, the Bruhat lattice of permutations, and more, are key to their structure. Mathematicians will learn in Part II their importance for social choice theory à la Arrow and incentive compatibility properties of voting rules. All will discover in Part I no less than five different families of Condorcet domains each with its own flavour and state of the art characterisation results.

This interdisciplinary volume is already a canonical reference in both communities.

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Introduction

In his celebrated "Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix" [1785] the mathematician, philosopher and public intellectual Marie Antoine Nicolas Caritat Marquis de Condorcet proposed and investigated pairwise majority voting as a method of reaching collective decisions. Both from a normative and from a positive standpoint, majority voting is a particularly attractive aggregation method, satisfying key desirable properties such as anonymity, neutrality, positive responsiveness and robustness against strategic manipulation. However, as already noticed by Condorcet himself, there are preference combinations in which pairwise majority voting results in cycles and therefore renders no choice justified at the collective level; this is commonly known as the 'Condorcet paradox.'

Condorcet *domains* are sets of linear orders on a given set of alternatives such that, if all voters are known to have preferences over alternatives represented by linear orders from that set, the pairwise majority relation is acyclic. In other words, Condorcet domains are the preference domains that systematically avoid the Condorcet paradox. This also means that preference aggregation on Condorcet domains is not affected by the difficulties that plague social choice theory otherwise; in particular, non-dictatorial aggregation of individual preferences into a social preference á la Arrow is possible on every Condorcet domain, while it is not possible in general according to his famous result [Arrow, 1951].

If we think of preference domains as representing the potential preferences of a society, the Condorcet domains thus correspond to the societies that are able to base all their collective decisions on one simple and fundamental principle, pairwise majority voting. In particular, in every voting situation in such a society there exists a unique alternative that beats every other alternative by a strict majority of votes, provided that the number of voters is odd. Such an alternative is called a Condorcet winner. Moreover, by the acyclicity of the pairwise majority relation, a Condorcet winner not only exists on the universal set of all alternatives but indeed on every non-empty subset, respectively. For real life applications, an important consequence of the well-behavedness of pairwise majority voting on Condorcet domains is the fact that in societies represented by Condorcet domains collective decision are path-independent and Pareto efficient. It is well-known that majority decisions are path-dependent in general, and can even lead to dominated outcomes, see e.g. Moulin [1988]. This cannot happen in 'Condorcet societies.' Thus, if we have reason to believe that a society's potential preferences form a Condorcet domain, pairwise majority voting offers a method of making collective decisions in an optimal way that avoids many potential flaws and mistakes. Of course, also on Condorcet domains any aggregation method is vulnerable to *individual* mistakes stemming, for instance, from insufficient information or wrong perceptions; the point is that pairwise majority voting on Condorcet domains avoids the *additional* problems stemming from the necessary shortcomings of collective choice procedures on preference domains that are less well-behaved than the Condorcet domains.

These attractive properties of Condorcet domains lend them naturally to applications in several disciplines, among others in economics, political and computer science. Indeed, one of prime examples of a Condorcet domain, the so-called 'single-peaked' domain [Black, 1948], plays a central role in social choice theory and in many models of political economy and theoretical political science. Another prominent Condorcet domain that has received considerable attention in collective choice theory is the so-called 'single-crossing' domain introduced by Roberts [1977]. One purpose of the present monograph is to make also other classes of Condorcet domains accessible to researchers working in the area of collective choice and its application in economics, political science and related disciplines. Of particular interest in this context are the symmetric Condorcet domains on the one hand, including the 'group-separable' domains [Inada, 1964], and the so-called 'peak-pit' domains on the other hand, including Fishburn's alternating scheme domains [Fishburn, 1997] and its generalizations.

Condorcet domains are also a fascinating subject to study from a purely mathematical perspective. Indeed, their analysis requires careful investigation of the deep connections with a number of branches of mathematics, most notably group theory, combinatorics and graph theory. Of evident importance is the analysis of the symmetric groups and, more generally, Coxeter groups which have been intensely studied in mathematics. From the mathematical point of view, the main aim of the present monograph is to uncover the rich and diverse structure of the class of Condorcet domains, by providing classifications of its members as well as nesting them in the rich mathematical environment that includes geometric objects such as the permutahedron and associahedron, combinatorial objects such as simple sequences, arrangements of pseudolines and rhombic tilings, and algebraic objects such as Coxeter groups, Bruhat orders, and distributive lattices.

While this is the first comprehensive monograph devoted to Condorcet domains in the literature (to the best of our knowledge), it builds on a number of articles published in recent years. Indeed, the study of Condorcet domains has been an active area of research over the last two decades, and significant progress has been made most recently. For the developments before the year 2010, we draw on the excellent survey by Monjardet [2009]. Some of the more recent results have been surveyed in Puppe and Slinko [2024b].

Historical overview of key contributions

As noted above, the subject area has grown from the intellectual contribution made by Condorcet [1785]. Similar ideas had in fact previously been proposed by Ramon Llull in the 13th century but this was not known until 2001 when some of his Lull's lost manuscripts were found.

Figure 1: Marquis de Condorcet (left) and Ramon Llull (right)

The period of the French Revolution and immediately after it was characterised by an explosion of various elections with the purpose of aggregating opinions of voters expressed in the form of various ballots. Often the input on these ballots is taken to be the voters' rankings of candidates. Mathematically, these rankings can be represented by linear orders on the set of candidates, i.e., transitive, complete and asymmetric binary relations.

The most interesting developments took place in the the French Academy of Sciences that served at that time as "a living laboratory where[in] to test, in the small, the virtues and the vices of different voting rules" [Barberà et al., 2021]. Condorcet's idea was that the candidate who shall win the election should be the candidate who wins a majority of the vote in every head-to-head contest against each other candidate. However, as Condorcet himself noticed, there are situations in which no such candidate exists. This is the so-called 'voting paradox' or 'Condorcet paradox.' Indeed, if preferences of voters are cyclic, the majority relation $\succ$ is not transitive. The six possible linear orders over three alternatives a, b, c can be split into two groups as follows (orders are shown as columns):

$$\begin{array}{ccc} 1 & 2 & 3 \\ \hline a & b & c \\ b & c & a \\ c & a & b \end{array} \qquad \begin{array}{ccc} 4 & 5 & 6 \\ \hline a & b & c \\ c & a & b \\ b & c & a \end{array} \tag{0.0.1}$$

As is easily seen, if each order in any of the two groups receives the support of 1/3 of the electorate, the corresponding pairwise majority relation is cyclic, hence in particular non-transitive. In case of the group $\{1, 2, 3\}$ the pairwise majority relation is

$$a \succ b \succ c \succ a,$$

which means that majority of voters will prefer a to b to c and c will be preferred to a closing the cycle. In case of the group $\{4, 5, 6\}$ the pairwise majority relation is

$$a \succ c \succ b \succ a.$$

In his monograph "Social Choice and Individual Values" (1951, second edition 1963), Kenneth Arrow asked if there are possibly other aggregation methods, different from pairwise majority voting, that would still satisfy some of its salient properties but avoid the voting paradox. The conclusion of Arrow's celebrated theorem is that on an unrestricted preference domain such a method does not exist. At the same time, it is been noticed by Duncan Black that on certain restricted preference domains the voting paradox does not occur [Black, 1948]. Specifically, if all voters' preferences are single-peaked with respect to some common axis, then the pairwise majority relation is always transitive (up to possible ties if the number of voters is even).

In the wake of this fundamental observation, researchers have investigated the general conditions under which pairwise majority voting (with an odd number of voters) would result in a transitive collective preference order. The best known of these conditions is Amartya Sen's 'value restriction' condition [Sen, 1966] according to which, in each triple of alternatives, there must exist at least one alternative that either does not occupy the first, or the second, or the third rank in any ranking restricted to the triple (see also Ward [1965]). The single-peaked domain, for instance, has the property that for any triple of alternatives, the middle alternative (with respect to the common axis) is never at the bottom of any preference order restricted to the triple; we thus say that the single-peaked domain satisfies, for any triple of alternatives a 'never-bottom' condition in the terminology of Fishburn [1997]. Similarly, the dual 'single-dipped' domain satisfies, for any triple of alternatives a 'never-top' condition. On the other hand, the 'group-separable' domains introduced by Inada [1964] satisfies, for any triple of alternatives, a 'never-middle condition,' i.e. in any triple of alternatives there is one alternative that never occupies the second rank in the restriction of any preference order to this triple. Thus, in other words, Sen's value restriction characterisation says that a preference domain is a Condorcet domain if and only if it satisfies, for each triple of alternatives at least one of a never-bottom, a never-top or a never-middle condition.[1]

After the simplest examples of Condorcet domains had been understood, much of the research of the 1970s to the 1990s was driven by the question of how 'large' Condorcet domains can get. The number of all single-peaked orders on a set of n alternatives is 2^{n-1}, and it has been conjectured that this was indeed the maximum number of orders in any Condorcet domain. However, this conjecture turned out to be correct only for $n = 2$ and $n = 3$. In 1980, Ki Hang Kim and Fred Roush proved that on $n = 4$ alternatives there exists a Condorcet domain containing 9 preference orders [Kim and Roush, 1980], and subsequently larger Condorcet domains have been found also for other values of n. The motivation underlying the quest for large Condorcet domains is aptly described by the title of Hervé Raynaud's unpublished working paper "The Individual Freedom Allowed by the Value Restriction Condition" [Raynaud, 1982]. This line of research culminated in Peter Fishburn's paper from 1997 in which he introduced his

[1]Observe that, unlike for preference *domains*, i.e. sets of potential preference orders, Sen's value restriction is clearly not a necessary condition for *fixed profiles* of individual preferences to yield a transitive majority relation. For instance, even if all preference orders from the group $\{1, 2, 3\}$ in (0.0.1) occur in a given profile, the majority relation will be transitive if the proportion of voters who hold the first preference order is larger than 50%. By contrast, Sen's value restriction becomes the necessary and sufficient condition if required *for all* possible profiles (with an odd number of voters) that can be composed of preferences from the domain.

'alternating scheme' Condorcet domain and proved that it achieves the maximum size $f(n)$ of a Condorcet domain on n alternatives for $n = 4, 5, 6$ with $f(4) = 9$, $f(5) = 20$ and $f(6) = 45$ [Fishburn, 1997]. In this paper, Fishburn also conjectured that $f(6) = 100$ and surveyed the pertinent contributions of the literature, among others the contributions by James Abello, Célestin Chameni-Nembua and Charles Johnson, see also Monjardet [2009]. The alternating scheme Condorcet domains have two key properties: (i) every triple of alternatives satisfies either a never-bottom, or a never-top condition, and (ii) they contain two orders that are complete reverses of each other. In the following, we will refer to Condorcet domains with these two properties as 'peak-pit' domains with maximal width; other peak-pit domains with maximal width are the classical single-peaked, single-dipped and single-crossing domains.

That Fishburn's alternating scheme achieves the maximum size of a Condorcet domain also for $n = 6$ (with $f(6) = 100$ as conjectured) was finally proven by Ádám Galambos and Victor Reiner who gave a general formula for the number of elements of the alternating scheme Condorcet domains and who introduced representations of peak-pit domains with maximal width in terms of reduced decompositions and simple arrangements of pseudolines [Galambos and Reiner, 2008]. A further breakthrough was achieved by Vladimir Danilov, Alexander Karzanov and Gleb Koshevoy in 2012 who presented yet another representation of peak-pit domains of maximal width in terms of rhombus tilings [Danilov et al., 2012]. This allowed these authors to give a counterexample to the long-standing conjecture that Fishburn's alternating scheme achieves the maximum cardinality for all n among the peak-pit domains with maximal width. (The fact that it does not achieve the maximum cardinality among *all* Condorcet domains for large n was noted by Fishburn [1997] himself.) In another paper, Danilov and Koshevoy characterised maximal Condorcet domains of maximal width in terms of an explicit lattice formula and investigated 'symmetric' Condorcet domains that contain with every order also its complete reverse order [Danilov and Koshevoy, 2013]. The maximal size of a symmetric Condorcet domain on n alternatives is 2^{n-1}, and Danilov and Koshevoy [2013] characterize these maximal Condorcet domains in terms of a certain composition operator. These authors also generalised a construction already described by Raynaud [1981] to obtain 'small' symmetric Condorcet domains of size 4 for all n that are nevertheless maximal in the sense that they are not contained in any larger Condorcet domain.

More recently, the authors of the present monograph have established a useful connection between Condorcet domains and the theory of median graphs [Puppe and Slinko, 2019]. Specifically, each Condorcet domain induces a median graph with its elements as vertices, and conversely, for each median graph, one can construct (non-uniquely) a Condorcet domain. Among other things, this connection can be exploited to obtain a characterisation of all Arrovian aggregators on ('closed') Condorcet domains besides pairwise majority voting, and to uncover a large class of strategy-proof social choice functions on these domains.

Many of the other recent contributions are devoted to shed further light on the structure of Condorcet domains by characterising sub-classes in terms of simple properties. For example, Puppe [2018] proved that the domain of all single-peaked preferences is the unique Condorcet domain with maximal width that is connected and minimally rich in the sense that it contains for every alternative at least one preference order that has it as top alternative; Slinko [2019] showed that the same conditions without the maximal width conditions characterise the locally single-peaked domains (or "Arrow single-peaked" domains); Slinko et al. [2021] characterise the domains that are both single-crossing and maximal Condorcet in terms of "relays;" and Karpov and Slinko [2023c] characterise the indecomposable symmetric maximal Condorcet domains in terms of simple sequences. Significant progress has also been made in the search for large Condorcet domains with contributions by A. Karpov, C. Leedham-Green, K. Markström, S. Riis, and others.

Finally, some new results have been obtained using computational methods that became available with the development of more powerful computing devices. Using a computational protocol, Dittrich [2018] has provided a complete list and partial classification of the 18 different (up to isomorphism) maximal Condorcet domains on $n = 4$ alternatives and the 688 different maximal Condorcet domains on $n = 5$ alternatives. Further computational results have been obtained by Akello-Egwell et al. [2025] who compiled a complete list of all Condorcet domains for up to $n \leq 7$ alternatives, and by Zhou and Riis [2023] who investigated Condorcet domains on $n = 10$ and $n = 11$ alternatives.

Perhaps the most surprising among the recent computational findings is the discovery of the largest Condorcet domain on $n = 8$ alternatives by Leedham-Green et al. [2024]. Remarkably, this domain does not have maximal width, in particular, it is not one of the Fishburn's alternating scheme domains (though it is a peak-pit domain); it contains 224 orders, hence two more than the 222 orders of the alternating scheme domain on $n = 8$ alternatives. Currently, this topic is an area of active research and any attempted summary will necessarily have premature character. By consequence, we abstain here from a systematic appraisal of the various computational results obtained recently.

Overview of this book

The present book is divided into two parts. The first and main part is devoted to the analysis of the classical case of maximal Condorcet domains of linear orders, i.e., permutations. Note that non-maximal Condorcet domains are generally of far less interest for obvious reasons; for instance, every set of two linear orders is a Condorcet domain. Chapter 1 presents the fundamental concepts, definitions and results. Section 1.1 introduces Condorcet domains, the notions of isomorphism and flip-isomorphism, the never conditions, and derives some fundamental properties of maximal Condorcet domains. Section 1.2 describes Condorcet domains as subsets of the permutohedron and introduces the central notions of connectedness and inversion triples, respectively. Section 1.3 establishes the close connection between 'closed' Condorcet domains and median graphs, offering a new self-contained proof of the main representation theorem. A Condorcet domain is *closed* if it contains with any odd numbered collection of orders also their pairwise majority relation. Section 1.4 introduces the weak Bruhat poset and reduced decompositions, and Section 1.5 the notion of an ideal of a Condorcet domain.

Chapter 2 is devoted to the analysis of the single-peaked Condorcet domains and its generalisations. Section 2.1 introduces Black's classical single-peaked domain and shows that it is the unique maximal Condorcet domain with maximal width that is connected and minimally rich. Section 2.2 introduces and analyses the "locally single-peaked" or "Arrow single-peaked" domains. Section 2.2.1 derives structural properties, Section 2.2.2 shows that the Arrow single-peaked domains are exactly the maximal Condorcet domains that are connected and minimally rich (without the maximal width assumption), Section 2.2.3 classifies them for $n = 4$ and $n = 5$ alternatives, and Section 2.2.6 enumerates them for $n \leq 9$. Section 2.3 considers complexity, and Section 2.4 generalisations of the single-peakedness property allowing the spectrum to be a tree and a circle.

Chapter 3 is devoted to the analysis of single-crossing domains. Section 3.1 presents the definition and derives basic properties. Section 3.2 shows that the single-crossing domains are (essentially) those that have the so-called "representative voter" property. Sections 3.3 and 3.4 present different characterisations of the single-crossing domains that are at the same time also maximal Condorcet ones. A consequence of these results and those from the previous chapter is that those domains cannot be minimally rich. In other words, for every single-crossing domain that is also a maximal Condorcet domain, there exists an alternative that is never on the top of any of its orders. Finally, Section 3.5 considers the generalisation of the single-crossing property to trees.

Chapter 4 studies the important class of peak-pit Condorcet domains and offers a new self-contained proof of the main characterisation theorem that characterises them in terms of arrangements of pseudolines. Section 4.1 shows that every peak-pit domain contains at most one pair of completely reversed orders. Sections 4.2 - 4.5 introduce the notions of separated systems of sets, arrangements of pseudolines, reduced decompositions and study their basic properties, while Section 4.6 contains the two main characterisation results, Section 4.6.1 characterises the peak-pit domains in terms of arrangements of pseudolines, and Section 4.6.2 shows that the tiling domains introduced by Danilov et al. [2012] can be understood as duals to domains of arrangements of pseudolines. Section 4.7 studies the representation of Arrow single-peaked domains. Section 4.8 presents classifications of small maximal peak-pit domains both with and without the maximal width assumption.

Chapter 5 investigates in more detail the subclass of (peak-pit) alternating schemes. Section 5.1 defines the Fishburn domains and their generalisations, Section 5.2 offers a combinatorial representation of generalised Fishburn domains (GF-domains), and Section 5.3 derives some of their key properties. Section 5.4 introduces the class of "set-alternating scheme" domains and derives some of their properties. Finally, Section 5.5 describes yet another class of domains, the so-called Cambrian domains Labbé and Lange [2020] and shows that this class of domains in fact coincides with the generalised Fishburn domains.

Chapter 6 studies different notions of compositions of Condorcet domains. After defining a suitable composition in Section 6.1, Section 6.2 investigates the role of clone sets and the notion of decomposability. The group separable domains introduced by Inada [1964] are shown to be the completely decomposable maximal Condorcet domains in Section 6.3. The final Section 6.4 introduces another notion of 'never-last' composition of Condorcet domains and shows that many maximal Condorcet domains can be obtained as never-last compositions of smaller Condorcet domains.

Chapter 7 studies symmetric Condorcet domains, i.e., domains that with any order also contain its completely reverse order. Section 7.1 studies the properties of these domains. Section 7.2 introduces a class of 'small' symmetric maximal Condorcet domains, here referred to as 'Raynaud domains' after Raynaud [1981]. Section 7.3 introduces the combinatorial notion of simple sequences, and Section 7.4 shows that the Raynaud domains are in one-to-one correspondence with the simple sequences. The final section 7.5 establishes yet another connection of (symmetric) Condorcet domains to a branch of combinatorics via the notion of 'associahedron.'

Chapter 8 is devoted to the construction of large maximal Condorcet domains. Section 8.1 introduces Fishburn's original replacement scheme that produces larger Condorcet domains than the alternating scheme for sufficiently large n. Sections 8.2 and 8.3 consider the constructions by Danilov-Karzanov-Koshevoy and Karpov-Slinko, respectively. The latter construction is used in Section 8.4 to construct large peak-pit domains, and a lower bound for the size of the largest of them. An upper bound is given in Section 8.5.

Chapter 9 describes Dittrich's classification of all maximal Condorcet domains on $n = 4$ alternatives.

The second part considers applications and extensions. Chapter 10 considers aggregation methods on closed Condorcet domains other than pairwise majority voting. Section 10.1 characterises those that satisfy a monotone version of Arrow's independence of irrelevant alternatives condition. In Section 10.2 this characterisation is used to derive a large class of strategy-proof social choice functions on every closed, hence in particular also on every maximal, Condorcet domain.

Chapter 11 extends parts of our analysis to the case of weak orders and the case of (strict) partial orders. The main result of Section 11.1 shows that the characterisation of the classical single-peaked domain is robust against the generalisation to weak orders: the (weakly) single-peaked domain remains the only maximal Condorcet domain with maximal width that is connected and satisfies a natural richness property requiring that each alternative be the *unique* top of some order in the domain. Section 11.2 investigates the extension to strict partial orders and shows that things change quite significantly here. First, there exist 'large' Condorcet domains that do not contain *any* linear order. Secondly, while there exists a unique maximal Condorcet domain that contains the set of all classically single-peaked linear orders, this domain is no longer characterised by the properties of connectedness and minimal richness. Thus, while some key results are robust with respect to the extension from linear to weak orders, this is no longer the case with respect to the extension to partial orders.

Part I

Maximal Condorcet domains of linear orders

Chapter 1

Basic concepts and results

1.1 Condorcet domains: examples and properties

Let A be a finite set of cardinality m and $\mathcal{L}(A)$ be the set of all linear orders on A. Arrow [1963] called $\mathcal{L}(A)$ the *universal domain* and argued that in many instances voters or economic agents cannot have arbitrary preferences. In such a case their preferences will come from a non-empty subset $\mathcal{D} \subseteq \mathcal{L}(A)$ which will be called a *domain*.

1.1.1 Majority relation

If $a_1 \succ a_2 \succ \cdots \succ a_m$ is a linear order on A, it will be denoted by a string $a_1a_2 \ldots a_m$. In the context of an election, we will often write $a \succ_v b$ if a is ranked higher than b in the linear order v associated with the voter v[1]. Let us also introduce notation for reversing orders, i.e., if $v = a_1a_2 \ldots a_n$, then $\bar{v} = a_na_{n-1} \ldots a_1$.

Any sequence $P = (v_1, \ldots, v_n) \in \mathcal{D}^n$ of linear orders from $\mathcal{D}$ is called a *profile* over $\mathcal{D}$. Unlike a domain it can contain several identical linear orders and the order of linear orders is important. Profiles usually represent a collection of opinions of members of a certain society so $v_1, \ldots, v_n$ are also called *voters*. If we disregard the order of linear orders in the profile and only count multiplicities for each of the linear order from $\mathcal{D}$ we get what is called a *voting situation*.

When all head-to-head contests are accomplished, the result can be mathematically characterised by the object we are now going to define.

Definition 1.1.1. *The* majority relation $\succeq_P$ *of a profile* $P = (\succ_1, \ldots, \succ_n)$ *is defined as*

$$a \succeq_P b \iff \#\{i \mid a \succ_i b\} \geq \#\{i \mid b \succ_i a\},$$

i.e., $a \succeq_P b$ *means that the number of voters who prefer* a *to* b *is at least as large as the number of voters who prefer* b *to* a*.*

The asymmetric part of the majority relation is defined as

$$a \succ_P b \iff (a \succeq_P b) \;\&\; \neg(b \succeq_P a).$$

[1] Voters and their linear orders are usually denoted with the same letter.

For an odd n we have $\succeq_P = \succ_P$. In such a case, this relation is a *tournament*, i.e., complete and asymmetric binary relation.

As McGarvey [1953] showed, this tournament can be arbitrary and, in particular, can be non-transitive. For example, the majority relation of profiles $P_1 = \{abcd, cdab, cabd\}$ and $P_2 = \{abcd, bcda, cdab\}$ are shown on Figure 1.1. An arrow from an alternative x to alternative y means that the majority prefer x to y. The first one is a linear order $cabd$ and the second one is not transitive since $a \succ_{P_2} b \succ_{P_2} c \succ_{P_2} d \succ_{P_2} a$.

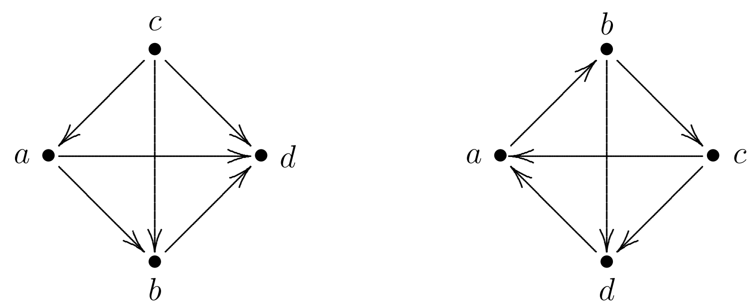


Figure 1.1: The majority relation of P_1 on the left and of P_2 on the right.

If the majority relation is transitive, its top element beats every other alternative by a majority, and if the number of voters is odd, even by a strict majority. Such an element is called a *Condorcet winner*. For instance, the Condorcet winner at profile P_1 is c, while there is no Condorcet winner at profile P_2.

Now suppose we decide the "winner" at the profile P_2 by a sequential pairwise majority vote according to the following sequence of comparisons: first (c vs. b), followed by (b vs. a) and finally (d vs. a). Then, c is eliminated by b, b is eliminated a, and a is eliminated by d, hence d remains as the unique winner. However, *all* voters agree that c is better than d, thus the winner of the sequential majority vote is a Pareto dominated alternative [Moulin, 1988, Nehring et al., 2016]. By contrast, a Condorcet winner is the outcome of *every* sequential pairwise majority vote and is evidently Pareto optimal.

By consequence, if the majority relation has cycles it is not at all clear how to identify the "most preferred" candidate. The simplest profiles for which the majority relation is cyclic are shown in (0.0.1). For instance, the profile $P = \{abc, bca, cab\}$, results in the cyclic majority relation $a \succ_P b \succ_P c \succ_P a$. We will therefore call P a *cyclic triple*.

The main object of our investigation are Condorcet domains for which cycles may never appear.

1.1.2 Two definitions of a Condorcet domain

Definition 1.1.2. *A domain $\mathcal{D} \subseteq \mathcal{L}(A)$ over a set of alternatives A is a* Condorcet domain *if the majority relation of any profile P over $\mathcal{D}$ with an odd number of voters is transitive. A Condorcet domain $\mathcal{D}$ is* maximal *if for any Condorcet domain $\mathcal{D}' \subseteq \mathcal{L}(A)$ the inclusion $\mathcal{D} \subseteq \mathcal{D}'$ implies $\mathcal{D} = \mathcal{D}'$.*

We say that a binary relation R on a set A is *acyclic*, if for the asymmetric part R° of R there is no positive integer $k \geq 3$ and a sequence $a_1, \ldots, a_k$ such that

$$a_1 R^\circ a_2 R^\circ \ldots R^\circ a_k R^\circ a_1.$$

An equivalent definition of a Condorcet domain requires the majority relation for every profile to be *acyclic*, see Monjardet [2009]; this is why Fishburn [1996a, 2002] refers to Condorcet domains also as acyclic sets of linear orders.

Acyclicity looks a stronger requirement but not really. This follows from the following proposition.

Proposition 1.1.1. *A domain $\mathcal{D} \subseteq \mathcal{L}(A)$ over a set of alternatives A is a Condorcet domain if and only if the majority relation $\succeq_P$ of any profile P over $\mathcal{D}$ is acyclic.*

Proof. For a profile $P \in \mathcal{D}^n$ with an odd number of voters we have $\succeq_P = \succ_P$ so acyclicity of the majority relation implies its transitivity which means that $\mathcal{D}$ is a Condorcet domain. Conversely, if $\mathcal{D}$ is a Condorcet domain and $\succeq_P$ is not acyclic, then P must have an even number of voters. Suppose

$$a_1 \succ_P a_2 \succ_P \cdots \succ_P a_k \succ_P a_1. \tag{1.1.1}$$

for some $a_1, \ldots, a_k$. We note that

$$\#\{i \mid a_s \succ_i a_{s+1}\} - \#\{i \mid a_{s+1} \succ_i a_s\} \geq 2$$

and

$$\#\{i \mid a_k \succ_i a_1\} - \#\{i \mid a_1 \succ_i a_k\} \geq 2.$$

Therefore adding an arbitrary linear order $u \in \mathcal{D}$ to P we will have a profile $P' = P \cup \{u\}$ with an odd number of voters and we will still have for this profile

$$a_1 \succ_{P'} a_2 \succ_{P'} \cdots \succ_{P'} a_k \succ_{P'} a_1.$$

This is because in (1.1.1) the advantages of a_s over a_{s+1} and a_k over a_1 under P were at least 2. Since P' has an odd number of orders, this cannot happen in a Condorcet domain, hence we get a contradiction. □

One obvious observation is in order and will be useful: since the majority relation is defined on pairs of alternatives, if $\mathcal{D} \subseteq \mathcal{L}(A)$ is a Condorcet domain and $B \subset A$, then the restricted domain $\mathcal{D}|_B \subseteq \mathcal{L}(B)$ is also a Condorcet domain.

1.1.3 Isomorphism and flip-isomorphism

As any other abstractly defined object in mathematics, a Condorcet domain can have several "material" appearances. To relate them together we need the concepts of an isomorphism and flip-isomorphism.

Let $\psi \colon A \to A'$ be a bijection between two sets of alternatives. It can be extended to a mapping $\psi \colon \mathcal{L}(A) \to \mathcal{L}(A')$ in two ways: by mapping linear order $u = a_1 a_2 \ldots a_m$ onto $\psi(u) = \psi(a_1)\psi(a_2)\ldots\psi(a_m)$[2] or to $\overline{\psi(u)} = \psi(a_m)\psi(a_{m-1})\ldots\psi(a_1)$.

[2] We use the same notation for both mappings since there can be no confusion.

Definition 1.1.3. *Let A and A' be two sets of alternatives (not necessarily distinct) of equal cardinality. We say that two domains, $\mathcal{D} \subseteq \mathcal{L}(A)$ and $\mathcal{D}' \subseteq \mathcal{L}(A')$ are* isomorphic *if there is a bijection $\psi\colon A \to A'$ such that $\mathcal{D}' = \{\psi(d) \mid d \in \mathcal{D}\}$ and flip-isomorphic if $\mathcal{D}' = \{\overline{\psi(d)} \mid d \in \mathcal{D}\}$.*

In the particular case, when $\overline{\mathcal{D}} = \{\bar{u} \mid u \in \mathcal{D}\}$, we call the domain $\overline{\mathcal{D}}$ as *flipped* $\mathcal{D}$; evidently, the flipped domain $\overline{\mathcal{D}}$ is flip-isomorphic to $\mathcal{D}$, but it is usually not isomorphic to it.

1.1.4 Maximal Condorcet domains over the set of three alternatives

We will now present some initial examples of Condorcet domains. There is only one maximal Condorcet domain on a set $A = \{a, b\}$ of two alternatives, namely $\{ab, ba\}$. Let us now consider the case of $m = 3$ with $A = \{a, b, c\}$.

Lemma 1.1.2. *Let $\mathcal{D}$ be a maximal domain on a set of three alternatives $A = \{a, b, c\}$ which does not contain cyclic triples. Then, up to an isomorphism, it is one of the following domains:*

$$\mathcal{D}_{3,1} = \{abc, cab, acb, cba\},\ \mathcal{D}_{3,2} = \{abc, bca, acb, cba\},\ \mathcal{D}_{3,3} = \{abc, bac, bca, cba\}. \quad (1.1.2)$$

Proof. Among the six orders presented in (0.0.1) one order from each group $\{1, 2, 3\}$ and $\{4, 5, 6\}$ must be not in $\mathcal{D}$. Let us consider domains with four linear orders that satisfy this requirement. We note that among the three pairs of completely reversed orders $\{abc, cba\}$, $\{acb, bca\}$, $\{bac, cab\}$ one must have both orders in $\mathcal{D}$. Up to an isomorphism we can consider that $\{abc, cba\} \subset \mathcal{D}$. From the other two pairs we must choose two orders one from each set $\{bca, cab\}$ and $\{acb, bac\}$.

Thus we can now add four pairs of orders to this subset, namely, $\{acb, cab\}$, $(acb, bca\}$, $\{bca, bac\}$, $\{bac, cab\}$. In the first three cases we obtain the domains $\mathcal{D}_{3,1}, \mathcal{D}_{3,2}, \mathcal{D}_{3,3}$, respectively. In the fourth case we obtain domain

$$\{abc, bac, cab, cba\},$$

which is isomorphic to $\mathcal{D}_{3,2}$ by $a \to c$, $b \to b$, $c \to a$. It is easy to see that no pair of domains among $\mathcal{D}_{3,1}, \mathcal{D}_{3,2}, \mathcal{D}_{3,3}$ are isomorphic. For example, in $\mathcal{D}_{3,1}$ only two alternatives may occupy top positions while in $\mathcal{D}_{3,2}$ and $\mathcal{D}_{3,3}$ we have three such alternatives. □

Theorem 1.1.3. *Up to an isomorphism, domains $\mathcal{D}_{3,1}, \mathcal{D}_{3,2}, \mathcal{D}_{3,3}$ listed in* (1.1.2) *are the only three maximal Condorcet domains on a set of three alternatives $A = \{a, b, c\}$. Moreover, $\mathcal{D}_{3,1}$ is flip-isomorphic to $\mathcal{D}_{3,3}$.*

Proof. All we need to prove is that each of the $\mathcal{D}_{3,1}, \mathcal{D}_{3,2}, \mathcal{D}_{3,3}$ is a Condorcet domain. Let us consider an odd profile P over $\mathcal{D}_{3,1}$ with a voting situation

n_1	n_2	n_3	n_4
a	a	c	c
b	c	a	b
c	b	b	a

where n_1, n_2, n_3 and n_4 are the number of voters with preferences *abc*, *acb*, *cab* and *cba*, respectively. Of course, $n_1 + n_2 + n_3 + n_4 = n$ and n is odd. Without loss of generality we assume that $n_1 + n_2 > n_3 + n_4$. Then $a \succ_P b$ and $a \succ_P c$. Then, regardless of the relation between b and c we will have $\succ_P$ transitive.

Let us consider also an odd profile P over $\mathcal{D}_{3,2}$ with a voting situation

n_1	n_2	n_3	n_4
a	a	b	c
b	c	c	b
c	b	a	a

If $n_1+n_2 > n_3+n_4$, then $a \succ_P b$ and $a \succ_P c$ which leads to transitivity. If $n_1+n_2 < n_3+n_4$, then $b \succ_P a$ and $c \succ_P a$ which leads to transitivity again.

$\mathcal{D}_3$ is a Condorset domain since it is flipped $\mathcal{D}_1$. In each case we have transitivity of the majority relation showing that $\mathcal{D}_{3,1}, \mathcal{D}_{3,2}, \mathcal{D}_{3,3}$ are Condorcet domains. □

1.1.5 Criteria of being Condorcet. Never conditions

To formulate the first criterion, let us introduce the following notation. Let $\mathcal{D} \subseteq \mathcal{L}(A)$ be a domain, $B \subset A$ and $u \in \mathcal{D}$. Then by $u|_B$ we denote the restriction of u onto B. For example, the restriction of $u = cabed$ onto $\{b, c, e\}$ is $u|_{\{b,c,e\}} = cbe$.

Proposition 1.1.4. *A domain $\mathcal{D} \subseteq \mathcal{L}(A)$ is Condorcet domain if and only if for no triple of alternatives a, b, c there exist linear orders $u, v, w \in \mathcal{D}$ such that the restrictions $u|_{\{a,b,c\}}, v|_{\{a,b,c\}}, w|_{\{a,b,c\}}$ onto $\{a, b, c\}$ form a cyclic triple.*

Proof. One direction is straightforward. If $u, v, w \in \mathcal{D}$ are such that $u|_{\{a,b,c\}}, v|_{\{a,b,c\}}, w|_{\{a,b,c\}}$ form a cyclic triple, then the profile $P = (u, v, w)$ has non-transitive majority relation and $\mathcal{D}$ is not a Condorcet domain.

Suppose now that $\mathcal{D}$ is a Condorcet domain and $a, b, c \in A$. Then the restriction $\mathcal{D}' = \mathcal{D}|_{\{a,b,c\}}$ of $\mathcal{D}$ onto $\{a, b, c\}$ is also a Condorcet domain. By Theorem 1.1.3 $\mathcal{D}'$ has no cyclic triple. This proves the criterion. □

To formulate the second criterion we need to make several observations. The domain $\mathcal{D}_{3,1}$ on the left in (1.1.2) contains all the linear orders on $\{a, b, c\}$ in which b is never ranked first, the domain $\mathcal{D}_{3,2}$ in the middle contains all the linear orders on $\{a, b, c\}$ in which a is never ranked second, and domain $\mathcal{D}_{3,3}$ on the right contains all the linear orders on $\{a, b, c\}$ in which b is never ranked last. Following Monjardet [2009], we denote these conditions as $bN_{\{a,b,c\}}1$, and $aN_{\{a,b,c\}}2$, and $bN_{\{a,b,c\}}3$, respectively.

Definition 1.1.4. *Any condition of type $xN_{\{a,b,c\}}i$ with $x \in \{a, b, c\}$ and $i \in \{1, 2, 3\}$ is called a* never condition *since it being applied to a domain $\mathcal{D}$ requires that in orders of $\mathcal{D}|_{\{a,b,c\}}$ alternative x never takes ith position. We say that a subset $\mathcal{N}$ of*

$$\{xN_{\{a,b,c\}}i \mid \{a, b, c\} \subseteq A,\ x \in \{a, b, c\} \text{ and } i \in \{1, 2, 3\}\} \tag{1.1.3}$$

is a complete set of never-conditions *if* $\mathcal{N}$ *contains exactly one never-condition for every triple* a, b, c *of elements of* A*.* $\mathcal{N}$ *is said to be* consistent*, if there exists at least one linear order satisfying all conditions from* $\mathcal{N}$*.*

Now we can formulate the second criterion.

Theorem 1.1.5. *A domain of linear orders* $\mathcal{D} \subseteq \mathcal{L}(A)$ *is a Condorcet domain if and only if it satisfies a complete set of never conditions.*

Proof. Since the property of a domain $\mathcal{D}$ of being Condorcet can be checked on triples of alternatives, this characterisation can be checked for each restriction $\mathcal{D}|_{\{x,y,z\}}$ for $x, y, z \in A$. Hence this criterion follows from Theorem 1.1.3. □

This is a well-known characterisation which goes back to Ward [1965], Sen [1966], Inada [1969]; see also Theorem 1(d) in Puppe and Slinko [2019] and references therein.

This proposition immediately implies that if a domain $\mathcal{D}$ does not satisfy any never-condition for a triple $\{a, b, c\} \subset A$, then the restriction $\mathcal{D}|_{\{a,b,c\}}$ of $\mathcal{D}$ onto this domain onto $\{a, b, c\}$ contains a cyclic triple.

Theorem 1.1.5 in particular means that the collection $\mathcal{D}(\mathcal{N})$ of all linear orders that satisfy a certain complete set of never conditions $\mathcal{N}$, if non-empty, is a Condorcet domain. Let us also denote by $\mathcal{N}(\mathcal{D})$ the set of all never conditions that are satisfied by all linear orders from $\mathcal{D}$.

Not all complete sets of never conditions are consistent, here is an example.

Example 1.1.1. *For* $A = \{a, b, c, d\}$ *the set*

$$aN_{\{a,b,c\}}3, \quad bN_{\{a,b,d\}}3, \quad cN_{\{a,c,d\}}3, \quad dN_{\{b,c,d\}}3$$

of never-bottom conditions is inconsistent, that is defines an empty domain. Indeed, if there exists a linear order satisfying these conditions, then no one of the four alternatives can be bottom-ranked.

Not all consistent complete sets of never conditions define a maximal Condorcet domain either.

Example 1.1.2. *Let* $A = \{1, 2, 3, 4\}$ *and a domain* $\mathcal{D}$ *is defined by the never conditions*

$$2N_{\{1,2,3\}}1, \qquad 2N_{\{1,2,4\}}3, \qquad 3N_{\{1,3,4\}}3, \qquad 3N_{\{2,3,4\}}1,$$

Then it is easy to see that $\mathcal{D} = \{e = 1234, \bar{e} = 4321\}$ *and, in particular, it is not maximal.*

It is often the case that the set $[n] = \{1, 2, \ldots, n\}$ is used as a set of alternatives. It is also convenient to consider (which always can be assumed up to an isomorphism) that the linear order $12 \ldots n$ belongs to the domain $\mathcal{D}$ under consideration in which case it is called *unital.* In such a case it is convenient to consider

$$1 \lhd 2 \lhd \cdots \lhd n$$

as a spectrum relative to which never conditions can be specified. For a triple $\{i, j, k\}$ with $i < j < k$ the unital domain $\mathcal{D}$ can satisfy only six never conditions

$$iN_{\{i,j,k\}}2, \quad iN_{\{i,j,k\}}3, \quad jN_{\{i,j,k\}}1, \quad jN_{\{i,j,k\}}3, \quad kN_{\{i,j,k\}}1, \quad kN_{\{i,j,k\}}2.$$

1.1.6 Properties of maximal Condorcet domains

We will say that a Condorcet domain $\mathcal{D}$ is *closed* if the majority relation corresponding to any odd profile over $\mathcal{D}$ is again an element of $\mathcal{D}$.

Proposition 1.1.6. *Let $\mathcal{D}$ be a Condorcet domain and $R \in \mathcal{L}(A)$ be the majority relation corresponding to an odd profile P over $\mathcal{D}$. Then $\mathcal{D} \cup \{R\}$ is again a Condorcet domain. In particular, every Condorcet domain is contained in a closed Condorcet domain and every maximal Condorcet domain is closed.*

Proof. It suffices to show that $\mathcal{D}' = \mathcal{D} \cup \{R\}$ does not admit three orders $u, v, w \in \mathcal{D}'$ and three elements $a, b, c \in A$ such that the restrictions of u, v, w onto $\{a, b, c\}$ satisfy $u|_{\{a,b,c\}} = abc$, $v|_{\{a,b,c\}} = bca$ and $w|_{\{a,b,c\}} = cab$ or $u|_{\{a,b,c\}} = abc$, $v|_{\{a,b,c\}} = cab$ and $w|_{\{a,b,c\}} = bca$. Assume on the contrary that it does and the former is true.

Then, evidently, not all three orders u, v, w belong to $\mathcal{D}$. Thus, one of them, say w, is the majority relation $R = \succ_P$. Then $c \succ_P a \succ_P b$. Consider the profile $P' = (nu, nv, P) \in \mathcal{D}^{3n}$ that consists of n copies of the order u, n copies of the order v and the n voters of the profile P. Then the voters of the subprofile (nu, nv) will unanimously prefer b to c, which forces the majority relation of P' to have the same ranking of b and c. At the same time, the voters of this subprofile are evenly split in the ranking of any other pair of alternatives from $\{a, b, c\}$ so the majority relation of P' will rank them as $\succ_P$ does. Hence, the majority relation P' yields the cycle $b \succ_{P'} c \succ_{P'} a \succ_{P'} b$, in contradiction to the assumption that $\mathcal{D}$ is a Condorcet domain. □

Let us formulate several properties of Condorcet domains. Let $x = x_1 x_2 \dots x_n \in \mathcal{L}(A)$ be a linear order on A. We recap that $\bar{x} = x_n \dots x_2 x_1$, i.e., the bar reverses the order of alternatives in the linear order. We say that a domain $\mathcal{D}$ has *maximal width* [Puppe, 2018] if for some linear order u this domain also contains $\bar{u}$. In such a case, up to an isomorphism, we can assume that $\mathcal{D}$ contains $e = 12 \dots n$ and $\bar{e} = n \dots 21$[3].

The property of maximal width plays an important role in the analysis of Condorcet domains and simplifies matters sometimes considerably. This is also reflected by the fact that a maximal Condorcet domain can be naturally endowed with the structure of a distributive lattice if and only if it has maximal width, see Corollary 3.2 in Puppe and Slinko [2019] and the references there.

A domain that, for any triple $\{a, b, c\} \subseteq A$, satisfies a condition $xN_{\{a,b,c\}}1$ with $x \in \{a, b, c\}$ is called *never-top domain*, a domain that for any triple $\{a, b, c\} \subseteq A$ satisfies a condition $xN_{\{a,b,c\}}2$ with $x \in \{a, b, c\}$ is called *never-middle domain*, and a domain that for any triple $\{a, b, c\} \subseteq A$ satisfies a condition $xN_{\{a,b,c\}}3$ with $x \in \{a, b, c\}$ is called *never-bottom domain*. A domain that for any triple satisfies either never-top or never-bottom condition is called a *peak-pit domain* [Danilov et al., 2012]. Both never-top and never-bottom conditions are called *peak-pit conditions*. The peak-pit domains with maximal width are the most studied class of Condorcet domains; we will devote Chapter 4 to them. Several classes of peak-pit domains of maximal width are especially interesting: single-peaked domains, single-crossing domains and Fishburn (alternating scheme) domains.

[3] Danilov and Koshevoy [2013] called such domains "normal."

Guttmann and Maucher [2006] show that, given a set of never-middle conditions, it is NP-complete problem to decide on whether or not the corresponding Condorcet domain is empty or not. We also conjecture

Conjecture 1. *Given a set of peak-pit conditions, it is an NP-complete problem to decide whether or not the corresponding Condorcet domain is empty or not.*

The never-bottom domains are also known as *Arrow single-peaked domains*; they have been characterised by Slinko [2019]. The Arrow single-peaked domains with maximal width correspond to the standard (Black) single-peaked domains and have been characterised by Puppe [2018]. We will devote Chapter 2 to single-peaked and Chapter 3 to single-crossing domains—two classes of peak-pit domains which represent considerable interest to economists. Much less known about the structure of Condorcet domains that satisfy a mixture of peak-pit and never-middle conditions.

Definition 1.1.5 (Slinko [2019])**.** *A Condorcet domain $\mathcal{D}$ is called* copious *if for any triple of alternatives $a, b, c \in A$ the restriction $\mathcal{D}|_{\{a,b,c\}}$ of this domain to this triple has four distinct orders, that is, $|\mathcal{D}|_{\{a,b,c\}}| = 4$.*

We note that, if a Condorcet domain is copious, then it satisfies a unique complete set of never conditions (1.1.3).

Proposition 1.1.7. *For any complete set of never conditions $\mathcal{N}$ the domain $\mathcal{D}(\mathcal{N})$, if non-empty, is a closed Condorcet domain. Every maximal Condorcet domain is of this form.*

Proof. Let us prove closedness of $\mathcal{D}(\mathcal{N})$. Let $a, b, c \in A$ and P be an odd profile. Suppose $\mathcal{N}$ contains a never condition $bN_{\{a,b,c\}}1$. Then the restriction $\mathcal{D}|_{\{a,b,c\}}$ of $\mathcal{D} = \mathcal{D}(\mathcal{N})$ onto $\{a, b, c\}$ is contained in $\mathcal{D}_{3,1}$ given in (1.1.2). Let $Q = P|_{\{a,b,c\}}$. Then the majority relation $\succ_Q$ is transitive and belongs to $\mathcal{D}_{3,1}$. But the restriction of $\succ_P$ onto $\{a, b, c\}$ is exactly $\succ_Q$ which means that linear order $\succ_P$ satisfies $bN_{\{a,b,c\}}1$ too. Thus $\succ_P$ belongs to $\mathcal{D}(\mathcal{N})$. □

However, we have to note that not every complete set of never conditions $\mathcal{N}$ defines a maximal Condorcet domain $\mathcal{D}(\mathcal{N})$. This is demonstrated in Example 1.1.2.

Proposition 1.1.8. *Let $\mathcal{N}$ be a complete set of never conditions such that $\mathcal{D}(\mathcal{N})$ is copious. Then $\mathcal{D}(\mathcal{N})$ is a maximal Condorcet domain.*

Proof. Suppose $\mathcal{D}(\mathcal{N})$ is copious but not maximal. Then there exists a linear order u such that $\mathcal{D}' = \mathcal{D}(\mathcal{N}) \cup \{u\}$ is a Condorcet domain. Since $u \notin \mathcal{D}(\mathcal{N})$ for a certain triple of alternatives a, b, c the domain $\mathcal{D}'|_{\{a,b,c\}}$ contains an order on a, b, c which is not in $\mathcal{D}|_{\{a,b,c\}}$. But then $\mathcal{D}'|_{\{a,b,c\}}$ contains five orders on a, b, c which is not possible. □

In Chapter 4 we will see that all maximal peak-pit Condorcet domains of maximal width as well as Arrow's single-peaked domains are copious. However, the following is still unclear.

Conjecture 2. *All maximal peak-pit Condorcet domains are copious.*

As to maximal never-middle domains, they may not be copious starting from $m = 5$ alternatives. This will be noted in Observation 7.2.3.

A weaker condition is ampleness.

Definition 1.1.6. *We call a Condorcet domain $\mathcal{D}$* ample *if for any pair of alternatives $a, b \in A$ the restriction $\mathcal{D}|_{\{a,b\}}$ of this domain to $\{a, b\}$ has two distinct orders, that is, $\mathcal{D}|_{\{a,b\}} = \{ab, ba\}$.*

In general, a maximal Condorcet domain does not have to be even ample. Here is the smallest non-ample maximal Condorcet domain discovered by Akello-Egwell et al. [2025]:

1	1	1	1	1	1	1	1	4	4	4	4
2	2	2	3	3	3	4	4	1	1	2	3
3	3	5	2	2	5	2	3	2	3	1	1
4	5	3	4	5	2	3	2	3	2	3	2
5	4	4	5	4	4	5	5	5	5	5	5

Quite a few facts are also known about never-middle domains that we describe in Chapter 7. We also say that a domain is *symmetric* if together with any order u it also contains $\bar{u}$. As we will see, symmetric domains intimately related to never-middle domains.

Proposition 1.1.9. *Any symmetric Condorcet domain $\mathcal{D} \subseteq \mathcal{L}(A)$ satisfies a complete set of never-middle conditions. A maximal never-middle Condorcet domain is symmetric.*

Proof. Suppose that a domain $\mathcal{D}$ is symmetric and satisfies $aN_{\{a,b,c\}}3$ for some triple $a, b, c \in A$. Then $\mathcal{D}|_{\{a,b,c\}} \subseteq \{abc, bac, acb, cab\}$. Since $\overline{abc}$ and $\overline{acb}$ are not in $\mathcal{D}|_{\{a,b,c\}}$, then abc and acb cannot actually be found there in which case $\mathcal{D}$ also satisfies $bN_{\{a,b,c\}}2$. If $\mathcal{D}$ satisfies $aN_{\{a,b,c\}}1$, a similar argument holds.

On the other hand, suppose $\mathcal{D}$ satisfies a complete set of never-middle conditions. But every never-middle condition is itself symmetric. Hence, if x satisfies a never-middle condition so does $\bar{x}$. But $\mathcal{D}$ is maximal, hence $\bar{x}$ must also belong to $\mathcal{D}$. So if a Condorcet domain $\mathcal{D}$ satisfies a never-middle condition for each triple and is maximal, then it is symmetric. □

1.2 Graphs of Condorcet domains and their connectedness

In mathematics, the universal domain $\mathcal{L}(A)$ has many representations. One of the most useful ones is by the *permutohedron* of order n, which is an $(n-1)$-dimensional polytope embedded in an n-dimensional space [Ziegler, 2012, Monjardet, 2009]. Its vertices are labeled by the permutations of $\{1, 2, \ldots, n\}$ from the symmetric group S_n. Two permutations are connected by an edge if they differ in only two neighbouring places. For our purposes geometry is not important so for us the permutohedron is the skeleton of this polytope, that is, the graph whose vertices are permutations with edges inherited from the edges of the aforementioned polytope.

The universal domain $\mathcal{L}(A)$ is naturally endowed with the following betweenness structure [Kemeny, 1959]. An order v is *between* orders u and w if $v \supseteq u \cap w$, i.e., if v agrees with all binary comparisons of alternatives in which u and w agree (see also Kemeny and Snell [1960]). The set of all orders that are between u and w is called the *interval* spanned by u and w and is denoted by $[u, w]$.

Given a domain $\mathcal{D}$, for any $u, w \in \mathcal{D}$ we define the induced interval as $[u, w]_\mathcal{D} = [u, w] \cap \mathcal{D}$. Puppe and Slinko [2019] defined a graph $G_\mathcal{D} = (V_\mathcal{D}, E_\mathcal{D})$ associated with $\mathcal{D}$ as follows. The set of linear orders from $\mathcal{D}$ is the set of vertices $V_\mathcal{D}$, and for two orders $u, w \in \mathcal{D}$ we draw an edge connecting them if they have no other vertex between them. Thus $E_\mathcal{D}$ is the set of edges so defined. As established in Puppe and Slinko [2019], for every Condorcet domain $\mathcal{D}$ the graph $G_\mathcal{D}$ is a median graph [Mulder, 1978] and any median graph can be obtained in this way. This will be shown in Section 1.3.

Example 1.2.1. *Up to an isomorphism, there are only three maximal Condorcet domains on the set of alternatives* $\{a, b, c\}$ *presented at* (1.1.2)*, namely,*

$$\mathcal{D}_{3,1} = \{abc, acb, cab, cba\},\ \mathcal{D}_{3,2} = \{abc, bca, acb, cba\},\ \mathcal{D}_{3,3} = \{abc, bac, bca, cba\}.$$

The graphs of these Condorcet domains are shown on Figure 1.2.

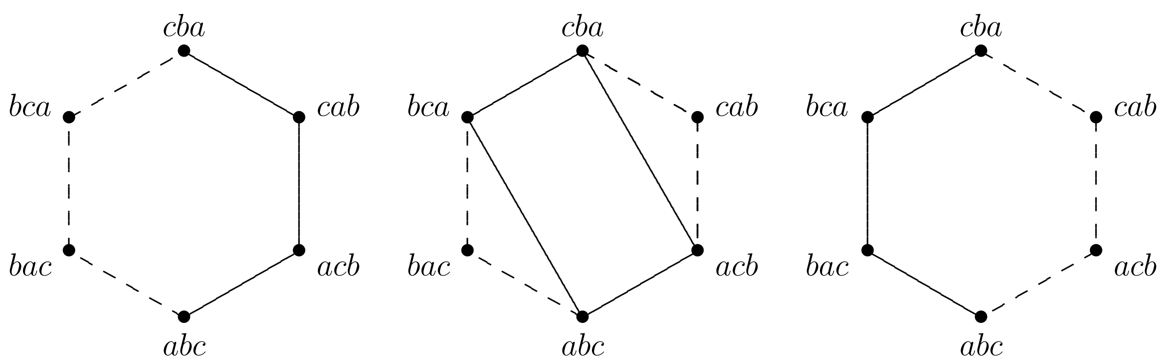


Figure 1.2: Graphs of $\mathcal{D}_{3,1}$, $\mathcal{D}_{3,2}$ and $\mathcal{D}_{3,3}$

Note that the graph $G_\mathcal{D}$ is always connected in the graph-theoretic sense. However, historically connectedness of domains was defined differently.

Definition 1.2.1. *A domain* $\mathcal{D}$ *is called* connected *if its graph* $G_\mathcal{D}$ *is a subgraph of the permutohedron.*

In Figure 1.2 the universal domain $\mathcal{L}(A)$ for $A = \{a, b, c\}$ is represented by the permutohedron on this set (which in this case a hexagon) and we see that the domains $\mathcal{D}_{3,1}$ and

$\mathcal{D}_{3,3}$ are connected while the domains $\mathcal{D}_{3,2}$ is not connected. Sometimes to reflect the fact that the edges connecting *abc* and *bac* and also *acb* and *cba* connect vertices at distance 2 we show them as double lines as shown on Figure 1.3.

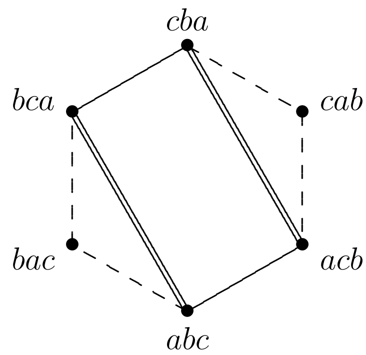


Figure 1.3: A refined graph of $\mathcal{D}_{3,2}$

Besides the single-crossing and the single-peaked domains, to which we will devote separate chapters, the best-known peak-pit domains are Fishburn domains. They are defined by the so-called alternating scheme of never conditions. There are two types of them.

Definition 1.2.2. *Let* $A = [n]$*. A complete set of never-conditions* (1.1.3) *is said to be the* alternating scheme*, if for all* $1 \le i < j < k \le n$ *either*

(i) $jN_{\{i,j,k\}}1$*, if* j *is even, and* $jN_{\{i,j,k\}}3$*, if* j *is odd, or*

(ii) $jN_{\{i,j,k\}}3$*, if* j *is even, and* $jN_{\{i,j,k\}}1$*, if* j *is odd.*

Following Galambos and Reiner [2008] we denote these domains as F_n and $\overline{F_n}$, respectively. The second domain is flip-isomorphic to the first. In particular, $F_2 = \{12, 21\}$, $F_3 = \{123, 132, 312, 321\}$ and

$$F_4 = \{1234, 1324, 3124, 1342, 3142, 3412, 3421, 4312, 4321\}. \tag{1.2.1}$$

The following picture shows how this domain is located in the permutohedron.

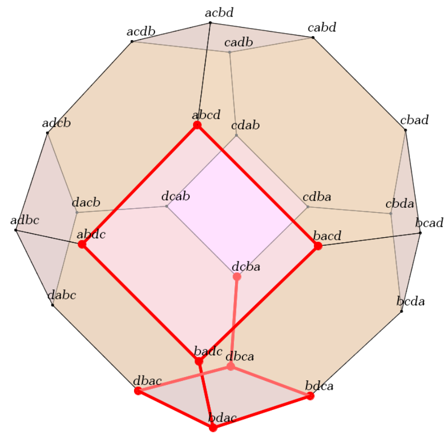


Figure 1.4: Fishburn's domain F_4 embedded in the permutohedron

The latter has the following median graph associated with it:

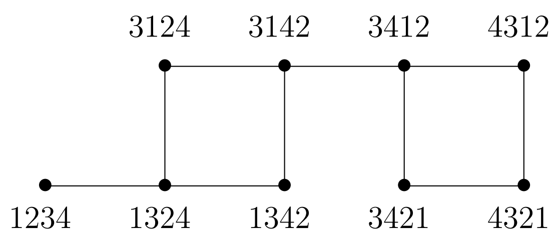


Figure 1.5: Graph of Fishburn's domain F_4 on four alternatives

Note that F_4 has cardinality 9 and is in fact the (uniquely) largest Condorcet domain on a set of four alternatives [Raynaud, 1982]. Fishburn domains are one of the largest for an arbitrary number of alternatives too. Galambos and Reiner [2008] gave the exact formula for the cardinality of F_n:

$$|F_n| = (n+3)2^{n-3} - \begin{cases} (n-\frac{3}{2})\binom{n-2}{\frac{n}{2}-1} & \text{for even } n; \\ (\frac{n-1}{2})\binom{n-1}{\frac{n-1}{2}} & \text{for odd } n. \end{cases} \tag{1.2.2}$$

Fishburn's domains are maximal, we will prove this in Chapter 5 for a larger class of Condorcet domains. We will also discuss constructions of large Condorcet domains in Chapter 8 where Fishburn's domains play a big role. The following table list the first 16 values of $|F_n|$:

2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	4	9	20	45	100	222	488	1069	2324	5034	10840	23226	49704	105884

Finally, in a very important paper Danilov et al. [2012] the following definition was introduced.

Definition 1.2.3. *A domain $\mathcal{D}$ of maximal width is called* semi-connected *if the two completely reversed orders e and $\bar{e}$ can be connected by a shortest (geodesic) path in the permutohedron whose all vertices belong to $\mathcal{D}$.*

A stronger version of connectedness is the following one.

Definition 1.2.4. *We call a domain $\mathcal{D}$ of linear orders* directly connected *if any two orders of this domain can be connected by a shortest (geodesic) path in the permutohedron whose all vertices are in $\mathcal{D}$.*

Clearly, any directly connected Condorcet domain is connected and semi-connected (if of maximal width). For example, the domains $\mathcal{D}_{3,1}$ and $\mathcal{D}_{3,3}$ above are directly connected, but $\mathcal{D}_{3,2}$ is not even connected or semi-connected.

Proposition 1.2.1. *Any semi-connected Condorcet domain $\mathcal{D}$ of maximal width is a copious peak-pit domain.*

Proof. Let $e = 12\ldots n$ and $\bar{e} = n \ldots 21$. For any triple $\{a, b, c\} \subseteq [n]$ with $a < b < c$ we will have $e|_{\{a,b,c\}} = abc$ and $\bar{e}|_{\{a,b,c\}} = cba$. Moving along the geodesic path connecting e with $\bar{e}$ we will have a chain of suborders

$$abc \to bac \to bca \to cba \quad \text{or} \quad abc \to acb \to cab \to cba,$$

that is $\{a, b, c\}$ is either never-bottom or never-top triple with $|\mathcal{D}|_{\{a,b,c\}}| = 4$. □

In a semi-connected domain the order $12\ldots n$ can be transformed into $n \ldots 21$ by a sequence of swaps of neighbouring alternatives. If $i < j < k$, as we have seen in Proposition 1.2.1, then there are two ways of converting ijk into kji, namely,

$$ijk \to jik \to jki \to kji \quad \text{and} \quad ijk \to ikj \to kij \to kji. \tag{1.2.3}$$

In the second case the triple $[i, j, k]$ is called an *inversion triple*. Inversion triples characterise maximal semi-connected Condorcet domains uniquely [Galambos and Reiner, 2008] and provide a convenient description of these domains. For example, Fishburn's domain F_4 is defined by the inversions $[1, 2, 3]$ and $[1, 2, 4]$.

Proposition 1.2.2. *A triple $[i, j, k]$ is an inversion for $\mathcal{D}$ if and only if $jN_{\{i,j,k\}}1$ is satisfied and a non-inversion if and only if $jN_{\{i,j,k\}}3$ is satisfied.*

Proof. Follows directly from (1.2.3). □

1.3 Median and Condorcet domains

In this section we will show that the associated graph $G_{\mathcal{D}}$ of any closed Condorcet domain $\mathcal{D}$ is a median graph and every median graph can be obtained this way.

It will be useful to formulate properties of intervals for further reference.

Proposition 1.3.1. *Let $\mathcal{D}$ be a domain and $u, v \in \mathcal{D}$. Let also $w, z \in [u, v]_D$. Then $[z, w]_D \subseteq [u, v]_D$.*

Proof. As $w, z \in [u, v]_D$, we have $w \supseteq u \cap v$ and $z \supseteq u \cap v$. Suppose $t \in [z, w]_D$. Then $t \supseteq z \cap w \supseteq u \cap v$ and $t \in [u, v]_D$. □

1.3.1 Median domains

A subset $\mathcal{D} \subseteq \mathcal{L}(A)$ is called *median domain* if, for any triple of elements $v_1, v_2, v_3 \in \mathcal{D}$, there exists an element $m(v_1, v_2, v_3) \in \mathcal{D}$, called the *median order* of v_1, v_2, v_3, such that

$$m(v_1, v_2, v_3) \in [v_1, v_2]_{\mathcal{D}} \cap [v_1, v_3]_{\mathcal{D}} \cap [v_2, v_3]_{\mathcal{D}}.$$

Proposition 1.3.2. *The median order of a triple $v_1, v_2, v_3 \in \mathcal{L}(A)$, if exists, is unique.*

Proof. If a triple v_1, v_2, v_3 admits two different median orders, say m and m', these must differ on the ranking of at least one pair of alternatives. Suppose they disagree on the ranking of a and b. In this case, not all three orders of the triple v_1, v_2, v_3 agree on the ranking of a and b. Hence, exactly two of them, say v_1 and v_2, must agree on the ranking of a versus b; but then, either m or m' is not between v_1 and v_2, a contradiction. □

The close connection between Condorcet domains and median domains that we are going to establish stems from the following simple but fundamental observation.

Proposition 1.3.3. *A triple $v_1, v_2, v_3 \in \mathcal{L}(A)$ admits a median order $m(v_1, v_2, v_3)$ if and only if the majority relation $\succ_\rho$ of the profile $\rho = (v_1, v_2, v_3)$ is acyclic, in which case the median order and the majority relation of ρ coincide.*

Proof. If the majority relation $\succ_\rho$ is acyclic, and hence is an element of $\mathcal{L}(A)$, it belongs to each interval $[v_i, v_j]$ for all distinct $i, j \in \{1, 2, 3\}$. Indeed, if both v_i and v_j rank a higher than b, then so does the majority relation. So $\succ_\rho = m(v_1, v_2, v_3)$.

Conversely, if $m = m(v_1, v_2, v_3)$ is the median of the triple v_1, v_2, v_3, then for any pair $a, b \in A$, at least two orders, say v_i and v_j from this triple agree on ranking of a and b. Then m must be between them and hence agree with them. This means it is the majority relation for the profile $\rho = (v_1, v_2, v_3)$. □

Corollary 1.3.4. *Any closed Condorcet domain is a median domain.*

Proof. Suppose $\mathcal{D}$ is a closed Condorcet domain, and let v_1, v_2, v_3 be any triple of orders from $\mathcal{D}$. The majority relation $\succ_\rho$ corresponding to the profile $\rho = (v_1, v_2, v_3) \in \mathcal{D}^3$ is an element of $\mathcal{L}(A)$, and by the assumed closedness it is in fact an element of $\mathcal{D}$. By Proposition 1.3.3,

$$\succ_\rho \in [v_1, v_2] \cap [v_1, v_3] \cap [v_2, v_3] \cap \mathcal{D} = [v_1, v_2]_{\mathcal{D}} \cap [v_1, v_3]_{\mathcal{D}} \cap [v_2, v_3]_{\mathcal{D}}.$$

Thus, $\succ_\rho$ is the median order for the triple v_1, v_2, v_3 in domain $\mathcal{D}$. □

Definition 1.3.1. *A family $\mathbb{F}$ of subsets of a set is said to have the* Helly property *if the sets in any subfamily $\mathbb{F}' \subseteq \mathbb{F}$ have a non-empty intersection whenever all their pairwise intersections are non-empty, i.e., $\mathcal{C} \cap \mathcal{C}' \neq \emptyset$ for each pair $\mathcal{C}, \mathcal{C}' \in \mathbb{F}'$ implies $\cap \mathbb{F}' \neq \emptyset$.*

This property will be important in relation to the concept of convexity that will be now defined.

Definition 1.3.2. *A subset $\mathcal{C} \subseteq \mathcal{D}$ of a domain $\mathcal{D} \subseteq \mathcal{L}(A)$ will be called* convex *(in $\mathcal{D}$) if with any pair $v, v' \in \mathcal{C}$ this set contains the entire interval spanned by v and v', that is, $\mathcal{C}$ is convex if*

$$\{v, v'\} \subseteq \mathcal{C} \;\Rightarrow\; [v, v']_{\mathcal{D}} \subseteq \mathcal{C}.$$

A typical examples of convex sets are intervals themselves. This follows from Proposition 1.3.1.

Proposition 1.3.5. *A domain $\mathcal{D} \subseteq \mathcal{L}(A)$ is a median domain if and only if $\mathcal{D}$ has the Helly property for the family of convex subsets.*

Proof. Let $\mathcal{D}$ be a median domain and $\mathbb{F}$ be a family of convex subsets with pairwise non-empty intersections. We proceed by induction over $m = |\mathbb{F}|$. If $m = 2$, there is nothing to prove, thus let $m = 3$, i.e., $\mathbb{F} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3\}$. Choose any triple of orders $v_1 \in \mathcal{C}_1 \cap \mathcal{C}_2$, $v_2 \in \mathcal{C}_2 \cap \mathcal{C}_3$ and $v_3 \in \mathcal{C}_3 \cap \mathcal{C}_1$, and consider the median order $m = m(v_1, v_2, v_3) \in \mathcal{D}$ for which

$$m \in [v_1, v_2]_{\mathcal{D}} \cap [v_1, v_3]_{\mathcal{D}} \cap [v_2, v_3]_{\mathcal{D}}.$$

By convexity of the sets $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ we have $[v_1, v_2]_{\mathcal{D}} \in \mathcal{C}_2$, $[v_2, v_3]_{\mathcal{D}} \in \mathcal{C}_3$ and $[v_1, v_3]_{\mathcal{D}} \in \mathcal{C}_1$, so $m \in \mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{C}_3$. This, in particular, shows that $\cap \mathbb{F}$ is non-empty. Now consider $\mathbb{F} = \{\mathcal{C}_1, \ldots, \mathcal{C}_k\}$ with $k > 3$ elements, and assume that the assertion holds for all families with less than k elements. Then, by the induction. hypothesis $\mathcal{C}_3 \cap \cdots \cap \mathcal{C}_k \neq \emptyset$ and the family $\{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \cap \cdots \cap \mathcal{C}_k\}$ constitutes a family of three convex subsets with pairwise non-empty intersections. By the preceding argument, we thus have $\cap \mathbb{F} \neq \emptyset$.

Conversely, consider a domain $\mathcal{D}$ such that any family of convex subsets of $\mathcal{D}$ has the Helly property. Consider any three orders $v_1, v_2, v_3 \in \mathcal{D}$. Since, evidently, all intervals are convex, the Helly property applies to the intervals $[v_1, v_2]_{\mathcal{D}}$, $[v_1, v_3]_{\mathcal{D}}$, $[v_2, v_3]_{\mathcal{D}}$ and implies the existence of a median. □

For any domain $\mathcal{D}$ and any pair $x, y \in A$ of alternatives, denote by $\mathcal{V}^{\mathcal{D}}_{xy}$ the set of orders in $\mathcal{D}$ that rank x above y, i.e.,

$$\mathcal{V}^{\mathcal{D}}_{xy} := \{v \in \mathcal{D} \mid x \succ_v y\}. \tag{1.3.1}$$

Note that, for all distinct $x, y \in A$, the sets $\mathcal{V}^{\mathcal{D}}_{xy}$ and $\mathcal{V}^{\mathcal{D}}_{yx}$ form a partition of $\mathcal{D}$. Also observe that the sets of the form $\mathcal{V}^{\mathcal{D}}_{xy}$ are convex for all pairs $x, y \in A$. We will now apply the Helly property to this family of convex sets to show that every median domain is a closed Condorcet domain.

Theorem 1.3.6. *The classes of median domains and closed Condorcet domains coincide, i.e., a domain is a median domain if and only if it is a closed Condorcet domain.*

Proof. In the light of Corollary 1.3.4, it suffices to show that every median domain is a closed Condorcet domain. Thus, let $\mathcal{D}$ be a median domain and consider an odd profile $\rho = (v_1, \ldots, v_n) \in \mathcal{D}^n$. For any two alternatives $x, y \in A$, let $\mathcal{U}_{xy} = \{v_i \mid x \succ_{v_i} y\}$, and observe that obviously, $\mathcal{U}_{xy} \subseteq \mathcal{V}^{\mathcal{D}}_{xy}$. Let z, w be alternatives in A, not necessarily distinct from x and y. If $x \succ_\rho y$ and $z \succ_\rho w$, then $\mathcal{U}_{xy} \cap \mathcal{U}_{zw} \neq \emptyset$ and hence $\mathcal{V}_{xy}^{\mathcal{D}} \cap \mathcal{V}_{zw}^{\mathcal{D}} \neq \emptyset$. By Proposition 1.3.5 by the Helly property

$$\bigcap_{x,y \in A\,:\, x \succ_\rho y} \mathcal{V}_{xy}^{\mathcal{D}} \neq \emptyset,$$

hence there is a linear order in $\mathcal{D}$ which coincides with the majority relation of ρ. Thus $\mathcal{D}$ is a closed Condorcet domain. □

1.3.2 Median graphs

Let $\Gamma = (V, E)$ be a connected graph. The *distance* $d(u, v)$ between two vertices $u, v \in V$ is the smallest number of edges that a path connecting u and v may contain. While the distance is uniquely defined, there may be several shortest paths from u to v. A *(geodesically) convex* set in a graph $\Gamma = (V, E)$ is any subset $C \subseteq V$ such that for every two vertices $u, v \in C$ all vertices on every shortest path between u and v in Γ lie in C.

Definition 1.3.3. *A connected graph $\Gamma = (V, E)$ is called a* median graph *if, for any three vertices $u, v, w \in V$, there is a unique vertex $m(u, v, w) \in V$ which lies simultaneously on some shortest paths from u to v, from u to w and from v to w.*

Trees and hypercubes are examples of median graphs.

Definition 1.3.4. *A* tree *is a connected graph $T = (V, E)$ with the set of vertices V and the set of edges E such that $|E| = |V| - 1$. A subgraph $T_1 = (V_1, E_1)$ is a* subtree *of T, if it is a tree in its own right.*

A tree can be defined in a variety of ways shown in the following proposition.

Proposition 1.3.7. *For a graph $T = (V, E)$ the following conditions are equivalent:*

1. *T is a tree;*
2. *T is connected and acyclic;*
3. *T is a graph where there is a unique path between every two vertices.*

These equivalences can be found in any book on elementary graph theory, e.g, in Diestel [2005].

Since there is a unique path between two vertices u and v we can define the distance $d(u, v)$ between them as the number of edges in the unique path between u and v. Intersection of any two subtrees is a subtree. If any edge in a tree $T = (V, E)$ is removed, the tree becomes a union of two disconnected subtrees.

Example 1.3.1. *A tree is a median graph.*

Proof. Let $u, v, w \in V$. Let us consider the path from u to v and the path from u to w. Let u' be the the farthermost vertex from u which is common to both paths. Since $uu'v$ and $uu'w$ are both paths, we have a path $vu'w$ from v to w which is the only path between these two vertices. Thus u' is the median of u, v and w. □

Example 1.3.2. *A hypercube is a median graph.*

Proof. The n-cube has the set of vertices $V = \{A \mid A \subseteq [n]\}$ with two vertices A and B joined by an edge if and only if $|(A \setminus B) \cap (B \setminus A)| = 1$. Alternatively, representing every subset A of $[n]$ as an n-dimensional characteristic vector $\chi(A)$ we can view the n-cube as the set of all binary n-dimensional vectors with neighbours having Hamming distance 1.

Let $u, v, w \in V$. Without loss of generality we may consider $u = \chi(\emptyset) = (0, 0, \dots, 0)$, $v = \chi(A)$ and $w = \chi(B)$. The shortest path from u to v is characterised by having ones in positions corresponding to $A \cap B$ and zeros in positions corresponding to $[n] \setminus (A \cup B)$. The shortest path from u to v is characterised with zeros in positions of $[n] \setminus A$ and the shortest path from u to w is characterised with zeros in positions of $[n] \setminus B$. Thus the unique vertex on the three paths is the vertex $m = \chi(A \cap B)$. □

Bandelt [1984] characterised arbitrary median graph as a retract of a hypercube. However we need another characterisation. To characterise the structure of an arbitrary median graph $\Gamma = (V, E)$ we need to introduce the concept of *convex expansion*. For any two subsets $S, T \subseteq V$ of the set of vertices of the graph Γ, let $E(S, T) \subseteq E$ denote the set of edges that connect vertices in S and vertices in T.

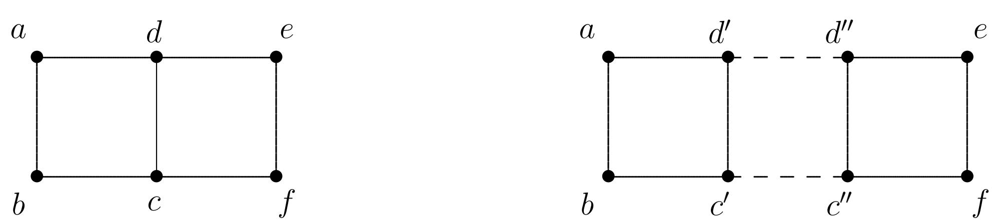


Figure 1.6: Convex expansion of a median graph

Definition 1.3.5. *Let $\Gamma = (V, E)$ be a graph. Let $W_1, W_2 \subset V$ be two subsets with a non-empty intersection $W_1 \cap W_2 \neq \emptyset$ such that $W_1 \cup W_2 = V$ and $E(W_1 \setminus W_2, W_2 \setminus W_1) = \emptyset$. The* expansion *of Γ with respect to W_1 and W_2 is the graph Γ' constructed as follows:*

- *each vertex $v \in W_1 \cap W_2$ in Γ is replaced by two vertices v^1, v^2 in Γ' joined by an edge;*
- *v^1 is joined to all the neighbours of v in $W_1 \setminus W_2$ and v^2 is joined to all the neighbours of v in $W_2 \setminus W_1$;*
- *if $v, w \in W_1 \cap W_2$ and $vw \in E$, then v^1 is joined to w^1 and v^2 is joined to w^2;*

- *if $v, w \in W_1 \setminus W_2$ or if $v, w \in W_2 \setminus W_1$, they will be joined by an edge in Γ' if and only if they were joined by an edge in Γ; if $v \in W_1 \setminus W_2$ and $w \in W_2 \setminus W_1$, they remain not joined in Γ'.*

If W_1 and W_2 are convex, then Γ' will be called a convex expansion *of Γ.*

Example 1.3.3. *In the graph Γ shown on the left of Figure 1.6 we set $W_1 = \{a, b, c, d\}$ and $W_2 = \{c, d, e, f\}$. These are convex and their intersection $W_1 \cap W_2 = \{c, d\}$ is not empty. On the right we see the graph Γ' obtained by the convex expansion of Γ with respect to W_1 and W_2.*

The following important theorem about median graphs is due to Mulder [1978].

Theorem 1.3.8 (Mulder's convex expansion theorem)**.** *A graph is median if and only if it can be obtained from a trivial one-vertex graph by repeated convex expansions.*

The following property of median graphs will be important.

Definition 1.3.6. *A* chordless cycle *in a graph is a cycle such that no two vertices of the cycle are connected by an edge that does not itself belong to the cycle.*

Proposition 1.3.9. *The only positive integer k for which a chordless k-cycle can exist in a median graph is $k = 4$.*

Proof. Clearly, a 3-cycle obviously cannot exist in a median graph. Moreover, for any $k \geq 5$, one can find three vertices v_1, v_2, v_3 on a chordless k-cycle such that the three shortest paths between any pair from v_1, v_2, v_3 cover the entire cycle; this implies that no vertex can simultaneously lie on all three shortest paths, i.e., that the triple v_1, v_2, v_3 does not admit a median. □

1.3.3 Median graphs and closed Condorcet domains

As we already discussed in Section 1.2, with every domain $\mathcal{D} \subseteq \mathcal{L}(A)$ one can associate a graph $G_{\mathcal{D}}$ on $\mathcal{D}$. Note that the graph $G_{\mathcal{D}}$ is always connected, i.e., any two orders in $\mathcal{D}$ are connected by a path in $G_{\mathcal{D}}$. Moreover, any two neighbours v, v' in the permutohedron belonging to $\mathcal{D}$ are always neighbours in $G_{\mathcal{D}}$. However, two neighbours in $G_{\mathcal{D}}$ need not be neighbours in $\mathcal{L}(A)$, so, if $\mathcal{D} \neq \mathcal{L}(A)$, the associated graph $G_{\mathcal{D}}$ need not be a subgraph of the permutohedron.

We will now define another concept of betweenness, different from Kemeny's one.

Definition 1.3.7. *Given a domain $\mathcal{D} \subseteq \mathcal{L}(A)$, the order $w \in \mathcal{D}$ is* geodesically between *the orders $u, v \in \mathcal{D}$ in $G_{\mathcal{D}}$ if w lies on a shortest $G_{\mathcal{D}}$-path that connects u and v.*

As a first example, consider Figure 1.2 which depicts three domains on the set $A = \{a, b, c\}$ with their corresponding graphs. Note that the permutohedron is given by the entire 6-cycle, and that the graphs associated with the two domains $\mathcal{D}_{3,1}$ and $\mathcal{D}_{3,3}$ are linear subgraphs of the permutohedron. In particular, both domains are directly connected and the Kemeny betweenness relation on $\mathcal{D}_{3,1}$ and $\mathcal{D}_{3,3}$ translates into the geodesic betweenness of their associated graphs. More generally, in the universal domain $\mathcal{L}(A)$ both concepts of betweenness coincide.

Proposition 1.3.10. *Let $u, v \in \mathcal{L}(A)$. Then $w \in \mathcal{L}(A)$ is geodesically between u and v in the permutohedron $G_{\mathcal{L}(A)}$ if and only if it is between u and v in the Kemeny sense.*

Proof. Firstly, we observe that a shortest path in the permutohedron between two vertices u and v (from u to v) consists of a series of swaps of neighbouring alternatives a and b such that $a \succ_u b$ and $b \succ_v a$. For this we have to note that, if $u \neq v$, then such a pair of alternatives exists. If not, consider $u = \ldots a \ldots b \ldots$, where the distance between a and b is minimal provided that $b \succ_v a$. As a and b are not neighbours, there is an alternative c such that $u = \ldots a \ldots c \ldots b \ldots$. Then in v we must have either $c \succ_v a$ or $b \succ_v c$, a contradiction to the minimality. This implies that at no time on a shortest path between u and v we swap alternatives on ranking of which u and v agree. Thus all orders on a shortest path are between u and v in the Kemeny sense.

On the other hand, if w is between u and v in the Kemeny sense, let us consider a shortest path π between u and w and a shortest path π' between w and v. Then we claim that $\pi \cup \pi'$ is a shortest path between u and v. Indeed, at no time any pair of alternatives will be swapped on ranking of which u and v agree. If, they disagree on ranking of a pair of alternatives, say $a \succ_u b$ and $b \succ_v a$, then w will agree with one of them and disagree with the other. Hence the pair of alternatives a and b will be swapped either on π or on π' but not on both. This shows that $\pi \cup \pi'$ is a shortest path in the permutohedron. □

It is important to note that for an arbitrary domain $\mathcal{D}$ the Kemeny betweenness on $\mathcal{D}$ and the geodesic betweenness on $G_{\mathcal{D}}$ may not correspond to each other. To illustrate this, consider the three domains with their associated graphs depicted in Figure 1.7. Evidently, none of the three graphs is a subgraph of the permutohedron, hence none of the corresponding domains is connected. As is easily verified, the domain $\{abc, acb, cba, bca\}$ in the middle of Figure 1.7 is a median domain, but the two other domains are not; for instance, the domain $\{abc, cab, cba, bca\}$ on the left of Figure 1.7 contains a cyclic triple of orders abc, bca and cab, and the 5-element domain $\{acb, cab, cba, bca, bac\}$ on the right of Figure 1.7 contains the cyclic triple acb, cba and bac.

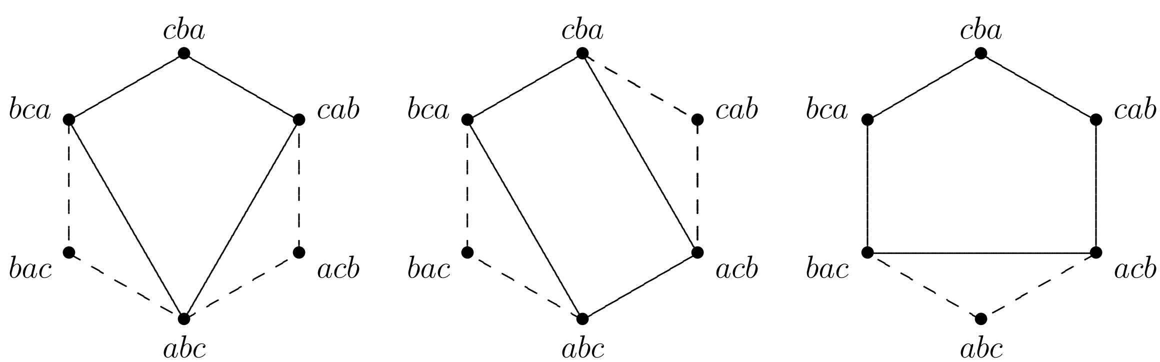


Figure 1.7: Three domains on $\{a, b, c\}$ with their associated graphs

In case of the domain in the middle, which is a Condorcet domain, the Kemeny betweenness on $\mathcal{D}$ and the geodesic betweenness on $G_{\mathcal{D}}$ agree. By contrast, for the domain on the left we have $abc \notin [cab, bca]$ in the permutohedron but evidently abc is geodesically between the orders cab and bca in the associated graph $G_{\mathcal{D}}$. On the other hand, in this example the Kemeny betweenness implies geodesic betweenness. But this also does not hold in general, as the domain on the right of Figure 1.7 shows: here, both orders cab and cba are elements of $[acb, bca]$, but neither of them is geodesically between acb and bca.

Thus, for any domain $\mathcal{D} \subseteq \mathcal{L}(A)$, we now have two betweenness relations on $\mathcal{D}$: the Kemeny betweenness and the geodesic betweenness in the associated graph $G_{\mathcal{D}}$. We will show that in important classes of domains these concepts coincide, that is, if $\mathcal{D}$ is in such a class, for all $u, v, w \in \mathcal{D}$, we have

$$w \in [u, v]_{\mathcal{D}} \Leftrightarrow w \text{ is geodesically between } u \text{ and } v \text{ in } G_{\mathcal{D}}. \tag{1.3.2}$$

Given two linear orders $u, v \in \mathcal{L}(A)$ we define the set of inversions

$$\text{Inv}(u, v) := \{(a, b) \mid a \succ_u b \text{ but } b \succ_v a\}.$$

The following simple observation can be useful.

Proposition 1.3.11. *$w \in [u, v]$ if and only if*

$$Inv(u, w) \cup Inv(w, v) = Inv(u, v).$$

Definition 1.3.8. *Let $\mathcal{D}$ be a Condorcet domain. We say that $u = t_0, t_1, \ldots, t_m = v$ is a* chain *connecting $u, v \in \mathcal{D}$ if $\{t_0, t_1, \ldots, t_m\} \subseteq [u, v]_{\mathcal{D}}$ and $[t_i, t_{i+1}]_{\mathcal{D}} = \{t_i, t_{i+1}\}$ for each $i \in [m-1]$, i.e., if t_i and t_{i+1} are neighbours in $G_{\mathcal{D}}$, and, moreover,*

$$Inv(u, t_1) \subset Inv(u, t_2) \subset \cdots \subset Inv(u, t_{m-1}) \subset Inv(u, v).$$

The chain is maximal *if it cannot be extended to a longer chain.*

If $d(u, v)$ is the distance between u and v in the graph $G_{\mathcal{D}}$, then we can say that the length of a maximal chain between u and v is $d(u, v)$.

Proposition 1.3.12. *Let $\mathcal{D}$ be a Condorcet domain, $u, v, w \in \mathcal{D}$ and $w \in [u, v]_{\mathcal{D}}$. Then w belongs to a chain connecting u and v. All maximal chains connecting u and v have the same length.*

Proof. To prove the first statement we notice that if u, w, v is not a chain, then either u and w or w and v (or both) are not neighbours.

Suppose now that there are two maximal chains $u = t_0, t_1, \ldots, t_m = v$ and $u = s_0, s_1, \ldots, s_n = v$ between u and v of lengths m and n, respectively. Suppose $m < n$.

Consider the representation of $\text{Inv}(u, v)$ as unions of sets in two different ways:

$$\text{Inv}(u, t_1) \cup \text{Inv}(t_1, t_2) \cup \ldots \cup \text{Inv}(t_{m-1}, v) = \text{Inv}(u, v)$$

and

$$\text{Inv}(u, s_1) \cup \text{Inv}(s_1, s_2) \cup \ldots \cup \text{Inv}(s_{n-1}, v) = \text{Inv}(u, v).$$

Since $m < n$ there must be two pairs (a, b) and (c, d) that belong to the same set in the first sequence but are in different sets in the second. This means that for some $i \in [m-1]$ and certain four alternatives $a, b, c, d \in A$ we have $a \succ_{t_i} b$ and $c \succ_{t_i} d$ for all $i = 1, \ldots, q$ and $b \succ_{t_i} a$ and $d \succ_{t_i} c$ for all $i = q+1, \ldots, m$ while for a certain $j \in [n-1]$ we will have $a \succ_{s_j} b$ and $d \succ_{s_j} c$.

Consider now the median $m = m(t_q, t_{q+1}, s_j)$ that exists according to Theorem 1.3.6. By Proposition 1.3.3 it is the majority relation of the profile $\rho = (t_q, t_{q+1}, s_j)$. Hence $a \succ_m b$ and $d \succ_m c$. On the other hand, since $[t_q, t_{q+1}] = \{t_q, t_{q+1}\}$ we must have $m = t_q$ or $m = t_{q+1}$, which is a contradiction. □

The following lemma is central to our approach.

Theorem 1.3.13. *The Kemeny betweenness on a closed Condorcet domain $\mathcal{D} \subseteq \mathcal{L}(A)$ coincides with the geodesic betweenness on the associated graph $G_\mathcal{D}$. In particular, convexity in $\mathcal{D}$ coincides with the geodesic convexity in $G_\mathcal{D}$.*

Proof. For $u, v \in \mathcal{D}$ let $I(u, v)$ be the set of orders w which are geodesically between u and v. Let us prove first, that $I(u, v) \subseteq [u, v]_\mathcal{D}$. The proof will be by induction on $d(u, v)$, where d is the distance in the graph $G_\mathcal{D}$. If $d(u, v) = 1$, then obviously $I(u, v) = \{u, v\} \subseteq [u, v]_\mathcal{D}$. Suppose that for all pairs of orders u, v for which $d(u, v) < n$ the inclusion $I(u, v) \subseteq [u, v]_\mathcal{D}$ holds and suppose $d(u, v) = n$.

Let us assume on the contrary that there exists a sequence of orders $w_0, w_1, \ldots, w_n \in \mathcal{D}$ on a shortest path from u to v, i.e., such that $w_0 = u$, $w_n = v$, and w_i and w_{i+1} are neighbours in $G_\mathcal{D}$ which means $[w_i, w_{i+1}]_\mathcal{D} = \{w_i, w_{i+1}\}$. Suppose $\{w_1, \ldots, w_{n-1}\} \not\subseteq [u, v]_\mathcal{D}$ and let us consider w_{n-1}. If $w_{n-1} \in [u, v]_\mathcal{D}$, then by induction hypothesis $I(u, w_{n-1}) \subseteq [u, w_{n-1}]_\mathcal{D} \subseteq [u, v]_\mathcal{D}$ so $\{w_1, \ldots, w_{n-2}\} \subseteq [u, v]_\mathcal{D}$. We would then have $\{w_1, \ldots, w_{n-1}\} \subseteq [u, v]_\mathcal{D}$, a contradiction. So $w_{n-1} \notin [u, v]_\mathcal{D}$.

As $\mathcal{D}$ by Theorem 1.3.6 is also a median domain, consider now $x = m(u, v, w_{n-1}) \in \mathcal{D}$. In the triple x, v, w_{n-1} we have $[w_{n-1}, v]_\mathcal{D} = \{w_{n-1}, v\}$. Moreover, $w_{n-1} \notin [u, v]_\mathcal{D}$ and hence by Proposition 1.3.1 $w_{n-1} \notin [x, v]_\mathcal{D}$. As all chains from u to w_{n-1} by Proposition 1.3.12 have the same lengths we have $v \notin [x, w_{n-1}]_\mathcal{D}$ since in this case we would have $d(u, v) < n$. Thus

$$[w_{n-1}, v]_\mathcal{D} \cap [x, w_{n-1}]_\mathcal{D} \cap [x, v]_\mathcal{D} = \emptyset,$$

which contradicts to the Helly property which must be satisfied according to Proposition 1.3.5.

Next, we have to prove that $[u, v]_\mathcal{D} \subseteq I(u, v)$. For this not to happen, the distance in the graph $G_\mathcal{D}$ between u and v must be smaller than the common length k of maximal chains between u and v. Hence by Proposition 1.3.12 there will be a chain connecting u and v shorter than k, a contradiction. □

Theorem 1.3.14. *Let $\mathcal{D}$ be a closed Condorcet domain. Then its graph $G_\mathcal{D}$ is median graph.*

Proof. By Corollary 1.3.4 $\mathcal{D}$ is a median domain. By Theorem 1.3.13 the Kemeny betweenness on $\mathcal{D}$ coincides with the geodesic betweenness on the associated graph $G_\mathcal{D}$. Hence $G_\mathcal{D}$ is a median graph. □

Is every median graph induced by some closed Condorcet domain? The following result gives an affirmative answer.

Theorem 1.3.15. *For every (finite) median graph $G = (V, E)$ there exists a closed Condorcet domain $\mathcal{D} \subseteq \mathcal{L}(Y)$ on a finite set of alternatives Y with $|Y| \leq |V|$ such that $G_\mathcal{D}$ is isomorphic to G.*

Proof. We apply Mulder's theorem (Theorem 1.3.8). Since the statement is true for the trivial graph consisting of a single vertex and no edges, arguing by induction, we assume that the statement is true for all median graphs with k vertices or less. Let $G' = (V', E')$ be a median graph with $|V'| = k + 1$. By Mulder's theorem G' is a convex expansion of some median graph $G = (V, E)$ relative to convex subsets W_1 and W_2, where $|V| = \ell \leq k$. By induction there exists a domain $\mathcal{D} \subseteq \mathcal{L}(A)$ with $|A| \leq \ell$ such that G is isomorphic to $G_\mathcal{D}$ via the bijection mapping $\psi\colon v \mapsto \psi(v)$ associating a vertex $v \in V$ with a linear order $\psi(v) \in \mathcal{D}$.

To obtain a new domain $\mathcal{D}'$ such that $G_{\mathcal{D}'}$ is isomorphic to G' we, firstly extend the set of alternatives A to $A' = A \cup \{b\}$, where $b \notin A$. The number of alternatives has increased by one only, so it is not greater than $|V'| = \ell + 1 \leq k + 1$.

Our $\mathcal{D}'$ will be a domain in $\mathcal{L}(A')$. Let us define a bijection $\psi'\colon V' \to \mathcal{D}'$. We note that those elements of V' that are in $W_1 \setminus W_2$ or $W_2 \setminus W_1$ are inherited from V. We fix an alternative $a \in A$. If $v \in W_1 \setminus W_2$ we set $\psi'(v) := \psi(v)(a \to ab)$. If $v \in W_2 \setminus W_1$ we set $\psi'(v) := \psi(v)(a \to ba)$. This means that if v is a vertex of $W_1 \setminus W_2$, we replace a with ab in $\psi(v)$ and for $u \in W_2 \setminus W_1$ we replace a by ba.

Let v now be in $W_1 \cap W_2$. In the convex expansion this vertex is split into v^1 and v^2 and these two will replace v in V'. To obtain $\psi'(v^1)$ we replace a with ab and to obtain $\psi'(v^1)$ we replace a with ba.

To prove that $G_{\mathcal{D}'}$ is isomorphic to G' we must prove that $\psi'(u)$ and $\psi'(v)$ are neighbours if and only if $u, v \in V'$ are.

Firstly, we need to show that there is no edge between $\psi'(u)$ and $\psi'(v)$ if $v \in W_1 \setminus W_2$ and $u \in W_2 \setminus W_1$ since u and v are not neighbours in G'. This follows from the fact that u and v were not neighbours in G and hence $\psi(u)$ and $\psi(v)$ would not be neighbours in $G_\mathcal{D}$. Indeed, if w was between u and v, then both $w(a \to ab)$ and $w(a \to ba)$ will be between $\psi'(u)$ and $\psi'(v)$ in $G'_\mathcal{D}$ as they do not agree on ranking of a and b.

Next, we have to check that $\psi'(v^1)$ and $\psi'(v^2)$ are linked by an edge since v^1 and v^2 are neighbours in G'. This holds because these orders differ in the ranking of just one pair of alternatives, namely a and b, hence they are neighbours in $G_{\mathcal{D}'}$. If $w \in W_1 \cap W_2$ was a neighbour of $v \in W_1 \setminus W_2$ in G, then w^1 is a neighbour of v in G' but w^2 is not.

We then have that $\psi(w)$ is a neighbour of $\psi(v)$ in $G_\mathcal{D}$. Then $\psi'(w^1)$ will obviously be a neighbour of $\psi'(v)$ in $G_{\mathcal{D}'}$ while $\psi'(w^2)$ will not be a neighbour of $\psi'(v)$ in $G_{\mathcal{D}'}$ since $\psi'(w^1)$ will be between them. The remaining cases are considered similarly.

To complete the induction step we need to prove that $\mathcal{D}'$ is a Condorcet domain. We need to consider only triples of elements of A' that contain a and b. Let $\{a, b, c\}$ be such a triple. Since in orders of $\mathcal{D}'$ elements a and b are always standing together, c can never be in the middle. Hence conditions $cN_{\{a,b,c\}}2$ are satisfied and $\mathcal{D}'$ is a Condorcet domain. □

Clearwater et al. [2015] showed that for a median graph with k vertices we might need exactly k alternatives to construct a closed Condorcet domain that has the given graph as associated graph, with the star-graph representing the worst-case scenario. Thus, Theorem 1.3.15 cannot be improved in this respect.

We conclude with the following observation. In Definition 1.2.1 we defined the concept of a connected domain. A stronger version of it is a directly connected domain defined in Definition 1.2.4.

Theorem 1.3.16. *Let $\mathcal{D}$ be a closed Condorcet domain which is either connected or a semi-connected domain of maximal width. Then it is directly connected.*

Proof. By contradiction, assume that $\mathcal{D}$ is not directly connected in which case there exist two orders $u, v \in \mathcal{D}$ such that no shortest path connecting u and v exists in $G_{\mathcal{D}}$ is a shortest path in the permutohedron. Let π be one of those shortest paths in $G_{\mathcal{D}}$. We can then identify orders u_1 and v_1 on that path which are neighbours in $G_{\mathcal{D}}$ but not neighbours in the permutohedron and such that each shortest path between u_1 and v_1 in the permutohedron has only u_1 and v_1 in common with π.

Since u_1 and v_1 are not neighbours in the permutohedron we have two distinct pairs of alternatives (although these pairs may have one alternative in common) (a, b) and (c, d) such that $a \succ_{u_1} b$ and $c \succ_{u_1} d$ but $b \succ_{v_1} a$ and $d \succ_{v_1} c$.

If $\mathcal{D}$ is semi-connected, there is a shortest path in the permutohedron, all belonging to $\mathcal{D}$, that connects e and $\bar{e}$. Then there exist a linear order $w \in \mathcal{D}$ on this path such that either $a \succ_w b$ and $d \succ_w c$ or $b \succ_w a$ and $c \succ_w d$. If $\mathcal{D}$ is connected, then we have a path in the permutohedron connecting u_1 and v_1 that is entirely within $\mathcal{D}$, hence an element $w \in \mathcal{D}$ with the above properties can also be found.

Without loss of generality, suppose $a \succ_w b$ and $d \succ_w c$. Then the median $m = m(u_1, v_1, w) \in \mathcal{D}$ must be either u_1 or v_1 (as they are neighbours in $G_{\mathcal{D}}$). By Proposition 1.3.3 m is also the majority relation of the profile (u_1, v_1, w) in which case $a \succ_m b$ and $d \succ_m c$. So $m \neq u_1$ and $m \neq v_1$, a contradiction. □

1.3.4 Further properties of median graphs

In median graphs, convex sets can be viewed in another useful way through the notion of a gate. Let $G = (V, E)$ be any graph. For $W \subseteq V$ and $x \in V$, the vertex $z \in W$ is a *gate* for x in W if $z \in [x, w]$ for all $w \in W$ or, alternatively, $d(x, w) = d(x, z) + d(z, w)$.

Note that a vertex x has at most one gate in any set W. Indeed, if z_1 and z_2 are two gates for W, then $z_1 \in [x, z_2]$ and $z_2 \in [x, z_1]$, which is impossible. Also, if x has a gate $z \in W$, then z is the unique nearest vertex to x in W.

Definition 1.3.9. *The set of vertices $W \subseteq V$ is* gated *if every vertex in V has a gate in W.*

Proposition 1.3.17. *In any graph $G = (V, E)$, a gated set is convex and in a median graph a set is gated if and only if it is convex.*

Proof. If $W \subseteq V$ is gated and not convex, then there exists $x \in [w_1, w_2]$ while $w_1, w_2 \in W$ but $x \notin W$. Then $d(w_1, w_2) = d(w_1, x) + d(x, w_2)$. Let z be a gate for x in W. Then z is between x and w_1 and between x and w_2, hence

$$d(w_1, w_2) = d(w_1, x)+d(x, w_2) = d(w_1, z)+d(z, x)+d(w_2, z)+d(z, x) = d(w_1, w_2)+2d(z, x),$$

whence $d(z, x) = 0$ and $x = z \in W$, a contradiction.

Suppose now G is a median graph and W is convex. Let $x \notin W$. Choose $z \in W$ which is one of the closest to x in W. Let $w \in W$ and consider the median m of the triple $\{x, w, z\}$. Due to convexity of W we have $m \in [w, z] \subseteq W$. On the other hand, $d(x, z) = d(x, m) + d(m, z)$ from which $d(m, z) = 0$ and $z = m$, and $d(x, w) = d(x, z) + d(z, w)$. Thus z is the gate of x for W. □

Definition 1.3.10. *Let $G = (V, E)$ be a graph. We say that $H \subseteq V$ is a* half-space *of G if both H and $V \setminus H$ are convex.*

Proposition 1.3.18. *Let $G = (V, E)$ be a median graph and consider two neighbours $v_1, v_2 \in V$, i.e., such that $d(v_1, v_2) = 1$. Then*

$$W_1 = \{w \in V \mid d(w, v_1) < d(w, v_2)\}, \qquad W_2 = \{w \in V \mid d(w, v_2) < d(w, v_1)\}. \quad (1.3.3)$$

are both convex and complements of each other.

Proof. Firstly, suppose there exist $x \in V$ such that $d(x, v_1) = d(x, v_2)$. Then the median of the triple $\{x, v_1, v_2\}$ must be either v_1 or v_2. Suppose it is v_1. Then $d(x, v_2) = d(x, v_1)+1$, a contradiction. Hence $W_1 \cup W_2 = V$ and $W_1 \cap W_2 = \emptyset$.

Let $u, v \in W_1$ and $w \in [u, v]$ with $w \in W_2$. Consider the triple $\{u, v_1, v_2\}$. The median of this triple must be v_1, hence $d(u, v_2) = d(u, v_1) + 1$. Also, $d(v, v_2) = d(v, v_1) + 1$. Similarly, $d(w, v_1) = d(u, v_2) + 1$. Thus, $d(u, w) = d(u, v_1) + 1 + d(v_2, w)$ and $d(v, w) = d(v, v_1) + 1 + d(v_2, w)$. Whence

$$d(u, v) = d(u, w) + d(w, v) > d(u, v_1) + d(v_1, v) \geq d(u, v),$$

a contradiction. □

Let $\pi = (v_1, \ldots, v_n)$ be a multiset of vertices of a graph $G = (V, E)$. We then denote

$$\mathrm{Med}(\pi) = \arg\min_{x \in V} \sum_{i=1}^{n} d(x, v_i). \quad (1.3.4)$$

In general, this set may contain several vertices.

Lemma 1.3.19. *Let G be a median graph and let $v \in Med(\pi)$ for a multiset of vertices $\pi = (v_1, \ldots, v_n)$ of odd cardinality, $n = 2k + 1$. Then every half-space in G that contains v also contains more than half of vertices of π.*

Proof. Let H be a half-space containing at least $k+1$ vertices from π. Suppose $v \in G \setminus H$. As H is convex, by Proposition 1.3.17 it is gated. Let z be a gate of v for H. Let us also choose $v' \in [v, z]$ such that $d(v, v') = 1$ (it is possible that $v' = z$).

Obviously, $d(v', v_i) = d(v, v_i) - 1$ for $v_i \in H$ and $d(v', v_i) \leq d(v, v_i) + 1$ for $v_i \in G \setminus H$. Hence

$$\sum_{i=1}^{n} d(v', v_i) = \sum_{v_i \in H} d(v', v_i) + \sum_{v_i \in G \setminus H} d(v', v_i) \leq$$

$$\sum_{v_i \in H} d(v, v_i) - k - 1 + \sum_{v_i \in G \setminus H} d(v, v_i) + k < \sum_{i=1}^{n} d(v, v_i),$$

which contradicts to the fact that $v \in \mathrm{Med}(\pi)$. □

Lemma 1.3.20. *Let G be a median graph and $\pi = (v_1, \dots, v_n)$ be a multiset of vertices of odd cardinality. Then*

$$|Med(\pi)| = 1.$$

Proof. Since π is finite, $\mathrm{Med}(\pi)$ is non-empty. Let $m_1, m_2 \in \mathrm{Med}(\pi)$ with $m_1 \neq m_2$. Let us consider any shortest path between m_1 and m_2 and choose on this path two neighbours v_1 and v_2. Let W_1 and W_2 be the sets defined in (1.3.3). They are convex by Proposition 1.3.18 with $m_1 \in W_1$ and $m_2 \in W_2$. By Lemma 1.3.19 both W_1 and W_2 contain more than half of elements of π which is impossible. □

Finally, we will prove a statement which is true for any distance function. Let us denote by $v = (v_1, \dots, v_n)$ and by (v_{-i}, z) the vector v with v_i replaced by z.

Proposition 1.3.21. *For every distance function $d\colon X \times X \to \mathbb{N}$, and any sequence $v = (v_1, \dots, v_n) \in X^n$ the correspondence*

$$C(v) \;:=\; \arg\min_{x \in X} \sum_{i=1}^{n} d(v_i, x)$$

is self-supporting*, that is, if $x \in C(v)$ then $x \in C(v_{-i}, x)$.*

Proof. Suppose that $x \in C(v)$; then, for all $y \in X$. For $z \in X$ let us denote $\Delta_v(z) = \sum_{i=1}^{n} d(v_i, z)$. Then

$$\Delta_v(x) \;=\; \sum_{j \neq i} d(v_j, x) + d(v_i, x) \;\leq\; \sum_{j \neq i} d(v_j, y) + d(v_i, y) \;=\; \Delta_v(y),$$

hence

$$\sum_{j \neq i} d(v_j, x) \;\leq\; \sum_{j \neq i} d(v_j, y) + d(v_i, y) - d(v_i, x).$$

By the triangle inequality, $d(v_i, y) - d(v_i, x) \leq d(x, y)$, hence

$$\Delta_{(v_{-i},x)}(x) \;=\; \sum_{j \neq i} d(v_j, x) \;\leq\; \sum_{j \neq i} d(v_j, y) + d(x, y) \;=\; \Delta_{(v_{-i},x)}(y),$$

i.e., $x \in C(v_{-i}, x)$. □

In particular, we have

Corollary 1.3.22. *Let G be a median graph and $\pi = (v_1, \ldots, v_n)$ be a multiset of vertices of odd cardinality. Then $Med(\pi_{-i}, Med(\pi)) = Med(\pi)$.*

Proposition 1.3.23. *Let $\mathcal{D}$ be a closed Condorcet domain and let $\pi = (v_1, \ldots, v_n) \in \mathcal{D}^n$ be a profile with an odd number of voters. Then $m = Med(\pi)$ coincides with the majority relation of π.*

Proof. Let us consider a partition of the median graph $\Gamma_{\mathcal{D}}$ into two convex sets $\Gamma_{\mathcal{D}} = \mathcal{V}^{\mathcal{D}}_{ab} \cup \mathcal{V}^{\mathcal{D}}_{ba}$, where $\mathcal{V}^{\mathcal{D}}_{xy} = \{u \in \mathcal{D} \mid x \succ_u y\}$. Suppose $a \succ_m b$, i.e., $m \in \mathcal{V}^{\mathcal{D}}_{ab}$. By Lemma 1.3.19 then $\mathcal{V}^{\mathcal{D}}_{ab}$ contains more than half of linear orders of π, hence the majority relation will also rank a above b. □

1.4 Linear orders, weak Bruhat poset and maximal Condorcet domains

In this section we will talk only about inversions for the pair of linear orders relative to the order $e = 12 \ldots n$, and denote $\mathrm{Inv}(v) = \mathrm{Inv}(e, v)$.

1.4.1 Maximal Condorcet domains and pairwise compatible linear orders

Let $\Omega = \Omega_n = \{(i, j), 1 \le i < j \le n\}$ be the set of pairs of distinct numbers from $[n]$ where the first number is smaller than the second. A pair (i, j) from Ω is called an *inversion* of a linear order v if $j \succ_v i$ while $i < j$, that is if v inverses the natural relation $i < j$. The set of all inversions of v is denoted $\mathrm{Inv}(v)$; it is a subset of Ω. For example, $\mathrm{Inv}(e)$ is the empty subset, whereas $\mathrm{Inv}(\bar{e})$ is the whole Ω. For a set $S \subseteq \Omega$ we denote $\bar{S} = \Omega \setminus S$. Then

$$\mathrm{Inv}(\bar{v}) = \Omega \setminus \mathrm{Inv}(v) = \overline{\mathrm{Inv}(v)}. \tag{1.4.1}$$

We need to characterise those subsets of Ω which are the inversion sets of linear orders.

Definition 1.4.1. *A subset $S \subseteq \Omega$ is* transitive *if $(i, j) \in S$ and $(j, k) \in S$ implies $(i, k) \in S$. A subset $S \subseteq \Omega$ is* co-transitive *if $(i, k) \in S$ and $i < j < k$ implies either $(i, j) \in S$ or $(j, k) \in S$ (or both).*

Example 1.4.1. *Let $v \in \mathcal{L}([n])$ be a linear order. Then the inversion set $Inv(v)$ is transitive and co-transitive.*

Proof. If v is a linear order, then transitivity of $\mathrm{Inv}(v)$ is implied from the transitivity of v. Indeed, if $(i, j) \in \mathrm{Inv}(v)$ and $(j, k) \in \mathrm{Inv}(v)$, then $i < j < k$ and $j \succ_v i$ and $k \succ_v j$. By transitivity of v we have $k \succ_v i$ and $(i, k) \in \mathrm{Inv}(v)$.

Suppose now that $(i,k) \in \mathrm{Inv}(v)$. Then $i < k$ and $k \succ_v i$. If there is no j such that $i < j < k$, then co-transitivity is trivially true. Suppose such j exists. Then we can have three options:

$$k \succ_v i \succ_v j, \quad k \succ_v j \succ_v i, \quad j \succ_v k \succ_v i,$$

from which we have $k \succ_v j$ in the first two cases and $j \succ_v i$ in the last two. This means that $(j,k) \in \mathrm{Inv}(v)$ or $(i,j) \in \mathrm{Inv}(v)$ which means $\mathrm{Inv}(v)$ is co-transitive. □

Note that the union of co-transitive subsets is co-transitive, and the intersection of transitive subsets is transitive. Also

Proposition 1.4.1. *If $S \subseteq \Omega$ is transitive, then $\bar{S}$ is co-transitive. If $S \subseteq \Omega$ is co-transitive, then $\bar{S}$ is transitive.*

Proof. Suppose S is transitive $i < j < k$ and suppose $(i,k) \in \bar{S}$. If $(i,j) \notin \bar{S}$ and $(j,k) \notin \bar{S}$, then $(i,j) \in S$ and $(j,k) \in S$ which by transitivity implies $(i,k) \in S$, a contradiction. □

Yanagimoto and Okamoto [1969] used these concepts to characterise inversion sets.

Theorem 1.4.2. *A subset $S \subseteq \Omega$ is the inversion set Inv(v) for some linear order v if and only if S is transitive and co-transitive.*

Proof. If v is a linear order, then, as we saw in Example 1.4.1 the set $\mathrm{Inv}(v)$ is transitive and co-transitive.

On the other hand, if S is transitive and co-transitive, then for any pair of numbers i and j with $i < j$ let us define $i \succ j$ if $(i,j) \in \bar{S}$ and $j \succ i$ if $(i,j) \in S$. Let us prove that $\succ$ is a linear order. This relation is obviously complete since for $i,j \in [n]$ we have either $(i,j) \in S$ or $(i,j) \in \bar{S}$ resulting in $j \succ i$ or $i \succ j$. It suffices to prove its transitivity of $\succ$. Suppose $i \succ j$ and $j \succ k$. These could result from:

- $(i,j) \in \bar{S}$ and $(j,k) \in \bar{S}$. Then $i < j < k$ and suppose $k \succ i$ which means $(i,k) \in S$. But due to transitivity of S and co-transitivity of $\bar{S}$ we have $(i,k) \in \bar{S}$. This cannot be true, contradiction.

- $(i,j) \in \bar{S}$ and $(k,j) \in S$. Then $i < j$ and $k < j$. Suppose $k \succ i$. This could mean $i < k$ and $(i,k) \in S$. Then due to transitivity $(i,j) \in S$, which is a contradiction. Alternatively $k < i$ and $(k,i) \in \bar{S}$. Then we have $k < i < j$ and due to co-transitivity of S and transitivity of $\bar{S}$ we have $(k,j) \in \bar{S}$, a contradiction again.

- $(j,i) \in S$ and $(j,k) \in \bar{S}$. This case is similar to the previous one.

- $(j,i) \in S$ and $(k,j) \in S$ with $k < j < i$. Then by transitivity $(k,i) \in S$ which contradicts to $k \succ i$.

Theorem is proved. □

Definition 1.4.2. *Linear orders u and v are* compatible *if the set Inv$(u) \cup$ Inv(v) is transitive and the set Inv$(u) \cap$ Inv(v) is co-transitive.*

In particular, e and $\bar{e}$ are compatible with any linear order. For any u, the linear orders u and $\bar{u}$ are compatible.

Lemma 1.4.3. *Linear orders u and v are compatible if and only if $\bar{u}$ and $\bar{v}$ are compatible.*

Proof. Due to (1.4.1) we have

$$\mathrm{Inv}(\bar{u}) \cap \mathrm{Inv}(\bar{v}) = \overline{\mathrm{Inv}(u)} \cap \overline{\mathrm{Inv}(v)} = \overline{\mathrm{Inv}(u) \cup \mathrm{Inv}(v)}$$

is co-transitive by Proposition 1.4.1. The transitivity of $\mathrm{Inv}(\bar{u}) \cup \mathrm{Inv}(\bar{v})$ is similar. □

To relate this with the previous two section, let us prove the following.

Proposition 1.4.4. *In a domain $\mathcal{D}$ of maximal width, linear orders u and v are compatible if and only if the set $\{e, \bar{e}, u, v\}$ is a Condorcet domain.*

Proof. By Theorem 1.1.5 it is sufficient to prove this for the case of just three alternatives. Suppose now that u and v are compatible and consider, for example, the subset $\{1, 2, 3\} \subseteq [n]$. If $u = 132$, then $\mathrm{Inv}(u) = \{(2, 3)\}$. Then $\mathrm{Inv}(u) \cap \mathrm{Inv}(v)$ will always be co-transitive. Then if v is different from e and $\bar{e}$, then we have the following options for $\mathrm{Inv}(v)$:

$$\{(1, 2)\}, \quad \{(1, 2), (1, 3)\}, \quad \{(1, 2), (2, 3)\}, \quad \{(1, 3), (2, 3)\}.$$

Only the second and the fourth produce compatible orders $\bar{u} = 231$ and 312, respectively. This leads to Condorcet domains

$$\{123, 132, 231, 321\} \quad \text{and} \quad \{123, 132, 312, 321\},$$

respectively. The other cases are similar. □

Corollary 1.4.5. *In a symmetric domain $\mathcal{D}$ of maximal width, if the linear orders u and v are compatible and $v \notin \{e, \bar{e}, u\}$ then $v = \bar{u}$.*

Finally we have the following characterisation of Condorcet domains of maximal width.

Theorem 1.4.6. *Maximal Condorcet domains of maximal width are exactly maximal sets of pairwise compatible linear orders.*

Proof. Due to Proposition 1.4.4 it is only sufficient to note that maximal set of pairwise compatible linear orders always contain e and $\bar{e}$ as they are compatible with any order. □

1.4.2 Bruhat lattice

Definition 1.4.3. *For linear orders $u, v \in \mathcal{L}([n])$, we write $u \ll v$ if $Inv(u) \subseteq Inv(v)$. The relation $\ll$ is called the* weak Bruhat order*.*

Together with the weak Bruhat order the universal domain $\mathcal{L}([n])$ becomes a partially ordered set. We note that, if $u \ll v$, then u and v are compatible.

A partially ordered set $(L, \leq)$ such that for every pair of elements $x, y \in L$ there is a unique supremum $x \vee y$ (also called a least upper bound or join) and a unique infimum $x \wedge y$ (also called a greatest lower bound or meet) is said to be a *lattice*, often denoted $(L, \vee, \wedge)$.

Theorem 1.4.7. *$(\mathcal{L}([n]), \ll)$ is a lattice.*

To prove this theorem we need the following concept and the following lemma. Let $S \subseteq \Omega$. A pair $(i, k) \in \Omega$ belongs to the transitive closure $\hat{S}$ if there exists a sequence

$$i = i_0 < i_1 < \cdots < i_p = k \tag{1.4.2}$$

such that every neighbouring pair (i_s, i_{s+1}) belongs to S.

Lemma 1.4.8. *Let S be a co-transitive subset in Ω. Then the transitive closure $\hat{S}$ of S is co-transitive as well.*

Proof. Suppose that $(i, k) \in \hat{S}$ and $i < j < k$. We have to prove that either $(i, j) \in \hat{S}$ or $(j, k) \in \hat{S}$. We have a sequence (1.4.2) with (i_s, i_{s+1}) belongs to S for $s = 0, 1, \ldots, p-1$. If $j \in \{i_1, \ldots, i_{p-1}\}$, then both (i, j) and (j, k) belong to $\hat{S}$. If not, then $i_s < j < i_{s+1}$ for some s. As (i_s, i_{s+1}) belongs to S and S is co-transitive, we have either (i_s, j) or (j, i_{s+1}) in $\hat{S}$. Suppose the former, then we have the sequence

$$i < i_1 < \cdots < i_s < j$$

where each neighbouring pair belongs to S. Thus $(i, j) \in \hat{S}$. □

Now we can prove Theorem 1.4.7.

Proof of Theorem 1.4.7. For linear orders u and v, denote $S = \mathrm{Inv}(u)$ and $T = \mathrm{Inv}(v)$. Let R be the transitive closure of $S \cup T$. Since $S \cup T$ is co-transitive as a union of two co-transitive sets, the set R is transitive and co-transitive due to Lemma 1.4.8. Therefore R is the inversion set for some linear order w.

Obviously, $u \ll w$ and $v \ll w$. Now, suppose that z is a linear order such that $u \ll z$ and $v \ll z$. Let $Q = \mathrm{Inv}(z)$. Due to $u \ll z$ and $v \ll z$ we have $S \subseteq Q$ and $T \subseteq Q$, hence $S \cup T \subseteq Q$. Since Q is a transitive set, Q contains the transitive closure of $S \cup T$, whence $w \ll z$.

Thus we proved the existence of the join $w = u \vee v$ in the poset $(\mathcal{L}([n]), \ll)$. The meet of u and v also exists due to duality. □

1.4.3 Maximal chains in Bruhat lattice and Condorcet domains

A lattice $(L, \vee, \wedge)$ is *distributive* if the following identity holds for all $x, y, z \in L$:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

It is a basic fact of lattice theory that the above condition is equivalent to its dual:

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

Theorem 1.4.9. *Let $\mathcal{D}$ be a maximal Condorcet domain of maximal width and $u, v \in \mathcal{D}$. Then $u \vee v = u \cup v$ and $u \wedge v = u \cap v$ are linear orders belonging to $\mathcal{D}$ and $\mathcal{D}$ is a distributive sublattice in the Bruhat lattice $(\mathcal{L}[n], \ll)$.*

Proof. By Theorem 1.4.6 $\mathcal{D}$ consists of pairwise compatible linear orders. We know that if u and v are compatible then $\mathrm{Inv}(u) \cup \mathrm{Inv}(v)$ is automatically co-transitive as a union of two co-transitive sets and $\mathrm{Inv}(u) \cap \mathrm{Inv}(v)$ is automatically transitive so both sets are balanced and each correspond to a linear order. This means that $u \cup v$ and $u \cap v$ are both linear orders and hence $u \vee v = u \cup v$ and $u \wedge v = u \cap v$. If $\mathrm{Inv}(u) = S$ and $\mathrm{Inv}(v) = T$ we have

$$\mathrm{Inv}(u \vee v) = S \cup T, \qquad \mathrm{Inv}(u \wedge v) = S \cap T.$$

As the lattice of subsets of a finite set is distributive, $\mathcal{D}$ is a distributive sublattice in the Bruhat lattice $(\mathcal{L}[n], \ll)$. □

In the theory of Condorcet domains an important role is played by the maximal chains in the Bruhat lattice $(\mathcal{L}[n], \ll)$.

Proposition 1.4.10. *Let $(\mathcal{E}, \ll)$ be a maximal chain in the Bruhat lattice $(\mathcal{L}[n], \ll)$. Then $\mathcal{E}$ is a copious peak-pit Condorcet domain of size $\frac{1}{2}n(n-1)+1$.*

Proof. Since all members of $\mathcal{E}$ are pairwise compatible, $\mathcal{E}$ is a Condorcet domain. It has maximal width and semi-connected. By Proposition 1.2.1 it is peak-pit and copious. The size of $\mathcal{E}$ is equal to $|\mathrm{Inv}(\bar{e})| = \frac{1}{2}n(n-1)+1$. □

Lemma 1.4.11. *Let $\mathcal{E}_1, \ldots, \mathcal{E}_k$ be maximal chains in the Bruhat lattice. Then $\mathcal{E} = \bigcup_{i=1}^{k} \mathcal{E}_i$ is a Condorcet domain if and only if each maximal chain satisfies the same set of inversion triples.*

Proof. Let $i < j < k$. If for all of the maximal chains $[i, j, k]$ is a non-inversion, then $\mathcal{E}|_{\{i,j,k\}} = \{ijk, jik, jki, kji\}$ and $\mathcal{E}$ satisfies $jN_{\{i,j,k\}}3$. If it is an inversion, then $\mathcal{E}|_{\{i,j,k\}} = \{ijk, ikj, kij, kji\}$ and $\mathcal{E}$ satisfies $jN_{\{i,j,k\}}1$. If for one of the maximal chains $[i, j, k]$ is an inversion and for another it is non-inversion, then $\mathcal{E}|_{\{i,j,k\}} = \{ijk, jik, jki, ikj, kij, kji\}$ and $\mathcal{E}$ cannot be a Condorcet domain. □

Theorem 1.4.12 (Galambos and Reiner [2008])**.** *Any semi-connected maximal Condorcet domain $\mathcal{D}$ is a union of maximal chains satisfying a common set of inversion triples.*

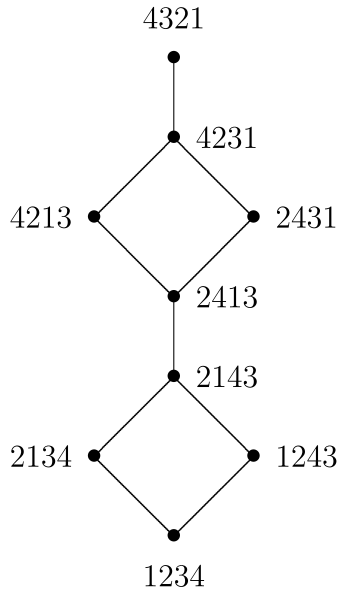


Figure 1.8: Four maximal chains in Fishburn's domain

Proof. Since $\mathcal{D}$ is semi-connected, it contains a maximal chain, say $\mathcal{E}$. By Lemma 1.4.11 $\mathcal{D} \supset \bigcup_{i=1}^{k} \mathcal{E}_i$, where $\{\mathcal{E}_i \mid i \in [k]\}$ is the set of all maximal chains that satisfy the same set of inversion triples as $\mathcal{E}$ (in particular, $\mathcal{E}$ is one of them). Suppose $\mathcal{D}$ is larger than the union $\bigcup_{i=1}^{k} \mathcal{E}_i$ and contains a linear order u which does not belong to this union. By Theorem 1.3.16 $\mathcal{D}$ is directly connected, hence u can be connected by two shortest paths with e and $\bar{e}$, i.e., it belongs to a maximal chain $\mathcal{F}$ which is satisfied by a set of inversion triples different from that of $\mathcal{E}$. This, however, contradicts to Lemma 1.4.11. $\square$

Example 1.4.2. *Fishburn's domain is the union of four maximal chains in* $\mathcal{L}([4])$*:*

$$1234 \to 2134 \to 2143 \to 2413 \to 4213 \to 4231 \to 4321$$
$$1234 \to 1243 \to 2143 \to 2413 \to 4213 \to 4231 \to 4321$$
$$1234 \to 2134 \to 2143 \to 2413 \to 2431 \to 4231 \to 4321$$
$$1234 \to 1243 \to 2143 \to 2413 \to 2431 \to 4231 \to 4321$$

shown on Figure 1.8.

As a side comment we note that in Computer Science the weak order is known as a sorting network [Knuth, 1997]. A sorting network is a series of comparisons and exchanges to apply on a list of entries on the list in order to sort the list. Each descending path from the maximal permutation to the identity "sorts" the permutation so each of these paths corresponds to a possible implementations of the well known (inefficient) bubble sort algorithm.

1.4.4 Maximal chains and reduced decompositions

We remind the reader that the set of adjacent transpositions of S_n, namely,

$$\{s_i = (i, i+1) \mid 1 \le i \le n-1\},$$

generate S_n. Moreover, S_n, as a group, is defined by the so-called *Coxeter relations* [Kassel and Turaev, 2008, Chapter 4.1]:

$$\begin{aligned} s_i^2 &= 1, \\ s_i s_j &= s_j s_i \text{ for all } |i-j| > 1, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} \text{ for all } 1 \le i \le n-2, \end{aligned}$$

where 1 stands for the identity permutation. This, in particular, means that, if any two permutations are equal in S_n, they can be transformed one into another by the Coxeter relations alone. The second equation is called the *short braid relation*, and last equation is the *long braid relation*.

Permutations from S_n act on linear orders from $\mathcal{L}([n])$ as follows. If v is a linear order then vs_i interchanges the alternatives in positions i and $i+1$ in v. That is, if $v = v_1 \dots v_i v_{i+1} \dots v_n$, then $vs_i = v_1 \dots v_{i+1} v_i \dots v_n$.

We say that $r = (i_1, \dots, i_p)$ is a *reduced decomposition* of $w \in S_n$, if

$$w = s_{i_1} \dots s_{i_p},$$

where p is minimal, in which case

$$p = \ell(w) = \#\{(i,j) \mid i < j \text{ and } w(i) > w(j)\},$$

i.e., p is the *number of inversions* in w, denoted $\mathrm{inv}(w)$. In this case w can be obtained from $e = 12 \dots n$ by making swaps of adjacent alternatives $s_{i_1}, \dots, s_{i_p}$. We note that $\ell(\bar{e}) = \frac{1}{2}n(n-1)$. We will be especially interested in the reduced decompositions of $\bar{e} = n \dots 21$ as this will be related to maximal chains in Bruhat order (and later in Chapter 4 to arrangements of pseudolines).

Let $e = u_0 \ll u_1 \ll \dots \ll u_p = \bar{e}$, where $p = \frac{1}{2}n(n-1)+1$ be a maximal chain $(\mathcal{E}, \ll)$ in the Bruhat lattice $(\mathcal{L}[n], \ll)$, where e is the identity permutation and $\bar{e}$ is the reversing permutation. Then, since u_{k+1} is obtained from u_k by a swap of adjacent alternatives, we have $u_{k+1} = u_k s_{i_k}$ and hence

$$\bar{e} = e s_{i_1} \dots s_{i_p} = s_{i_1} \dots s_{i_p}. \tag{1.4.3}$$

where $\bar{e}$ is the reversing permutation whose set of inversions is $\mathrm{Inv}(\bar{e}) = \Omega_n$.

Example 1.4.3 (Example 1.4.2 continued)**.** *In this case we have four maximal chains and, respectively, four reduced decompositions of $\bar{e}$:*

$$\bar{e} = s_1 s_3 s_2 s_1 s_3 s_2 = s_3 s_1 s_2 s_1 s_3 s_2 = s_1 s_3 s_2 s_3 s_1 s_2 = s_3 s_1 s_2 s_3 s_1 s_2.$$

The following result, although intuitively obvious, is non-trivial and important.

Theorem 1.4.13 (Tits [1969]). *Two reduced decompositions of $w \in S_n$ can be transformed one into another using only short and long braid relations*

The proof can also be found in [Kassel and Turaev, 2008, Lemma 4.11].

Definition 1.4.4. *Two reduced decompositions of $w \in S_n$ are* equivalent *if they can be transformed one into the other using only the short braid relation.*

Theorem 1.4.14. *Two maximal chains in $\mathcal{L}(A)$ satisfy the same set of inversion triples if and only if the corresponding reduced decompositions are equivalent.*

Proof. If the reduced decompositions are obtained one from another by using the short braid relation $s_i s_j = s_j s_i$ with $|j-i| > 1$, then obviously this does not change the inversion triples as s_i and s_j act on disjoint pairs of alternatives. On the other hand, if the long braid relation $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ is used, then for some triple of alternatives $a, b, c \in A$ we change the respective subsequence

$$\ldots abc \ldots \ll \ldots bac \ldots \ll \ldots bca \ldots \ll \ldots cba \ldots$$

in the first chain into subsequence

$$\ldots abc \ldots \ll \ldots acb \ldots \ll \ldots cab \ldots \ll \ldots cba \ldots$$

in the second chain which changes a non-inversion on $\{a, b, c\}$ into an inversion. □

1.5 Ideals of Condorcet domains

Definition 1.5.1. *Let $A = \{a_1, \ldots, a_n\}$ be a finite set with $|A| = n$. Every subset $I \subseteq 2^A$ is* graded *in the sense that $I = \bigcup_{k=0}^{n} I_k$, where $I_k = \{J \in I \mid |J| = k\}$. A graded set $I \subseteq 2^A$ is called an* ideal *in A if:*

(I0) $I_0 = \emptyset$;

(I1) For all $k = 1, 2, \ldots, n$, if $X \in I_k$, then there exist $a \in X$ such that $X \setminus \{a\} \in I_{k-1}$.

(I2) For all $k = 0, 1, \ldots, n-1$, if $X \in I_k$, then there exist $a \in A$ such that $X \cup \{a\} \in I_{k+1}$.

Let us show that every domain of linear orders (not necessarily Condorcet) determines an ideal and vice versa. Let $u = x_1 x_2 \ldots x_n \in \mathcal{L}(A)$ be a linear order on $n = |A|$ alternatives. By $u_k = x_1 \ldots x_k$ we will denote the initial segment of u of length k. We set $\mathrm{Id}_k(u) = \{x_1, \ldots, x_k\}$.

Definition 1.5.2. *Let $\mathcal{D} \subseteq \mathcal{L}(A)$ be any domain. By the* ideal *of this domain we mean the set of subsets of A*

$$Id(\mathcal{D}) := \bigcup_{k=0}^{n} Id_k(\mathcal{D}),$$

where $Id_k(\mathcal{D}) = \{Id_k(u) \mid u \in \mathcal{D}\}$.

Indeed, it is easy to see that $\mathrm{Id}(\mathcal{D})$ is an ideal for any domain $\mathcal{D}$. The converse construction that for any ideal $I \subseteq 2^A$ outputs a domain $\mathrm{Dom}(I) \subseteq \mathcal{L}(A)$ also exists and will be presented shortly. In some important cases—most notably for peak-pit domains—the domain can be reconstructed uniquely from its ideal. We will explain it later and now we give a few examples for $A = \{a, b, c\}$:

- $\mathrm{Id}(\mathcal{D}_{3,1}) = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\} = 2^A \setminus \{b\}$;
- $\mathrm{Id}(\mathcal{D}_{3,2}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\} = 2^A$;
- $\mathrm{Id}(\mathcal{D}_{3,3}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}\} = 2^A \setminus \{a, c\}$.

Proposition 1.5.1. *For any domain $\mathcal{D} \subseteq \mathcal{L}([n])$ of maximal width its ideal $Id(\mathcal{D})$ contains all of the following subsets*

$$\emptyset, \{1\}, \{1, 2\}, \ldots, \{1, 2, \ldots, n\}, \{2, \ldots, n-1, n\}, \ldots, \{n-1, n\}, \{n\}. \tag{1.5.1}$$

Proof. Indeed, $\mathrm{Id}_k(e) = \{1, 2, \ldots, k\}$ and $\mathrm{Id}_k(\bar{e}) = \{n-k+1, n-k, \ldots, n\}$. □

Definition 1.5.3. *We call a sequence $F = (F_0, F_1, \ldots, F_n)$ of subsets of A a* flag *if elements of A can be ordered in a sequence $a_1, \ldots, a_n$ so that $F_k = \{a_1, \ldots, a_k\}$. The linear order $a_1 a_2 \ldots a_n$ is said to* correspond *to the flag F.*

We say that a flag $F = (F_0, F_1, \ldots, F_n)$ *belongs* to an ideal $I = \bigcup_{i=0}^{n} I_i$ iff $F_k \in I_k$ for all $k = 0, 1, \ldots, n$. We write $F \in I$ to denote this.

Proposition 1.5.2. *Any ideal I is the union of its flags, i.e., $I_k = \{F_k \mid F \in I\}$.*

Proof. It is sufficient to show that any set $X \subseteq I$ of cardinality k is the k-th component of a flag belonging to I. This follows from conditions (I1) and (I2) of Definition 1.5.1. □

Definition 1.5.4. *For any ideal $I \subseteq 2^A$, we denote by $Dom(I)$ the* domain *consisting of all linear orders corresponding to the flags that are contained in I.*

It is easy to note that in general the following inclusion holds but not necessarily equality.

Proposition 1.5.3. *$Dom(Id(\mathcal{D}))) \supseteq \mathcal{D}$.*

Example 1.5.1. *In particular, $Dom(Id(\mathcal{D}_{3,2})) = \mathcal{L}(A) \neq \mathcal{D}_{3,2}$ while*

$$Dom(Id(\mathcal{D}_{3,1})) = \mathcal{D}_{3,1} \quad \text{and} \quad Dom(Id(\mathcal{D}_{3,3})) = \mathcal{D}_{3,3}.$$

Example 1.5.2. *Let $I = \bigcup_{i=0}^{4} I_i \subseteq 2^{[4]}$, where*

$$\begin{aligned} I_0 &= \{\emptyset\}, \\ I_1 &= \{\{1\}, \{3\}, \{4\}\}, \\ I_2 &= \{\{1, 2\}, \{1, 3\}, \{3, 4\}\}, \\ I_3 &= \{\{1, 2, 3\}, \{1, 3, 4\}, \{2, 3, 4\}\}, \\ I_4 &= \{\{1, 2, 3, 4\}\}. \end{aligned}$$

(Note that this ideal coincides with $Id(F_4)$.) The flags contained in I are

$1 \subset 12 \subset 123 \subset 1234$
$1 \subset 13 \subset 132 \subset 1324$
$1 \subset 13 \subset 134 \subset 1342$
$3 \subset 31 \subset 312 \subset 3124$
$3 \subset 31 \subset 314 \subset 3142$

$3 \subset 34 \subset 341 \subset 3412$
$3 \subset 34 \subset 342 \subset 3421$
$4 \subset 43 \subset 431 \subset 4312$
$4 \subset 43 \subset 432 \subset 4321$

and we can see that $Dom(I) = F_4$.

We also note the following.

Proposition 1.5.4. *Let $\mathcal{D}$ be a domain on the set of alternatives A of cardinality n. Let $I = Id(\mathcal{D}) = \bigcup_{k=0}^{n} I_k$ be the ideal of $\mathcal{D}$. Then $\bar{I} = \bigcup_{k=0}^{n} \bar{I}_k$, where $\bar{I}_k = \{A \setminus J \mid J \in I_{n-k}$, is the ideal of the flipped domain $\bar{\mathcal{D}}$.*

The proof is straightforward.

Chapter 2

Single-peaked domains and generalisations

2.1 Classical single-peaked domains

Black [1948, 1958] was the first to come up with a sequence of maximal Condorcet domains—one for each set of alternatives—which he called the *domains of the single-peaked preferences.* Imagine that, in an election, parties a, b, c, d, e are lined up on the Left-Right political spectrum so that a is a left party, b is centre-left, c is right at the centre, d is centre-right and e is a right party. We can denote this as

$$a \lhd b \lhd c \lhd d \lhd e,$$

where $x \lhd y$ means that x is to the left of y. It is natural to assume that every voter has his political views represented on the same spectrum. Then a voter comparing two parties will like the party that is 'closer' to her political views. For example, three voters $1, 2, 3$ may have the following preferences over the set of those five parties:

$$\begin{aligned} &a \succ_1 b \succ_1 c \succ_1 d \succ_1 e, \\ &b \succ_2 c \succ_2 d \succ_2 e \succ_2 a, \\ &d \succ_3 e \succ_3 c \succ_3 b \succ_3 a. \end{aligned}$$

If we assume that when a voter ranks an alternative kth, then the utility of this alternative for her is $6 - k$, then these preferences can be presented in graphical form in Figure 2.1. We can see the graphs of all three preferences have a unique peak.

Let us now define single-peakedness formally. For a domain $\mathcal{D} = \{v_1, \ldots, v_n\}$ to be *single-peaked* Black required the existence of a societal axis (spectrum) on the set of alternatives A, i.e., he assumed that all alternatives can be put into a sequence $a_1 \lhd a_2 \lhd \ldots \lhd a_m$ so that every order v_i of $\mathcal{D}$ has its peak $a_i^* \in A$ such that

$$\begin{aligned} b \lhd c \unlhd a_i^* \quad &\Rightarrow \quad c \succ_i b, \\ a_i^* \unrhd b \rhd c \quad &\Rightarrow \quad b \succ_i c, \end{aligned}$$

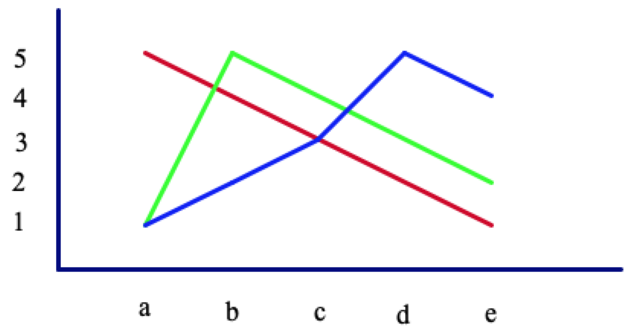


Figure 2.1: Single-peaked preferences of voters 1,2,3

where $a \trianglerighteq b$ means $a \triangleright b$ or $a = b$. The domain of all possible linear orders on A, single-peaked relative to $\triangleleft$ is sometimes denoted $\mathcal{SP}(\triangleleft, A)$.

Example 2.1.1. *Let us consider a unique maximal single-peaked domain $\mathcal{SP}(\triangleleft, A)$ on four alternatives with $A = \{a, b, c, d\}$ relative to spectrum $a \triangleleft b \triangleleft c \triangleleft d$:*

$$\mathcal{SP}(\triangleleft, A) = \{abcd,\ bacd,\ bcad,\ cbad,\ bcda,\ cbda,\ cdba,\ dcba\},$$

whose graph is presented on Figure 2.2. Figure 2.3 depicts this domain on the permutahedron.

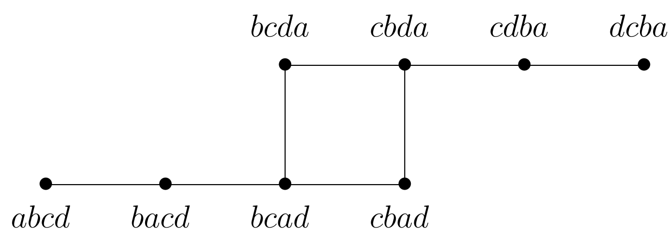


Figure 2.2: Graph of the single-peaked domain $\mathcal{SP}(\triangleleft, A)$ on four alternatives

We have $\mathcal{SP}(\triangleleft, A) = \mathcal{D}(\mathcal{N})$, where

$$\mathcal{N} = \{bN_{\{a,b,c\}}3,\ bN_{\{a,b,d\}}3,\ cN_{\{a,c,d\}}3,\ cN_{\{b,c,d\}}3\}.$$

This domain will also be denoted as $\mathcal{D}_{4,4}$ in the classification of Dittrich [2018].

An alternative definition of single-peakedness can be given in terms of upper contour sets. Given a linear order $v \in \mathcal{L}(A)$ and $a \in A$ we define the *upper contour set* of a in v as

$$U(v, a) = \{x \in A \mid x \succ_v a\}.$$

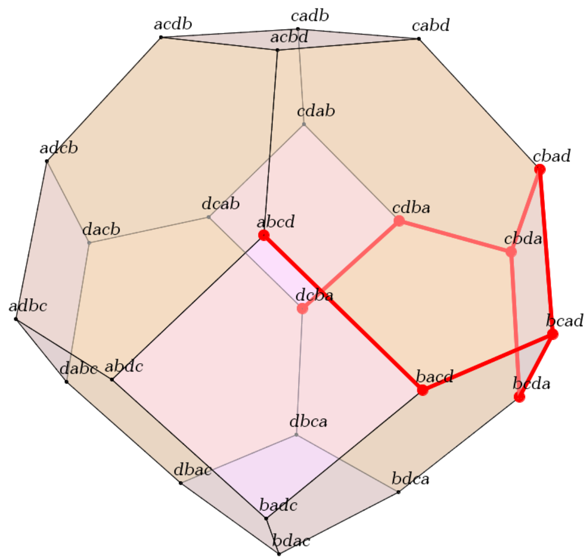


Figure 2.3: The domain $\mathcal{D}_{4,4}$ embedded in the permutahedron

A linear order v is single-peaked relative to the societal axis

$$a_1 \lhd a_2 \lhd \cdots \lhd a_n \tag{2.1.1}$$

if all upper contour sets of v are convex relative to this axis which means that for any $a \in A$ the upper contour set $U(v, a)$ is an interval of the given axis.

Given a societal axis, single-peaked order $v = x_1 x_2 \ldots x_n$ can be associated with the following flag of convex sets

$$\{x_1\} = U(v, x_2) \subset U(v, x_3) \subset \cdots \subset U(v, x_n) \subset [n].$$

On the other hand, to every flag of convex subsets of $[n]$

$$X_1 \subset X_2 \subset \cdots \subset X_{n-1} \subset X_n = [n], \tag{2.1.2}$$

with $|X_i| = i$, there corresponds a single-peaked order $x_1 x_2 \ldots x_n$, where $\{x_i\} = X_i \setminus X_{i-1}$ taking as usual $X_0 = \emptyset$.

Proposition 2.1.1. *Let $\mathcal{SP}(\lhd, A)$ be the single-peaked domain relative to the axis $\lhd$, $a \in A$. Let $\lhd'$ be this axis without a and $A' = A \setminus \{a\}$. Then $\mathcal{SP}(\lhd, A)_{-a} = \mathcal{SP}(\lhd', A')$.*

Proof. If a set X is convex relative to $\lhd$, then obviously $X \setminus \{a\}$ is convex relative to $\lhd'$. So a flag of convex subsets remains a flag of convex subsets after removal of a. This shows $\mathcal{SP}(\lhd, A)_{-a} \subseteq \mathcal{SP}(\lhd', A')$. On the other hand, every flag

$$X_1 \subset X_2 \subset \cdots \subset X_{n-2} \subset X_{n-1} = [n] \setminus \{a\},$$

of convex subsets relative to $\triangleleft'$ can be obtained this way. Indeed, suppose X_i is the first convex subset in this flag containing one of the neighbours of a in $\triangleleft$. Then we define $X'_k = X_{k-1} \cup \{a\}$ for $k = i+1, \ldots, n$ and obtain a flag

$$X_1 \subset \cdots \subset X_i \subset X'_{i+1} \cdots \subset X'_n = [n],$$

relative to $\triangleleft$. Hence we get $\mathcal{SP}(\triangleleft, A)_{-a} \supseteq \mathcal{SP}(\triangleleft', A')$ which proves the statement. □

Corollary 2.1.2. *$\mathcal{D} = \mathcal{SP}(\triangleleft, A)$ is a copious domain.*

Proof. For $n = 3$ we have four linear orders as required. Suppose $n > 3$, and $x, y, z \in A$. Then there exists $a \in A \setminus \{x, y, z\}$. Let $\triangleleft'$ and A' be as in Proposition 2.1.1. By the induction hypothesis the domain $\mathcal{SP}(\triangleleft', A')$ is copious, hence $\mathcal{D}|_{\{x,y,z\}}$ has four suborders which proves the induction step. □

We will prove a more general statement in Theorem 2.2.6.

Definition 2.1.1. *A domain $\mathcal{D} \subseteq \mathcal{L}(A)$ of linear orders on A is* minimally rich *if every alternative from A is the top alternative of at least one linear order in $\mathcal{D}$.*

We note that in Example 1.2.1 the domains $\mathcal{D}_{3,2}$ and $\mathcal{D}_{3,3}$ are minimally rich while $\mathcal{D}_{3,1}$ is not. The domain in Examples 2.1.1 is minimally rich.

Theorem 2.1.3. *$\mathcal{D} = \mathcal{SP}(\triangleleft, A)$ is a copious and minimally rich maximal Condorcet domain of maximal width such that each triple of alternatives a, b, c with $a \triangleleft b \triangleleft c$ satisfies $bN_{\{a,b,c\}}3$.*

Proof. By Corollary 2.1.2 $\mathcal{D}$ is copious. It is minimally rich since a flag of convex subsets of $[n]$ can be started from any one-element set $\{a\} \subset [n]$.

Let $a, b, c \in A$ with $a \triangleleft b \triangleleft c$. Then, due to convexity of upper contour sets the restriction $\mathcal{D}|_{\{a,b,c\}}$ will not contain orders acb and cab, hence $\mathcal{D}|_{\{a,b,c\}} \subseteq \{abc, cba, bac, bca\}$. Thus, $\mathcal{D}$ satisfies $bN_{\{a,b,c\}}3$, and $\mathcal{D}$ is a Condorcet domain. If the societal spectrum is $a_1 \triangleleft a_2 \triangleleft \cdots \triangleleft a_n$, then orders $a_1 a_2 \ldots a_n$ and $a_n \ldots a_2 a_1$ are in $\mathcal{D}$ so it has maximal width.

Suppose now that $u \notin \mathcal{D}$, then it is not single-peaked and not all upper contour sets of u are convex. Suppose $U(u, d)$ is not convex for $d \in A$. Then there exist $a, b, c \in A$ such that $a \triangleleft b \triangleleft c$ but $b \notin U(u, d)$ while $a, c \in U(u, d)$. Then $\mathcal{D}|_{\{a,b,c\}}$ contains order acb or order cab. However $\mathcal{D}|_{\{a,b,c\}}$ contains already four suborders abc, cba and also bac or bca (as b is a peak of some order), which leads to a contradiction. □

Minimal richness is about top-ranked alternatives, let us now pay attention to the bottom-ranked ones. For a linear order $v = a_1, \ldots, a_n \in \mathcal{L}(A)$ we will denote $\mathrm{pos}_v(a_i) = i$ indicating position of alternative a_i in v, i.e., the top preference has position 1 and the bottom one has position n.

Definition 2.1.2. *If $\mathcal{D} \subseteq \mathcal{L}(A)$ is any domain over the set of alternatives A with $|A| = n$, then the alternatives from the set*

$$\mathit{Term}(\mathcal{D}) = \{x \in A \mid \exists_{v \in \mathcal{D}}\, pos_v(x) = n\}$$

are called terminal.

In other words, terminal alternatives are those which are bottom-ranked in at least one linear order of the domain.

Proposition 2.1.4. *The terminal alternatives of single-peaked domain* $\mathcal{SP}(\lhd, A)$ *are the leftmost and the rightmost ones on the societal axis* $\lhd$.

Proof. Suppose $\lhd$ is as in (2.1.1). Obviously, only a_1 or a_n can be added at the last step $X_{n-1} \subset X_n = [n]$ of the flag (2.1.2). □

For $A = [n]$ and $<$ is the natural order on $[n]$ we denote this domain as $\mathcal{SP}_n$=$\mathcal{SP}(<, [n])$.

Proposition 2.1.5. *For every positive integer* n *all maximal single-peaked domains over the set of* n *alternatives are isomorphic to* $\mathcal{SP}_n$.

Proof. Let A be the set of alternatives with $|A| = n$, and $\mathcal{SP}(\lhd, A)$ be domain of single-peaked preferences with

$$a_1 \lhd a_2 \lhd \cdots \lhd a_n,$$

being the societal axis for $\mathcal{SP}(\lhd, A)$. Then the isomorphism ψ between $\mathcal{SP}_n$ and $\mathcal{SP}(\lhd, A)$ can be defined by $\psi(i) = a_i$, $i = 1, \ldots, n$. It is easy to check that it is indeed an isomorphism. □

Theorem 2.1.6. $\mathcal{D} = \mathcal{SP}(\lhd, A)$ *is a connected and semi-connected Condorcet domain.*

Proof. We prove only semi-connectedness, connectedness will be proved in a more general setting in Theorem 2.2.6. Due to Proposition 2.1.5 we may consider that $\mathcal{D} = SP_n$. As we noticed earlier for $n = 3$ domain $\mathcal{D}$ is semi-connected. Suppose now that SP_{n-1} is semi-connected and consider $\mathcal{D} = SP_n$. As we mentioned in Theorem 2.1.3 $\mathcal{D}$ is of maximal width with $e = 12\ldots n$ and $\bar{e} = n \ldots 21$ belonging to $\mathcal{D}$.

Consider now domain $\mathcal{D}_1 = \{v \in \mathcal{L}([n-1]) \mid vn \in \mathcal{D}\}$ on $[n-1]$. Firstly, we note that $\mathcal{D}_1 = \mathcal{SP}_{n-1}$. Indeed, a flag

$$X_1 \subset X_2 \subset \cdots \subset X_{n-1} \subset X_n = [n],$$

of convex sets has $X_n \setminus X_{n-1} = \{n\}$ if and only if

$$X_1 \subset X_2 \subset \cdots \subset X_{n-1} = [n-1], \tag{2.1.3}$$

a flag of convex sets on $[n-1]$. By induction hypothesis there is a shortest path in $\mathcal{SP}_{n-1}$ connecting $12\ldots(n-1)n$ and $(n-1)\ldots 21n$. From the latter there is a sequence of swaps

$$(n-1)\ldots 21n \to (n-1)\ldots 2n1 \to (n-1)n\ldots 21 \to n(n-1)\ldots 21.$$

The reason why all these swaps are possible is that the flag of convex subsets (2.1.3) corresponding to $(n-1)\ldots 21$ has $X_i = \{n{-}1, \ldots, n{-}i\}$, hence n can be added to each of the $X_i, \ldots, X_{n-1}$ for every $i \geq 0$ (where again $X_0 = \emptyset$). Thus $\mathcal{D}$ is semi-connected. □

We will now prove the converse of Theorems 2.1.3 and 2.1.6 which is due to Puppe [2018].

Theorem 2.1.7. *The following statements for a Condorcet domain $\mathcal{D} \subseteq \mathcal{L}(A)$ are equivalent:*

(i) $\mathcal{D} = \mathcal{SP}(\triangleleft, A)$ *for a certain line spectrum* $\triangleleft$*;*

(ii) $\mathcal{D}$ *is a minimally rich and semi-connected maximal Condorcet domain;*

(iii) $\mathcal{D}$ *is a maximal Condorcet domain such that there exists a line spectrum* $\triangleleft$ *with the property that each triple of alternatives* $a, b, c \in A$ *with* $a \triangleleft b \triangleleft c$ *satisfies* $bN_{\{a,b,c\}}3$*.*

Proof. The fact that (i) implies (ii) and (iii) follows from Theorems 2.1.3 and 2.1.6.

Let us prove that (ii) implies (iii). Suppose $\mathcal{D}$ is a minimally rich and semi-connected maximal Condorcet domain. Then $\mathcal{D}$ contains two completely reversed orders $v, \bar{v} \in \mathcal{D}$. By Proposition 1.2.1 $\mathcal{D}$ is copious peak-pit domain. Since the restriction $\mathcal{D}|_{\{a,b,c\}}$ is minimally rich, the triple $\{a, b, c\}$ must be a never-bottom triple. Let $v = a_1 a_2 \dots a_n$. We then consider the spectrum

$$a_1 \triangleleft a_2 \triangleleft \cdots \triangleleft a_n,$$

Suppose $a \triangleleft b \triangleleft c$. Then $\mathcal{D}|_{\{a,b,c\}}$ contains abc and cba and, since it is a never-bottom triple, it satisfies $bN_{\{a,b,c\}}3$ so (iii) follows.

Let us prove now that (iii) implies (i). Let $v \in \mathcal{D}$ be arbitrary. Consider upper contour sets $U(v, d)$ and assume, for the sake of contradiction, that for some $d \in A$ it is not convex. Then there exist $a \triangleleft b \triangleleft c$ such that $a, c \in U(v, d)$ but $b \notin U(v, d)$. Then in $\mathcal{D}|_{\{a,b,c\}}$ we have either acb or cab, each contradicts to $bN_{\{a,b,c\}}3$. □

Black's single-peaked domain also appears to be optimal for the local diversity which is a measure of abundance of distinct suborders of alternatives [Karpov et al., 2024].

Finally we introduce the sign representation of single-peaked preferences of domain $\mathcal{SP}_n$ due to Zhan [2022]. Given a preference order $v = a_1, \dots, a_n \in \mathcal{L}([n])$, then a_1 is a peak of it. If the second most preferred alternative a_2 is such that $a_1 < a_2$, then the first sign in the sequence corresponding to v is a $+$. If $a_1 > a_2$, then it is a $-$. Suppose that we have already constructed a string of $k - 1$ signs from the set $\{+, -\}$ for the suborder $a_1 \dots a_k$.

Since $\{a_1 \dots a_k\}$ is an interval, for the next alternative a_{k+1} we have either $a_{k+1} < a_j$ for all $j = 1, \dots, k$, in which case the kth sign in the sequence is $-$, or $a_{k+1} > a_j$ for all $j = 1, \dots, k$, in which case the kth sign in the sign sequence is $+$. Continuing this way we will obtain a sequence of $+$ and $-$ of length $n - 1$ that uniquely encodes v.

For example, single-peaked orders 34251 and 43251 with $n = 5$ can also be expressed as $+ - +-$ and $- - +-$, respectively. Conversely, $+ + -+$ denotes the single-peaked preference 23415.

In particular, we obtain

Proposition 2.1.8. $|\mathcal{SP}_n| = 2^{n-1}$.

Many properties of single-peaked domains, such as their maximality, connectedness and copiousness, can be proved in a more general setting; this is what we will do in the next section.

We conclude this section with the following observations. The subdomain $\mathcal{D}_{4,1} \setminus \{bcda\}$ of $\mathcal{D}_{4,1}$ (in Dittrich's classification presented in Chapter 9) is closed and copious Condorcet domain, which is of course not maximal. We learn two conclusions:

- It is not true that every closed Condorcet domain is of the form $\mathcal{D}(\mathcal{N})$ for some set of never conditions $\mathcal{N}$;
- A copious Condorcet domain is not necessarily maximal.

2.2 Arrow's single-peaked domains

Inada [1964] noticed that a much weaker requirement than global single-peakedness introduced in the previous subsection is still sufficient for obtaining a Condorcet domain. He imposed the requirement of single-peakedness only on triples, i.e., he required that the restriction $\mathcal{D}|_{\{a,b,c\}}$ of a domain $\mathcal{D}$ on any triple $\{a, b, c\}$ must be single-peaked relative to some societal axis specific for this triple. This condition appeared briefly in Arrow [1963] which gave Monjardet [2009] the reason to call this condition Arrow-Black's single-peakedness (so do Raynaud and Arrow [2011]). We, however, prefer to follow Raynaud [1981] and call it Arrow's single-peakedness while we often call Black's single-peakedness as simply single-peakedness to comply with the majority of the literature.[1]

Obviously, any Black's single-peaked domain is Arrow's single-peaked but we will see that there are many interesting Condorcet domain which are Arrow's single-peaked but not Black's single-peaked. In terms of never-condition Arrow's single-peakedness can be characterised as follows.

Proposition 2.2.1. *A domain $\mathcal{D} \subset \mathcal{L}(A)$ is Arrow's single-peaked if and only if it satisfies a complete set of never-conditions $xN_{\{a,b,c\}}3$ where $x \in \{a, b, c\}$.*

Apart from Black's single-peaked domains, described in the previous section, there is only one known construction of Arrow's single-peaked domains which is due to Romero [1978] (also translated into English by Raynaud and Arrow [2011] and mentioned by Monjardet [2009]).

Given two linear orders

$$p = x_1 \succ_1 x_2 \succ_1 \cdots \succ_1 x_m, \qquad q = y_1 \succ_2 y_2 \succ_2 \cdots \succ_2 y_m$$

on the set A of alternatives, we construct a domain $\mathcal{D}(p, q)$ by generating its orders as follows. By generating a particular order v we start with choosing a bottom-ranked alternative which can be either x_m or y_m, i.e., $\text{pos}_v(x_m) = m$ or $\text{pos}_v(y_m) = m$. Then we remove the chosen alternative from both orders and repeat the procedure to choose the alternative whose position in v will be $m-1$, etc. Unfortunately, as the following theorem shows, this construction never gives us new maximal Condorcet domains.

[1] Raynaud and Arrow [2011] sometimes call it Blackian single-peakedness.

Proposition 2.2.2. *$\mathcal{D}(p,q)$ is Arrow's single-peaked domain. This domain is maximal only if p and q are completely reversed in which case $\mathcal{D}(p,q)$ is a Black's single-peaked domain.*

Proof. If p and q are completely reversed, i.e., $q = \bar{p}$, we get Black's single-peaked domain which is maximal by Theorem 2.1.7. To see that we firstly assume that $A = [n]$ and $p = e = 12\dots n$, $q = \bar{e} = n\dots 21$. Obviously we can do this up to an isomorphism. Then the spectrum will be

$$1 \lhd 2 \lhd \cdots \lhd n.$$

The terminal alternative in any order of $\mathcal{D}(p,q)$ will be either 1 or n as in $\mathcal{SP}_n$. Indeed, if the bottom alternative in v is chosen to be n, then $v = v'n$, where $v' \in \mathcal{D}(p',q')$, where $p' = 12\dots n-1$, $q' = n-1\dots 21$. By the induction hypothesis $\mathcal{D}(p',q') = \mathcal{SP}_{n-1}$. Similarly, if the bottom alternative in v was chosen to be 1, then $v = v'1$, where $v' \in \mathcal{D}(p',q')$, where $p' = 23\dots n$, $q' = n\dots 32$. By the induction hypothesis $\mathcal{D}(p',q') = \mathcal{SP}(\lhd', [n]\setminus\{1\})$, where $2 \lhd' 3 \lhd' \cdots \lhd' n$. Combining the two classes of orders we get $\mathcal{SP}_n$.

Let us show that if p and q are not completely reversed, then $\mathcal{D}(p,q)$ is not ample. Indeed, in such case we will have a pair of alternatives a and b for whicch $a >_1 b$ and $a >_2 b$. Then in any order of $\mathcal{D}(p,q)$ alternative b will be ranked lower than a. However, as we will see in Theorem 2.2.6 every maximal Arrow's single-peaked domain is copious, hence ample. □

Example 2.2.1. *If $p = abcd$ and $q = dbca$, then*

$$\mathcal{D}(p,q) = \{abcd, bacd, bcad, bcda, bdca, dbca\},$$

which is a proper subset of a maximal Arrow's single-peaked domain $\mathcal{D}$ from Example 2.2.2 that will be introduced later.

Here is an example of a maximal Arrow's single-peaked domain which is not Black's single-peaked.

Example 2.2.2. *Let us consider an Arrow's single-peaked maximal Condorcet domain on four alternatives:*

$$\mathcal{D}_{4,5} = \{abcd,\ bacd,\ bcad,\ cbad,\ bcda,\ cbda,\ bdca,\ dbca\},$$

whose graph is presented on Figure 2.4. We have $\mathcal{D}_{4,5} = \mathcal{D}(\mathcal{N})$, where

$$\mathcal{N} = \{bN_{\{a,b,c\}}3,\ bN_{\{a,b,d\}}3,\ cN_{\{a,c,d\}}3,\ bN_{\{b,c,d\}}3\}$$

and $\mathcal{D}_{4,5}$ is copious. This domain is not single-peaked (for example, because it does not have two completely reversed orders).

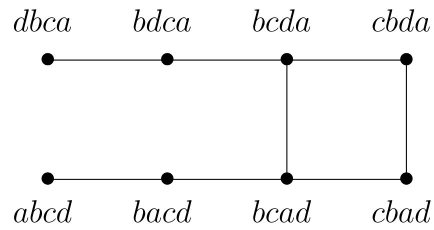


Figure 2.4: Graph of a non-classical Arrow's single-peaked domain.

2.2.1 Structural properties of Arrow's single-peaked Condorcet domains

As Example 2.2.2 shows Black's single-peaked domains are not the only Arrow's single-peaked domains (and later we will see that there is a large family of the latter). However, not surprisingly, Arrow's single-peaked domains share some common features with Black's single-peaked domains. This is reflected in their structural properties.

Lemma 2.2.3. *An Arrow's single-peaked domain $\mathcal{D}$ has at most two terminal alternatives. If $\mathcal{D}$ is maximal, it has exactly two of them.*

Proof. If $a, b, c \in A$ are distinct bottom-ranked alternatives, then the triple $\{a, b, c\}$ does not satisfy a never-bottom condition. Hence $|\mathrm{Term}(\mathcal{D})| \leq 2$.

If $|\mathrm{Term}(\mathcal{D})| = 1$ and, say, $\mathrm{Term}(\mathcal{D}) = \{z\}$, then z is the bottom ranked alternative in any order of $\mathcal{D}$. Consider any triple with participation of z, say $\{a, b, z\}$. Both a and b are never-bottom in this triple, hence both $aN_{\{a,b,z\}}3$ and $bN_{\{a,b,z\}}3$ will be satisfied. If we move z in any order one position up, then for the new order one of the two never-bottom condition will still be satisfied, hence we can add this new order to $\mathcal{D}$ and still have an Arrow's single-peaked domain. In this case $\mathcal{D}$ cannot be maximal. □

Now we start looking into the internal structure of Arrow's single-peaked maximal Condorcet domains. Suppose now that $\mathcal{D} \subseteq A$ is such a domain with $|A| = m$. Then by Lemma 2.2.3 $|\mathrm{Term}(\mathcal{D})| = 2$, and let $\mathrm{Term}(\mathcal{D}) = \{a_1, a_2\}$. For $i \in \{1, 2\}$, let us consider subdomain $\widetilde{\mathcal{D}}_i = \{v \in \mathcal{D} \mid \mathrm{pos}_v(a_i) = m\}$, and domain $\mathcal{D}_i$ which is the restriction of $\widetilde{\mathcal{D}}_i$ onto $A_i = A \setminus \{a_i\}$. An important role will be also played by the following subdomains of $\widetilde{\mathcal{D}}_1$ and $\widetilde{\mathcal{D}}_2$:

$$\begin{aligned}
\widehat{\mathcal{D}}_1 &= \{v \in \mathcal{D} \mid \mathrm{pos}_v(a_1) = m \text{ and } \mathrm{pos}_v(a_2) = m-1\} \subset \widetilde{\mathcal{D}}_1, \\
\widehat{\mathcal{D}}_2 &= \{v \in \mathcal{D} \mid \mathrm{pos}_v(a_2) = m \text{ and } \mathrm{pos}_v(a_1) = m-1\} \subset \widetilde{\mathcal{D}}_2.
\end{aligned}$$

The next several statements will use these assumptions and this notation.

Lemma 2.2.4. *There exists an isomorphism σ between $\widehat{\mathcal{D}}_1$ and $\widehat{\mathcal{D}}_2$ such that $\sigma(a_1) = a_2$, $\sigma(a_2) = a_1$ and σ being the identity mapping on $A \setminus \{a_1, a_2\}$.*

Proof. Suppose that for some order $u \in \mathcal{L}(A \setminus \{a_1, a_2\})$ we have $ua_1a_2 \in \widehat{\mathcal{D}}_2$. Then we claim that $ua_2a_1 \in \widehat{\mathcal{D}}_1$. Since ua_2a_1 ends with a_2a_1, it is sufficient to notice that ua_2a_1 satisfies all the never-bottom conditions that $\mathcal{D}$ satisfied (since a_1 being terminal cannot serve as the never-bottom alternative in any triple). Then due to maximality of $\mathcal{D}$ ua_2a_1 will be in $\mathcal{D}$ and hence in $\widehat{\mathcal{D}}_1$. □

Lemma 2.2.5. *$\mathcal{D}_1$ and $\mathcal{D}_2$ are Arrow's single-peaked maximal domains on the sets A_1 and A_2, respectively.*

Proof. It is obvious that $\mathcal{D}_1$ and $\mathcal{D}_2$ are Arrow's single-peaked domains. To show that they are maximal we will firstly prove that for any triple of distinct alternatives $\{b, c, d\} \subseteq A \setminus \{a_1, a_2\}$ (if such exists) we have

$$\mathcal{D}_1|_{\{b,c,d\}} = \mathcal{D}_2|_{\{b,c,d\}} = \mathcal{D}|_{\{b,c,d\}}. \tag{2.2.1}$$

Suppose we have a linear order $u = ...x...y...z...a_1$ in $\widetilde{\mathcal{D}}_1$ with $\{x, y, z\} = \{b, c, d\}$ and a_2 ranked somewhere (but not at the bottom as $a_2 \neq a_1$). We then move a_2 to the last mth position without changing the order of other alternatives to obtain order $v = ...x...y...z...a_1a_2$. Since a_2 is always a bottom alternative in any triple so moving it down means that v still satisfies all the never-bottom conditions that u satisfied. Hence it satisfies all defining never-bottom conditions that $\mathcal{D}$ satisfied. Due to maximality of $\mathcal{D}$ the order v is in $\mathcal{D}$, hence in $\mathcal{D}_2$. Thus $\mathcal{D}_1|_{\{b,c,d\}} = \mathcal{D}_2|_{\{b,c,d\}} = \mathcal{D}|_{\{b,c,d\}}$.

Suppose now $\mathcal{D}_1$ is not maximal and there is a linear order $u \in \mathcal{L}(A_1)$ such that $\mathcal{D}_1 \cup \{u\}$ is a larger Arrow's single-peaked domain on A_1. Let us consider then $\tilde{u} = ua_1$. We claim that $\mathcal{D} \cup \{\tilde{u}\}$ is also a Condorcet domain on A. For this we show that $\tilde{u}$ satisfies all never-bottom conditions not involving terminal alternatives. Let *bcd* be such alternatives and $u|_{\{b,c,d\}}$ be the restriction of u onto $\{b, c, d\}$. Then $\mathcal{D}_1|_{\{b,c,d\}} \cup u|_{\{b,c,d\}}$ is a Condorcet domain which, due to (2.2.1) implies that $\mathcal{D}|_{\{b,c,d\}} \cup u|_{\{b,c,d\}}$ is a Condorcet domain.

We know also that $\mathcal{D}$ and hence $\widetilde{\mathcal{D}_1}$ satisfy some never-bottom condition $xN_{\{a_2,x,y\}}3$ for every triple $\{a_2, x, y\}$ with $x \neq a_2$ and $y \neq a_2$. Hence the restriction of $\widetilde{\mathcal{D}_1}$ onto $\{a_2, x, y\}$ is contained in $\{xa_2y, a_2xy, xya_2, yxa_2\}$ and the restriction of $\tilde{u}$ onto $\{a_2, x, y\}$ must be within this set. But the restriction of $\widetilde{\mathcal{D}_2}$ onto $\{a_2, x, y\}$ is contained only in $\{xya_2, yxa_2\}$. Hence, if the addition of $\tilde{u}$ does not violate $xN_{\{a_2,x,y\}}3$ for $\mathcal{D}_1$, it would not violate the same never-bottom condition for $\mathcal{D}$.

Finally, any triple $\{a_1, x, y\}$ with $x \neq a_1$ and $y \neq a_1$ in $\mathcal{D}$ satisfies a never-bottom condition $xN_{\{a_1,x,y\}}3$ with $x \neq a_1$ so adding $\tilde{u}$ with a_1 at the bottom will not violate any never-bottom conditions of $\mathcal{D}$ involving a_1. Thus $\tilde{u}$ satisfies all the never-bottom conditions that $\mathcal{D}$ satisfies. This contradicts to maximality of $\mathcal{D}$ and proves the lemma. □

Definition 2.2.1. *Suppose a domain $\mathcal{D} \subseteq \mathcal{L}(A)$ has $Term(\mathcal{D}) = \{a_1, a_2\}$ for some $a_1, a_2 \in A$. Then a linear order $v \in \mathcal{D}$ such that $pos_v(a_1) = 1$ and $pos_v(a_2) = m$ or $pos_v(a_2) = 1$ and $pos_v(a_1) = m$ will be called* extremal *for $\mathcal{D}$.*

In Examples 2.1.1 and 2.2.2 the pairs of extremal orders are

$$abcd,\ dcba \quad \text{and} \quad abcd,\ dbca,$$

respectively. Note that in the second case they are not completely reversed.

Theorem 2.2.6. *Any Arrow's single-peaked maximal Condorcet domain $\mathcal{D}$ on the set A of m alternatives satisfies the following properties:*

(a) $\mathcal{D}$ contains 2^{m-1} orders;

(b) $\mathcal{D}$ is copious;

(c) $\mathcal{D}$ is connected;

(d) $\mathcal{D}$ is minimally rich;

(e) $\mathcal{D}$ contains exactly two extremal linear orders.

Proof. (a) For $m = 3$ the unique (up to an isomorphism) Arrow's single-peaked domain (domain $\mathcal{D}_3$ on Figure 1.2) contains $4 = 2^{3-1}$ orders, this will be a basis for the induction. Let $\mathcal{D}$ be a maximal Arrow's single-peaked domain on m alternatives. We have $\mathcal{D} = \widetilde{\mathcal{D}}_1 \cup \widetilde{\mathcal{D}}_2$, where $|\widetilde{\mathcal{D}}_1| = |\mathcal{D}_1|$ and $|\widetilde{\mathcal{D}}_2| = |\mathcal{D}_2|$. As by Lemma 2.2.5 $\mathcal{D}_1$ and $\mathcal{D}_2$ are maximal on the set of $m-1$ alternatives, by induction hypothesis we now deduce that $|\mathcal{D}_1| = |\mathcal{D}_2| = 2^{m-2}$, whence $|\mathcal{D}| = 2^{m-1}$.

(b) We will prove this also by induction. We note that for $m = 3$ all maximal Condorcet domains are copious. By induction hypothesis both $\mathcal{D}_1$ and $\mathcal{D}_2$ are copious, so the only triples $\{x, y, z\} \subseteq A$ in $\mathcal{D}$ for which it is not clear that $|\mathcal{D}|_{\{x,y,z\}}| = 4$ are those which contain both a_1 and a_2. Let $\{x, a_1, a_2\} \subset A$ be such a triple.

Since $\mathcal{D}_1$ and $\mathcal{D}_2$ are copious, they are ample, hence $\mathcal{D}|_{\{x,a_1\}} = \{xa_1, a_1x\}$ and $\mathcal{D}|_{\{x,a_2\}} = \{xa_2, a_2x\}$. From this we see that $\mathcal{D}|_{\{x,a_1,a_2\}}$ contains xa_1a_2, a_1xa_2 and xa_2a_1, a_2xa_1. Thus $|\mathcal{D}|_{\{x,a_1,a_2\}}| = 4$ and $\mathcal{D}$ is copious.

(c) By the induction hypothesis $\mathcal{D}_1$ and $\mathcal{D}_2$ are connected (i.e., subgraphs of their respective permutahedra), which implies that $\widetilde{\mathcal{D}}_1$ and $\widetilde{\mathcal{D}}_2$ are connected as well. Since the order of a_1 and a_2 in $\widetilde{\mathcal{D}}_1$ and $\widetilde{\mathcal{D}}_2$ are different, the one edge between them that is also an edge of the permutohedron is the edge between $...x...y...z...a_2a_1$ and $...x...y...z...a_1a_2$. Let us show that there are no other edges between these two sets of vertices in $G_\mathcal{D}$. Suppose $x = ua_1 \in \widetilde{\mathcal{D}}_1$ is connected by an edge in $G_\mathcal{D}$ with $y = va_2 \in \widetilde{\mathcal{D}}_2$. Let us move a_2 down in x to obtain $x' = u'a_2a_1$ and also move a_1 down in y to obtain $y' = v'a_1a_2$. Both x' and y' are between x and y, hence $x = x' = u'a_2a_1$ and $y = y' = v'a_1a_2$. If only $v' \neq u'$, then $u'a_1a_2$ is between x and y and different from both of them. Thus $v' = u'$ and the edge between x and y exists in the permutohedron.

(d) By Lemma 2.2.5 $\mathcal{D}_1$ and $\mathcal{D}_2$ are maximal and by induction hypothesis both are minimally rich. Hence $\mathcal{D}_1$ has all alternatives of $A_1 = A \setminus \{a_1\}$ on top in some rankings and $\mathcal{D}_2$ has all alternatives of $A_2 = A \setminus \{a_2\}$ on top. As the union of A_1 and A_2 is A, we have proved (c).

(e) Let us consider $\mathrm{Term}(\mathcal{D}_1) = \{b_1, b_2\}$ and $\mathrm{Term}(\mathcal{D}_2) = \{c_1, c_2\}$. We claim that $a_1 \in \{c_1, c_2\}$ and $a_2 \in \{b_1, b_2\}$. The reason is that, if among b_1, b_2, c_1, c_2 there were three distinct elements that are also different from a_1 and a_2, then for this triple no never-bottom condition is satisfied. If $\mathrm{Term}(\mathcal{D}_1) = \mathrm{Term}(\mathcal{D}_1) = \{b, c\}$, then no never-bottom condition is satisfied for triples $\{a_i, b, c\}$, $i = 1, 2$. Thus we may consider that $\mathrm{Term}(\mathcal{D}_1) = \{a_2, b\}$ and $\mathrm{Term}(\mathcal{D}_2) = \{a_1, c\}$. By Lemma 2.2.5 $\mathcal{D}_1$ and $\mathcal{D}_2$ are maximal Arrow's single-peaked domains on sets of $m - 1$ alternatives so by induction hypothesis $\mathcal{D}_1$ contains a linear order u with a_2 on the top and $\mathcal{D}_2$ contains linear order v with a_1 on the top. This ua_1 and va_2 are two extremal orders of $\mathcal{D}$ sought for.

To show the uniqueness of those linear orders suppose we have two different orders w_1 and w_2 satisfying $\mathrm{pos}_{w_1}(a_1) = \mathrm{pos}_{w_2}(a_1) = 1$ and $\mathrm{pos}_{w_1}(a_2) = \mathrm{pos}_{w_2}(a_2) = m$. Then $w_1 = a_1 \ldots x \ldots y \ldots a_2$ and $w_2 = a_1 \ldots y \ldots x \ldots a_2$ for some pair of alternatives $x, y \in A$. Then for the triple $\{a_1, x, y\}$ only a_1 can be the never-bottom alternative but it cannot be. This contradiction proves the statement. □

We note that Theorem 2.2.6(a) is quite a surprising result. The local Arrow's single-peakedness is much weaker than the global Black's single-peakedness but the former does not allow more individual freedom than the latter. This was discovered already by Raynaud [1981] that the cardinality of Arrow's single-peaked domain cannot exceed 2^{m-1}.

As we have seen in Theorem 2.1.3 Black's single-peaked maximal domain contains two completely reversed orders. Arrow's single-peaked domains may not have these. However, Theorem 2.2.6(e) shows that in the case of Arrow's single-peakedness we can salvage at least something, namely, the two extremal linear orders. These two extremal linear orders in Black's single-peaked domain become completely reversed. But, in general, we can guarantee only that their top and bottom preferences are reversed. Example 2.2.2 shows that we cannot do any better.

Let us now summarise the information about the structure of the graph of a maximal Arrow's single-peaked domain. It has a recursive structure.

Theorem 2.2.7. *The graph of a maximal Arrow's single-peaked Condorcet domain on m alternatives is constructed from graphs of two maximal Arrow's single-peaked Condorcet domains $\mathcal{D}_1 = \mathcal{D}|_{A\setminus\{a_1\}}$ and $\mathcal{D}_2 = \mathcal{D}|_{A\setminus\{a_2\}}$ on $m - 1$ alternatives (which may not be isomorphic) each of which has a subgraph isomorphic to a graph of a maximal Arrow's single-peaked Condorcet domain on $m-2$ alternatives which is the same for both domains, i.e., $\mathcal{D}|_{A\setminus\{a_1,a_2\}} = \mathcal{D}|_{A\setminus\{a_1,a_2\}}$.*

Corollary 2.2.8. *Let $\mathcal{D}$, $\widetilde{\mathcal{D}}_1$, $\widetilde{\mathcal{D}}_2$, $\widehat{\mathcal{D}}_1$, $\widehat{\mathcal{D}}_2$ be as before. Then the structure of the graph $G_\mathcal{D} = (V_\mathcal{D}, E_\mathcal{D})$ is as follows:*

- $V_\mathcal{D} = V_{\widetilde{\mathcal{D}}_1} \cup V_{\widetilde{\mathcal{D}}_2}$;
- $E_\mathcal{D} = E_{\widetilde{\mathcal{D}}_1} \cup E_{\widetilde{\mathcal{D}}_2} \cup E(\widehat{\mathcal{D}}_1, \widehat{\mathcal{D}}_2)$, *where $E(\widehat{\mathcal{D}}_1, \widehat{\mathcal{D}}_2)$ contains edges connecting the vertices that correspond to each other under the isomorphism σ from Lemma 2.2.4.*

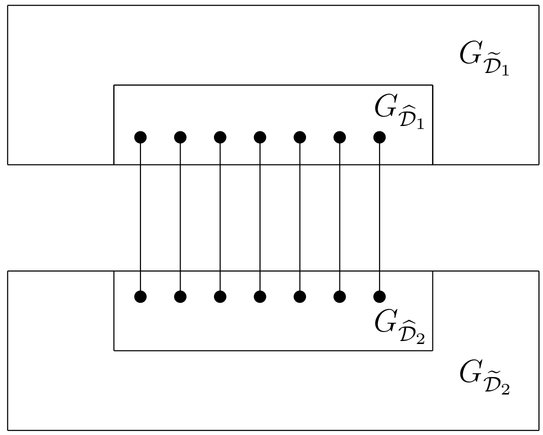


Figure 2.5: The anatomy of the graph of an Arrow's single-peaked domain

Proof. According to Lemma 2.2.4 subdomains $\widehat{\mathcal{D}}_1$ and $\widehat{\mathcal{D}}_2$ are isomorphic and the isomorphism only swaps a_1 and a_2 which means that any order in $\widehat{\mathcal{D}}_1$ is linked by an edge of permutohedron to its image in $\widehat{\mathcal{D}}_2$. And we showed in the proof of Theorem 2.2.6(c) that there are no other edges between $\widehat{\mathcal{D}}_1$ and $\widehat{\mathcal{D}}_2$. □

For example, for the two already presented maximal Arrow's single-peaked Condorcet domains on four alternatives presented on Figures 2.2 and 2.4 the horizontal lines are edges in the two single-peaked domains on three alternatives and the vertical lines show the isomorphism between isomorphic subdomains on two alternatives.

2.2.2 A Characterisation of Arrow's single-peaked domains

There are several characterisations of Black's single-peaked domains of which the most notable ones are Ballester and Haeringer [2011] and Puppe [2018]. In the latter it was shown that a maximal Condorcet domain is single-peaked if and only if it is semi-connected (and, in particular, contains two completely reversed orders) and minimally rich. Theorem 2.2.6 allows us to obtain a similar characterisation of Arrow's single-peaked domains.

Proposition 2.2.9. *Any Condorcet domain that is connected and minimally rich is Arrow's single-peaked.*

Proof. If a Condorcet domain $\mathcal{D} \subseteq \mathcal{L}(A)$ is connected and minimally rich, then so is its restriction to any triple $\{a, b, c\} \subseteq A$. In particular, $\mathcal{D}|_{\{a,b,c\}}$ must be isomorphic to $\mathcal{D}_{3,3}$ of Example 1.2.1 (neither $\mathcal{D}_{3,1}$ nor $\mathcal{D}_{3,2}$ nor any subdomain of $\mathcal{D}_{3,3}$ is connected and minimally rich) which means $\mathcal{D}|_{\{a,b,c\}}$ is single-peaked. Hence $\mathcal{D}$ is Arrow's single-peaked. □

Theorem 2.2.10. *A maximal Condorcet domain is Arrow's single-peaked domain if and only if it is connected and minimally rich.*

Proof. If a Condorcet domain $\mathcal{D} \subseteq \mathcal{L}(A)$ is connected and minimally rich, then $\mathcal{D}$ is Arrow's single-peaked by Proposition 2.2.9. (We do not need maximality of $\mathcal{D}$ in this direction.) The converse follows from Theorem 2.2.6 (c) and (d). □

Due to Theorem 2.2.10, the following is equivalent to Puppe's characterisation.

Corollary 2.2.11. *A maximal Condorcet domain is Black's single-peaked if and only if it is an Arrow's single-peaked and contains two completely reversed orders.*

Proof. It is enough to show that an Arrow's single-peaked domain $\mathcal{D}$ with two completely reversed orders v and $\bar{v}$ is Black's single-peaked relative the axis $\lhd$ defined by any of these orders, i.e., we define $a \lhd b$ if $b \succ_v a$. Suppose that an upper contour set $U(u,d)$ of a certain linear order $u \in \mathcal{D}$ is not convex relative to $\lhd$ and there exist $a \lhd b \lhd c$ with $a, c \in U(u,d)$ but $b \notin U(u,d)$. Then in the restriction $\mathcal{D}|_{\{a,b,c\}}$ we have suborders acb or cab and also abc and cba occurring in v and $\bar{v}$. Thus in the triple $\{a,b,c\}$ every alternative occurs in the bottom position which contradicts Arrow's single-peakedness. □

The criterion of Ballester and Haeringer [2011] also immediately follows from our structural results.

Corollary 2.2.12 (Ballester and Haeringer [2011])**.** *A maximal Condorcet domain $\mathcal{D} \subseteq A$ is Black's single-peaked if and only if*

1. *$\mathcal{D}$ is an Arrow's single-peaked domain*[2]*, and*
2. *There do not exist two orders, $\succ_1, \succ_2 \in \mathcal{D}$, and four alternatives $x, y, z, t \in A$ such that the following four conditions simultaneously hold:*

$$x \succ_1 y \succ_1 z, \quad z \succ_2 y \succ_2 x, \quad t \succ_1 y, \quad t \succ_2 y.$$

Proof. Consider the two extremal orders of an Arrow's single-peaked domain $\mathcal{D}$. If the second condition is satisfied, then they are completely reversed and $\mathcal{D}$ is Black's single-peaked by Corollary 2.2.11. □

2.2.3 Classification of small Arrow's single-peaked maximal Condorcet domains

2.2.4 Four alternatives

Theorem 2.2.13. *Up to an isomorphism there are two maximal Arrow's single-peaked domains on the set of 4 alternatives*

$$\begin{aligned}\mathcal{D}_{4,4} &= \{abcd, bacd, bcad, cbad, bcda, cbda, cdba, dcba\},\\ \mathcal{D}_{4,5} &= \{abcd, bacd, bcad, cbad, bcda, cbda, bdca, dbca\}.\end{aligned}$$

with their median graphs shown on Figures 2.2 and 2.4, respectively. $\mathcal{D}_{4,4}$ is the only maximal Black's single-peaked domain.

[2]Ballester and Haeringer [2011] call it worst-restricted.

Proof. Let $A = \{a, b, c, d\}$ and $\mathcal{D} \subset \mathcal{L}(A)$ be a maximal Arrow's single-peaked domain. By Lemma 2.2.3, up to an isomorphism, we assume that $\mathrm{Term}(\mathcal{D}) = \{a, d\}$. By Theorem 2.2.6(e) we know that there are two extremal linear orders and without loss of generality we may assume that one of them is *abcd*. By Theorem 2.2.6(a) $\mathcal{D}$ contains $2^{4-1} = 8$ linear orders. Let us list all eight linear orders of $\mathcal{D}$ as columns of a 4×8 table. Then $\mathcal{D}$ satisfies $bN_{\{a,b,c\}}3$ since $cN_{\{a,b,c\}}3$ is not satisfied for the *abcd*. By Theorem 2.2.6(b) we know that $\mathcal{D}$ is copious hence $bN_{\{a,b,c\}}3$ leads to $\mathcal{D}$ containing the following linear orders

$$abcd,\ bacd,\ bcad,\ cbad.$$

Moreover, Lemma 2.2.4 states that the two terminals can be swapped when they occupy the last two positions, so *bcda* and *cbda* should be also in $\mathcal{D}$. Hence we obtain the table

a	b	b	c	c	b		
b	a	c	b	b	c		
c	c	a	a	d	d	$\star$	$\star$
d	d	d	d	a	a	a	a

In the remaining two spots of the third row marked by the stars we cannot have both b and c since otherwise the triple b, c, d will not satisfy the never-bottom condition. Hence the only two options to complete the table are as follows:

a	b	b	c	c	b	c	d
b	a	c	b	b	c	d	c
c	c	a	a	d	d	b	b
d	d	d	d	a	a	a	a

a	b	b	c	c	b	b	d
b	a	c	b	b	c	d	b
c	c	a	a	d	d	c	c
d	d	d	d	a	a	a	a

These are indeed Arrow's single-peaked domains $\mathcal{D}_{4,4}$ and $\mathcal{D}_{4,5}$ which we have already met them in Examples 2.1.1 and 2.2.2. □

2.2.5 Five alternatives

Let $A = \{a, b, c, d, e\}$ and let us assume that $\mathrm{Term}(\mathcal{D}) = \{d, e\}$. The idea of the construction is to put two isomorphic copies of a maximal Arrow's single-peaked domain $\{a, b, c\}$—without loss of generality let us assume that it satisfies $bN_{\{a,b,c\}}3$—each of them appended by two terminals d and e (but in different order). Then try extending each of the isomorphic copies to a maximal Arrow's single-peaked domain on four alternatives. Hence we start with the half-filled table

				a	b	b	c	a	b	b	c				
K				b	a	c	b	b	a	c	b	L			
				c	c	a	a	c	c	a	a				
I				d	d	d	d	e	e	e	e	J			
e	e	e	e	e	e	e	e	d	d	d	d	d	d	d	d

Initially, all focus is on areas I and J of the table. Due to the never-bottom property we can have no more than two alternatives different from d and e in the second to last row of the table. Moreover, all alternatives in I must be the same and so is in J.

Case 1. Both areas I and J are occupied by the same alternative. Due to the choice of $bN_{\{a,b,c\}}3$, it can be either a or c. Up to an isomorphism, we can assume that this alternative is a. Then, we have the following possibilities:

	K	L
Case 1.1	$cN_{\{b,c,d\}}3$	$bN_{\{b,c,e\}}3$
Case 1.2	$cN_{\{b,c,d\}}3$	$cN_{\{b,c,e\}}3$
Case 1.3	$bN_{\{b,c,d\}}3$	$bN_{\{b,c,e\}}3$
Case 1.4	$bN_{\{b,c,d\}}3$	$cN_{\{b,c,e\}}3$

Let us consider these cases one by one.

Case 1.1. We have the table of the domain $\mathcal{D}_{5,1}$

d	c	c	b	a	b	b	c	a	b	b	c	b	c	b	e
c	d	b	c	b	a	c	b	b	a	c	b	c	b	e	b
b	b	d	d	c	c	a	a	c	c	a	a	e	e	c	c
a	a	a	a	d	d	d	d	e	e	e	e	a	a	a	a
e	e	e	e	e	e	e	e	d	d	d	d	d	d	d	d

and its graph is presented in Figure 2.6.

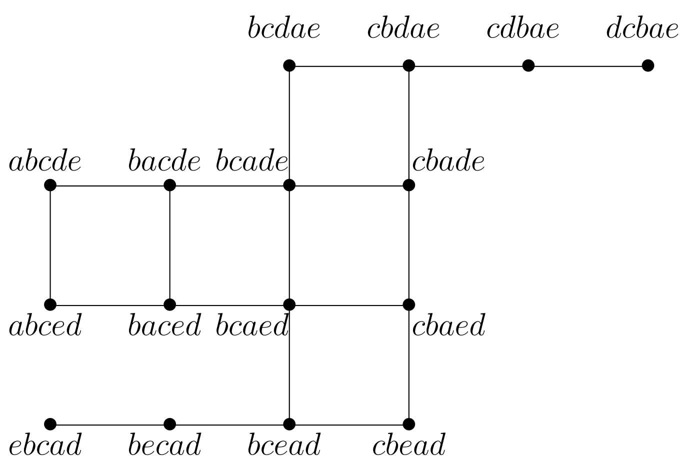


Figure 2.6: Graph of the Arrow's single-peaked domain $\mathcal{D}_{5,1}$ (Case 1.1).

Case 1.2. The table of the domain $\mathcal{D}_{5,2}$ in this case would be

d	c	c	b	a	b	b	c	a	b	b	c	b	c	c	e
c	d	b	c	b	a	c	b	b	a	c	b	c	b	e	c
b	b	d	d	c	c	a	a	c	c	a	a	e	e	b	b
a	a	a	a	d	d	d	d	e	e	e	e	a	a	a	a
e	e	e	e	e	e	e	e	d	d	d	d	d	d	d	d

with the graph depicted in Figure 2.7.

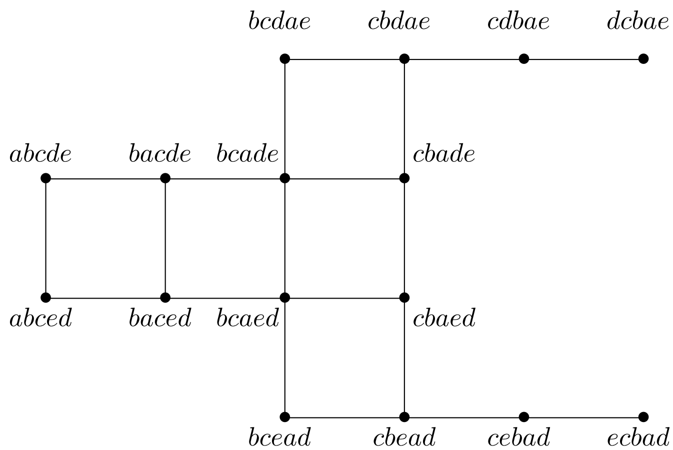


Figure 2.7: Graph of the Arrow's single-peaked domain $\mathcal{D}_{5,2}$ (Case 1.2).

Case 1.3. In this case the table of domain $\mathcal{D}_{5,3}$ would be

d	b	c	b	a	b	b	c	a	b	b	c	b	c	b	e
b	d	b	c	b	a	c	b	b	a	c	b	c	b	e	b
c	c	d	d	c	c	a	a	c	c	a	a	e	e	c	c
a	a	a	a	d	d	d	d	e	e	e	e	a	a	a	a
e	e	e	e	e	e	e	e	d	d	d	d	d	d	d	d

with the graph shown in Figure 2.8.

Case 1.4. This domain is isomorphic to $\mathcal{D}_{5,1}$ in Case 1.1 under the isomorphism that swaps d and e and fixes all other alternatives.

Case 2. Areas I and J are occupied by different alternatives. These must be a and c since, if b occupies one of them, we would be able to find a triple, namely $\{a, b, c\}$, which does not satisfy never-bottom condition. So we get the following possibilities:

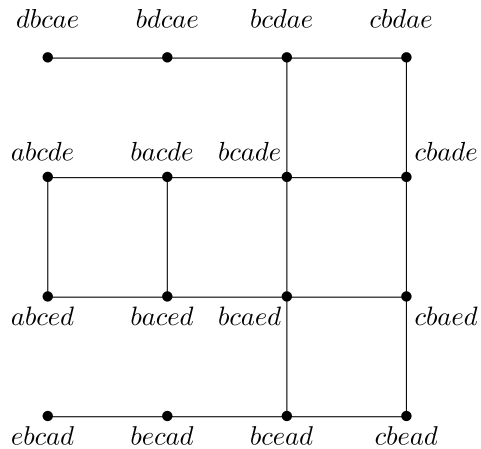


Figure 2.8: Graph of the Arrow's single-peaked domain $\mathcal{D}_{5,3}$ (Case 1.3).

	K	L
Case 2.1	$cN_{\{b,c,d\}}3$	$aN_{\{a,b,e\}}3$
Case 2.2	$cN_{\{b,c,d\}}3$	$bN_{\{a,b,e\}}3$
Case 2.3	$bN_{\{b,c,d\}}3$	$bN_{\{a,b,e\}}3$
Case 2.4	$bN_{\{b,c,d\}}3$	$aN_{\{a,b,e\}}3$

Case 2.1. The table of $\mathcal{D}_{5,4}$ would be

d	c	c	b	a	b	b	c	a	b	b	c	a	b	a	e
c	d	b	c	b	a	c	b	b	a	c	b	b	a	e	a
b	b	d	d	c	c	a	a	c	c	a	a	e	e	b	b
a	a	a	a	d	d	d	d	e	e	e	e	c	c	c	c
e	e	e	e	e	e	e	e	d	d	d	d	d	d	d	d

This is the maximal Black's single-peaked domain with two completely reversed orders with the graph shown in Figure 2.9.

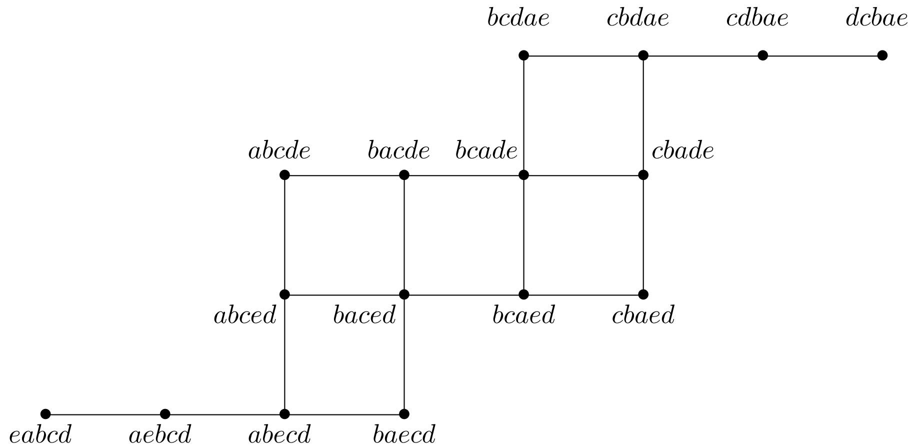


Figure 2.9: Graph of the Arrow's single-peaked domain $\mathcal{D}_{5,4}$ (Case 2.1).

Case 2.2. We get the domain $\mathcal{D}_{5,5}$ with the table

d	c	c	b	a	b	b	c	a	b	b	c	a	b	b	e
c	b	d	c	b	a	c	b	b	a	c	b	b	a	e	b
b	d	b	d	c	c	a	a	c	c	a	a	e	e	a	a
a	a	a	a	d	d	d	d	e	e	e	e	c	c	c	c
e	e	e	e	e	e	e	e	d	d	d	d	d	d	d	d

the graph of which is shown on Figure 2.10.

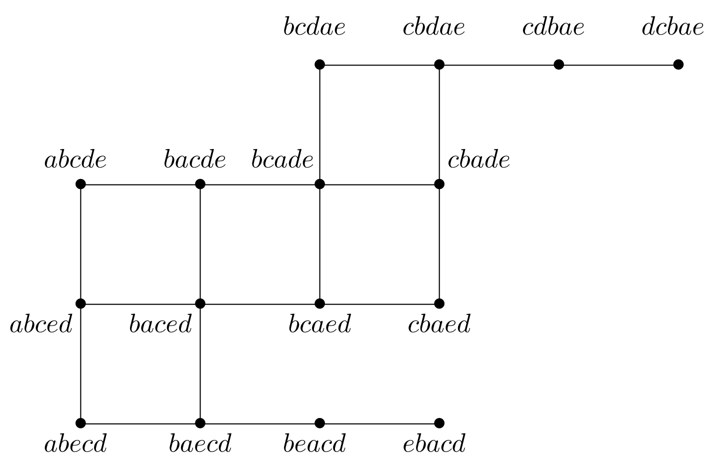


Figure 2.10: Graph of the Arrow's single-peaked domain $\mathcal{D}_{5,5}$ (Case 2.2).

Case 2.3. We get the domain $\mathcal{D}_{5,6}$ with the table

d	b	c	b	a	b	b	c	a	b	b	c	a	b	b	e
b	d	b	c	b	a	c	b	b	a	c	b	b	a	e	b
c	c	d	d	c	c	a	a	c	c	a	a	e	e	a	a
a	a	a	a	d	d	d	d	e	e	e	e	c	c	c	c
e	e	e	e	e	e	e	e	d	d	d	d	d	d	d	d

the graph of which is shown on Figure 2.11.

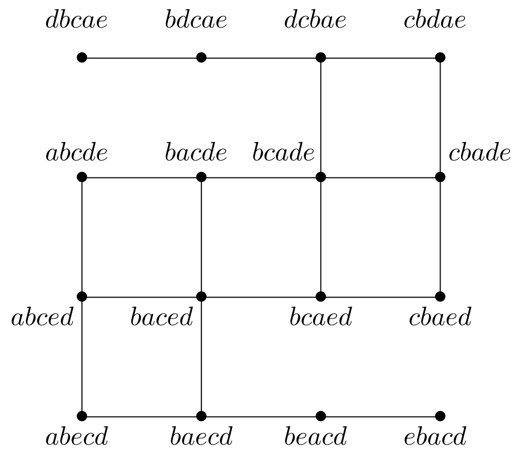


Figure 2.11: Graph of the Arrow's single-peaked domain $\mathcal{D}_{5,6}$ (Case 2.3).

Case 2.4. The table in this case would be

d	b	b	c	a	b	b	c	a	b	b	c	b	a	a	e
b	d	c	b	b	a	c	b	b	a	c	b	a	b	e	a
c	c	d	d	c	c	a	a	c	c	a	a	e	e	b	b
a	a	a	a	d	d	d	d	e	e	e	e	c	c	c	c
e	e	e	e	e	e	e	e	d	d	d	d	d	d	d	d

This domain is isomorphic to $\mathcal{D}_{5,5}$ in Case 2.2. under the map which swaps a and c and also d and e.

2.2.6 Enumeration of Arrow's single-peaked domains

Let $D(n)$ be the number of Arrow single-peaked Condorcet domains on n alternatives. The current known values of $D(n)$ are in the following table

n	3	4	5	6	7	8	9
$D(n)$	1	2	6	40	560	17024	1066496

For $n \leq 5$ the values have been obtained by Slinko [2019], for $6 \leq n \leq 8$ by Liversidge [2020], and for $n = 9$ the value $D(9)$ has been calculated by Markström et al. [2024].

2.3 Complexity considerations

In Example 1.1.1 we saw that some sets of never-bottom conditions are inconsistent and do not define a Condorcet domain. Naturally, the question arises: does there exist a polynomial-time algorithm that, given a complete set of never-bottom conditions, determines if this set is actually consistent? We suggest the following algorithm. Let

$$\mathcal{N} = \{xN_{\{a,b,c\}}3 \mid \{a,b,c\} \subseteq A,\ x \in \{a,b,c\}\ \}$$

be a set of never-bottom conditions on a set of alternatives A. The key idea in constructing a linear order satisfying all never-bottom conditions from $\mathcal{N}$ is to look recursively for terminal elements. We proceed as follows. The algorithm gradually builds an order $\succ$ and reduces subsets $A' \subseteq A$ and $\mathcal{N}' \subseteq \mathcal{N}$. Start with a trivial order $\succ$ on an empty set of alternatives $\emptyset$ and with $A' = A$ and $\mathcal{N}' = \mathcal{N}$. The step of the algorithm is as follows: suppose that at some iteration we have A', $\mathcal{N}'$ and $\succ'$. Then

1. If $|A'| = 2$, say $A' = \{a, b\}$, then $\mathcal{N}' = \emptyset$. In such a case output "$\mathcal{N}$ is consistent." prepend both elements of A' to $\succ'$ obtain $\succ = ab \succ'$ and halt. If $|A'| > 2$, calculate the set
$$X = \{x \in A' \mid \forall\ a, b \in A' \text{ it holds that } xN_{\{x,a,b\}}3 \notin \mathcal{N}'\}.$$
X is the set of alternatives that are not restricted in any way by the remaining never-conditions in $\mathcal{N}'$.

2. If X is empty, then output "there is no order" and halt. Alternatively, if x is randomly chosen element from X, prepend x to the order $\succ'$, update $\succ' = x \succ'$, remove x from A', update $\mathcal{N}'$ by removing from it all never conditions with participation of x and go to step 1.

We justify this algorithm as follows. After selection of an alternative x all never-bottom conditions $yN_{\{x,y,z\}}3$, with $y \neq x$ and $y \neq z$, are removed because y and z will be higher in the constructed order, hence it will automatically satisfy $yN_{\{x,y,z\}}3$. If it halts when $|A'| \geq 3$, then there are at least three alternatives and none of them can be placed lower than the rest of them. So the subset $\mathcal{N}' \subseteq \mathcal{N}$ is clearly inconsistent. On the other hand, it is easy to see that, if this algorithm outputs a linear order, then it satisfies all never conditions from $\mathcal{N}$. To prove this suppose that $\succ = \ldots a \ldots b \ldots c \ldots$ but one of the never conditions from $\mathcal{N}$ is not satisfied for $\{a, b, c\}$. This can be only if $cN_{\{a,b,c\}}3 \in \mathcal{N}$. By the construction of $\succ$ alternative c cannot be added earlier than b and a as $cN_{\{a,b,c\}}3 \in \mathcal{N}$ means c cannot be found in X while a and b still exist there. Clearly it runs in is polynomial time.

It is worth noting that analogous question for never-middle domains is NP-complete which follows from Opatrny [1979] and Guttmann and Maucher [2006].

2.4 Generalising the single-peaked property

Various generalisations of single-peakedness usually involve a different notion of the spectrum. For example, Demange [1982] considered single-peakedness on a tree. In such a case the alternatives are attached to vertices of a tree.

Definition 2.4.1. *Suppose the alternatives from the set of alternatives A are associated with the vertices of a tree $\mathcal{T}$. A linear order $u \in \mathcal{L}(A)$ is said to be single-peaked on $\mathcal{T}$ if for arbitrary $a \in A$ the upper contour set $U(u,a)$ is a subtree.*

In such a case, if $\mathcal{T}$ is not a line, we immediately arrive at an intransitivity.

Example 2.4.1. *Consider the set of alternatives $A = \{a, b, c, d\}$ arranged as vertices of the following graph $\mathcal{T}$:*

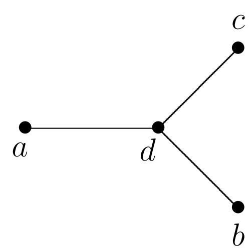


Figure 2.12: Graph $\mathcal{T}$ with alternatives attached to vertices.

Obviously, any order with d as the top preference is single-peaked. Consider the following profile:

$$\begin{array}{ccc} d & d & d \\ a & b & c \\ b & c & a \\ c & a & b \end{array}$$

Here d is a Condorcet winner but the majority relation $\succ$ on $\{a, b, c\}$ is obviously intransitive: $a \succ b \succ c \succ a$.

However, there is still some positive feature of single-peakedness on a tree.

Theorem 2.4.1. *Let $\mathcal{D} \subseteq \mathcal{L}(A)$ be a domain of single-peaked preferences on a tree $\mathcal{T} = (A, E)$, where E is the set of edges of $\mathcal{T}$, and $\succ$ be the corresponding majority relation. Then there exist an alternative $m \in A$ (Condorcet winner) such that $m \succ a$ for every $a \in A$.*

Proof. Let us introduce the relation

$$x \succ_0 y \Longleftrightarrow x \succ y \text{ and } (x, y) \in E.$$

This relation is acyclic since trees do not have cycles. Hence $\succ_0$ has at least one maximum, let us call it m. Let us prove that this is a global maximum, i.e., a Condorcet winner. Let $y \neq m$ be another alternative. Then there is a path $\mathcal{P}$ in $\mathcal{T}$ going through m and y and $\mathcal{D}$ is a classical single-peaked on this path. If m and y are neighbours we know $m \succ y$. If not, let x be a neighbour of m on the path $\mathcal{P}$ towards y. Then more than half orders $v \in \mathcal{D}$ have $m \succ_v x$. Due to single-peakedness on $\mathcal{P}$ they will also have $m \succ_v y$. □

Another and maybe more productive idea is to assume that alternatives can be placed on a circle, with agents' preferences again being decreasing on both sides of their peaks. Intuitively, a domain is single-peaked on a circle if, for every order of the domain, we can 'cut' the circle once so that the given order is single-peaked on the resulting line spectrum. Crucially, the location of the cutting point may differ for different orders of the domain. On the circle the difference between single-peaked and single-dipped orders disappears so it is, thus, clear that the set of all linear orders that are single-peaked on a circle is not a Condorcet domain. For example, for three alternatives the universal domain $\mathcal{L}([3])$ is obviously single-peaked on a circle. For $n = 4$ the domain of orders of single-peaked on a circle has 16 orders out of 24 in the universal domain.

However, the domain of single-peaked orders on a circle contains a number of interesting maximal Condorcet domains. Obviously, the classical single-peaked domain is a subset of the domain of orders single-peaked on a circle. As we will see later so does Fishburn's domain but not single-crossing domain.

Definition 2.4.2 (Peters and Lackner [2020]). *A linear order $v \in \mathcal{L}(A)$ is said to be* single-peaked on a circle*, if alternatives from A can be placed on a circle*

$$a_1 \lhd a_2 \lhd \cdots a_n \lhd a_1$$

in anticlockwise order so that for every alternative $a \in A$ the upper counter set $U(a, v) = \{b \in A \mid b \succ_v a\}$ is a contiguous segment (arc) of the circle. A domain $\mathcal{D} \subseteq \mathcal{L}(A)$ is said to be single-peaked on a circle *(SPOC) if there exists an arrangement of alternatives on a circle (same for each order of $\mathcal{D}$) such that each order of $\mathcal{D}$ is single peaked on a circle relative to their common spectrum.*

Peters and Lackner [2020] formulated necessary and sufficient criterion of being SPOC in terms of forbidden configurations. For sets $B, C \subseteq A$ of alternatives, for a linear order $\succ$ we write $B \succ C$ to mean that $b \succ c$ for all $b \in B$ and $c \in C$.

Theorem 2.4.2 (Peters and Lackner [2020]). *A domain $\mathcal{D}$ of linear orders on a finite set of alternatives A is not SPOC if and only if one of the following three configurations occurs:*

(i) There exist alternatives $a, b, c, d, e \in A$ and orders $\succ_i$ and $\succ_j$ in $\mathcal{D}$ such that

$$\{a, b\} \succ_i c \succ_i \{d, e\},$$
$$\{a, e\} \succ_j c \succ_j \{d, b\}.$$

(ii) There exist alternatives $a, b, c, d \in A$ and orders $\succ_i$, $\succ_j$ and $\succ_k$ in $\mathcal{D}$ such that

$$\{a, b\} \succ_i \{c, d\},$$
$$\{a, c\} \succ_j \{b, d\},$$
$$\{a, d\} \succ_k \{b, c\}.$$

(ii) There exist alternatives $a, b, c, d \in A$ and orders $\succ_i$, $\succ_j$ and $\succ_k$ in $\mathcal{D}$ such that

$$\begin{aligned} \{a, b\} \succ_i \{c, d\}, \\ \{b, c\} \succ_j \{a, d\}, \\ \{c, a\} \succ_k \{b, d\}. \end{aligned}$$

Example 2.4.2. *The domain whose orders are written as columns of the following matrix*

$$\begin{bmatrix} 1 & 2 & 2 & 2 & 2 & 4 & 4 \\ 2 & 1 & 3 & 3 & 4 & 2 & 3 \\ 3 & 3 & 1 & 4 & 3 & 3 & 2 \\ 4 & 4 & 4 & 1 & 1 & 1 & 1 \end{bmatrix},$$

is not SPOC since it contains configuration (ii):

$$\begin{aligned} \{2, 1\} \succ_i \{3, 4\}, \\ \{2, 3\} \succ_j \{1, 4\}, \\ \{2, 4\} \succ_k \{1, 3\}. \end{aligned}$$

As we will see in the next section it is a single-crossing domain.

Chapter 3

Single-crossing domains

Another frequently useful sufficient condition for transitivity of the majority relation is the following single-crossing property. A domain is said to be *single-crossing* if linear orders in it can be put in a sequence so that along this sequence the relative positions of any pair of alternatives is reversed at most once.

Roberts [1977] and Gans and Smart [1996] provide a number of economic applications of single-crossingness. All of them represent situations when preferences of individuals depend on a single parameter. For example, in Roberts' seminal paper voters' preferences on the level of taxation depend solely on their income: the lower the income the higher taxation this individual prefers. However, if the state provides subsidies for families with children, this condition may not be satisfied. But, if we fix the income, we will find single-crossing condition relative to the second parameter, i.e., the more children the person has the higher the level of taxation she prefers. So there are compelling economic reasons which prompt us to consider single-crossingness on graphs more general than a line.

Single-crossing domains have many nice properties that have attracted the attention of researchers. In particular, single-crossing domains are Condorcet domains. In fact, for a single-crossing domain in any profile over this domain with an odd number of voters there is always a voter in the group whose preference coincides with the group preference aggregated by means of the pairwise majority voting — this fact is known as the Representative Voter Theorem [Grandmont, 1978, Rothstein, 1991, Demange, 2012]. Moreover, the collective choice predicted by the Representative Voter Theorem can be implemented in dominant strategies through a simple mechanism [Tohmé and Saporiti, 2006], among the many social choice rules implementable in dominant strategies on single-crossing domains Saporiti [2009].

Recent research has revealed that understanding single-crossing domains could be crucial to understanding Condorcet domains in general. Indeed, as we saw in Theorem 1.4.12 any connected maximal Condorcet domain of maximal width is a union of maximal chains and each maximal chain, as we will see, is a single-crossing domain. An important question of the same spirit, then, is the following: when is a maximal Condorcet domain also by itself single-crossing? One answer to this question was given by Puppe and Slinko [2019] who gave a characterisation in terms of a property called the *pairwise concatenation property*. We will show that this fact is an easy consequence of the results in Section 3.2.

Bredereck et al. [2013] provided an efficient way to verify this property for domains in general.

3.1 Single-crossingness: definition and simple consequences

Let us start with a rigorous definition.

Definition 3.1.1. *A domain $\mathcal{D} \subseteq \mathcal{L}(A)$ is said to be a* single-crossing domain *if the orders from $\mathcal{D}$ can be written in a sequence $(\succ_1, \ldots, \succ_{|\mathcal{D}|})$ so that $i \succ_1 j$ implies either $i \succ_s j$ for every s, or there exists k for which $i \succ_s j$ for every $s \leq k$ and $j \succ_s i$ for every $s > k$. Simply put, traveling along $\succ_1, \succ_2, \ldots$ the relative positions of i and j in these orders swap at most once.*

Example 3.1.1. *Domain $\mathcal{D} = \{abc, bac, bca, cba\}$ is single-crossing since we can put these orders in a sequence as shown on Figure 3.1.*

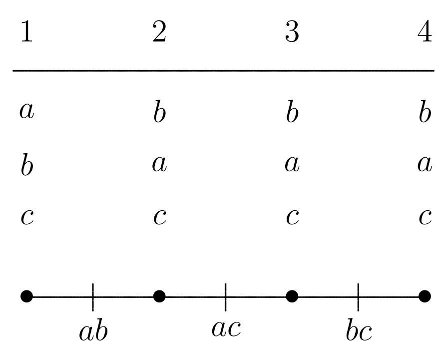


Figure 3.1: An example of single-crossing domain

The change in positions of a and b occurs when moving from the first order to the second, then we have the change of positions of a and c and finally the change in positions of b and c.

Proposition 3.1.1. *Any single-crossing domain is a peak-pit domain.*

Proof. We note that a restriction of a single-crossing domain $\mathcal{D}$ to a set of a smaller number of alternatives is also a single-crossing domain. Hence the restriction $\mathcal{D}_{\{a,b,c\}}$ to a set of three alternatives must be either a subdomain of the domain in Example 3.1.1 or flip-isomorphic to it. Indeed, without loss of generality we may assume that the first order in the sequence is abc and the first switch of alternatives is $ab \to ba$. Then the only choice for the second switch is $ac \to ca$ and the last possible switch is $bc \to cb$. $\square$

The graph of a single-crossing domain is especially simple.

Proposition 3.1.2. *A Condorcet domain $\mathcal{D} \subseteq \mathcal{L}(A)$ has the single-crossing property if and only if its associated graph $G_{\mathcal{D}}$ is a chain.*

Proof. It is easily seen that, if $\mathcal{D} = \{v_1, \ldots, v_m\}$ is single-crossing with respect to the sequence $(v_1, v_2, \ldots, v_m)$, then the interval $[v_i, v_k]$ for $i < k$ consists of linear orders $v_i, v_{i+1}, \ldots, v_k$. Indeed, if v_i and v_k agree on the ranking of, say a and b, then no order from $\{v_i, v_{i+1}, \ldots, v_k\}$ can rank them differently. Hence, v_j and v_{j+1} are neighbours in the graph $G_{\mathcal{D}}$ and it is given by the chain connecting v_j and v_{j+1} by an edge for all $j = 1, \ldots, m-1$.

Conversely, suppose that the associated graph $G_{\mathcal{D}}$ is a chain that connects v_j with v_{j+1} by an edge for all $j = 1, \ldots, m-1$. We show that $\mathcal{D}$ is single-crossing with respect to the sequence $(v_1, v_2, \ldots, v_m)$. By Lemma 1.3.13, the geodesic betweenness in $G_{\mathcal{D}}$ coincides with the Kemeny betweenness of $\mathcal{D}$ as a subset of $\mathcal{L}(A)$. Let $a, b \in A$ and $a \succ_{v_1} b$. Let ℓ be the largest positive integer for which $a \succ_{v_\ell} b$. Then Kemeny betweenness and maximality og ℓ imply that $a \succ_{v_h} b$ for all $h \in \{1, \ldots, \ell\}$ and $b \succ_{v_k} a$ for all k such that $k \in \{\ell+1, \ldots, m\}$. This means that $\mathcal{D}$ has the single-crossing property relative to the specified sequence. □

Proposition 3.1.3. *Any maximal single-crossing domain*[1] *is of maximal width.*

Proof. Let $\mathcal{D} = \{v_1, \ldots, v_n\}$ and without loss of generality we may assume that $v_1 = e$. Suppose $v_n \neq \bar{e}$. Then there are numbers i and j with $i < j$ that with are not switched in v_n. Assume that $j - i$ is minimal with this property. Then all numbers between i and j will be switched with i so in v_n we will have i and j standing together, say $v_n = \ldots ij \ldots$. However then we can switch i and j and add at the end linear order $u = \ldots ji \ldots$ and obtain a larger single-crossing domain. This contradicts to maximality of $\mathcal{D}$. □

We can now characterise maximal single-crossing domains.

Theorem 3.1.4. *Any maximal single-crossing domain $\mathcal{D}$ on a set A of n alternatives is isomorphic to a maximal chain in the Bruhat lattice $(\mathcal{L}([n]), \ll)$.*

Proof. Up to an isomorphism we can assume that $A = [n]$ and due to Proposition 3.1.3 we can think that orders of $\mathcal{D}$ can be listed $e = v_1, v_2, \ldots, v_n = \bar{e}$, so that $\mathcal{D}$ is single-crossing with respect to the sequence $(v_1, v_2, \ldots, v_n)$. Let us prove that

$$e = v_0 \ll v_1 \ll \cdots \ll v_p = \bar{e} \tag{3.1.1}$$

is a maximal chain in $\mathcal{L}([n])$.

If, for every $i \in [n-1]$, order v_{i+1} differs from v_i by a switch of two neighbouring alternatives j and k with $j < k$, so that $v_i = u_i jk w_i$ and $v_{i+1} = u_i kj w_i$, then we have a maximal chain in the Bruhat lattice (3.1.1).

Suppose, for some $i \in [n-1]$ from v_i to v_{i+1} at least two pairs of alternatives get switched. Consider any geodesic path in $\mathcal{L}([n])$ connecting v_i and v_{i+1} and suppose the first pair of alternatives switched on that path is the pair (j, k) with $j < k$. Then

[1] Not a single-crossing maximal domain.

$v_i = u_i jkw_i$ and $v_{i+1} = \ldots k \ldots j \ldots$. We claim that in such a case we can add $v = u_i kjw_i$ between v_i and v_{i+1}. Indeed, we see that v is between v_i and v_{i+1} in $G_{\mathcal{D}}$ which were previously neighbours. Thus the graph $G_{\mathcal{D}\cup\{v\}}$ of domain $\mathcal{D} \cup \{v\}$ is also a line. Hence by Theorem 3.1.2 $\mathcal{D} \cup \{v\}$ is single-peaked and $\mathcal{D}$ was not maximal. □

As we have seen in Theorem 3.1.4, maximal chains in the Bruhat lattice and maximal single-crossing domains are the same things expressed in a different language.

Theorem 1.4.12 now gets a new meaning stating that every semi-connected maximal Condorcet domain is a union of its single-crossing subdomains. In Section 3.3 we will address the following question: when a single-crossing domain is itself a maximal Condorcet domain.

Corollary 3.1.5. *The size of any maximal single-crossing domain on the set of n alternatives is $\frac{1}{2}n(n-1)+1$.*

Proof. The size of any maximal single-crossing domain is the size of a maximal chain in $(\mathcal{L}([n], \ll)$ which by Proposition 1.4.10 is $\frac{1}{2}n(n-1)+1$. □

3.2 The representative voter property

The single-crossing condition can naturally be also applied to profiles over a domain $\mathcal{D}$. In the context of profiles we call linear orders *voters*. A profile (for the purpose of majority voting) is effectively a multiset of orders from $\mathcal{D}$, that means it may contain several identical linear orders.

A *representative voter* for a given profile of linear orders is a voter, present in this profile, whose preference coincides with the majority relation of this profile. Rothstein [1991] was the first to notice the validity of the following theorem.

Theorem 3.2.1 (Representative Voter Theorem)**.** *Let n be an odd positive integer, $n^* = \lfloor \frac{n+1}{2} \rfloor$ and $v_1, \ldots, v_n$ be linear orders over the set of alternatives A. Suppose a profile $P = (v_1, \ldots, v_n)$ is a single-crossing profile relative to the sequence $(v_1, v_2, \ldots, v_n)$. Then v_{n^*} coincides with the majority relation of P.*

Proof. Suppose that $a \succ_{n^*} b$. Then there are two options: 1) $a \succ_j b$ for $j = 1, \ldots, n^*$; 2) $a \succ_j b$ for $j = n^*, \ldots, n$. In both cases the majority of voters prefer a to b. □

This property can be also formulated for domains and it becomes very much stronger. We say that a domain $\mathcal{D}$ has the *representative voter property* if any odd profile composed of linear orders from $\mathcal{D}$ contains a representative voter. The following result shows that the single-crossingness is in fact 'almost' necessary for the representative voter property.

Theorem 3.2.2. *Let $\mathcal{D} \subseteq \mathcal{L}(A)$ be a domain of linear orders on A. Then, $\mathcal{D}$ has the representative voter property if and only if $\mathcal{D}$ is either a single-crossing domain, or $\mathcal{D}$ is a closed Condorcet domain with exactly four elements such that the associated graph $G_{\mathcal{D}}$ is a 4-cycle.*

Proof. Suppose that $\mathcal{D}$ has the representative voter property. Evidently, in this case $\mathcal{D}$ is a closed Condorcet domain. By Theorem 1.3.14, the associated graph $G_\mathcal{D}$ is median, and the geodesic betweenness in $G_\mathcal{D}$ agrees with the Kemeny betweenness on $\mathcal{D}$. We will show that all vertices of $G_\mathcal{D}$ have degree at most 2. Suppose by way of contradiction, that $G_\mathcal{D}$ has a vertex, say v, of degree at least 3. Consider any profile $\rho = (v_1, v_2, v_3)$ consisting of three distinct neighbours of v. The majority relation corresponding to ρ is the median $m(v_1, v_2, v_3)$. Since the median graph $G_\mathcal{D}$ by Proposition 1.3.9 does not have 3-cycles, v_i and v_j are not neighbours for all distinct $i, j \in \{1, 2, 3\}$, hence $v_i v v_j$ is the shortest path between v_i and v_j yielding $m(v_1, v_2, v_3) = v$. Since v is not an element of $\{v_1, v_2, v_3\}$ the representative voter property is violated, a contradiction.

Since $G_\mathcal{D}$ is always connected (as a graph), the absence of vertices of degree 3 or more implies that $G_\mathcal{D}$ is either a chordless cycle or a chain. By Proposition 1.3.9 among all chordless cycles, only 4-cycles are a median graphs. On the other hand, if $G_\mathcal{D}$ is a chain, then $\mathcal{D}$ has the single-crossing property by Proposition 3.1.2.

To prove the converse, suppose, first, that $\mathcal{D}$ is a single-crossing domain. Then $G_\mathcal{D}$ is a chain by Proposition 3.1.2 and, evidently, the preference of the median voter in any odd profile coincides with the corresponding majority relation; this is Rothstein's observation (Rothstein [1991]). On the other hand, consider any odd profile over a closed Condorcet domain $\mathcal{D}$ such that the induced graph is a 4-cycle. In that case, the representative voter property holds trivially if the profile contains all four different orders (as $\mathcal{D}$ is closed); if it contains at most three distinct orders, the representative voter property follows as in the case of a chain with at most three elements. □

3.3 Pairwise concatenation property

By Theorem 3.1.4 any maximal single-crossing domain $\mathcal{D}$ on a set A of n alternatives is isomorphic to a maximal chain in the Bruhat lattice $(\mathcal{L}([n]), \ll)$. The question that we ask here is when such a domain will be maximal but not in the class of single-crossing domains but maximal in the class of all Condorcet domains.

Clearly, any maximal chain $e = v_0 \ll v_1 \ll \cdots \ll v_p = \bar{e}$ is characterised by switches of a sequence of $p = \frac{1}{2}n(n-1)$ pairs of alternatives

$$(i_1, j_1),\ (i_2, j_2),\ \ldots, (i_p, j_p) \tag{3.3.1}$$

from the set $\Omega_n = \{(i, j) \mid 1 \le i < j \le n\}$. The pair (i_s, j_s) means that i_s and j_s are neighbours in v_s and v_{s+1}, with $i_s \succ_{v_t} j_s$ for $t = 1, \ldots, s$, and $j_s \succ_{v_t} i_s$ for $t = s+1, \ldots, p$, while all other relations between alternatives in v_s and v_{s+1} coincide. Roughly speaking, the passage from v_s to v_{s+1} is a swap of neighbours i_s and j_s with $i_s < j_s$. Firstly, we will try to give the answer to the question in terms of the sequence (3.3.1).

Theorem 3.3.1 (Puppe and Slinko [2019]). *A maximal single-crossing domain $\mathcal{D}$ is a maximal Condorcet domain if and only if the sequence* (3.3.1) *characterising $\mathcal{D}$ satisfies the following property:*

$$\{i_s, j_s\} \cap \{i_{s+1}, j_{s+1}\} \ne \emptyset \text{ for every } s \in \{1, 2, \ldots, p-1\}. \tag{3.3.2}$$

Proof. As we learned in Section 1.4.4, maximal chains in Bruhat lattice and reduced decompositions are equivalent things. If we translate a maximal chain with pairwise concatenation property to the corresponding reduced decomposition we will have a decomposition

$$\bar{e} = s_{i_1} s_{i_2} \dots s_{i_p},$$

where for each $k = 1, \dots, p-1$ we have $|i_{k+1} - i_k| = 1$. But then by Theorem 1.4.14 the corresponding maximal chain in Bruhat lattice has no other equivalent chains (as it does not have neighbouring s_i and s_j with $|j - i| \geq 2$) and by Theorem 1.4.12 this chain itself is a maximal Condorcet domain. □

The condition (3.3.2) we call the *pairwise concatenation property.* Theorem 3.3.1 is the first characterisation of domains that are single-crossing and maximal Condorcet, and it will also be an important building block for further characterisations.

As we know, the inversion triples provide a very succinct and convenient characterisation of maximal Condorcet domains of maximal width.

Example 3.3.1. *Let us consider the maximal chain $\mathcal{D}$ whose orders are represented as columns of the following matrix*

$$\begin{bmatrix} 1 & 2 & 2 & 2 & 2 & 4 & 4 \\ 2 & 1 & 3 & 3 & 4 & 2 & 3 \\ 3 & 3 & 1 & 4 & 3 & 3 & 2 \\ 4 & 4 & 4 & 1 & 1 & 1 & 1 \end{bmatrix}$$

It can be characterised by the sequence of swapping pairs

$$(1,2),\ (1,3),\ (1,4),\ (3,4),\ (2,4),\ (2,3)$$

or by the corresponding reduced decomposition

$$\bar{e} = s_1 s_2 s_3 s_2 s_1 s_2.$$

Since the pairwise concatenation property is satisfied, this single-crossing domain is a maximal Condorcet domain. It is not too difficult to see that it is characterised by a single inversion triple $[2, 3, 4]$. *Later on, we will show that the main result of this section provides an easy way of finding the characterising set of inversion triples of a domain that is single-crossing and maximal Condorcet.*

We also note that the sequence of linear orders in $\mathcal{C}$ can be recovered if we know the sets $\mathcal{V}_{xy}^{\mathcal{D}}$, $x, y \in [4]$ defined in (1.3.1)*. Indeed, the following inclusions correspond to the switching pairs:*

$$\mathcal{V}_{12}^{\mathcal{D}} \subset \mathcal{V}_{13}^{\mathcal{D}} \subset \mathcal{V}_{14}^{\mathcal{D}} \subset \mathcal{V}_{34}^{\mathcal{D}} \subset \mathcal{V}_{24}^{\mathcal{D}} \subset \mathcal{V}_{23}^{\mathcal{D}} \subset \mathcal{L}([4]).$$

We will show that the pairwise concatenation property imposes on Condorcet domain a very rigid structure that will be described in the next subsection.

3.4 The structure of single-crossing maximal Condorcet domains

We will show that a domain is single-crossing and maximal Condorcet if and only if it can be represented by a structure that we call a *relay*. Let us first use an example to illustrate what a relay looks like. In this example $A = [7]$. The domain is represented by the following matrix where each column corresponds to a preference.

$$\begin{bmatrix} 1&2&2&2&2&2&2&2&2&2&2&7&7&7&7&7&7&7&7&7&7&7\\ 2&1&3&3&3&3&3&3&3&3&7&2&3&3&3&3&3&3&6&6&6&6\\ 3&3&1&4&4&4&4&4&4&7&3&3&2&4&4&4&4&6&3&4&4&5\\ 4&4&4&1&5&5&5&5&7&4&4&4&4&2&5&5&6&4&4&3&5&4\\ 5&5&5&5&1&6&6&7&5&5&5&5&5&5&2&6&5&5&5&5&3&3\\ 6&6&6&6&6&1&7&6&6&6&6&6&6&6&6&2&2&2&2&2&2&2\\ 7&7&7&7&7&7&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \end{bmatrix}.$$

The domain is a maximal chain satisfying the pairwise concatenation property, and by Theorem 3.3.1 is hence single-crossing and maximal Condorcet. In addition, with the help of the red-coloring, it is not difficult to see that the left-to-right progression of linear orders follows a distinct pattern that leaves behind an undulating trajectory like a damped wave. In particular, focusing on the red-colored alternatives, we see that the sequence of orders starts with the movement of 1 that keeps going down from the top until it reaches the bottom. Then 7, which occupies the bottom place just before arrival of 1, as if having received a relay baton from 1 as they meet, starts moving up until it reaches the top. As 7 reaches the top, the then the former top alternative, 2, starts to move down. However, instead of stopping at the bottom, 2 stops at the second-to-bottom place, handing the baton to the then second-to-bottom alternative, 6, which starts to go up until reaching the second-to-top place. This to-and-fro relay run continues, each leg ending with the initial kth-to-top alternative reaching the kth-to-bottom position, or the kth-to-bottom alternative reaching the kth-to-top position, until, eventually, the initial ranking is reversed. The red trajectory is undulating because of the to-and-fro relay motion, and it is damped because a later runner covers a shorter distance than an earlier runner. We call such a progression of linear orders a *top-down relay*, because it starts with the top alternative going down.

Here is a more formal definition. A sequence of preferences $(\succ_1, ..., \succ_m)$ over n alternatives is a *top-down relay* if and only if it can be represented (up to relabelling the alternatives) as the matrix $R_n(1, ..., n)$ recursively defined as follows:

$$R_1(1) = [1], \qquad R_2(1,2) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix},$$

$$\left[\begin{array}{ccccccc|ccc}
1 & 2 & \cdots & 2 & 2 & \cdots & 2 & n & \cdots & n \\
2 & 1 & \cdots & 3 & 3 & \cdots & n & & & \\
3 & 3 & \cdots & 4 & 4 & \cdots & 3 & & & \\
\vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & & R_{n-2}(2,\ldots,n-1) & \\
\vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & & & \\
n-1 & n-1 & \cdots & 1 & n & \cdots & n-1 & & & \\
n & n & \cdots & n & 1 & \cdots & 1 & 1 & \cdots & 1
\end{array}\right]. \tag{3.4.1}$$

It is straightforward to construct an analogous progression of linear orders that starts with the bottom alternative moving up, then followed by the top alternative moving down, and so on so forth. We call such a procession of preferences a *bottom-up relay*. Such relay is of course flip-isomorphic to the top down relay. Top-down relays and bottom-up ones are collectively called *relays*. We say that domain $\mathcal{D}$ *has a relay representation* if there is an ordering $(\succ_1, \ldots, \succ_m)$ of preferences in it that is a relay.

Theorem 3.4.1. *A domain $\mathcal{D} \subseteq \mathcal{L}(A)$ is single-crossing and maximal Condorcet if and only if it has a relay representation.*

Proof. If $\mathcal{D}$ has a relay representation, then it is a maximal chain in $\mathcal{L}(A)$, hence $\mathcal{D}$ is a maximal single-crossing domain that satisfies the pairwise concatenation property. Thus, by Theorem 3.3.1, $\mathcal{D}$ is also a maximal Condorcet domain.

Now we prove the other direction. Suppose $\mathcal{D}$ is a single-crossing and maximal Condorcet domain over alternatives $A = [n]$. It follows that $\mathcal{D}$ contains exactly one pair of completely reversed orders. Without loss of generality (by relabelling of the alternatives) assume they are $12\ldots n$ and $n(n-1)\ldots 1$. Let P be the matrix where preferences in $\mathcal{D}$ are written as columns in such an order that the neighbouring swapping pairs are linked as in (3.3.2). We will show that P is a relay. A couple of observations are helpful.

Lemma 3.4.2. *P has one of the following submatrices:*

$$Q_1 = \begin{bmatrix}
1 & \star & \cdots & \star & \star & \cdots & \star & n \\
\star & 1 & \cdots & \star & \star & \cdots & n & \star \\
\star & \star & \cdots & \star & \star & \cdots & \star & \star \\
\vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
\star & \star & \cdots & 1 & n & \cdots & \star & \star \\
n & n & \cdots & n & 1 & \cdots & 1 & 1
\end{bmatrix}, \quad
Q_2 = \begin{bmatrix}
1 & 1 & \cdots & 1 & n & \cdots & n & n \\
\star & \star & \cdots & n & 1 & \cdots & \star & \star \\
\star & \star & \cdots & \star & \star & \cdots & \star & \star \\
\vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
\star & n & \cdots & \star & \star & \cdots & 1 & \star \\
n & \star & \cdots & \star & \star & \cdots & \star & 1
\end{bmatrix}$$

where all the columns are the same after the removal of 1 *and* n.

Proof. Consider the corresponding sequence of swapping pairs (3.3.1) which we know has all possible pairs of distinct alternatives and any two neighbouring pairs are linked, i.e., satisfy (3.3.2). Let us consider the last column of P such that 1 occupies the top. That column is thus

$$[1, a_1, \ldots, a_{n-2}, a_{n-1}]^T. \tag{3.4.2}$$

In the next order (proceeding to the column next to the right), 1 will be swapped with a_1. Claim that in the subsequent orders, 1 goes straight to the bottom (each time moving down one position). Indeed, suppose at some point we had the column

$$[a_1, \ldots, a_{i-1}, 1, a_i, \ldots, a_{n-1}]^T.$$

where 1 has just swapped with a_{i-1}. If 1 does not swap with a_i in the next step, then it will be a_{i-1} swapping with a_{i-2}. If so, then 1 will then never has a chance to be swapped with a_i since for that to happen 1 must be involved in the previous swap with one of $a_1, \ldots, a_{i-1}$, but this is impossible since 1 has already been swapped with all these alternatives. Since 1 has to reach the bottom eventually, it follows that the next step must be 1 swapping with a_i, and hence 1 has to go down to the bottom continuously.

In a similar argument, we can now show that n has to continuously go all the way up to the top once it starts moving. Thus, if 1 starts moving before n, then P has Q_1 as a submatrix, whereas, if n starts to move before 1, then P has Q_2 as a submatrix. □

Lemma 3.4.3. *P has a submatrix $Q \in \{Q_1, Q_2\}$ that occupies either the leftmost $2n-1$ columns or the rightmost $2n-1$ columns.*

Proof. Suppose Q_1 is a submatrix of P. (The case where Q_2 is a submatrix of P can be established in a similar argument.) If the first column of Q_1, given in (3.4.2), is not the first column in P, then the previous column was

$$[1, a_2, a_1, \ldots, a_{n-2}, \underbrace{a_{n-1}}_{=n}]^T,$$

i.e., the previous swap was between a_1 and a_2. Similarly, if the rightmost column of Q_1

$$[n, a_1, a_2, \ldots, a_{n-2}, 1]^T.$$

is not the last column in P, then the next swap must also be between a_1 and a_2. However, a_1 and a_2 cannot be swapped more than once, hence either the first column of Q_1 is the first column in P or the last column of Q_1 is the last column in P. □

Now we are ready to show that P is a relay. Suppose Q_1 occupies the leftmost $2n-1$ columns of P. We want to show that P is a top-down relay, i.e., it satisfies the recursive definition given in (3.4.1). Consider the inductive hypothesis that for any $k < n$, a matrix with k rows is a top-down relay if it satisfies the following:

(a) It corresponds to a maximal chain on $\mathcal{L}(\{i, i+1, ..., i+k-1\})$ for some $i \in \mathbb{N}$,

(b) The columns are in such an order that the neighbouring swapping pairs are linked as in (3.3.2), and

(c) In the leftmost $2k-1$ columns the top alternative goes all the way down followed by the bottom alternative going all the way up (as in Q_1).

This hypothesis is obviously true for $k = 1$ or $k = 2$.

Since Q_1 occupies the leftmost part of P, the first column of Q_1 is $[1, 2, \ldots, n]^T$. All the remaining columns to the right of Q_1 will have n on the top and 1 at the bottom, and clearly, right after Q_1 the first pair to swap is $[2, 3]$, i.e., 2 starts to move down. Consider the submatrix P' of P enframed by n's on the top and 1's at the bottom: it obviously corresponds to a maximal chain on $\mathcal{L}(\{2, 3, ..., n-1\})$ and the columns are ordered so that the neighbouring swapping pairs are linked. Moreover, since 2, the top alternative among $\{2, 3, ..., n-1\}$, is the first to move in P', by an argument analogous to that used in Lemma 3.4.3, the leftmost $2n-3$ columns of the P' is analogous to Q_1 (2 moves all the way down, then $n-1$ moves all the way up). Therefore, by the inductive hypothesis P' is a top-down relay, i.e. $P' = R_{n-2}(2, \ldots, n-1)$ (defined in (3.4.1)). This implies that $P = R_n(1, 2, ..., n)$. Thus P is a top-down relay. If, on the other hand, Q_1 occupies the rightmost $2n-1$ columns of P, then it is clear that after relabelling every alternative i as $n+1-i$ and listing the preferences in a reverse order, the resulting matrix is also $R_n(1, 2, ..., n)$, a top-down relay. So the two cases lead to isomorphic domains. If Q_2 is a submatrix of P, then P is shown to be a bottom-up relay in an analogous argument. It is flip-isomorphic to a top-down relay. □

Corollary 3.4.4. *A single-crossing domain is a maximal Condorcet domain if and only if it is either isomorphic or flip-isomorphic to the Condorcet domain with matrix R_n.*

It is easy now to obtain a characterisation of single-crossing and maximal Condorcet domains in term of inversion triples.

Theorem 3.4.5. *For $m = 3$, we have $\mathcal{I}_3 = \emptyset$. Moreover, let $\mathcal{I}_n$ be the set of inversion triples that define domain R_n, then*

$$\mathcal{I}_n = \mathcal{I}_{n-2} \cup \{[i, j, n] \mid 1 < i < j < n\}.$$

Proof. Follows from the representation (3.4.1) by induction. Indeed, when we add 1 and n, then $[i, j, n]$, where $1 < i < j < n$, will be the inversion triples added. □

Example 3.4.1. *For $m = 4$, the unique single-crossing maximal Condorcet domain is characterised by a single inversion triple* $[2, 3, 4]$. *For $m = 5$ the inversion triples defining the domain are* $[2, 3, 5]$, $[2, 4, 5]$, $[3, 4, 5]$.

3.5 Generalising the single-crossing property

The classical single-crossing property [Mirrlees, 1971, Gans and Smart, 1996] naturally generalises to trees. Corresponding definitions were suggested by Kung [2015] and Clearwater et al. [2015]. We give a different but equivalent definition. But firstly, we remind the reader that basic facts about trees were introduced in Section 1.3.2.

As we saw in Example 1.3.1 any tree is a median graph. According to Definition 1.3.2 a subset of vertices of a tree is convex if and only if it is connected. Any tree satisfies the Helly property.

Definition 3.5.1. *Let $\mathcal{D} \subseteq \mathcal{L}(A)$ be a domain of cardinality n on the set of alternatives A and $T = (\mathcal{D}, E)$ be a tree on the set of orders $\mathcal{D}$. We say that the domain $\mathcal{D}$ is* single-crossing *with respect to T if for every pair of alternatives $a, b \in A$ one of the following holds:*

- *We can remove an edge e from T so that for the two resulting subtrees $T_1 = (\mathcal{D}_1, E_1)$ and $T_2 = (\mathcal{D}_2, E_2)$ all linear orders in $\mathcal{D}_1$ rate a above b and all orders in $\mathcal{D}_2$ rate b above a. In this case the edge e will be called an ab-*cut.

- *All linear orders in $\mathcal{D}$ rank a higher than b or all linear orders in $\mathcal{D}$ rank b higher than a. In this case we say that the ab-cut is* virtual.

We say that a domain is *generalised single-crossing* if it is single-crossing with respect to some tree.

Theorem 3.5.1. *Any generalised single-crossing domain is a closed Condorcet domain.*

Proof. Let $\mathcal{D} \subseteq \mathcal{L}(A)$ is single-crossing with respect to a tree $T = (\mathcal{D}, E)$ and P be an odd profile over $\mathcal{D}$ which can be considered as a multiset of linear orders from $\mathcal{D}$. Let V_{xy}^P be the multiset of linear orders in P which rank x higher than y and let $\mathcal{V}_{xy}^P$ be the set of unique linear orders in P that rank x higher than y.

Let the majority relation of P be $\succ_P$. Suppose $a \succ_P b$ and $b \succ_P c$ where $a, b, c \in A$. Then $|V_{ab}^P| > |V_{ba}^P|$ and $|V_{bc}^P| > |V_{cb}^P|$. In particular, $|V_{ab}^P| > \frac{1}{2}|P|$ and $|V_{bc}^P| > \frac{1}{2}|P|$, hence $V_{ab}^P \cap V_{bc}^P \neq \emptyset$ and also $\mathcal{V}_{ab}^P \cap \mathcal{V}_{bc}^P \neq \emptyset$. For any linear order v in $\mathcal{V}_{ab}^P \cap \mathcal{V}_{bc}^P$ we have $a \succ_v b$ and $b \succ_v c$, hence also $a \succ_v c$. Thus an ac-cut cannot be between linear orders of $\mathcal{V}_{ab}^P \cap \mathcal{V}_{bc}^P$. This means that either $|V_{ac}^P| > |V_{ab}^P|$ or $|V_{ac}^P| > |V_{bc}^P|$. In each case V_{ac}^P has more than half of all linear orders, hence $a \succ_P c$ which proves transitivity of $\succ_P$. □

The following proposition is almost obvious but still useful.

Proposition 3.5.2. *Let $\mathcal{D} \subseteq \mathcal{L}(A)$ be a domain.*

(a) If $\mathcal{D}$ is a closed Condorcet domain whose associated graph $G_{\mathcal{D}}$ is a tree, then $\mathcal{D}$ is single-crossing on that tree.

(b) On the other hand, if $\mathcal{D}$ is single-crossing with respect to a tree $T = (\mathcal{D}, E)$. Then $\mathcal{D}$ is a Condorcet domain and $G_{\mathcal{D}} = T$.

Proof. (a) Consider an edge uv in $G_{\mathcal{D}}$. There will be at least two alternatives a and b such that $a \succ_u b$ and $b \succ_v a$. On removal of this edge, $G_{\mathcal{D}}$ splits into two subtrees G_u and G_v. Then any path in $G_{\mathcal{D}}$ from $w_1 \in G_u$ to $w_2 \in G_v$ will go through uv. Then v is geodesically - and hence Kemeny - between u and w_2 and u is between w_1 and v. Hence we must have $a \succ_{w_1} b$ and $b \succ_{w_2} a$ which means that uv is an ab-cut.

(b) Firstly, by Theorem 3.5.1 $\mathcal{D}$ is a closed Condorcet domain. Two things are obvious:

(i) Every edge $e = uv$ in T is an ab-cut for a certain pair of alternatives (otherwise $u = v$ which cannot be the case);

(ii) The restriction of $\mathcal{D}$ to any path in T is a single-crossing domain.

Suppose v and u are two vertices which are neighbours in T. Then uv is an ab-cut for some $a, b \in A$ and removal of uv splits $\mathcal{D}$ into $\mathcal{V}_{ab}^{\mathcal{D}}$ and $\mathcal{V}_{ba}^{\mathcal{D}}$. If $w \neq u$ and $w \neq v$, then without loss of generality we may assume that $w \in \mathcal{V}_{ab}^{\mathcal{D}}$. Suppose w is closer to u than to v. Then, due to (i) there is a cd-cut between w and u, hence for some pair of alternatives $c, d \in A$ we have $c \succ_w d$ while $d \succ_u c$ and $d \succ_v c$. Thus w cannot be between u and v in the Kemeny sense, hence u and v are neighbours in $G_{\mathcal{D}}$. Thus, $G_{\mathcal{D}} = T$. □

Example 3.5.1 (The single-crossing property on a star tree). *Consider domain $\mathcal{D}$ on the set $A = \{a, b, c, d\}$ consisting of four orders: $v = abcd$, $v_1 = acbd$, $v_2 = abdc$, and $v_3 = bacd$. As is easily seen, $\mathcal{D}$ is a closed Condorcet domain. The associated graph $G_{\mathcal{D}}$ connects v with each of the other three orders by an edge and the graph has no other edges (cf. Figure 3.2). Thus, $G_{\mathcal{D}}$ is a tree and v is the median order of any triple of distinct elements of $\mathcal{D}$.*

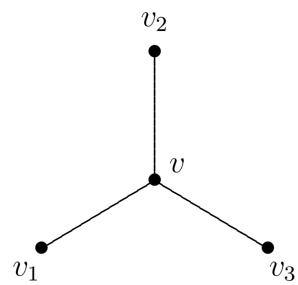


Figure 3.2: A generalised single-crossing domain on a star tree

The following generalisation of the classical Representative Voter Theorem [Rothstein, 1991] follows from [Demange, 2012] but we give a direct proof.

Theorem 3.5.3 (Representative Voter Theorem). *Let n be odd. If a profile $P = (P_1, \ldots, P_n)$ is single-crossing with respect to a tree $T = (V, E)$, then there exists $i \in \{1, \ldots, n\}$ such that preference order P_i coincides with the majority relation $\succ_P$ of P.*

Proof. Let, as above, V_{xy} be the multiset of voters who prefer x to y. These sets are convex. Consider the set of subsets $\mathcal{M} = \{H_{xy} \mid x \succ_P y\}$. Any two subsets $H_{xy}, H_{zt} \in \mathcal{M}$ have a nonempty intersection since each of them contains a majority of vertices. By the Helly property we have $\bigcap_{H_{xy} \in \mathcal{M}} H_{xy} \neq \emptyset$. Voters' preferences in this intersection coincide with the majority relation. □

The following result characterises the single-crossing property and its generalisation to trees directly in terms of the structure of the underlying domain, i.e., without explicit reference to the associated graph.

Theorem 3.5.4. *(a) A domain $\mathcal{D} \subseteq \mathcal{L}(A)$ has the single-crossing property with respect to some linear order on $\mathcal{D}$ if and only if, for any two distinct subsets $\{x, y\}$ and $\{z, w\}$ of A such that each of the sets $\mathcal{V}_{xy}^{\mathcal{D}}$, $\mathcal{V}_{yx}^{\mathcal{D}}$, $\mathcal{V}_{zw}^{\mathcal{D}}$, and $\mathcal{V}_{wz}^{\mathcal{D}}$ is non-empty, we have either*

$$\mathcal{V}_{xy}^{\mathcal{D}} \subseteq \mathcal{V}_{zw}^{\mathcal{D}} \text{ and } \mathcal{V}_{wz}^{\mathcal{D}} \subseteq \mathcal{V}_{yx}^{\mathcal{D}}. \tag{3.5.1}$$

or

$$\mathcal{V}^{\mathcal{D}}_{zw} \subseteq \mathcal{V}^{\mathcal{D}}_{xy} \text{ and } \mathcal{V}^{\mathcal{D}}_{yx} \subseteq \mathcal{V}^{\mathcal{D}}_{wz}. \tag{3.5.2}$$

(b) Let $|X| \geq 4$ and $\mathbb{V} = \{\mathcal{V}^{\mathcal{D}}_{xy} \mid x, y \in A\}$. A domain $\mathcal{D}$ has the generalised single-crossing property if and only if $\mathbb{V}$ has the Helly property and for any two distinct subsets $\{x, y\}$ and $\{z, w\}$ of A at least one of the following four sets is empty:

$$\mathcal{V}^{\mathcal{D}}_{xy} \cap \mathcal{V}^{\mathcal{D}}_{zw},\ \mathcal{V}^{\mathcal{D}}_{xy} \cap \mathcal{V}^{\mathcal{D}}_{wz},\ \mathcal{V}^{\mathcal{D}}_{yx} \cap \mathcal{V}^{\mathcal{D}}_{zw},\ \mathcal{V}^{\mathcal{D}}_{yx} \cap \mathcal{V}^{\mathcal{D}}_{wz}. \tag{3.5.3}$$

Proof. (a) Evidently, every single-crossing domain satisfies (3.5.1) or (3.5.2) depending on whether xy-cut is to the left of the zw-cut or the other way around.

To prove the converse, we notice that these conditions allows one to order the family of all sets of the form $\mathcal{V}^{\mathcal{D}}_{xy}$ so that, for an appropriate sequence of $m = \frac{1}{2}|A|(|A|-1)$ distinct pairs of of alternatives $(a_1, b_1), \ldots, (a_m, b_m)$, as in Example 3.3.1,

$$\mathcal{V}^{\mathcal{D}}_{a_1b_1} \subseteq \mathcal{V}^{\mathcal{D}}_{a_2b_2} \subseteq \ldots \subseteq \mathcal{V}^{\mathcal{D}}_{a_mb_m} \subseteq \mathcal{L}(A) \text{ and } \mathcal{L}(A) \supseteq \mathcal{V}^{\mathcal{D}}_{b_1a_1} \supseteq \mathcal{V}^{\mathcal{D}}_{b_2a_2} \supseteq \ldots \supseteq \mathcal{V}^{\mathcal{D}}_{b_ma_m}.$$

It is now easily verified that $\mathcal{D}$ has the single-crossing property with respect to any linear order of the members of $\mathcal{D}$ which lists the elements of $\mathcal{V}^{\mathcal{D}}_{a_1b_1}$ first, then the elements of $\mathcal{V}^{\mathcal{D}}_{a_2b_2} \setminus \mathcal{V}^{\mathcal{D}}_{a_1b_1}$ second, and further lists elements of $\mathcal{V}^{\mathcal{D}}_{a_jb_j} \setminus \mathcal{V}^{\mathcal{D}}_{a_{j-1}b_{j-1}}$ after listing elements of $\mathcal{V}^{\mathcal{D}}_{a_{j-1}b_{j-1}} \setminus \mathcal{V}^{\mathcal{D}}_{a_{j-2}b_{j-2}}$.

(b) Suppose that $\mathcal{D}$ has the generalised single-crossing property. Then, $G_{\mathcal{D}}$ is a tree which is a median graph (Example 1.3.1). By Lemma 1.3.13(iii), the betweenness on $\mathcal{D}$ coincides with the induced geodesic betweenness on $G_{\mathcal{D}}$. In particular, $\mathcal{D}$ is a median domain, and as such satisfies the Helly property for convex sets, and all elements of $\mathbb{V}$ are convex. To verify that in (3.5.3) all four intersections cannot be nonzero, assume by contradiction that

$$v_1 \in \mathcal{V}^{\mathcal{D}}_{xy} \cap \mathcal{V}^{\mathcal{D}}_{zw}, \quad v_2 \in \mathcal{V}^{\mathcal{D}}_{xy} \cap \mathcal{V}^{\mathcal{D}}_{wz}, \quad v_3 \in \mathcal{V}^{\mathcal{D}}_{yx} \cap \mathcal{V}^{\mathcal{D}}_{zw}, \quad v_4 \in \mathcal{V}^{\mathcal{D}}_{yx} \cap \mathcal{V}^{\mathcal{D}}_{wz},$$

for $v_1, v_2, v_3, v_4 \in \mathcal{D}$. Consider a shortest path between v_1 and v_3 and a shortest path between v_2 and v_4. The first path lies entirely in $\mathcal{V}^{\mathcal{D}}_{zw}$ and the second one lies entirely in $\mathcal{V}^{\mathcal{D}}_{wz}$ which do not intersect. But on each of them there is a switch from xy to yx, and thus there exist two pairs of neighbouring orders such that one of them is in $\mathcal{V}^{\mathcal{D}}_{xy}$ and the other one in $\mathcal{V}^{\mathcal{D}}_{yx}$. This contradicts the generalised single-crossing property as there can be only one such edge.

Conversely, suppose that a domain $\mathcal{D}$ satisfies the Helly property and among intersections (3.5.3) at most three are non-empty. Proposition 1.3.5 then implies that $\mathcal{D}$ is a median domain, hence by Lemma 1.3.13(i), the betweenness in $\mathcal{D}$ coincides with the geodesic betweenness in $G_{\mathcal{D}}$. We claim that $G_{\mathcal{D}}$ is acyclic, hence a tree, which implies the generalised single-crossing property, as desired. If median graph $G_{\mathcal{D}}$ has a cycle, by Proposition 1.3.9 it has a cycle of length 4 (see Figure 3.3).

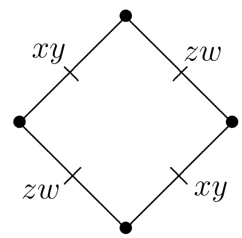


Figure 3.3: 4-cycle

However in such a case all four intersections in (3.5.3) will be non-empty. □

Finally we note that in generalised single-crossing domains we do not find new maximal Condorcet domains. To show this we need to do some preliminary work.

Lemma 3.5.5. *Suppose $\mathcal{D}$ is a generalised single-crossing domain and uv is an edge in $G_{\mathcal{D}}$. Suppose that $w \in [u, v]$ is different from u and v (thus $w \notin \mathcal{D}$). Then $\mathcal{D}' = \mathcal{D} \cup \{w\}$ is also a generalised single-crossing domain and, in particular, $\mathcal{D}$ is not a maximal Condorcet domain.*

Proof. Obviously, $G_{\mathcal{D}'}$ is still a tree. As $w \neq u$ and $w \neq v$, there are alternatives $a, b \in A$ such that $a \succ_u b$ and $a \succ_w b$ but $b \succ_v a$. The edge uv, then was an ab-cut in $\mathcal{D}$. It is easy to see that wv is the new ab-cut in $\mathcal{D}'$. Hence $\mathcal{D}'$ is generalised single-crossing and $\mathcal{D}$ is not maximal. □

Corollary 3.5.6. *Any maximal generalised single-crossing domain on X is connected.*

Now we can prove the promissed result.

Theorem 3.5.7. *Let $\mathcal{D}$ be a maximal Condorcet domain. If $G_{\mathcal{D}}$ is a tree, it is, in fact, a chain.*

Proof. Let $\mathcal{D}$ be a maximal Condorcet domain, and assume that $G_{\mathcal{D}}$ is a tree but not a chain. Then there exists a vertex u in $G_{\mathcal{D}}$ of degree at least 3. Consider now any three neighbours of u in $G_{\mathcal{D}}$, say u_1, u_2 and u_3. Since, by Corollary 3.5.6, $\mathcal{D}$ is connected, there are three distinct ordered pairs (x_i, y_i), $i = 1, 2, 3$, of alternatives such that $u_i = (u \setminus \{(x_i, y_i)\}) \cup \{(y_i, x_i)\}$. We will say that u_i is obtained from u by switching the pair of adjacent alternatives (x_i, y_i). Moreover, since in every pair (x_i, y_i), $i = 1, 2, 3$, the alternatives are adjacent in u, there must exist at least two pairs that have no alternative in common, say $\{x_1, y_1\} \cap \{x_2, y_2\} = \emptyset$.

Now let u' be the order that coincides with u except that both pairs (x_1, y_1) and (x_2, y_2) in u' are switched, i.e., $x_1 \succ_u y_1$ and $x_2 \succ_u y_2$ but $y_1 \succ_{u'} x_1$ and $y_2 \succ_{u'} x_2$. Consider the domain $\mathcal{D} \cup \{u'\}$. Since x_1, y_1 and x_2, y_2 are neighbours in each of the orders u, u_1, u_2, u', for every three alternatives $\{a, b, c\}$ no new order among them appears in u' which has not yet occurred in u, u_1, or u_2. Hence $\mathcal{D} \cup \{u'\}$ satisfies all never conditions that $\mathcal{D}$ satisfied and is therefore a Condorcet domain. By the maximality of $\mathcal{D}$, this implies $u' \in \mathcal{D}$. But in this case, the graph $G_{\mathcal{D}}$ evidently contains the 4-cycle $\{u, u_1, u', u_2\}$, contradicting the assumed acyclicity $G_{\mathcal{D}}$. Hence, there cannot exist a vertex of degree 3 or larger, i.e., $G_{\mathcal{D}}$, being connected, is a chain. □

Figure 3.2 (cf. Example 3.5.1 above) illustrates the proof of Theorem 3.5.7: one easily verifies that the order *badc* can be added to the depicted domain $\mathcal{D}$, creating a 4-cycle in the associated graph $G_{\mathcal{D}}$; in particular, $\mathcal{D}$ is not maximal.

Chapter 4

Peak-pit Condorcet domains

One of the most interesting and most investigated classes of Condorcet domains is the peak-pit domains. It includes Arrow's single peaked, single-crossing, GF-domains and set-alternating scheme domains that will be introduced in Chapter 5. One our key technique in investigating this type of domains is the use of the notion of the *ideal* of a domain introduced in Subsection 1.5.

The plan for this section is as follows. Firstly, in Subsection 4.2 we relate peak-pittedness of a Condorcet domain of maximal width to separated systems of sets and study their properties. In Subsection 4.3 we relate arrangements of pseudolines to reduced decompositions introduced in Subsection 1.4.4. Subsection 4.4 relates arrangements of pseudolines to Condorcet domains. Finally, in Subsection 4.5 we relate arrangements of pseudolines to separated systems of sets and in Subsection 4.6 we prove the main representation theorems stating that every peak-pit maximal Condorcet domain of maximal width can be obtained as a domain obtained from an arrangement of pseudolines and from rhombus tiling. We then drew rich consequences from this theorem. Finally, in Subsection 4.7 we investigate a possibility of representation by arrangements of pseudolines of peak-pit domains without maximal width. We discover that for such arepresentation of Arrow's single-peaked domains we need to generalise the concept of arrangement of pseudolines allowing pseudolines to have more than one intersection. Finally we in Subsection 4.8 we classify all maximal peak-pit domains of maximal width on 4 and 5 alternatives.

4.1 Uniqueness of two completely reversed orders

In this Section we consider domains of maximal width that contain a linear order u and its flipped order $\bar{u}$. In peak-pit domains such orders, if exist, are unique.

Proposition 4.1.1. *In a peak-pit domain $\mathcal{D} \subseteq \mathcal{L}(A)$ let $u, v \in \mathcal{D}$ be such that $u \neq v$ and $u \neq \bar{v}$. Then $\bar{u}$ and $\bar{v}$ cannot be both in $\mathcal{D}$.*

Proof. Clearly $|A| \geq 3$. Without loss of generality we may consider that $u = e = 12\ldots n$. Consider a subdomain $\mathcal{C} = \{u, \bar{u}\}$. We have $\mathcal{C}|_{\{a,b,c\}} = \{abc, cba \mid a < b < c\}$. It is easy to see that if $w|_{\{a,b,c\}} \in \mathcal{C}|_{\{a,b,c\}}$ for every triple a, b, c, then $w = u$ or $w = \bar{u}$. Indeed, w

can start with either 1 or n and the second alternative can be only 2 in the first case and $n-1$ in the second, etc.

Since $u \neq v$ and $u \neq \bar{v}$, we see that $v|_{\{a,b,c\}} \notin \mathcal{C}|_{\{a,b,c\}}$, for some $a, b, c \in A$ with $a < b < c$, hence $v|_{\{a,b,c\}}$ contains a suborder on a, b, c where b is not in the middle. Suppose it is bac. Then $\bar{v}|_{\{a,b,c\}}$ contains also cab. If $\mathcal{D}$ contains $\{u, \bar{u}, v, \bar{v}\}$, then $\mathcal{D}|_{\{a,b,c\}} = \{abc, cba, bac, cab\}$, which means that $\{a, b, c\}$ is a never-middle triple, a contradiction. □

Due to this proposition, up to an isomorphism, in this Section we consider domains of maximal width with the set of alternatives $A = [n]$ that contain $e = 12\ldots n$ and $\bar{e} = n\ldots 21$.

4.2 Peak-pit domains and separated systems of sets

Definition 4.2.1. *We say that a system S of subsets of $[n]$ is* weakly separated *if for any triple $\{a, b, c\} \subseteq [n]$, there exists an $x \in \{a, b, c\}$, such that either $\{x\} \notin S|_{\{a,b,c\}}$ or $\{a, b, c\} \setminus \{x\} \notin S|_{\{a,b,c\}}$.*

Lemma 4.2.1. *Let $\mathcal{D} \subseteq \mathcal{L}([n])$ be a domain of linear orders. Then $\mathcal{D}$ is a peak-pit Condorcet domain if and only if its ideal $Id(\mathcal{D})$ is weakly separated.*

Proof. Suppose $\mathcal{D}$ is a peak-pit domain and $a, b, c \in [n]$. Suppose that $\mathcal{D}$ satisfies $aN_{\{a,b,c\}}3$. Then $\mathcal{D}|_{\{a,b,c\}} \subseteq \{abc, acb, bac, cab\}$ and

$$\mathrm{Id}(\mathcal{D})|_{\{a,b,c\}} \subseteq \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}.$$

If $\mathcal{D}$ satisfies $aN_{\{a,b,c\}}1$. Then $\mathcal{D}|_{\{a,b,c\}} \subseteq \{bca, cba, bac, cab\}$ and

$$\mathrm{Id}(\mathcal{D})|_{\{a,b,c\}} \subseteq \{\{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}.$$

In both cases $\mathrm{Id}(\mathcal{D})$ is weakly separated.

Suppose now $\mathcal{D}$ is a domain for which $\mathrm{Id}(\mathcal{D})$ is weakly separated. For any $a, b, c \in [n]$ there are two options for $\mathrm{Id}(\mathcal{D})|_{\{a,b,c\}}$, let us consider the first one:

$$\mathrm{Id}(\mathcal{D})|_{\{a,b,c\}} \subseteq \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}.$$

Then $\mathcal{D}|_{\{a,b,c\}} \subseteq \{abc, acb, bac, cab\}$ which means $aN_{\{a,b,c\}}3$. The second option is similar. □

Lemma 4.2.2. *An ideal $I \subseteq 2^{[n]}$ is weakly separated if and only if $Dom(I)$ is a Condorcet peak-pit domain.*

Proof. Let $a, b, c \in [n]$. If I is weakly separated, then either $a \notin I|_{\{a,b,c\}}$ or $\{b, c\} \nsubseteq I|_{\{a,b,c\}}$. These are equivalent to $\mathrm{Dom}(I)|_{\{a,b,c\}}$ not containing abc and acb in the first case and bca and cba in the second, which means $aN_{\{a,b,c\}}3$ and $aN_{\{a,b,c\}}1$, respectively. □

Corollary 4.2.3. *Let $\mathcal{D}$ be a peak-pit maximal Condorcet domain. Then $Dom(Id(\mathcal{D})) = \mathcal{D}$.*

Proof. Since $\mathcal{D}$ is a peak-pit domain, by Lemma 4.2.1 its ideal $\mathrm{Id}(\mathcal{D})$ is weakly separated. By Lemma 4.2.2 $\mathrm{Dom}(\mathrm{Id}(\mathcal{D}))$ is a peak-pit domain. However $\mathcal{D} \subseteq \mathrm{Dom}(\mathrm{Id}(\mathcal{D}))$, hence $\mathcal{D} = \mathrm{Dom}(\mathrm{Id}(\mathcal{D}))$ due to maximality of $\mathcal{D}$. $\square$

Definition 4.2.2. *We say that a system S of subsets of $[n]$ is* separated *if for any triple $\{a, b, c\} \subseteq [n]$ with $a < b < c$, either $\{b\}$ or $\{a, c\}$ does not belong to $S|_{\{a,b,c\}}$.*

Theorem 4.2.4. *Let $\mathcal{D} \subseteq \mathcal{L}([n])$ be a domain of linear orders. If $\mathcal{D}$ is a peak-pit Condorcet domain of maximal width then its ideal Id($\mathcal{D}$) is separated. On the other hand, if Id($\mathcal{D}$) is separated, then $\mathcal{D}$ can be embedded into a peak-pit domain of maximal width.*

Proof. Let $\mathcal{D}$ be a peak-pit Condorcet domain of maximal width and $\{a, b, c\} \subseteq [n]$ with $a < b < c$. Then both abc and cba belong to $\mathrm{Id}(\mathcal{D})|_{\{a,b,c\}}$. Since by Lemma 4.2.2 it is weakly separated, we have either $\{b\}$ or $\{a, c\}$ not in $\mathrm{Id}(\mathcal{D})|_{\{a,b,c\}}$ which makes $\mathrm{Id}(\mathcal{D})$ separated.

To prove the second statement it is sufficient to show that if $\mathrm{Id}(\mathcal{D})$ is a separated system of subsets of $[n]$, then $\mathrm{Id}(\mathcal{D} \cup \{e\} \cup \{\bar{e}\})$ is also separated. Indeed, for any triple $\{a, b, c\}$ with $a < b < c$ neither e nor $\bar{e}$ can contribute $\{b\}$ or $\{a, c\}$ to $\mathrm{Id}(\mathcal{D} \cup \{e\} \cup \{\bar{e}\})|_{\{a,b,c\}}$ and by Lemma 4.2.1 $\mathcal{D} \cup \{e\} \cup \{\bar{e}\}$ is a peak-pit domain of maximal width. $\square$

Let us introduce a criterion for two sets to be separated. For that we will extend the natural relation $<$ for positive integers from $[n]$ to subsets of $[n]$. We write $I < J$ if $i < j$ for all $i \in I$ and $j \in J$. In particular, $\emptyset < J$ for all $J \neq \emptyset$.

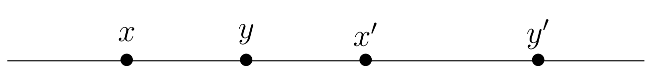
Definition 4.2.3. *Let I, J be subsets of $[n]$. We then define the relation*

$$I \ll J \iff I \setminus J < J \setminus I.$$

In particular, if $I \subset J$, then this condition is trivially satisfied and we have $I \ll J$ in this case. We can now reformulate the definition of separation in terms of $\ll$.

Lemma 4.2.5. *Two sets $X, Y \subseteq [n]$ are separated if and only if $X \ll Y$ or $Y \ll X$.*

Proof. Suppose X and Y are not separated and $X \cap \{a, b, c\} = \{b\}$ and $Y \cap \{a, b, c\} = \{a, c\}$ for some $a, b, c \in [n]$ with $a < b < c$. Then $b \in X \setminus Y$ and $a, c \in Y \setminus X$ which shows that neither $X \ll Y$ nor $Y \ll X$ can be satisfied. On the other hand, if neither $X \ll Y$ nor $Y \ll X$ are satisfied, we denote $x = \min X \setminus Y$, $x' = \max X \setminus Y$, $y = \min Y \setminus X$, $y' = \max Y \setminus X$. Then either y or y' must be between x and x'.



In the first case (on the picture) we have $x < y < x'$ with $y \in Y \setminus X$ and $x, x' \in X \setminus Y$ and in the second we have $x < y' < x'$. In such a case X and Y cannot be separated. $\square$

Proposition 4.2.6. *Relation $\ll$ on subsets of $[n]$ is transitive.*

Proof. Let $I, J, K \subseteq [n]$ with $I \ll J$ and $J \ll K$. Without loss of generality we may assume that $I \cap J \cap K = \emptyset$.

Let us define

$$I^0 = (I \setminus J) \setminus K, \quad J^0 = (J \setminus I) \setminus K, \quad K^0 = (K \setminus I) \setminus J.$$

These sets have pairwise empty intersections. Then obviously $I^0 < J^0 < K^0 \subseteq K \setminus I$. Also we have $I \cap J < I \cap K$ since $I \cap J \subseteq J \setminus K$ and $I \cap K \subseteq K \setminus J$. Moreover, $I \cap K < J \cap K$ since $I \cap K \subseteq I \setminus J$ and $J \cap K \subseteq J \setminus I$. Hence $I \cap J < J \cap K \subseteq K \setminus I$.

But then $I \setminus K = I^0 \cup (I \cap J) < K \setminus I$ so $I \ll K$. □

We need some more properties of separated systems of sets

Proposition 4.2.7. *Let $X, Y \subseteq [n-1]$ be two sets such that X and $Y \cup \{n\}$ are separated. Then $X \ll Y$ or $Y \subseteq X$ and, in particular, X and Y are separated.*

Proof. Since $n \notin X$, we have $X \setminus Y = X \setminus (Y \cup \{n\}) < (Y \cup \{n\}) \setminus X$. If $Y \setminus X \neq \emptyset$, then for every $y \in Y \setminus X$ and $X \setminus Y < \{y\}$, hence $X \setminus Y < Y \setminus X$, thus $X \ll Y$. If $Y \setminus X = \emptyset$, then $Y \subseteq X$. □

Lemma 4.2.8. *Let $S \subseteq 2^{[n]}$ be a collection of pairwise separated sets and $X, Y \subseteq [n-1]$. Suppose that four sets $X, Y, X \cup \{n\}, Y \cup \{n\}$ belong to S. Then $X \subseteq Y$ or $Y \subseteq X$.*

Proof. If neither $X \subseteq Y$ nor $Y \subseteq X$ is true, then by Proposition 4.2.7 we have both $X \ll Y$ and $Y \ll X$, a contradiction. □

Lemma 4.2.9. *Let $S \subseteq 2^{[n]}$ be a collection of pairwise separated sets. Let us define*

$$S' = \{X \subseteq [n-1] \mid X \in S\}, \qquad S'' = \{X \subseteq [n-1] \mid X \cup \{n\} \in S\}. \tag{4.2.1}$$

Then $\overline{S} = S' \cup S''$ is again a collection of pairwise separated sets in $2^{[n-1]}$ such that for any $X \in S'$ and $Y \in S''$ we have $X \ll Y$ or $Y \subseteq X$. Moreover, $S' \cap S''$ is a chain of subsets relative to the set inclusion.

Proof. Let $X \in S'$ and $Y \in S''$ for some $X, Y \subseteq [n-1]$. By Proposition 4.2.7 we know that X and Y are separated with $X \ll Y$ or $Y \subseteq X$, in which case they are also separated. Also, $S' \cap S''$ is a chain of subsets relative to set inclusion due to Lemma 4.2.8. □

Example 4.2.1. *Let us consider the following graded set $\mathcal{C} = \bigcup_{i=0}^{5} \mathcal{C}_i$ of pairwise separated subsets of $[5]$:*

$$\begin{aligned}
\mathcal{C}_0 &= \{\emptyset\},\\
\mathcal{C}_1 &= \{\{1\}, \{2\}, \{4\}, \{5\}\},\\
\mathcal{C}_2 &= \{\{1,2\}, \{2,4\}, \{4,5\}\},\\
\mathcal{C}_3 &= \{\{1,2,3\}, \{1,2,4\}, \{2,3,4\}, \{2,4,5\}, \{3,4,5\}\},\\
\mathcal{C}_4 &= \{\{1,2,3,4\}, \{2,3,4,5\}\},\\
\mathcal{C}_5 &= \{\{1,2,3,4,5\}\}.
\end{aligned}$$

Then $\mathcal{C}'$ and $\mathcal{C}''$ will be, respectively,

$$\begin{aligned}
&\mathcal{C}'_0 = \{\emptyset\}, && \mathcal{C}''_0 = \{\emptyset\},\\
&\mathcal{C}'_1 = \{\{1\},\{2\},\{4\}\}, && \mathcal{C}''_1 = \{\{4\}\},\\
&\mathcal{C}'_2 = \{\{1,2\},\{2,4\}\}, && \mathcal{C}''_2 = \{\{2,4\},\{3,4\}\},\\
&\mathcal{C}'_3 = \{\{1,2,3\},\{1,2,4\},\{2,3,4\}\}, && \mathcal{C}''_3 = \{\{2,3,4\}\},\\
&\mathcal{C}'_4 = \{\{1,2,3,4\}\}. && \mathcal{C}''_4 = \{\{1,2,3,4\}\}.
\end{aligned}$$

Note that it is not true that $\{1,2,4\} \ll \{2,4\}$ despite $\{1,2,4\} \in \mathcal{C}' \setminus \mathcal{C}''$ and $\{2,4\} \in \mathcal{C}''$. Also it is not true that $\{2,3,4\} \ll \{3,4\}$ despite $\{2,3,4\} \in \mathcal{C}'$ and $\{3,4\} \in \mathcal{C}'' \setminus \mathcal{C}'$. We note that there is a flag of subsets

$$\emptyset \subset \{4\} \subset \{2,4\} \subset \{2,3,4\} \subset \{1,2,3,4\}$$

common to both sets which is exactly the intersection $\mathcal{C}' \cap \mathcal{C}''$. Our goal will be to show that such a flag always exists.

4.3 Arrangements of pseudolines and reduced decompositions

A *pseudoline* is the abscissa monotone graph $\{(x, f(x)) \mid x \in \mathbb{R}\}$ of a continuous function $f\colon \mathbb{R} \to \mathbb{R}$. Each pseudoline has an orientation (from left to right).

An *arrangement of pseudolines* is a family of pseudolines with the property that each pair of pseudolines has a unique point of intersection. An arrangement is *simple* if no two distinct pairs of pseudolines intersect at the same point. In this manuscript the term *arrangement* will always denote a simple arrangement of pseudolines[1]. The *size* of an arrangement is the number of its pseudolines. Given an arrangement $\mathbf{a}$ of size n we label the pseudolines $L_1, \ldots, L_n$ with elements of $[n]$ so that they cross a vertical line to the left of all intersections in increasing order from top to bottom and write $\mathbf{a} = \{L_1, \ldots, L_n\}$. This vertical line will be denoted as L. Then the vertical line R to the right of all intersections will be crossed in the reverse order (as every two lines must cross exactly once), that is in increasing order from bottom to top. The points of intersection are called *vertices*. The strip between the two lines L and R will be called *LR-strip* and denoted by E. It is endowed with the topology induced from $\mathbb{R}^2$. From now on a pseudoline is a piece of the graph of a continuous function connecting a point on L with a point on R.

An arrangement $\mathbf{a} = \{L_1, \ldots, L_n\}$ can be viewed as a cell complex $\mathcal{C}(\mathbf{a})$ with cells of dimensions 0, 1, or 2, namely, the vertices, edges, and faces of the arrangement. The vertices are $L_i \cap L_j$ for $i \neq j$, the edges are connected components of $L_i \setminus \bigcup_{j \neq i} L_j$, $i = 1, 2, \ldots, n$, and the faces are connected components of $E \setminus \bigcup_{i=1}^n L_i$. Thus $\mathcal{C}(\mathbf{a}) = \bigcup_{i=0}^2 \mathcal{C}_i(\mathbf{a})$, where $\mathcal{C}_i(\mathbf{a})$ is the set of cells of dimension i. The faces of this complex belonging to $\mathcal{C}_2(\mathbf{a})$ are usually called *chambers*.

[1]more precisely simple labeled arrangement of pseudolines

The chambers are labeled by subsets of $[n]$, namely, a chamber C gets label $\Lambda(C) = \{i_1, \dots, i_k\}$ if lines $L_{i_1}, \dots, L_{i_k}$ and only they go above this chamber. These labels will be also called *chamber sets*. Instead of writing labels as sets we will write them as strings, e.g., the aforementioned label will be written as $i_1 i_2 \dots i_k$.

Two arrangements are considered to be *isomorphic* if their cell complexes are isomorphic under the correspondence induced by some labelling. More formally, arrangements $\mathbf{a} = \{L_1, \dots, L_n\}$ and $\mathbf{b} = \{M_1, \dots, M_n\}$ are isomorphic if there is a bijection $\sigma\colon [n] \to [n]$ under which the chamber sets of $\mathbf{a}$ becoming the chamber sets of $\mathbf{b}$. For example, the two arrangements on Figure 4.1 cannot be isomorphic since we have three cells with labels of cardinality 1 on the left and only two of these on the right.

Recall that a pseudoline L'_i is *homotopic* to pseudoline L_i if L'_i can be continuously deformed into L_i. More precisely, if $f\colon [a,b] \to L_i$ and $g\colon [a,b] \to L'_i$ be two parameterisations of L_i and L'_i, respectively, then there exists a continuous function $F\colon [a,b] \times [0,1] \to \mathbb{R}^2$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$. We will say that L_i can be *moved* to become L'_i. Following Ringel [1956] such move will be called a 0-*move* if no vertex of the arrangement is crossed during this move and 1-*move* if one vertex is crossed. Example of 1-move of line L_2 is shown on Figure 4.1. If we transform the configuration on the left into the configuration on the right, we say that we are doing the *lowering triangular flip*. If we are doing the reverse transformation, it is the *rising triangular flip*.

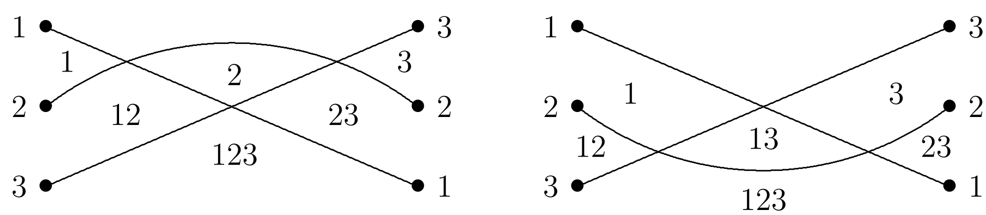


Figure 4.1: Lowering triangle flip

Obviously, after a sequence of 0-moves of one or several lines the arrangement remains isomorphic to the original one but every 1-move changes its combinatorial structure. On Figure 4.1 line L_2 being continuously deformed from its position on the left to its position on the right must pass through the intersection of Lines 1 and 3.

Finally, an arrangement is called *stretchable* if it is isomorphic to an arrangement of straight lines. Not all arrangements of pseudolines are stretchable [Goodman and Pollack, 1980] but the next best thing to stretchability is the so-called *wiring diagrams* in which pseudolines are horizontal except the neighbourhood of each crossing. Any arrangement of pseudolines is isomorphic to a wiring diagram [Goodman, 1980]. An example of wiring diagram can be seen on Figure 4.2. We note that the vertices in a wiring diagram are situated on $n-1$ levels, we count these levels in downward order.

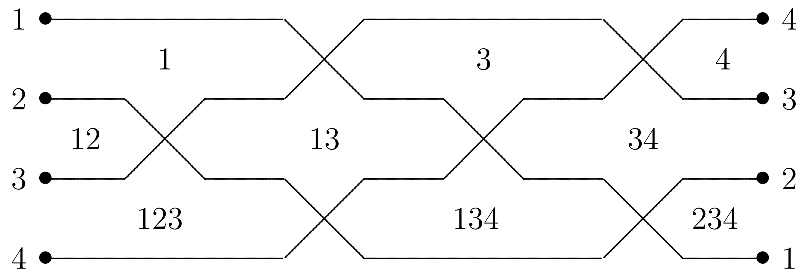


Figure 4.2: Example of a wiring diagram. As we will see later it is intimately related to Fishburn's domain F_4.

Reduced decompositions of $\bar{e}$, introduced in Subsection 1.4.4, are closely related to arrangements of pseudolines. Given a reduced decomposition $r = (i_1, \ldots, i_p)$

$$\bar{e} = s_{i_1} \cdots s_{i_p}, \tag{4.3.1}$$

of $\bar{e}$ in S_n we start with n parallel lines emanating from points $1, 2, \ldots, n$ on L. Firstly, we implement s_{i_1} by crossing line $L_{i_1 - 1}$ and line L_{i_1} and continue with the set of parallel lines where line $L_{i_1 - 1}$ and line L_{i_1} are interchanged. We claim that after implementation of all transpositions $s_{i_1}, \ldots, s_{i_p}$ our lines will be in the reversing order. Since permutation $\bar{e}$ acting on $12 \ldots n$ converts it into $n \ldots 21$, the sequence of lines (counted from top to bottom) $(L_1, L_2, \ldots, L_n)$ will meet R the sequence $(L_n, \ldots, L_2, L_1)$. This way we obtain an arrangement of pseudolines denoted $\mathbf{a}(r)$.

On the other hand, given an arrangement of n pseudolines $\mathbf{a}$ (assume without loss of generality that no vertical line passes through more than one vertex of $\mathbf{a}$ which can be achieved with 0-moves) we can do the following. When a vertical straight line moves from left to right (this is called *sweeping*) it meets vertices at levels $i_1, \ldots, i_p$, respectively, which leads to the reduced decomposition $r = (i_1, \ldots, i_p)$ of

$$\bar{e} = s_{i_1} \ldots s_{i_p},$$

where $p = \frac{1}{2}n(n-1)$. We denote it $r(\mathbf{a})$.

For example, sweeping the arrangement on Figure 4.2 the vertical line meets vertices at levels $1, 3, 2, 3, 1, 2$, giving us reduced decomposition $(1, 3, 2, 3, 1, 2)$. On the move this line meets wires in orders

$$1234, 2143, 2413, 2431, 4231, 4321, \tag{4.3.2}$$

This is equivalent to the representation of the reversing permutation $\bar{e}$ as

$$\bar{e} = (1, 4)(2, 3) = s_1 s_3 s_2 s_3 s_1 s_2.$$

Theorem 4.3.1. *(Ringel 1956,1957) Every pair of simple arrangements of pseudolines from $\mathcal{A}_n(\bar{e})$ can be converted into one another by a sequence of* 0*-moves and* 1*-moves.*

Proof. Let **a** and **b** be two arrangements of pseudolines and $r(\mathbf{a})$ and $r(\mathbf{b})$ be their corresponding reduced decompositions. By Theorem 1.4.13 $r(\mathbf{a})$ can be converted into $r(\mathbf{b})$ by a sequence of Coxeter transformations using short and long braid relations. If we switch s_i with s_j with $|j-i| \geq 2$ in $r(\mathbf{a})$ towards $r(\mathbf{b})$, then we change in **a** the order of two neighbouring crossings at levels i and j which are not next to each other. But they involve two pairs of distinct pseudolines and this change can be easily accomplished by 0-moves.

The replacement of $s_i s_{i+1} s_i$ with $s_{i+1} s_i s_{i+1}$ can be achieved by a lowering flip while the converse can be done by a rising flip (See Figure 4.3 (a) and (b)). □

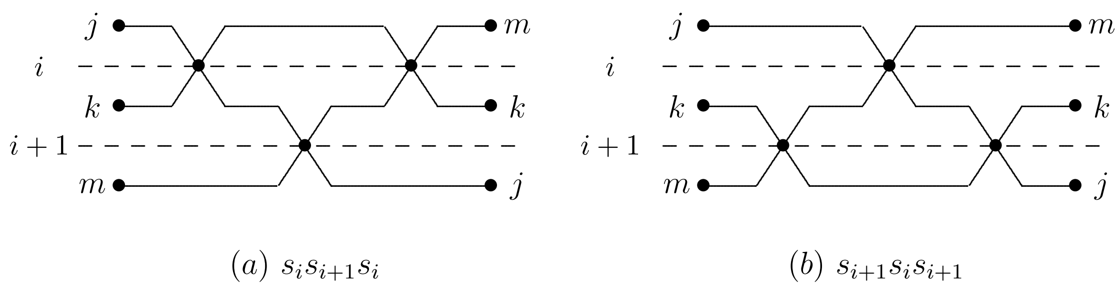


Figure 4.3: 1-moves.

Definition 4.3.1. *Two reduced decompositions of $\bar{e} \in S_n$ are* equivalent *if they can be transformed one into the other using only the short braid relation.*

The following theorem is well-known with most proofs being geometric. However, we give a different proof based on group theory.

Theorem 4.3.2. *Let* **a** *and* **b** *be two arrangements with reduced decompositions* $r(\mathbf{a})$ *and* $r(\mathbf{b})$*, respectively. Then* **a** *and* **b** *are isomorphic if and only if* $r(\mathbf{a})$ *and* $r(\mathbf{b})$ *are equivalent.*

Proof. It is easy to see that if $r(\mathbf{a})$ and $r(\mathbf{b})$ are equivalent, then **a** and **b** are isomorphic by converting **a** to **b** by 0-moves only which do not change the combinatorial structure of our complexes. Indeed, if we switch s_i with s_j with $|j-i| \geq 2$ in $r(\mathbf{a})$ towards $r(\mathbf{b})$, then we change in **a** the order of two neighbouring crossings at levels i and j which are not next to each other. They involve two pairs of four distinct pseudolines and this change is easy to get distorting one of the pairs homotopically. The proof finishes by induction on the number of usage of the short braid relation in transition from $r(\mathbf{a})$ to $r(\mathbf{b})$.

To prove the converse we assume that **a** and **b** are isomorphic and suppose $r(\mathbf{a})$ and $r(\mathbf{b})$ are not equivalent. Then by Theorem 1.4.13 reduced decomposition $r(\mathbf{a})$ can be transformed into $r(\mathbf{b})$ using both short and long braid relations. However, the long braid relation changes the combinatorial structure of the cell complex $\mathcal{C}_2(\mathbf{a})$. We see that when we replace $s_i s_{i+1} s_i$ with $s_{i+1} s_i s_{i+1}$ chamber $X \cup \{k\}$ disappears and chamber $X \cup \{j, m\}$ appears (see Figures 4.3(a) and 4.3(b)). This new arrangement cannot be isomorphic to the old one. □

4.4 Condorcet domains of arrangements of pseudolines

Proposition 4.4.1. *For any arrangement of pseudolines* $\mathbf{a} \in \mathcal{A}_n$ *the collection of chamber sets* $C_2(\mathbf{a})$ *is an ideal and* $Dom(C_2(\mathbf{a}))$ *is a copious peak-pit Condorcet domain of maximal width.*

Proof. For any chamber set C which is not $\emptyset$ or $[n]$ there are two pseudolines, one below and one above, say i and j, respectively, that form parts of the border of C. Hence both $C \setminus \{i\}$ and $C \cup \{j\}$ that lie across those borders belong to $C_2(\mathbf{a})$, hence conditions (I1) and (I2) are satisfied and $C_2(\mathbf{a})$ is an ideal.

We see that in the restriction of $C_2(\mathbf{a})$ onto $\{i, j, k\}$ can be of two different kinds, they are shown on Figures 4.3(a) and 4.3(b). We have either linear orders $\{ijk, jik, jki, kji\}$ or $\{ijk, ikj, kij, kji\}$ which satisfy $jN_{\{i,j,k\}}3$ and $jN_{\{i,j,k\}}1$, respectively. Thus, $\mathrm{Dom}(C_2(\mathbf{a}))$ is a copious peak-pit Condorcet domain.

This domain will always have maximal width as chamber sets

$$\{1\}, \{1, 2\}, \ldots, \{1, 2, \ldots, n-1\}, [n],$$

and

$$\{n\}, \{n, n-1\}, \ldots, \{n, n-1, \ldots, 2\}, [n]$$

belong to $\mathcal{D}(\mathbf{a})$. □

Condorcet domain $\mathrm{Dom}(C_2(\mathbf{a}))$ will be denoted $\mathcal{D}(\mathbf{a})$ and called *domain of arrangement of pseudolines* $\mathbf{a}$.

Proposition 4.4.2. *Let* $\mathcal{D} = \mathcal{D}(\mathbf{a})$ *be the Condorcet domain of arrangement of pseudolines* $\mathbf{a} \in \mathcal{A}_n$. *Then*

$$|Id(\mathcal{D}(\mathbf{a}))| = \binom{n+1}{2} + 1.$$

Proof. Let us count chambers that a vertical line crosses moving from L to R in the LR-strip. Initially it crosses $n+1$ chamber. Since any two pseudolines cross exactly once, we have $\binom{n+1}{2}$ vertices in the arrangement. Each of them adds an additional chamber set. Overall we have

$$n + 1 + \binom{n}{2} = \binom{n+1}{2} + 1$$

chambers. □

Proposition 4.4.3. *Let* $\mathbf{a}$ *be an arrangement of* n *pseudolines. Then Condorcet domain* $\mathcal{D}(\mathbf{a})$ *is semi-connected.*

Proof. Suppose a vertical line crosses wires in the order $i_1, \ldots, i_n$. This means that in $\mathcal{C}_2(\mathbf{a})$ there is a flag

$$\emptyset \subset \{i_1\} \subset \cdots \subset \{i_1, \ldots, i_k\} \subset \cdots \subset [n]$$

and the order $i_1 i_2 \ldots i_n$ exists in $\mathcal{D}(\mathbf{a})$. Sweeping the arrangement $\mathbf{a}$ from left to right the vertical line originally meets wires in orders $1, 2, \ldots, n$ and at the end in orders

$n, n-1, \dots, 1$. After crossing a vertex of the arrangement, the order of two neighbouring alternatives changes. Thus e can be transformed into $\bar{e}$ by a sequence of swaps of neighbouring alternatives. This proves the statement. □

As we will see later Condorcet domain $\mathcal{D}(\mathbf{a})$ is maximal for any arrangement of pseudolines $\mathbf{a}$.

Proposition 4.4.4. *Let* $\mathbf{a}$ *be an arrangement of pseudolines,* X *and* Y *be chamber sets such that* $X \subseteq Y$ *with* $|X| = k$ *and* $|Y| = m$, $k < m$. *Then there exists a flag of chamber sets*

$$\emptyset = A_0 \subset A_1 \subset \cdots \subset A_{n-1} = [n-1]$$

such that $X = A_k$ *and* $Y = A_m$.

Proof. Let $Y \setminus X = \{i_1, \dots, i_k\}$. This means there are exactly k pseudolines $L_{i_1}, \dots, L_{i_k}$ that go above Y and below X. This means that X borders with at least one chamber set $X \cup \{i_1\}, \dots, X \cup \{i_k\}$. Induction on $k = |Y \setminus X|$ completes the proof. □

4.5 Arrangements of pseudolines and separated systems of sets

Lemma 4.5.1. *Let* X *and* Y *be two chamber sets of the same cardinality of an arrangement of pseudolines* $\mathbf{a} \in \mathcal{A}_n$. *Then* $X \ll Y$ *if and only if* X *is to the left of* Y.

Proof. By Proposition 4.2.6 it is sufficient to prove this statement in case X and Y are neighbours. Then they have a common vertex which is a crossing of some lines L_i and L_j with $i < j$. In such a case L_i goes above X but below Y and L_j goes below X but above Y, hence $Y = (X \setminus \{i\}) \cup \{j\}$. As

$$X \setminus Y = \{i\} < \{j\} = Y \setminus X,$$

the statement is clear. □

Lemma 4.5.2. *For any arrangement of* n *pseudolines* $\mathbf{a}$, *the collection of chamber sets* $\mathcal{C} = \mathcal{C}_2(\mathbf{a})$ *form an ideal of separated subsets of* $2^{[n]}$. *Let*

$$\mathcal{C}' = \{X \subseteq [n-1] \mid X \in \mathcal{C}\}, \qquad \mathcal{C}'' = \{X \subseteq [n-1] \mid X \cup \{n\} \in \mathcal{C}\}.$$

Then $\mathcal{C}' \cup \mathcal{C}'' = \mathcal{C}(\mathbf{a_{-n}})$, *the arrangement of pseudolines* $\mathbf{a_{-n}}$ *formed by the first* $n-1$ *lines of arrangement* $\mathbf{a}$. *Moreover, the collection of chamber sets* $\mathcal{C}' \cap \mathcal{C}''$ *is a flag*

$$\emptyset = A_0 \subset A_1 \subset \cdots \subset A_{n-1} = [n-1] \tag{4.5.1}$$

with

$$\mathcal{C}' = \{X \in \mathcal{C}(\mathbf{a_{-n}}) \mid X \ll A_{|X|} \text{ or } X = A_{|X|}\},$$
$$\mathcal{C}'' = \{Y \in \mathcal{C}(\mathbf{a_{-n}}) \mid Y = A_{|Y|} \text{ or } A_{|Y|} \ll Y\}.$$

Proof. By Proposition 4.4.1 and Theorem 4.2.4 $\mathcal{C}$ is an ideal and a separated system of sets.

Let us consider the arrangement $\mathcal{C}(\mathbf{a_{-n}})$ formed by the first $n-1$ lines. Then line L_n cuts all the way from the bottom to the top through a flag of subsets (4.5.1). If we consider the chamber set A_k from this flag of cardinality k, then the chamber sets of $\mathcal{C}(\mathbf{a_{-n}})$ of cardinality k will be of two kinds. Those to the left of A_k will remain as they were—they will belong to $\mathcal{C}'$—and those to the right of A_k will get an additional symbol n—these are those from $\mathcal{C}''$. Thus the last statement of this lemma will follow from Lemma 4.5.1. □

The following lemma is in a sense the converse of Lemma 4.5.2.

Lemma 4.5.3. *Let* $\mathbf{a}$ *be an arrangement of* n *pseudolines and* (4.5.1) *be a flag of chamber sets. We define*

$$\mathcal{C}' = \{X \in \mathcal{C}(\mathbf{a}) \mid X \ll A_{|X|} \text{ or } X = A_{|X|}\}, \qquad \mathcal{C}'' = \{Y \in \mathcal{C}(\mathbf{a}) \mid Y = A_{|Y|} \text{ or } A_{|Y|} \ll Y\}.$$

Then $\mathcal{C} := \mathcal{C}' \cup \{Y \cup \{n+1\} \mid Y \in \mathcal{C}''\}$ *is the set of chamber sets of an arrangement of* $n+1$ *pseudolines* $\mathbf{a}'$*.*

Proof. We draw line L_{n+1} through chamber sets of the flag (4.5.1). Indeed, A_k has with A_{k-1} a common segment so you can start line L_{n+1} in chamber set $12\ldots n$ and finish in chamber set $\emptyset$. Then all chamber sets of $\mathcal{C}''$ will acquire an additional $n+1$. □

Theorem 4.5.4. *Any separated collection of subsets* S *of* $[n]$ *can be embedded into the set of chambers* $\mathcal{C}_2(\mathbf{a})$ *for some arrangement of* n *pseudolines* $\mathbf{a}$*.*

Proof. Without loss of generality we may assume that S is a maximal separated collection of sets. Let $n = 3$. Then it is easy to see that there are only two maximal pairwise separated collections of subsets

$$S_1 = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,2,3\}\},$$

$$S_2 = \{\emptyset, \{1\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

as you can include either $\{2\}$ or $\{1,3\}$ but not both. Both S_1 and S_2 can be represented by arrangements of pseudolines pictured on Figure 4.1. This gives us a basis for the induction.

Let $n \geq 4$ and consider $\overline{S} = S' \cup S'' \subseteq 2^{[n-1]}$ as defined in (4.2.1). Lemma 4.2.9 asserts that $\overline{S}$ is again a collection of pairwise separated sets and, hence, by the induction hypothesis, can be embedded into the collection of chamber sets $\mathcal{C} = \mathcal{C}_2(\mathbf{a}')$ of an arrangement $\mathbf{a}'$ of $n-1$ pseudolines.

We need to check that $S' \cap S''$ is a flag in $\overline{S}$ or that S can be enlarged remaining separated so that $S' \cap S''$ is a flag in $\overline{S}$. Suppose $S' \cap S''$ does not have a set in some cardinalities. Let A_k and A_m be two closest chamber sets in $S' \cap S''$ of cardinalities k and m with $m - k \geq 2$. By Proposition 4.4.4 we have a tower of chamber sets of $\mathbf{a}'$

$$A = A_k \subset A_{k+1} \subset \cdots \subset A_m = B.$$

Being a chamber sets of $\mathbf{a}'$, all of these are separated from the chamber sets in S'. This means they—and A_{m-1} in particular—are not separated from a certain set $Y \cup \{n\}$ for $Y \in S''$ while being separated from Y. Let $A_{m-1} = B \setminus \{x\}$. This implies that $(B \setminus \{x\}) \setminus Y < Y \setminus (B \setminus \{x\})$ and hence

$$(B \setminus \{x\}) \setminus (Y \cup \{n\}) = (B \setminus \{x\}) \setminus Y < Y \setminus (B \setminus \{x\}) < (Y \cup \{n\}) \setminus (B \setminus \{x\}).$$

This means $A_{m-1} \in S' \cap S''$ which contradicts to A_k and A_m being the closest. The statement now follows from Lemma 4.5.3. □

Corollary 4.5.5. *Any maximal system of separated sets S in $[n]$ is an ideal of cardinality $\binom{n+1}{2} + 1$.*

Proof. Due to Theorem 4.5.4 $S = \mathcal{C}_2(\mathbf{a})$ for some arrangement of pseudolines $\mathbf{a}$ and the result follows from Propositions 4.4.1 and 4.4.2. □

Corollary 4.5.6. *For any arrangement of pseudolines $\mathbf{a}$ the Condorcet domain $\mathcal{D}(\mathbf{a})$ is maximal.*

Proof. The cardinality of the ideal $\mathrm{Id}(\mathcal{D}(\mathbf{a})$ is $\binom{n+1}{2}+1$ which by Corollary 4.5.5 is maximal possible. If there existed a Condorcet domain $\mathcal{D}'$ strictly containing $\mathcal{D}(\mathbf{a})$, then its ideal would have larger cardinality which is not possible. □

Corollary 4.5.7. *Let $\mathcal{D} \subseteq \mathcal{L}([n])$ be a peak-pit maximal Condorcet domain of maximal width. Then the ideal Id($\mathcal{D}$) is a maximal system of separated subsets of $[n]$.*

Proof. By Theorem 4.2.4 $\mathrm{Id}(\mathcal{D})$ is a separated system of sets. If $\mathrm{Id}(\mathcal{D})$ is not maximal, it can be imbedded into a maximal separated system S of subsets of $[n]$ that strictly contains $\mathrm{Id}(\mathcal{D})$. By Corollary 4.5.5 S is an ideal. But then $\mathrm{Dom}(S)$ by Theorem 4.2.2 is a Condorcet domain strictly containing $\mathcal{D}$, a contradiction. □

Theorem 4.5.8. *Any peak-pit maximal domain $\mathcal{D} \subseteq \mathcal{L}([n])$ of maximal width is directly connected.*

Proof. Due to Theorem 1.3.16 we need to prove semi-connectedness. We know that $\mathcal{D} = \mathrm{Dom}(C(\mathbf{a}))$ for some arrangement of n pseudolines $\mathbf{a}$. Let $(i_1, \ldots, i_p)$ where $p = \frac{1}{2}n(n-1)$ be any reduced decomposition from the equivalence class of reduced decompositions corresponding to $\mathbf{a}$. Then as we know

$$\bar{e} = e s_{i_1} \ldots s_{i_p},$$

This means that e can be transformed into $\bar{e}$ by performing transpositions $s_{i_1}, \ldots, s_{i_p}$ which means that $\mathcal{D}$ is semi-connected. □

The condition of maximality in Theorem 4.5.8 cannot be dropped as we can see from the following example.

Example 4.5.1. *Let $A = \{a, b, c, d\}$ and*

$$\mathcal{N} = \{bN_{\{a,b,c\}}1,\ bN_{\{a,b,d\}}3,\ cN_{\{a,c,d\}}1,\ cN_{\{b,c,d\}}3\}.$$

It is easily verified that $\mathcal{D}(\mathcal{N}) = \{abcd, dcba\}$, which is a peak-pit domain with maximal width; evidently, it is not connected.

4.6 Two main theorems on peak-pit domains

4.6.1 Representation by arrangements of pseudolines

We arrive at the main theorem of this section.

Theorem 4.6.1 (Main Representation Theorem)**.** *A domain $\mathcal{D} \subseteq \mathcal{L}([n])$ is a peak-pit maximal Condorcet domain of maximal width if and only if $\mathcal{D} = \mathcal{D}(\mathbf{a})$ for an arrangement of pseudolines $\mathbf{a} \in \mathcal{A}_n$.*

Proof. Let $\mathcal{D} \subseteq \mathcal{L}([n])$ be a peak-pit maximal Condorcet domain of maximal width. By Corollary 4.5.7 $\mathrm{Id}(\mathcal{D})$ is a maximal system of separated subsets of $[n]$ and by Theorem 4.5.4 $\mathrm{Id}(\mathcal{D}) = \mathcal{C}_2(\mathbf{a})$ for some arrangement of pseudolines $\mathbf{a} \in \mathcal{A}_n$. By Corollary 4.2.3

$$\mathcal{D} = \mathrm{Dom}(\mathrm{Id}(\mathcal{D})) = \mathrm{Dom}(\mathcal{C}_2(\mathbf{a})) = \mathcal{D}(\mathbf{a}). \qquad \square$$

The converse follows from Corollary 4.5.6 and Proposition 4.4.1.

We can derive now the following important conclusion.

Corollary 4.6.2. *Let $\mathcal{D}$ be a maximal peak-pit Condorcet domain on n alternatives of maximal width. Then*

$$|Id(\mathcal{D})| = \binom{n+1}{2} + 1.$$

Proof. Due to Theorem 4.6.1 we have $\mathcal{D} = \mathcal{D}(\mathbf{a})$ for a certain arrangement of pseudolines $\mathbf{a}$. The result now follows from Corollary 4.5.5. $\square$

In Section 4.8.1, we will see that maximal peak-pit domains of maximal width can have quite different sizes. It is quite remarkable that the size of their ideals is constant. Let us continue extracting consequences from the main representation theorem.

Theorem 4.6.3. *Let $\mathcal{D} \subset \mathcal{L}([n])$ be a peak-pit maximal Condorcet domain of maximal width and $k \in [n]$. Then $\mathcal{D}_{-k}$ is also peak-pit maximal Condorcet domain of maximal width. In particular, $\mathcal{D}$ is copious.*

Proof. By Theorem 4.6.1 we may assume that $\mathcal{D} = \mathcal{D}(\mathbf{a})$ for some arrangement $\mathbf{a} \in \mathcal{A}_n$. Let $\mathbf{a}_{-k}$ be this arrangement with line L_k removed. But then $\mathcal{D}_{-k} = \mathcal{D}(\mathbf{a}_{-k})$ which by Theorem 4.6.1 is also maximal and peak-pit. $\square$

Definition 4.6.1. *Let $\mathcal{D} \subseteq \mathcal{L}([n])$ be a Condorcet domain. A linear order $h(\mathcal{D}) = a_1 \ldots a_{n-1} \in \mathcal{D}_{-n} \subseteq \mathcal{L}([n-1])$, is called* pivotal *for $\mathcal{D}$, if linear orders $u_1, \ldots, u_n$ exist in $\mathcal{D}$ such that*

$$u_\ell = a_1 \ldots a_{\ell-1} n a_\ell \ldots a_{n-1}, \qquad \ell = 1, \ldots, n. \tag{4.6.1}$$

Theorem 4.6.4. *Let $\mathcal{D} \subset \mathcal{L}([n])$ be a maximal pick-pit Condorcet domain. Then a pivotal linear order $h(\mathcal{D}) = a_1 \ldots a_{n-1} \in \mathcal{D}_{-n} \subseteq \mathcal{L}([n-1])$ exists and it is unique.*

Proof. By Theorem 4.6.1 we can view $\mathcal{D}$ as $\mathcal{D}(\mathbf{a})$ for some arrangement of pseudolines $\mathbf{a}$. Line L_n of this arrangement cuts through the flag of chamber sets of $\mathbf{a}_{-n}$

$$A_0 \subset A_1 \subset \cdots \subset A_{\ell-1} \subset A_\ell \subset \cdots \subset [n-1]$$

where $A_\ell = \{a_1, \ldots, a_\ell\}$. Then we will have the following flag in $\mathbf{a}_n$:

$$A_0 \subset A_1 \subset \cdots \subset A_{\ell-1} \subset A_{\ell-1} \cup \{n\} \subset A_\ell \cup \{n\} \subset \cdots \subset [n]$$

(we cross line L_n after visiting $A_{\ell-1}$). This implies the existence of orders $u_1, \ldots, u_n$ and such flag, and hence such orders, are unique by Lemma 4.5.2. □

Example 4.6.1. *The Fishburn domain*

$$F_4 = \{1234, 1324, 3124, 1342, 3142, 3412, 3421, 4312, 4321\}$$

is the domain obtained from the arrangement of pseudolines on Figure 4.2. We have linear orders 3124, 3142, 3412, 4312 *in the domain so the pivotal order* $h(F_4) = 312$. *Indeed, we note that in the arrangement on Figure 4.2 line* L_4 *cuts through the flag* $\emptyset \subset \{3\} \subset \{1,3\} \subset \{1,2,3\}$ *of* $(F_4)_{-4}$.

4.6.2 Representation by rhombus tilings

A *rhombus tiling* (or simply a tiling) is a subdivision into rhombic tiles of a regular $2n$-gon formed by the points $\sum_i a_i\psi_i$, where $0 \le a_i \le 1$ and $\psi_1, \ldots, \psi_n$ are the unit vectors in the upper half-plane. This centre-symmetric $2n$-gon has its bottom vertex b at the origin and the top vertex $t = \psi_1 + \ldots + \psi_n$. An ij-tile is a rhombus congruent to the one formed by the points $\lambda\psi_i + \mu\psi_j$, where $0 \le \lambda, \mu \le 1$. A *snake* is a path from t to b along the boundaries of the tiles which for each $i = 1, \ldots, n$ contains a unique segment parallel to ψ_i. Each snake corresponds to a linear order on $\{1, \ldots, n\}$ in the following way. If a point traveling from t to b passes segments parallel to $\psi_{i_1}, \psi_{i_2} \ldots, \psi_{i_n}$, then the corresponding linear order will be $i_1 i_2 \ldots i_n$. The set of snakes of a rhombus tiling, thus, defines a domain which is called a *tiling domain*.

For $n = 3$ we have a hexagon and it can be split into rhombus tiles in two different ways shown on a Figure 4.4.

These lead to the following domains:

$$\{123, 213, 231, 321\}, \quad \{123, 132, 312, 321\}$$

which are the familiar to us single-peaked and single-dipped domains, respectively.

Tilings and arrangements of pseudolines are dual objects and there is a one-to-one correspondence between them. Given an arrangement we mark a vertex in each chamber (including both unbounded ones). Two vertices are connected by an edge if their corresponding chambers have a common piece of pseudoline. The emerging graph can be shown to be transformed into a rhombus tiling [Elnitsky, 1997, Felsner, 2012]. For example, starting from an arrangement of pseudolines corresponding to $\bar{F}_4$ on Figure 4.5 (one

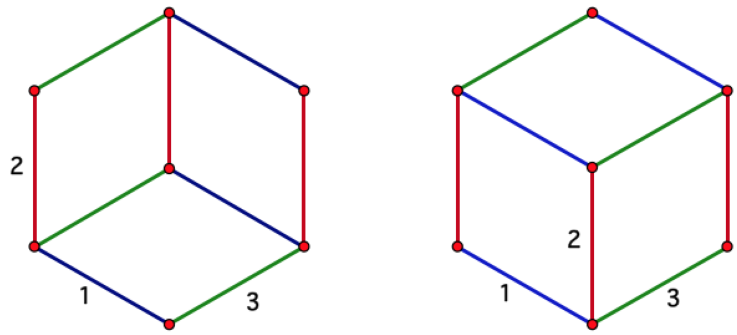


Figure 4.4: Two tiling domains

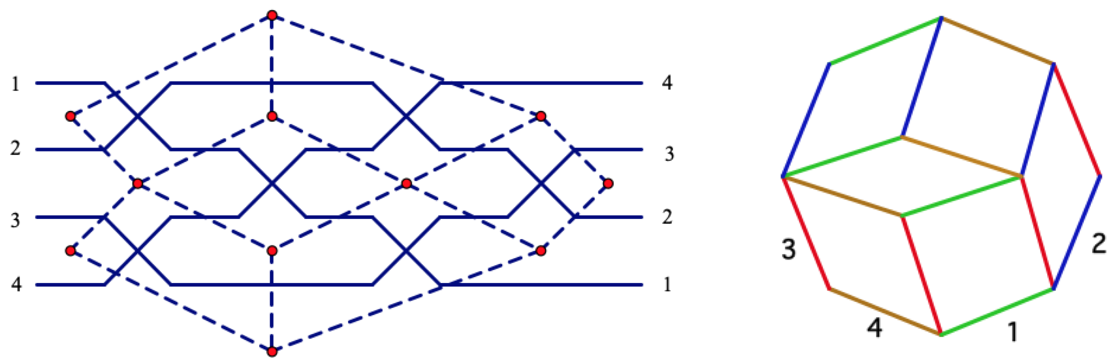


Figure 4.5: An arrangement of pseudolines and its dual tiling

of Fishburn's domains) we obtain a rhombus tiling for this domain. It is not hard to see that the snakes of a tiling domain correspond to flags in the corresponding arrangement.

Thus, as a consequence of Theorem 4.6.1 we obtain

Theorem 4.6.5 (Danilov et al. [2012])**.** *A maximal Condorcet domain of maximal width is peak-pit if and only if it is a tiling domain.*

4.7 Representation of Arrow's single-peaked domains

The question we ask here is whether or not we can have for this class of Condorcet domains a combinatorial representation similar to the representation of peak-pit domains of maximal width by arrangements of pseudolines. Firstly, we have to deal with the absence of maximal width. Indeed, the two extremal orders in an Arrow's single-peaked domain do not have to be completely reversed. So we have to assume that in a generalisation of an arrangement of pseudolines $\mathbf{g} = \{L_1, \ldots, L_n\}$ these lines are allowed to cross the left vertical line L in the order $i_1, \ldots, i_n$ and the right vertical line in the order $j_1, \ldots, j_n$. The resulting permutation will be

$$\mathcal{P}(\mathbf{g}) = \begin{pmatrix} i_1 & i_2 & \cdots & i_n \\ j_1 & j_2 & \cdots & j_n \end{pmatrix}.$$

This permutation can now be arbitrary.[2]

Let us try to represent the domain $\mathcal{D}_{4,5}$ (from Dittrich's classification presented in Chapter 9) in the following example.

Example 4.7.1. *Let us consider a maximal Arrow's single-peaked domain on four alternatives:*

$$\mathcal{D}_{4,5} = \{1234,\ 2134,\ 2314,\ 3214,\ 2341,\ 3241,\ 2431,\ 4231\},$$

introduced in Example 2.2.2 whose graph is presented on Figure 2.4. This domain is not single-peaked and does not have two completely reversed orders. The extremal orders of this domain are 1234 *and* 4231*. Thus we set* $(i_1, i_2, i_3, i_4) = (1, 2, 3, 4)$ *and* $(j_1, j_2, j_3, j_4) = (4, 2, 3, 1)$*. Drawing pseudolines in the classical way, we get the arrangement of pseudolines on Figure 4.6 which represents only the subdomain of* $\mathcal{D}_{4,5}$*, namely*

$$\{1234, 2134, 2314, 2341, 2431, 4231\}.$$

As we see the orders 3214 *and* 3241 *are missing.*

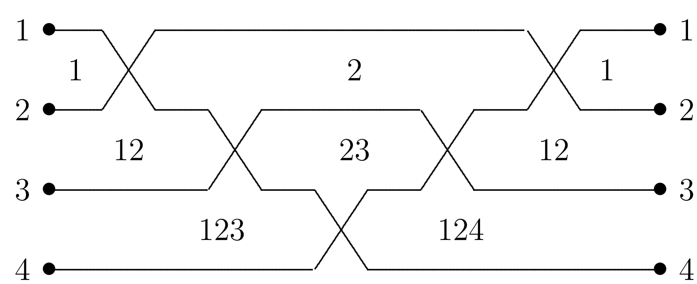


Figure 4.6: Representation of a subdomain of $\mathcal{D}_{4,5}$.

We need to do one trick to represent the whole $\mathcal{D}_{4,5}$*. Since we must have chambers* $\{3\}$ *and* $\{2, 3\}$ *the* $Line_3$ *must come to the top and intersect* $Line_2$*. However since the order of 2 and 3 on L and R coincide,* $Line_3$ *must intersect* $Line_2$ *twice. And the pseudoline representation would be as shown on Figure 4.7. It is easy to check that the flags of such arrangement correspond to the whole* $\mathcal{D}_{4,5}$ *as the missing orders now included.*

[2]In Li [2023] the definition of a generalised arrangement of pseudolines is narrower. Mistakes in this paper were outlined in Puppe and Slinko [2024a].

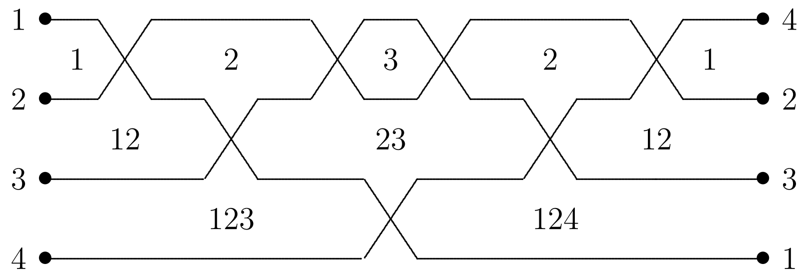


Figure 4.7: Representation of domain of $\mathcal{D}_{4,5}$.

We ask: under which conditions the set of flags of a set of pseudolines is a Condorcet domain?

As we did in Section 4.3 an arbitrary set of pseudolines $\mathbf{g} = \{L_1, \ldots, L_n\}$, which we will call a *generalised arrangement of pseudolines* can be viewed as a cell complex $\mathcal{C}(\mathbf{g})$ with cells of dimensions 0, 1, or 2, namely, the vertices, edges, and faces. The vertices are $L_i \cap L_j$ for $i \neq j$, the edges are connected components of $L_i \setminus \bigcup_{j \neq i} L_j$, $i = 1, 2, \ldots, n$, and the faces are connected components of $E \setminus \bigcup_{i=1}^{n} L_i$. Thus $\mathcal{C}(\mathbf{g}) = \bigcup_{i=0}^{2} \mathcal{C}_i(\mathbf{g})$, where $\mathcal{C}_i(\mathbf{g})$ is the set of cells of dimension i. The faces of this complex belonging to $\mathcal{C}_2(\mathbf{g})$ we will still call *chambers*.

The chambers are labeled by subsets of $[n]$, namely, a chamber C gets label $\Lambda(C) = \{i_1, \ldots, i_k\}$ if lines $L_{i_1}, \ldots, L_{i_k}$ and only they go above this chamber. These labels will be also called *chamber sets*. Instead of writing labels as sets we will write them as strings, e.g., the aforementioned label will be written as $i_1 i_2 \ldots i_k$.

For any set of pseudolines $\mathbf{g}$ the collection of chamber sets $C_2(\mathbf{g})$ is an ideal and $\mathcal{D}(\mathbf{g}) = \mathrm{Dom}(C_2(\mathbf{g}))$ is a domain of linear orders. We need to find out when $\mathcal{D}(\mathbf{g})$ is a Condorcet domain. Firstly, we note that not for every generalised arrangement $\mathbf{g}$ the domain $\mathcal{D}(\mathbf{g})$ is a Condorcet domain.

Example 4.7.2. *Let us consider the set* $\mathbf{g} = \{L_i, L_j, L_k\}$ *of three pseudolines on Figure 4.8. The corresponding domain*

$$\mathcal{D}(\mathbf{g}) = \{ijk, jik, kij, kji\}$$

is obviously not Condorcet.

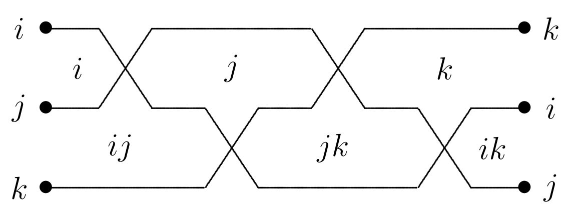


Figure 4.8: Non-Condorcet domain.

We note that Line_i and Line_j intersect twice at different levels and this is the culprit of the problem. We will try to turn this observation into a criterion.

Lemma 4.7.1. *Let* $\mathbf{g}$ *be a generalised arrangement of* n *pseudolines. Suppose there exist* $i, j \in [n]$ *such that* Line_i *and* Line_j *intersect twice at different levels. Then* $\mathcal{D}(\mathbf{g})$ *is not Condorcet.*

Proof. Let u and v be the two vertices at the intersections of Line_i and Line_j. Since these two vertices are at different levels - we may assume u is higher than v, - there must be a line Line_k which goes below u but above v. Let us restrict $\mathbf{g}$ to the subset of lines $\mathbf{g}' = \{L_i, L_j, L_k\}$. Then we obtain the configuration on Figure 4.9 or a similar configuration with i and j swapped.

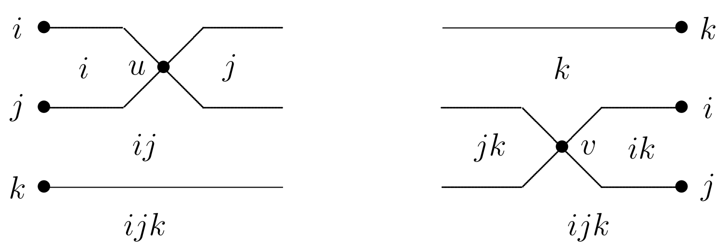


Figure 4.9: Line_i and Line_j intersect at two different levels.

Then $\mathcal{D}(\mathbf{g})$ contains orders ijk, jik, kij, kji which cannot occur in a Condorcet domain. If i and j are swapped on R, then this will not affect our reasoning as only the chambers ik and jk will be swapped. □

Definition 4.7.1. *We call the arrangement of pseudolines* tame *if for any two lines, if they intersect more than once, all intersections are on the same level.*

Theorem 4.7.2. *A generalised arrangement of* n *pseudolines* $\mathbf{g}$ *has domain* $\mathcal{D}(\mathbf{g})$ *Condorcet if and only if it is tame. In such a case* $\mathcal{D}(\mathbf{g})$ *is peak-pit.*

Proof. If $\mathbf{g}$ is not tame, by Lemma 4.7.1 $\mathcal{D}(\mathbf{g})$ is not Condorcet. Suppose it is tame. It is sufficient to consider the case when $\mathbf{g}$ has only three pseudolines.

Let us prove that it is enough to consider the case when any two pseudolines in $\mathbf{g}$ intersect at most twice. Suppose some two lines—which without loss of generality can be taken as Line_1 and Line_2—intersect three times. As $\mathbf{g}$ is tame, all three intersections occur at the same level, say level 1.

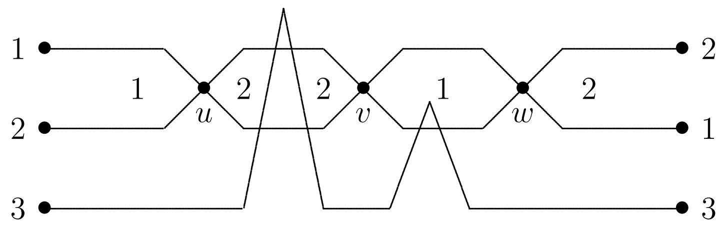


Figure 4.10: Line_1 and Line_2 intersect three times.

Let us show that two neighbouring intersections among u, v, w can be removed (straightened). We note that if Line_3 does not penetrate any of the two middle chambers labelled 1 and 2 between u and v or between v and w, then such chamber can be straightened. Also Line_3 must intersects both Line_1 and Line_2 somewhere (otherwise $\mathcal{D}(\mathbf{g})$ satisfies $3N_{\{1,2,3\}}1$) and return back before the next intersection of Line_1 and Line_2 (otherwise we have configuration shown at Figure 4.8).

For example, Line_3 intersect the chamber labelled by 2 between u and v as shown on Figure 4.10. Then we will have orders 123, 213, 321, 231. If only line 3 penetrates the chamber labelled 1 between v and w, as also shown on Figure 4.10, then we will immediately have order with 2 at the bottom and $\mathcal{D}(\mathbf{g})$ is not Condorcet. Thus, this chamber can be removed by removing the double crossing creating it. All other cases are similar.

Now let us show that, if no two lines in $\mathbf{g}$ intersect three times, the $\mathcal{D}(\mathbf{g})$ is peak-pit. If each pair of lines intersect once, this is the classical case and the result follows from Proposition 4.4.1.

Suppose now that a certain pair of pseudolines intersect twice. Up to an isomorphism and flip-isomorphism we may consider that these two lines are Line_1 and Line_2 which intersect at vertices u and v. If Line_3 does not intersect one of them, then $\mathbf{g}$ satisfies $3N_{\{1,2,3\}}1$. Line_3 can intersect Line_1 and Line_2 once before both u and v. We then have generalised arrangement $\mathbf{g}$ shown on Figure 4.11 with $\mathcal{D}(\mathbf{g})$ satisfying $2N_{\{1,2,3\}}1$.

The next possibility is for Line_3 to intersect both lines Line_1 and Line_2 twice before u and v. We then have the following generalised arrangement $\mathbf{g}$ shown on Figure 4.12 with $\mathcal{D}(\mathbf{g})$ satisfying $1N_{\{1,2,3\}}3$.

The last possibility is for Line_3 to intersect both lines Line_1 and Line_2 twice between u and v. We then have the following generalised arrangement $\mathbf{g}$ shown on Figure 4.13 with $\mathcal{D}(\mathbf{g})$ satisfying $2N_{\{1,2,3\}}3$. This proves the theorem.

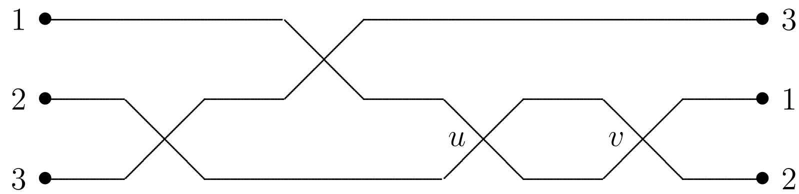


Figure 4.11: Line_1 and Line_2 intersect two times. Line_3 intersects both of them once.

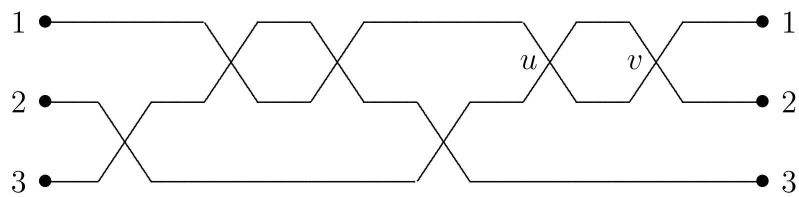


Figure 4.12: Line_1 and Line_2 intersect two times. Line_3 intersects both of them twice before u and v.

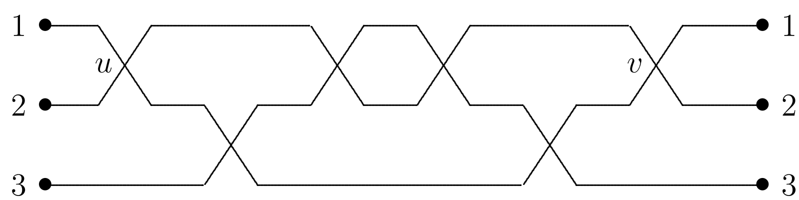


Figure 4.13: Line_1 and Line_2 intersect two times. Line_3 intersects both of them twice between u and v.

Theorem 4.7.3. *Any maximal Arrow's single-peaked Condorcet domain is representable as the domain* $\mathcal{D}(\mathbf{g})$ *associated with a generalised arrangement of pseudolines* $\mathbf{g}$.

Proof. Let $\mathcal{D} \subseteq \mathcal{L}([n])$ be an Arrow's single-peaked domain. We know that $\mathcal{D}$ has two terminal alternatives, which can be taken to be 1 and n, and two extremal linear orders, where terminal alternatives occupy the top and bottom positions.

We know that $\mathcal{D}_{-1}$ and $\mathcal{D}_{-n}$ which are the restrictions of $\mathcal{D}$ onto $[n-1]$ and $[n] \setminus \{1\}$, are two maximal Arrow's single-peaked domains on $n-1$ alternatives. Moreover $\mathcal{E} = (\mathcal{D}_{-1})_{-n} = (\mathcal{D}_{-n})_{-1}$ is a maximal Condorcet domain on $[n] \setminus \{1, n\}$ common for $\mathcal{D}_{-1}$ and

$\mathcal{D}_{-n}$. We note that to move from $\mathcal{D}_{-1}$ to $\mathcal{D}_{-n}$ we need only one switch of 1 and n.

$$\left[\begin{array}{c|c|c|c} 1 & & & n \\ & \mathcal{E} & \mathcal{E} & \\ \cline{2-3} & 1 \ \cdots \ 1 & n \ \cdots \ n & \\ \hline n \quad \cdots & n & 1 \quad \cdots & 1 \end{array}\right]. \qquad (4.7.1)$$

By the induction hypothesis each of $\mathcal{D}_{-1}$ and $\mathcal{D}_{-n}$ can be represented as $\mathcal{D}(\mathbf{g}_{-1})$ and $\mathcal{D}(\mathbf{g}_{-n})$, respectively, on sets of pseudolines $\mathbf{g}_{-1}$ and $\mathbf{g}_{-n}$, respectively. Now we create $\mathbf{g}$ representing $\mathcal{D}$ as shown on Figure 4.14.

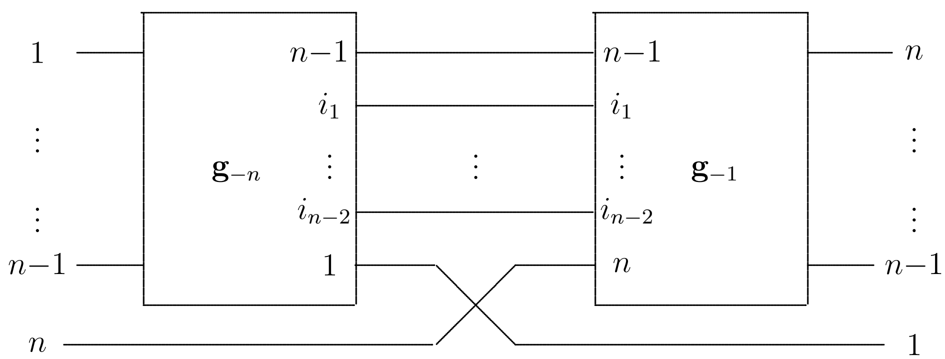


Figure 4.14: Method of combining $\mathbf{g}_{-1}$ and $\mathbf{g}_{-n}$.

Line$_n$ is added to $\mathbf{g}_{-n}$ trivially staying all the time at the bottom of the arrangement—this reflects n being the bottom alternative in (4.7.1). Similarly Line$_1$ is added to $\mathbf{g}_{-1}$. Then these lines swap before entering $\mathbf{g}_{-1}$ and $\mathbf{g}_{-n}$, respectively. □

4.8 Classification of small maximal peak-pit domains

4.8.1 Classification of small maximal peak-pit domains of maximal width

In this section we will demonstrate the power of the theory developed above by showing how to classify peak-pit maximal Condorcet domains of maximal width for $n = 4$ and $n = 5$. We will firstly formulate a few lemmas for the chamber sets of arrangements of pseudolines which, due to Theorem 4.5.4, are applicable to any maximal separated ideal.

Lemma 4.8.1. *In each row of chamber sets of an arrangement of pseudolines the chambers are situated in the lexicographically increasing order from left to right.*

Proof. Any two neighbouring chambers have a common vertex in common. At this vertex for some $i < j$ Line i and Line j cross and an inversion happens, i.e., Line i from being above Line j goes under Line j. Hence the chamber $X \cup \{i\}$ will have its right neighbour $X \cup \{j\}$ which is lexicographically greater. □

Lemma 4.8.2. *Let $C(\mathbf{a}) = \bigcup_{i=0}^{n} I_i$ be an ideal of an arrangements of pseudolines $\mathbf{a}$. If for some $X \in I_{k-1}$ the sets $X \cup \{i\} \in I_k$ and $X \cup \{j\} \in I_k$, then $X \cup \{i,j\} \in I_{k+1}$ if and only if $X \cup \{i\}$ and $X \cup \{j\}$ are neighbours in I_k.*

Proof. If $X \cup \{i\}$ and $X \cup \{j\}$ are neighbours, this means Line i and Line j cross between them making a vertex. Below this vertex we will have set $X \cup \{i,j\}$. Suppose now that $X \cup \{i\}$ and $X \cup \{j\}$ are not neighbours and $X \cup \{\ell\}$ is between them. Then by Lemma 4.8.1 we have $i < \ell < j$ and $X \cup \{i,j\}$ cannot be in I_{k+1} since it would not be separated from $X \cup \{\ell\}$. □

Our strategy for classifying peak-pit maximal domain of maximal width is as follows. We know by Theorems 4.5.2 and 4.6.2 that for such a domain $\mathcal{D} \subset \mathcal{L}([n])$ its ideal $\mathrm{Id}(\mathcal{D})$ is a separated system of sets of cardinality $\binom{n+1}{2} + 1$. By Corollary 4.5.7 such system is maximal. Hence we need to classify maximal systems of separated subsets of $[n]$ which is not too difficult to achieve for $n = 4$ and $n = 5$.

4.8.2 Four alternatives

As we discussed above, it is sufficient to classify maximal separated ideals in $2^{[4]}$. By Theorem 4.6.2 they have cardinality 11. By Proposition 1.5.1 such an ideal will always contain sets

$$\emptyset, \{1\}, \{1,2\}, \{1,2,3\}, \{1,2,3,4\}, \{2,3,4\}, \{3,4\}, \{4\}, \tag{4.8.1}$$

so only three other sets can be added to this list. Some choices may preclude some other ones. Let us list incompatible pairs of sets (omitting curly brackets around sets):

$$(2,13), (2,14), (2,134), (3,14), (3,24), (3,124),$$
$$(13,24), (23,14), (24,134), (23,134), (23,124).$$

We have the following choices:

- **The ideal has four singletons.** $I_1 = \{\{1\},\{2\},\{3\},\{4\}\}$. In this case by Lemma 4.8.2 we have to add $\{2,3\}$ in I_2 and after inclusion of it I becomes maximal. Thus, the ideal is

$$\begin{aligned} I_0 &= \{\emptyset\}, \\ I_1 &= \{\{1\},\{2\},\{3\},\{4\}\}, \\ I_2 &= \{\{1,2\},\{2,3\},\{3,4\}\}, \\ I_3 &= \{\{1,2,3\},\{2,3,4\}\}, \\ I_4 &= \{\{1,2,3,4\}\}. \end{aligned}$$

This is the ideal of a maximal single-peaked domain on four alternatives:

$$SP_4 = \{1234,\ 2134,\ 2314,\ 3214,\ 2341,\ 3241,\ 3421,\ 4321\}.$$

(see Section 2.1).

This is the only domain with four singletons in I_1.[3]

- **The ideal has three singletons.** Because of the isomorphism $\sigma(i) = 5 - i$, we may assume without loss of generality that $I_1 = \{\{1\}, \{2\}, \{4\}\}$. As $\{2\}$ belongs to the ideal, at level 2 we cannot have $\{1, 3\}$ or $\{1, 4\}$, nor $\{1, 3, 4\}$ at level 3. However, by Lemma 4.8.2 we must include $\{2, 4\}$; for the inclusion of the remaining 11th set we are left with two choices: (a) $\{2, 3\}$ or (b) $\{1, 2, 4\}$. The case (b), as we saw in Example 1.5.2, leads to Fishburn's domain $\bar{F}_4$ and in case (a) we have the ideal

$$\begin{aligned}
I_0 &= \{\emptyset\},\\
I_1 &= \{\{1\}, \{2\}, \{4\}\},\\
I_2 &= \{\{1, 2\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\},\\
I_3 &= \{\{1, 2, 3\}, \{2, 3, 4\}\},\\
I_4 &= \{\{1, 2, 3, 4\}\}.
\end{aligned}$$

which gives us the single-crossing domain (see Section 3.1)

$$SC_4 = \{1234, 2134, 2314, 2341, 2431, 4231, 4321\}.$$

displayed on Figure 4.15.

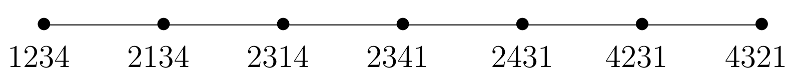


Figure 4.15: Graph of single-crossing domain SC_4 on four alternatives

- **The ideal has two singletons.** We claim that up to a flip-isomorphism this case has already been dealt with. The two singletons must be $\{1\}$ and $\{4\}$. Then by Lemma 4.8.2 we have to include $\{1, 4\}$. As $\{2, 3\}$ is not separated from $\{1, 4\}$ the remaining sets that can be added at level 2 are $\{1, 3\}$ and $\{2, 4\}$. We can add one of them or none. It does not matter which one among $\{1, 3\}$ and $\{2, 4\}$ we add as the domains would be isomorphic under $\sigma(i) = 5 - i$. If we add $\{2, 4\}$ by Lemma 4.8.2 we must also add $\{1, 2, 4\}$, hence we get the ideal

$$\begin{aligned}
I_0 &= \{\emptyset\},\\
I_1 &= \{\{1\}, \{4\}\},\\
I_2 &= \{\{1, 2\}, \{1, 4\}, \{2, 4\}, \{3, 4\}\},\\
I_3 &= \{\{1, 2, 3\}, \{1, 2, 4\}, \{2, 3, 4\}\},\\
I_4 &= \{\{1, 2, 3, 4\}\}.
\end{aligned}$$

[3] Having all singletons present means that the domain is *minimally rich* in the sense of Puppe [2018] who showed that the single-peaked domain is in fact the only semi-connected Condorcet domain with that property.

which by Proposition 1.5.4 is flip-isomorphic to the single-crossing domain, and if we do not add any set of cardinality 2 by Lemma 4.8.2 we must add $\{1, 3, 4\}$ hence we get the domain

$$\begin{aligned} I_0 &= \{\emptyset\}, \\ I_1 &= \{\{1\}, \{4\}\}, \\ I_2 &= \{\{1, 2\}, \{1, 4\}, \{3, 4\}\}, \\ I_3 &= \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}, \\ I_4 &= \{\{1, 2, 3, 4\}\}. \end{aligned}$$

which by Proposition 1.5.4 is flip-isomorphic to the single-peaked one.

Theorem 4.8.3. *Up to an isomorphism and flip-isomorphism there exist only three peak-pit maximal Condorcet domains of maximal width: the single-peaked, the single-crossing, and the Fishburn domains.*

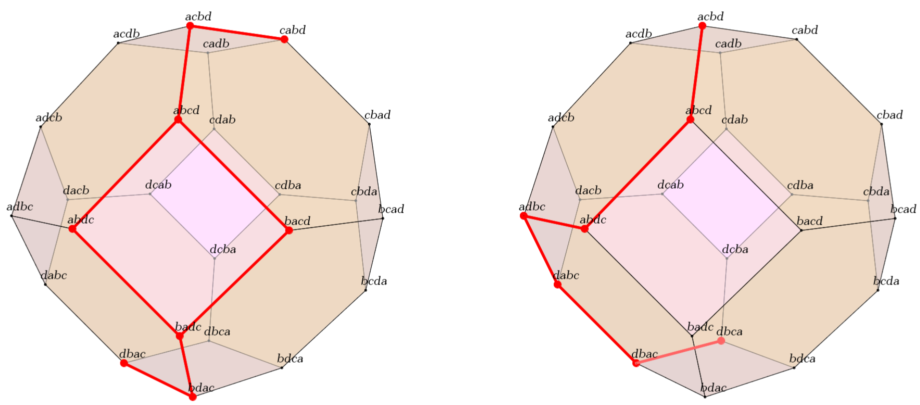


Figure 4.16: The single-peaked (left) and single-crossing (right) maximal CDs with maximal width in the permutohedron.

Fishburn's domain was shown on Figure 1.4.

4.8.3 Five alternatives

By our preceding analysis, we know that the ideal of any maximal peak-pit domain of maximal width on a set of $n = 5$ alternatives has cardinality 16 and will include the following 10 sets:

$$\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}, \{1, 2, 3, 4, 5\}, \{2, 3, 4, 5\}, \{3, 4, 5\}, \{4, 5\}, \{5\}.$$

A singleton $\{i\}$ is not compatible with any set containing $\{j,k\}$ with $j<i<k$. Also, the following pairs and triples of alternatives are incompatible:

$$\begin{aligned}
(1,3) &\text{ is incompatible with } (2,4),\ (2,5) \text{ and } (2,4,5);\\
(1,4) &\text{ is incompatible with } (2,5),(3,5) \text{ and } (2,3,5);\\
(1,5) &\text{ is incompatible with } (2,3),\ (3,4),\ (2,4) \text{ and } (2,3,4);\\
(2,4) &\text{ is incompatible with } (1,3),\ (3,5) \text{ and } (1,3,5);\\
(2,5) &\text{ is incompatible with } (1,3),\ (1,4)\ (3,4) \text{ and } (1,3,4);\\
(3,5) &\text{ is incompatible with } (1,4),\ (2,4) \text{ and } (1,2,4).
\end{aligned}$$

The following observation will simplify the classification.

Lemma 4.8.4. *Let $\mathcal{D}$ be a peak-pit maximal domain of maximal width over five alternatives such that its ideal $I(\mathcal{D})=\bigcup_{i=0}^{5} I_i$ contains exactly two singletons in I_1. Then I_4 must contains at least three sets.*

Proof. Suppose the ideal of $\mathcal{D}$ has two singletons in I_1. Then $I_1=\{\{1\},\{5\}\}$ and by Lemma 4.8.2, I_2 contains the set $\{1,5\}$. Hence there must be a flag containing $\{1,5\}$. The set of cardinality four from this flag belongs to I_4 and it is different from $\{1,2,3,4\}$ and $\{2,3,4,5\}$ which are always in I_4 due to maximal width. Hence I_4 contains at least three sets. □

Corollary 4.8.5. *Any peak-pit Condorcet domain of maximal width is isomorphic or flip-isomorphic to a peak-pit domain of maximal width whose ideal contains at least three singletons.*

Proof. By Lemma 4.8.4 and Proposition 1.5.4 if a domain has two singletons, then its flipped domain contains at least three. □

I. Five singletons. Suppose we have a maximal ideal I such that I_1 contains all five singletons.

Domain 1. By Lemma 4.8.2 all other options are forced and we cannot add anything anywhere:

$$\begin{aligned}
I_0 &= \{\emptyset\},\\
I_1 &= \{\{1\},\{2\},\{3\},\{4\},\{5\}\},\\
I_2 &= \{\{1,2\},\{2,3\},\{3,4\},\{4,5\}\},\\
I_3 &= \{\{1,2,3\},\{2,3,4\},\{3,4,5\}\},\\
I_4 &= \{\{1,2,3,4\},\{2,3,4,5\}\},\\
I_5 &= \{\{1,2,3,4,5\}\}.
\end{aligned}$$

This is the single-peaked domain. It contains 16 linear orders and is characterised by zero ‘inversion triples’.

II. Four singletons. Suppose we have a maximal ideal I such that I_1 contains four singletons. Then, up to an isomorphism, we have either $I_1 = \{\{1\},\{2\},\{3\},\{5\}\}$ or $I_1 = \{\{1\},\{2\},\{4\},\{5\}\}$.

Domain 2. Let $I_1 = \{\{1\},\{2\},\{3\},\{5\}\}$. Lemma 4.8.2 forces us to choosing $\{2,3\},\{3,5\}$ to I_2. If no other sets are added to I_2, we must, again by Lemma 4.8.2, add $\{2,3,5\}$ to I_3 and $\{1,2,3,5\}$ to I_4. The resulting ideal is maximal since it contains 16 sets:

$$\begin{aligned}
I_0 &= \{\emptyset\},\\
I_1 &= \{\{1\},\{2\},\{3\},\{5\}\},\\
I_2 &= \{\{1,2\},\{2,3\},\{3,5\},\{4,5\}\},\\
I_3 &= \{\{1,2,3\},\{2,3,5\},\{3,4,5\}\},\\
I_4 &= \{\{1,2,3,4\},\{1,2,3,5\},\{2,3,4,5\}\},\\
I_5 &= \{\{1,2,3,4,5\}\}.
\end{aligned}$$

The corresponding domain contains 19 linear orders and is defined by the following inversion triples:

$$[1,4,5],\ [2,4,5],\ [3,4,5].$$

Domain 3. $I_1 = \{\{1\},\{2\},\{3\},\{5\}\}$. Unlike with Domain 2, we can add $\{3,4\}$ at level 2 which will force us to replace $\{2,3,5\}$ in Domain 2 with $\{2,3,4\}$ and this will preclude $\{1,2,3,5\}$ at level 4:

$$\begin{aligned}
I_0 &= \{\emptyset\},\\
I_1 &= \{\{1\},\{2\},\{3\},\{5\}\},\\
I_2 &= \{\{1,2\},\{2,3\},\{3,4\},\{3,5\},\{4,5\}\},\\
I_3 &= \{\{1,2,3\},\{2,3,4\},\{3,4,5\}\},\\
I_4 &= \{\{1,2,3,4\},\{2,3,4,5\}\},\\
I_5 &= \{\{1,2,3,4,5\}\}.
\end{aligned}$$

This ideal generates a domain containing 14 linear orders which is defined by a single inversion triple:

$$[3,4,5].$$

Domain 4. $I_1 = \{\{1\},\{2\},\{3\},\{5\}\}$. Also, instead of $\{3,4\}$, as in Domain 3, we can add $\{2,3,4\}$ and $\{2,3,5\}$ to I_3. This also precludes $\{1,2,3,5\}$ at level 4. Then,

$$\begin{aligned}
I_0 &= \{\emptyset\},\\
I_1 &= \{\{1\},\{2\},\{3\},\{5\}\},\\
I_2 &= \{\{1,2\},\{2,3\},\{3,5\},\{4,5\}\},\\
I_3 &= \{\{1,2,3\},\{2,3,4\},\{2,3,5\},\{3,4,5\}\},\\
I_4 &= \{\{1,2,3,4\},\{2,3,4,5\}\},\\
I_5 &= \{\{1,2,3,4,5\}\}.
\end{aligned}$$

This domain contains 15 linear orders and is defined by the inversion triples:

$$[2, 4, 5], \ [3, 4, 5].$$

Domain 5. Now let $I_1 = \{\{1\}, \{2\}, \{4\}, \{5\}\}$. This and Lemma 4.8.2 forces us to choosing $\{2, 4\}$ to I_2, $\{1, 2, 4\}$ and $\{2, 4, 5\}$ to I_3, and $\{1, 2, 4, 5\}$ to I_4. The resulting ideal is already maximal since it contains 16 sets.

$$\begin{aligned}
I_0 &= \{\emptyset\}, \\
I_1 &= \{\{1\}, \{2\}, \{4\}, \{5\}\}, \\
I_2 &= \{\{1, 2\}, \{2, 4\}, \{4, 5\}\}, \\
I_3 &= \{\{1, 2, 3\}, \{1, 2, 4\}, \{2, 4, 5\}, \{3, 4, 5\}\}, \\
I_4 &= \{\{1, 2, 3, 4\}, \{1, 2, 4, 5\}, \{2, 3, 4, 5\}\}, \\
I_5 &= \{\{1, 2, 3, 4, 5\}\}.
\end{aligned}$$

This is one of two Fishburn's domains. It contains 20 linear orders and is defined by the inversion triples:

$$[1, 3, 4], \ [1, 3, 5], \ [2, 3, 4], \ [2, 3, 5].$$

With the same level 1, we can either add $\{2, 3\}$ and/or $\{3, 4\}$ to I_2, and/or add $\{2, 3, 4\}$ to I_3. We note that adding $\{3, 4\}$ will give us a domain isomorphic to that when we add $\{2, 3\}$.

Domain 6. Let $I_1 = \{\{1\}, \{2\}, \{4\}, \{5\}\}$ and adding $\{2, 3\}$ to I_2 in addition to those that are forced by Lemma 4.8.2. Then

$$\begin{aligned}
I_0 &= \{\emptyset\}, \\
I_1 &= \{\{1\}, \{2\}, \{4\}, \{5\}\}, \\
I_2 &= \{\{1, 2\}, \{2, 3\}, \{2, 4\}, \{4, 5\}\}, \\
I_3 &= \{\{1, 2, 3\}, \{2, 3, 4\}, \{2, 4, 5\}, \{3, 4, 5\}\}, \\
I_4 &= \{\{1, 2, 3, 4\}, \{2, 3, 4, 5\}\}, \\
I_5 &= \{\{1, 2, 3, 4, 5\}\}.
\end{aligned}$$

This domain contains 15 linear orders defined by the inversion triples:

$$[2, 3, 4], \ [2, 3, 5].$$

Domain 7. Let $I_1 = \{\{1\}, \{2\}, \{4\}, \{5\}\}$ and adding both $\{2, 3\}$ and $\{3, 4\}$ to I_2: Then

$$\begin{aligned}
I_0 &= \{\emptyset\}, \\
I_1 &= \{\{1\}, \{2\}, \{4\}, \{5\}\}, \\
I_2 &= \{\{1, 2\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{4, 5\}\}, \\
I_3 &= \{\{1, 2, 3\}, \{2, 3, 4\}, \{3, 4, 5\}\}, \\
I_4 &= \{\{1, 2, 3, 4\}, \{2, 3, 4, 5\}\}, \\
I_5 &= \{\{1, 2, 3, 4, 5\}\}.
\end{aligned}$$

This domain contains 14 linear orders and is defined by a single inversion triple:

$$[2, 3, 4].$$

Domain 8. Let $I_1 = \{\{1\}, \{2\}, \{4\}, \{5\}\}$ and adding $\{2, 3, 4\}$ to I_3. Then

$$I_0 = \{\emptyset\},$$
$$I_1 = \{\{1\}, \{2\}, \{4\}, \{5\}\},$$
$$I_2 = \{\{1, 2\}, \{2, 4\}, \{4, 5\}\},$$
$$I_3 = \{\{1, 2, 3\}, \{1, 2, 4\}, \{2, 3, 4\}, \{2, 4, 5\}, \{3, 4, 5\}\},$$
$$I_4 = \{\{1, 2, 3, 4\}, \{2, 3, 4, 5\}\},$$
$$I_5 = \{\{1, 2, 3, 4, 5\}\}.$$

This domain contains 16 linear orders defined by the inversion triples:

$$[1, 3, 4], \ [2, 3, 4], \ [2, 3, 5].$$

It will be useful to have the following statement for future reference.

Lemma 4.8.6. *If an ideal I has $|I_4| \geq 4$, then the corresponding domain is flip-isomorphic to one of the domains 2–8.*

Proof. Follows from Proposition 1.5.4. □

III. Three singletons. Suppose we have a maximal ideal I such that I_1 contains three singletons. Then, up to an isomorphism, we have either $I_1 = \{\{1\}, \{2\}, \{5\}\}$ or $I_1 = \{\{1\}, \{3\}, \{5\}\}$. Let us consider the first case.

Let $I_1 = \{\{1\}, \{2\}, \{5\}\}$ and $I_2 = \{\{1, 2\}, \{2, 5\}, \{4, 5\}\}$ which is the minimum required by Lemma 4.8.2. Then, if we do not add anything to I_2 and I_3, then it is easy to see that we will have four elements in I_4 and thus this case is flip-isomorphic to cases considered already by Lemma 4.8.6.

Let us start with additions to level 2. The pairs that are compatible with I_1 and may be included are:

$$\{2, 3\}, \{2, 4\}, \{3, 5\}.$$

However we note that $\{2, 4\}$ and $\{3, 5\}$ are mutually incompatible.

Domain 9. Let $I_1 = \{\{1\}, \{2\}, \{5\}\}$ and we add only $\{2, 3\}$. Then everything is determined:

$$I_0 = \{\emptyset\},$$
$$I_1 = \{\{1\}, \{2\}, \{5\}\},$$
$$I_2 = \{\{1, 2\}, \{2, 3\}, \{2, 5\}, \{4, 5\}\},$$
$$I_3 = \{\{1, 2, 3\}, \{2, 3, 5\}, \{2, 4, 5\}, \{3, 4, 5\}\},$$
$$I_4 = \{\{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{2, 3, 4, 5\}\},$$
$$I_5 = \{\{1, 2, 3, 4, 5\}\}.$$

This domain contains 16 linear orders and is defined by the inversion triples:

$$[1,4,5],\ [2,3,4],\ [2,3,5],\ [2,4,5],\ [3,4,5].$$

Domain 10. Let $I_1 = \{\{1\},\{2\},\{5\}\}$ and we add only $\{2,4\}$. Then by Lemma 4.8.2 everything else is determined:

$$\begin{aligned}
I_0 &= \{\emptyset\},\\
I_1 &= \{\{1\},\{2\},\{5\}\},\\
I_2 &= \{\{1,2\},\{2,4\},\{2,5\},\{4,5\}\},\\
I_3 &= \{\{1,2,3\},\{1,2,4\},\{2,4,5\},\{3,4,5\}\},\\
I_4 &= \{\{1,2,3,4\},\{1,2,4,5\},\{2,3,4,5\}\},\\
I_5 &= \{\{1,2,3,4,5\}\}.
\end{aligned}$$

This domain contains 17 linear orders and is defined by the inversion triples:

$$[1,3,4],\ [1,3,5],\ [2,3,4],\ [2,3,5],\ [2,4,5].$$

Domain 11. Let $I_1 = \{\{1\},\{2\},\{5\}\}$ and we add both $\{2,3\}$ and $\{2,4\}$. Then by Lemma 4.8.2 everything is determined:

$$\begin{aligned}
I_0 &= \{\emptyset\},\\
I_1 &= \{\{1\},\{2\},\{5\}\},\\
I_2 &= \{\{1,2\},\{2,3\},\{2,4\},\{2,5\},\{4,5\}\},\\
I_3 &= \{\{1,2,3\},\{2,3,4\},\{2,4,5\},\{3,4,5\}\},\\
I_4 &= \{\{1,2,3,4\},\{2,3,4,5\}\},\\
I_5 &= \{\{1,2,3,4,5\}\}.
\end{aligned}$$

This domain contains 12 linear orders and is defined by the inversion triples:

$$[2,3,4],\ [2,3,5],\ [2,4,5].$$

Let $I_1 = \{\{1\},\{2\},\{5\}\}$ and we add only $\{3,5\}$. Then everything is determined:

$$\begin{aligned}
I_0 &= \{\emptyset\},\\
I_1 &= \{\{1\},\{2\},\{5\}\},\\
I_2 &= \{\{1,2\},\{2,5\},\{3,5\},\{4,5\}\},\\
I_3 &= \{\{1,2,3\},\{1,2,5\},\{2,3,5\},\{3,4,5\}\},\\
I_4 &= \{\{1,2,3,4\},\{1,2,3,5\},\{2,3,4,5\}\},\\
I_5 &= \{\{1,2,3,4,5\}\}.
\end{aligned}$$

This domain is flip-isomorphic to Domain 9.

Domain 12. Let $I_1 = \{\{1\}, \{2\}, \{5\}\}$ and we add both $\{2, 3\}$ and $\{3, 5\}$. Then:

$$\begin{aligned} I_0 &= \{\emptyset\}, \\ I_1 &= \{\{1\}, \{2\}, \{5\}\}, \\ I_2 &= \{\{1, 2\}, \{2, 3\}, \{2, 5\}, \{3, 5\}, \{4, 5\}\}, \\ I_3 &= \{\{1, 2, 3\}, \{2, 3, 5\}, \{3, 4, 5\}\}, \\ I_4 &= \{\{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{2, 3, 4, 5\}\}, \\ I_5 &= \{\{1, 2, 3, 4, 5\}\}. \end{aligned}$$

This domain contains 16 linear orders and is defined by the inversion triples:

$$[1, 4, 5],\ [2, 3, 5],\ [2, 4, 5],\ [3, 4, 5].$$

Now let us start adding triples as well. The triples that are compatible are $\{1, 2, 4\}$, $\{2, 3, 4\}$, $\{2, 3, 5\}$. We note that $\{1, 2, 4\}$ is incompatible with $\{2, 3\}$ and $\{3, 5\}$ and we have already added it with $\{2, 4\}$. If we add it alone, then

$$\begin{aligned} I_0 &= \{\emptyset\}, \\ I_1 &= \{\{1\}, \{2\}, \{5\}\}, \\ I_2 &= \{\{1, 2\}, \{2, 5\}, \{4, 5\}\}, \\ I_3 &= \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}, \{2, 4, 5\}, \{3, 4, 5\}\}, \\ I_4 &= \{\{1, 2, 3, 4\}, \{1, 2, 4, 5\}, \{2, 3, 4, 5\}\}, \\ I_5 &= \{\{1, 2, 3, 4, 5\}\}. \end{aligned}$$

From I_4 we observe that this domain is flip-isomorphic to the one whose ideal has $I_1 = \{\{1\}, \{3\}, \{5\}\}$. It will be flip-isomorphic to Domain 16 below.

We can also add $\{2, 3, 4\}$ but not alone as it is incompatible with $\{1, 5\}$.

Domain 13. Let $I_1 = \{\{1\}, \{2\}, \{5\}\}$ and we add both $\{2, 3\}$ and $\{2, 3, 4\}$. Then

$$\begin{aligned} I_0 &= \{\emptyset\}, \\ I_1 &= \{\{1\}, \{2\}, \{5\}\}, \\ I_2 &= \{\{1, 2\}, \{2, 3\}, \{2, 5\}, \{4, 5\}\}, \\ I_3 &= \{\{1, 2, 3\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 4, 5\}, \{3, 4, 5\}\}, \\ I_4 &= \{\{1, 2, 3, 4\}, \{2, 3, 4, 5\}\}, \\ I_5 &= \{\{1, 2, 3, 4, 5\}\}. \end{aligned}$$

This domain contains 12 linear orders and is defined by the inversion triples:

$$[2, 3, 4],\ [2, 3, 5],\ [2, 4, 5],\ [3, 4, 5].$$

Domain 14. Let $I_1 = \{\{1\},\{2\},\{5\}\}$ and we add both $\{2,4\},\{2,3,4\}$. Then

$$\begin{aligned}
&I_0 = \{\emptyset\},\\
&I_1 = \{\{1\},\{2\},\{5\}\},\\
&I_2 = \{\{1,2\},\{2,4\},\{2,5\},\{4,5\}\},\\
&I_3 = \{\{1,2,3\},\{1,2,4\},\{2,3,4\},\{2,4,5\},\{3,4,5\}\},\\
&I_4 = \{\{1,2,3,4\},\{2,3,4,5\}\},\\
&I_5 = \{\{1,2,3,4,5\}\}.
\end{aligned}$$

This domain contains 12 linear orders and is defined by the inversion triples:

$$[1,3,4],\ [2,3,4],\ [2,3,5],\ [2,4,5].$$

Domain 15. Let $I_1 = \{\{1\},\{2\},\{5\}\}$ and we add $\{2,3\}$, $\{3,5\}$ and $\{2,3,4\}$. Then

$$\begin{aligned}
&I_0 = \{\emptyset\},\\
&I_1 = \{\{1\},\{2\},\{5\}\},\\
&I_2 = \{\{1,2\},\{2,3\},\{2,5\},\{3,5\},\{4,5\}\},\\
&I_3 = \{\{1,2,3\},\{2,3,4\},\{2,3,5\},\{3,4,5\}\},\\
&I_4 = \{\{1,2,3,4\},\{2,3,4,5\}\},\\
&I_5 = \{\{1,2,3,4,5\}\}.
\end{aligned}$$

It contains 11 linear orders and is defined by the inversion triples:

$$[2,3,5], [2,4,5], [3,4,5].$$

This is a single-crossing domain.

As for $\{2,3,5\}$, we added it with $\{2,3\}$ and $\{3,5\}$ and it is not compatible with $\{2,4\}$. We can add it alone but this will be flip-isomorphic to Domain 12.

Alternatively, we can start with $I_1 = \{\{1\},\{3\},\{5\}\}$. We will then have to select $\{1,3\},\{3,5\}$ into I_2 due to Lemma 4.8.2. If we add nothing else, then:

$$\begin{aligned}
&I_0 = \{\emptyset\},\\
&I_1 = \{\{1\},\{3\},\{5\}\},\\
&I_2 = \{\{1,2\},\{1,3\},\{3,5\},\{4,5\}\},\\
&I_3 = \{\{1,2,3\},\{1,3,5\},\{3,4,5\}\},\\
&I_4 = \{\{1,2,3,4\},\{1,2,3,5\},\{1,3,4,5\},\{2,3,4,5\}\},\\
&I_5 = \{\{1,2,3,4,5\}\}.
\end{aligned}$$

This is one of Fishburn's domains, it is flip-isomorphic to Domain 5.

We can then add pairs or triples. Only the pairs $\{2,3\}$,$\{3,4\}$, and the triples $\{2,3,4\}$, $\{1,3,4\}$, $\{2,3,5\}$ are compatible to $\{1,3\},\{3,5\}$. However, $\{1,3,4\}$ and $\{2,3\}$ and also $\{2,3,5\}$ are incompatible.

Domain 16. Let $I_1 = \{\{1\}, \{3\}, \{5\}\}$ and we add $\{2, 3\}$ to I_2 and nothing else This would be isomorphic to adding $\{3, 4\}$. Then:

$$\begin{aligned}
I_0 &= \{\emptyset\},\\
I_1 &= \{\{1\}, \{3\}, \{5\}\},\\
I_2 &= \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 5\}, \{4, 5\}\},\\
I_3 &= \{\{1, 2, 3\}, \{2, 3, 5\}, \{3, 4, 5\}\},\\
I_4 &= \{\{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{2, 3, 4, 5\}\},\\
I_5 &= \{\{1, 2, 3, 4, 5\}\}.
\end{aligned}$$

This domain contains 17 linear orders and is defined by the inversion triples:

$$[1, 2, 3], \quad [1, 4, 5], \quad [2, 4, 5], \quad [3, 4, 5].$$

Domain 17. Let $I_1 = \{\{1\}, \{3\}, \{5\}\}$ and we add $\{2, 3\}$ and $\{3, 4\}$. Then:

$$\begin{aligned}
I_0 &= \{\emptyset\},\\
I_1 &= \{\{1\}, \{3\}, \{5\}\},\\
I_2 &= \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\},\\
I_3 &= \{\{1, 2, 3\}, \{2, 3, 4\}, \{3, 4, 5\}\},\\
I_4 &= \{\{1, 2, 3, 4\}, \{2, 3, 4, 5\}\},\\
I_5 &= \{\{1, 2, 3, 4, 5\}\}.
\end{aligned}$$

This domain contains 12 linear orders and is defined by the inversion triples:

$$[1, 2, 3], \ [3, 4, 5].$$

Domain 18. Let $I_1 = \{\{1\}, \{3\}, \{5\}\}$ and we add $\{2, 3\}$ and $\{2, 3, 4\}$, which would be isomorphic to adding $\{3, 4\}$ and $\{2, 3, 4\}$. Then:

$$\begin{aligned}
I_0 &= \{\emptyset\},\\
I_1 &= \{\{1\}, \{3\}, \{5\}\},\\
I_2 &= \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 5\}, \{4, 5\}\}\\
I_3 &= \{\{1, 2, 3\}, \{2, 3, 4\}, \{2, 3, 5\}, \{3, 4, 5\}\},\\
I_4 &= \{\{1, 2, 3, 4\}, \{2, 3, 4, 5\}\},\\
I_5 &= \{\{1, 2, 3, 4, 5\}\}.
\end{aligned}$$

This domain contains 12 linear orders with the inversion triple:

$$[1, 2, 3], [2, 4, 5], [3, 4, 5].$$

We are left with the case when $I_1 = \{\{1\}, \{3\}, \{5\}\}$ and we are adding $\{1, 3, 4\}$ to I_2, then

$$\begin{aligned}
I_0 &= \{\emptyset\},\\
I_1 &= \{\{1\}, \{3\}, \{5\}\},\\
I_2 &= \{\{1, 2\}, \{1, 3\}, \{3, 5\}, \{4, 5\}\},\\
I_3 &= \{\{1, 2, 3\}, \{1, 3, 4\}, \{1, 3, 5\}, \{3, 4, 5\}\},\\
I_4 &= \{\{1, 2, 3, 4\}, \{1, 3, 4, 5\}, \{2, 3, 4, 5\}\},\\
I_5 &= \{\{1, 2, 3, 4, 5\}\}.
\end{aligned}$$

This domain is flip-isomorphic to Domain 10.

IV. Two singletons. This case is covered by Lemma 4.8.4. In such a case I_4 must contains at least 3 sets and the corresponding domains are filp-isomorphic to one of the domains above.

Theorem 4.8.7. *Up to isomorphism and flip-isomorphism, there exist exactly 18 peak-pit maximal Condorcet domains of maximal width on a set of $n = 5$ alternatives.*

The number of domains of different sizes is given in the following table:

Size	11	12	13	14	15	16	17	18	19	20
The number of domains	1	5	0	2	2	4	2	0	1	1

4.8.4 Classification of small peak-pit maximal domains of non-maximal width

This classification is done by computerised search done by Dittrich [2018]. Here we encounter the following previously unseen creatures:

Example 4.8.1 (Ladder domain)**.** *It is defined by the following complete set of never-conditions:*

$$3N_{\{1,2,3\}}1, \quad 4N_{\{1,2,4\}}1, \quad 1N_{\{1,3,4\}}3, \quad 2N_{\{2,3,4\}}3.$$

It is copious but does not have maximal width.

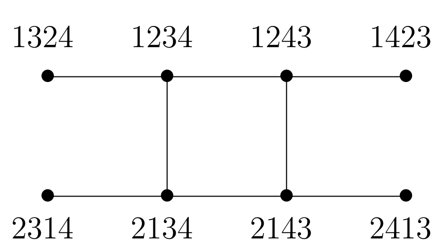


Figure 4.17: Graph of the Ladder domain.

Example 4.8.2 (Broken Ladder domain)**.** *It is defined by the following complete set of never-conditions:*

$$3N_{\{1,2,3\}}1, \quad 1N_{\{1,2,4\}}3, \quad 1N_{\{1,3,4\}}3, \quad 2N_{\{2,3,4\}}3.$$

It is copious but does not have maximal width.

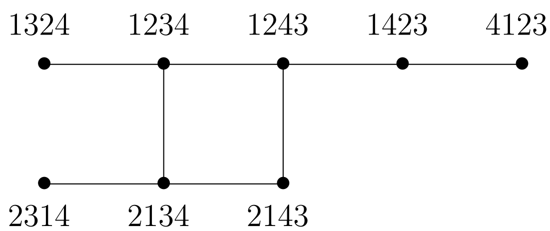


Figure 4.18: Graph of the Broken Ladder domain.

Theorem 4.8.8. *If $m = 4$, then any maximal connected Condorcet domain is either isomorphic or flip-isomorphic to one of the following:*

1. *Black's single-peaked domain;*
2. *Single crossing domain;*
3. *Fishburn's domain;*
4. *Arrow's single-peaked domain;*
5. *Ladder domain;*
6. *Broken ladder domain.*

Only the first three have maximal width.

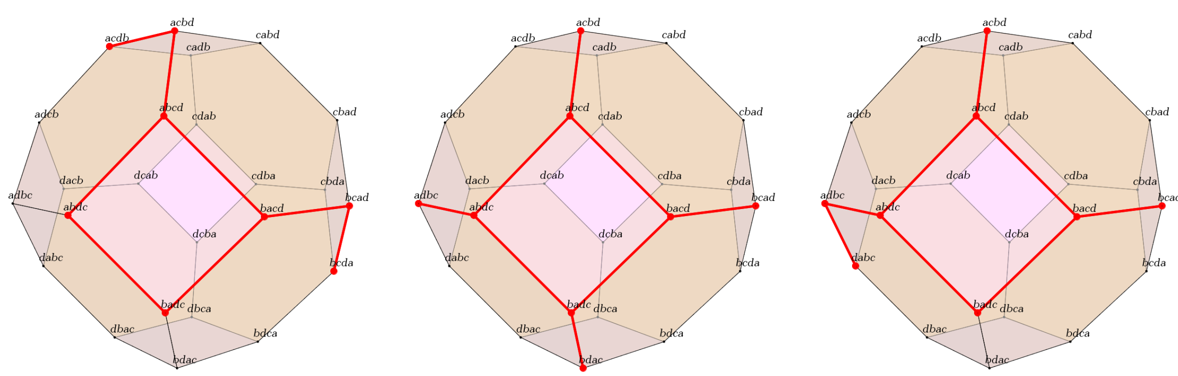


Figure 4.19: Arrow's single-peaked (left), Ladder domain (centre), Broken ladder domain (right).

Chapter 5

Domains defined by alternating schemes

5.1 Fishburn domains and their generalisations

Besides the single-crossing and the single-peaked domains, the best-known peak-pit domains are Fishburn's domains introduced in Definition 1.2.2. They are defined by the so-called *alternating scheme of never conditions* relative to some fixed order of alternatives (spectrum). Up to an isomorphism we can assume that the set of alternatives $A = [n]$ and the order is $1 < 2 < \ldots < n$. There are two types of Fishburn domains, one flip-isomorphic to another.

Fishburn's domains are maximal, we will prove this for a larger class of Condorcet domains that we are about to introduce.

Definition 5.1.1 (Karpov [2023])**.** *A complete set of never-conditions* (1.1.3) *is said to be a* generalised alternating scheme*, if for some subset* $K \subseteq [2, \ldots, n-1]$ *and for all* $1 \leq i < j < k \leq n$ *we have*

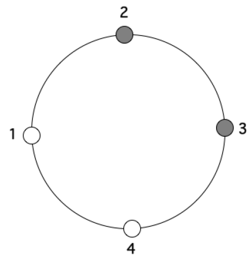
$$jN_{\{i,j,k\}}3, \text{ if } j \in K, \text{ and } jN_{\{i,j,k\}}1, \text{ if } j \notin K. \tag{5.1.1}$$

The domain which consists of all linear orders satisfying the generalised alternating scheme is called the generalised Fishburn's domain *or* GF-domain*.*

The GF-domain constructed using a subset $K \subseteq [2, \ldots, n-1]$ will be denoted as F_K. Every GF-domain has maximal width since orders $12 \ldots n$ and $n \ldots 21$ satisfy conditions (5.1.1).

Example 5.1.1. *The original Fishburn's alternating scheme has* K *to be the set of even numbers in* $[2, \ldots, n-1]$*. If* $K = [2, \ldots, n-1]$*, we have the classical single-peaked domain. This follows from Theorem 2.1.7(iii).*

With the introduction of the class of generalised Fishburn's domains we gain a new insight into the structure of the universe of Condorcet domains. We will show that Fishburn's domains and single-peaked domains are close relatives and are the two extremes of a certain family of Condorcet domains. In this paper we investigate the domains from this family and show that they are all single-peaked on a circle.



1	1	1	1	4	4	4	4
2	2	4	4	1	1	3	3
3	4	2	3	2	3	1	2
4	3	3	2	3	2	2	1

Figure 5.1: Necklace with two black beads

5.2 Combinatorial representation of GF-domains

The idea of this representation comes from an example in Danilov et al. [2010].

A set of n vertices on a circle, some white and some black, are numbered by integers $1, 2, \dots, n$. We will often identify the vertices with the numbers on them.

Definition 5.2.1 (Slinko [2024b]). *An arrangement of black and white vertices on a circle will be called a* necklace *and the vertices themselves will be called* beads.

Definition 5.2.2. *A set of beads $X \subseteq [n]$ is said to be* white convex *(w-convex) if*

(a) X is an arc of the circle;

(b) X does not consist of a single black bead;

(c) There does not exist $i < j < k$ such that $i, k \in X$, $j \notin X$ and j is white.

Definition 5.2.3. *A* flag *of w-convex sets is a sequence $X_1, \dots, X_n$ of w-convex sets*

$$X_1 \subset X_2 \subset \cdots \subset X_n = [n], \tag{5.2.1}$$

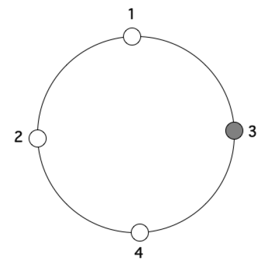
where $|X_k| = k$.

Any flag (5.2.1) of w-convex sets defines a linear order $v = x_1 x_2 \dots x_n$ on $[n]$, where $\{x_i\} = X_i \setminus X_{i-1}$ (for convenience we assume that $X_0 = \emptyset$).

Definition 5.2.4. *Given a necklace S, the domain $\mathcal{D}(S) \subseteq \mathcal{L}([n])$ is the set of all linear orders corresponding to flags of w-convex sets in $[n]$.*

Example 5.2.1. *Consider now the necklace S presented on Figure 5.1 and the corresponding domain $\mathcal{D}(S)$. This is a single-dipped domain relative to the spectrum $1 \triangleleft 2 \triangleleft 3 \triangleleft 4$ or $4 \triangleleft 3 \triangleleft 2 \triangleleft 1$.*

Example 5.2.2. *Consider now the necklace S presented on the Figure 5.2 and the corresponding domain $\mathcal{D}(S)$. Then domain $\mathcal{D}(S)$ is given by the array on the right. This is the Fishburn domain relative to the spectrum $1 \triangleleft 2 \triangleleft 3 \triangleleft 4$.*



1	1	2	2	2	2	4	4	4
2	2	1	1	4	4	2	2	3
3	4	3	4	1	3	1	3	2
4	3	4	3	3	1	3	1	1

Figure 5.2: Necklace with one black bead

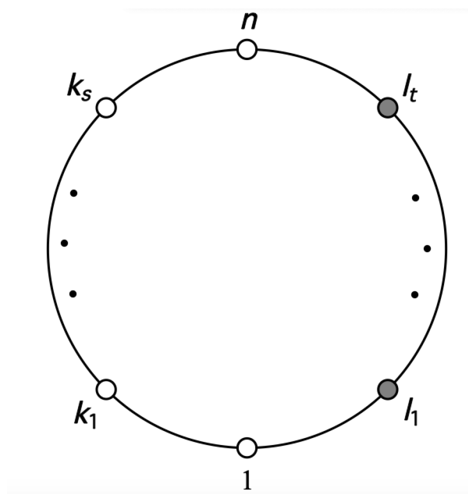


Figure 5.3: KL-spectrum

Our examples show that the construction is promising as we could generate two maximal GF-domains. Let us generalise these examples and offer a new combinatorial representation of GF-domains from which we will deduce their maximality.

Let $K \subseteq [2, \ldots, n-1]$ and $L = [2, \ldots, n-1] \setminus K$ be two complementary subsets of $[2, \ldots, n-1]$. Let $k_1 < \ldots < k_s$ and $\ell_1 < \ldots < \ell_t$ be ordered elements of K and L, respectively, where $s + t = n - 2$. Consider the following spectrum on the circle

$$1 \lhd k_1 \lhd \ldots \lhd k_s \lhd n \lhd \ell_t \lhd \ldots \lhd \ell_1 \lhd 1. \tag{5.2.2}$$

Mark beads $1, k_1, \ldots, k_s, n$ white and $\ell_1, \ldots, \ell_t$ black to obtain a necklace S_K as shown on Figure 5.3.

Example 5.2.3. *For $n = 3$ we have two options: one with $K_1 = \emptyset$ and another with $K_2 = \{2\}$. Respectively we have two necklaces S_{K_1} and S_{K_2} shown on Figure 5.4:*

Then $\mathcal{D}(S_{K_1}) = \{123, 213, 231, 321\}$ and $\mathcal{D}(S_{K_2}) = \{123, 132, 312, 321\}$ which are F_{K_1} and F_{K_2}, respectively. These are single-peaked and single-dipped triples.

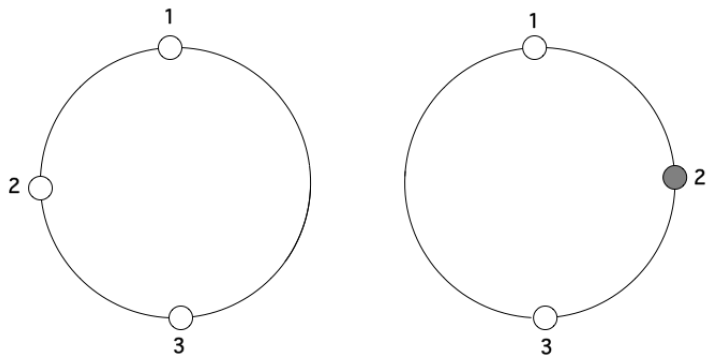


Figure 5.4: Necklaces with three bids

Proposition 5.2.1. *If* $K = [2, \dots, n-1]$*, then* $\mathcal{D}(S_K) = F_K$ *is the classical single-peaked domain.*

Proof. Since $L = \emptyset$ in S_K there are no black beads and beads 1 and n are neighbours on the circle. The only w-convex set containing both of them is the longer arc Z with endbeads 1 and n, that is, $\{1, 2, \dots, n\}$. Any arc $X \subseteq Z$ is w-convex. So w-convex sets coincide with the upper contour sets of the classical single-peaked domain with the spectrum $1 \lhd 2 \lhd \dots \lhd n-1 \lhd n$. □

Lemma 5.2.2. *Let* S_K *be a necklace given by* (5.2.2)*. In domain* $\mathcal{D}(S_K)$*:*

(i) A black bead a *satisfies the never-top condition in any triple* $\{a, b, c\}$ *such that* $b < a < c$*, that is* $aN_{\{a,b,c\}}1$*.*

(ii) A white bead a *satisfies the never-bottom condition in any triple* $\{a, b, c\}$ *such that* $b < a < c$*, that is* $aN_{\{a,b,c\}}3$*.*

Proof. (i) Suppose $a \in L \subseteq [2, \dots, n]$ is black and there is a linear order v in $\mathcal{D}(S_K)$ whose restriction to subset $\{a, b, c\}$ is abc with $b < a < c$. Let X be the first w-convex set from the flag corresponding to v that contains a. Then $a = \ell_i$ and $X = \{\ell_i, \dots, \ell_1, 1, k_1, \dots, k_j\}$ does not contain b and c (here it is possible that $j = 0$). We note that $b \notin L$ as $b < \ell_i$ and not in X, thus $b = k_s$ with $s > j$ (and $j \neq 0$). But then $k_j < k_s < \ell_i$ (or $1 < k_s < \ell_i$ if $j = 0$) and X is not w-convex.

(ii) Suppose $a \in K \subseteq [2, \dots, n]$ is white and a certain linear order v in $\mathcal{D}(S_K)$ has restriction bca to subset $\{a, b, c\}$ is with $b < a < c$. Let X be the largest w-convex set in the flag corresponding to v that still does not contain a. Then it contains b and c which contradicts to w-convexity of X. □

Theorem 5.2.3 (Slinko [2024b])**.** $\mathcal{D}(S_K) = F_K$*.*

Proof. By Lemma 5.2.2 we know that $\mathcal{D}(S_K) \subseteq F_K$ as $\mathcal{D}(S_K)$ satisfies all conditions (5.1.1). Let us prove the converse. Let $v = a_1 \dots a_n \in F_K$. Then the k-th ideal of v is

$\mathrm{Id}_k(v) = \{a_1, \ldots, a_k\}$. It is enough to show that for any $k \in [n]$ the set $\mathrm{Id}_k(v)$ is w-convex. We need to check conditions (a)–(c) of Definition 5.2.2.

As elements of L must satisfy the never first condition they cannot be first in v, thus a_1 must be white. Hence condition (b) of Definition 5.2.2 is satisfied.

Next, we prove (c). Suppose the contrary. Then there exist $a, b, c \in [n]$ with $a, c \in \mathrm{Id}_k(v)$ and $b \notin \mathrm{Id}_k(v)$ satisfying $a < b < c$ with $b \in K$ being white. Then the restriction of v onto $\{a, b, c\}$ is acb or cab in violation of $bN_{\{a,b,c\}}3$.

To prove (a) suppose, first, that both 1 and n are not in $\mathrm{Id}_k(v)$. Then no black bead can be in $\mathrm{Id}_k(v)$. Indeed, if $\ell \in \mathrm{Id}_k(v)$ is black, then either $\ell 1 n$ or $\ell n 1$ is in the restriction of v onto $\{1, \ell, n\}$ which contradicts $\ell N_{\{1,\ell,n\}}1$. Then, due to (c), $\mathrm{Id}_k(v)$ is an arc.

Now without loss of generality we assume that $1 \in \mathrm{Id}_k(v)$ and $n \notin \mathrm{Id}_k(v)$. Then for some $k_j \in \mathrm{Id}_k(v)$ the white beads in $\mathrm{Id}_k(v)$ form an arc $\{1, \ldots, k_j\}$. Let k_j be maximal with this property. The white arc $\{k_{j+1}, \ldots, n\}$ has no intersection with $\mathrm{Id}_k(v)$ which implies that the black arc $\{\ell_p \mid k_j < \ell_p < n\}$ has also empty intersection with $\mathrm{Id}_k(v)$.

If $\mathrm{Id}_k(v)$ is not an arc, then we must have $\ell_s \in \mathrm{Id}_k(v)$ such that for some $\ell_i \notin \mathrm{Id}_k(v)$ with $\ell_i < \ell_s < n$. But such a case would contradict to $\ell_s N_{\{\ell_i,\ell_s,n\}}1$. This contradiction proves (a) and, hence, the theorem. □

5.3 Properties of GF-domains

Lemma 5.3.1. *For any $K \subseteq [2, \ldots, n-1]$ the domain F_K is copious.*

Proof. We will use Theorem 5.2.3 and consider $\mathcal{D}(S_K)$ instead of F_K. Let $a, b, c \in [n]$ with $a < b < c$. We need to consider several cases.

1. a, b, c are all white. Then $a = k_p$, $b = k_s$, $c = k_r$ with $p < s < r$. The following sets are w-convex:

$$\{k_p\}, \{k_s\}, \{k_r\}, \{k_p, \ldots, k_s\}, \{k_s, \ldots, k_r\}, \{k_p, \ldots, k_s, \ldots, k_r\}.$$

   Thus, abc, cba, bac, bca all belong to the restriction of $\mathcal{D}(S_K)$ onto $\{a, b, c\}$.

2. a, b, c are all black. Note that every arc containing K is w-convex. Suppose $a = \ell_p$, $b = \ell_q$, $c = \ell_r$ with $p < q < r$. Let

$$K' = K \cup \{1\} \cup \{n\} \cup \{\ell_1, \ldots, \ell_{p-1}\} \cup \{\ell_t, \ldots, \ell_{r-1}\}.$$

   Then the sequence of $K' \cup \{\ell_r\} \subset K' \cup \{\ell_r, \ell_p\} \subset [n]$ gives us cab and the sequence $K' \cup \{\ell_p\} \subset K' \cup \{\ell_r, \ell_p\} \subset [n]$ gives us acb. Hence we have four suborders in $\mathcal{D}(S_K)|_{\{a,b,c\}}$. Also, the sequence $K' \cup \{\ell_r\} \subset K' \cup \{\ell_r, \ldots, \ell_q\} \subset [n]$ gives cba and the sequence $K' \cup \{\ell_p\} \subset K' \cup \{\ell_p, \ldots, \ell_q\} \subset [n]$ gives abc.

3. a is white; b, c are black. Then obviously, abc and acb belong to the restriction of $\mathcal{D}(S_K)$ onto $\{a, b, c\}$. Since $a \neq n$, in the restriction of $\mathcal{D}(S_K)$ onto $\{n, a, b, c\}$ we have $ncba$ and $ncab$, hence cba and cab belong to $\mathcal{D}(S_K)|_{\{a,b,c\}}$, so this restriction has four suborders.

4. b is white; a, c are black. Then bac and bca are in $\mathcal{D}(S_K)|_{\{a,b,c\}}$ and since $1 \neq a$ and $c \neq n$, we observe that $ncba$ and $1abc$ belong to the restrictions of $\mathcal{D}(S_K)$ onto $\{n, a, b, c\}$ and $\{1, a, b, c\}$, respectively. Hence cba and abc are in $\mathcal{D}(S_K)|_{\{a,b,c\}}$ and this restriction has four suborders as well.

5. a is black; b, c are white. Then bac, bca, cab, cba belong to the respective restriction, so four suborders exist.

6. If a and b are black and c is white. Then cba and cab belong to $\mathcal{D}(S_K)|_{\{a,b,c\}}$ together with $1abc$ and $1acb$ so four suborders exist.

These are all possible cases. □

Combining Proposition 1.1.8 with Lemma 5.3.1 we get

Theorem 5.3.2. *For any $K \subseteq [2, \dots, n-1]$ the domain F_K is a maximal Condorcet domain.*

Maximality of GF-domains has a number of profound consequences.

Theorem 5.3.3. *Every GF-domain $\mathcal{D}$ is a directly connected domain of maximal width.*

Proof. Due to the maximality of $\mathcal{D}$ the proof follows from Corollary 4.5.8. □

Our Theorem 5.2.3 as a corollary provides a constructive proof of the following result.

Corollary 5.3.4 (Karpov [2023]). *Every GF-domain F_K is single-peaked on a circle.*

Proof. By Theorem 5.2.3 F_K is isomorphic to $\mathcal{D}(S_K)$. The statement now follows from the fact that any upper contour set of this domain is a contiguous arc of the necklace. □

Original Karpov's proof was based on the characterisation of single-peaked on a circle domains by means of forbidden configurations given in Peters and Lackner [2020].

The question may be asked: Are all peak-pit maximal Condorcet domains of maximal width single-peaked on a circle? The answer is negative.

Theorem 5.3.5. *For any $n \geq 4$ maximal single-crossing Condorcet domain are not single-peaked on a circle.*

Proof. In Theorem 3.4.1 we characterised all single-crossing maximal Condorcet domains in terms of the relay structure. In this structure linear orders are arranged in a sequence so that moving from left to right 1 initially moves from top to bottom being swapped with $2, 3, 4, \dots, n$. Then n is swapped with $n-1, n-2, \dots, 2$ and moves to the top. The following three orders can then be found: $u = 21\dots$, $v = 23\dots$, $w = 2n\dots$. Then $\mathrm{Id}_2(u) = \{1, 2\}$, $\mathrm{Id}_2(v) = \{2, 3\}$ and $\mathrm{Id}_2(w) = \{2, 4\}$. But it is impossible to have such three arcs on a circle. □

In this proof effectively we spotted in any single-crossing maximal Condorcet domain one of the forbidden configurations described in Theorem 2.4.2.

5.4 Set-alternating scheme domains

Let us introduce another interesting complete set of never-conditions.

Definition 5.4.1 (Karpov et al. [2023]). *A complete set of never-conditions* (1.1.3) *is said to be a* set-alternating scheme, *if for some subset* $K \subseteq [2, \ldots, n-1]$ *and for all* $1 \leq i < j < k \leq n$ *we have*

$$iN_{\{i,j,k\}}3, \text{ if } j \in K, \text{ and } kN_{\{i,j,k\}}1, \text{ if } j \notin K. \tag{5.4.1}$$

The domain which consists of all linear orders satisfying the set of never conditions (5.4.1) *is called the* set-alternating scheme domain *or* SA-domain *and denoted* $SA_{n,K}$.

Example 5.4.1. *Let* $n = 4$ *and* $K = \{2\}$. *Then the corresponding set of never-conditions for* $SA_{4,\{2\}}$ *would be*

$$1N_{\{1,2,3\}}3, \quad 1N_{\{1,2,4\}}3, \quad 4N_{\{1,3,4\}}1, \quad 4N_{\{2,3,4\}}1.$$

We see that $SA_{4,\{2\}}$ *is isomorphic to Fishburn's domain* F_4 *relative to the axis* $2 \lhd 1 \lhd 4 \lhd 3$ *under* $\sigma \in S_4$ *such that* $\sigma(1) = 2$, $\sigma(2) = 1$, $\sigma(3) = 4$, $\sigma(4) = 3$.

Example 5.4.2. *Let* $n = 4$ *and* $K = \emptyset$. *Then the corresponding set of never-conditions would be*

$$1N_{\{1,2,3\}}3, \quad 1N_{\{1,2,4\}}3, \quad 1N_{\{1,3,4\}}3, \quad 2N_{\{2,3,4\}}3.$$

We see that in such a case $SA_{4,\emptyset}$ *is isomorphic to Arrow single-peaked domain* $\mathcal{D}_{4,5}$ *from Example 2.2.2 under* $\sigma \in S_4$ *such that* $\sigma(1) = 2$, $\sigma(2) = 3$, $\sigma(3) = 1$, $\sigma(4) = 4$.

We see that this scheme has potential since it delivers two maximal Condorcet domains for $n = 4$. One quick observation that we can make is the following: for any K the domain $SA_{n,K}$ contains the order $12 \ldots n$ but may not contain $n \ldots 21$ (as, for example, in Example 5.4.2). Domains containing $12 \ldots n$ some researchers call *unital*.

Lemma 5.4.1. *Let* $\mathcal{D} = SA_{n,K}$ *be a set-alternating scheme domain for* $K \subseteq [n]$.

(i) Suppose $v = v'a1v''$ *is a linear order from* $\mathcal{D}$. *Then* $u = v'1av''$ *also belongs to* $\mathcal{D}$;

(ii) Suppose $v = v'nav''$ *is a linear order from* $\mathcal{D}$. *Then* $u = v'anv''$ *also belongs to* $\mathcal{D}$.

Proof. We will prove only (i) since (ii) is similar. We have to pay attention only to restrictions of u to subsets $\{1, a, b\}$, where $a, b \in [n]$ since restrictions of u on other subsets coincide with the corresponding restrictions of v.

Let us assume $1 < a < b$. If $a \in K$, then the never condition for this triple would be $1N_{\{1,a,b\}}3$. This gives us allowable suborders $1ab$, $1ba$, $a1b$, $b1a$. The only option for v is, thus, $v|_{\{1,a,b\}} = a1b$. Since $1ab$ is also allowed, we have $u = v'1av''$ in $\mathcal{D}$.

If $a \notin K$, then the never condition for this triple would be $bN_{\{1,a,b\}}1$. This gives us allowable suborders $1ab$, $1ba$, $a1b$, $ab1$. Thus we have $v|_{\{1,a,b\}} = a1b$. Since $1ab$ is also allowed, we have $u = v'1av''$ in $\mathcal{D}$.

Now assume that $1 < b < a$. If $b \in K$, then the never condition for the triple $\{1, a, b\}$ is $1N_{\{1,a,b\}}3$ and the argument is the same as in the first case. If $b \notin K$, then the never condition is $aN_{\{1,a,b\}}1$. The allowable suborders are $1ab$, $1ba$, $ba1$, $b1a$. Thus we have $v|_{\{1,a,b\}} = ba1$. Since $b1a$ is also available we can switch 1 and a in v so $u \in \mathcal{D}$. □

Lemma 5.4.2. *Let $K \subseteq [2, \dots, n-1]$, then*

(i) $(SA_{n,K})_{-1} = SA_{n-1,K\setminus\{2\}}$, *where the latter domain is defined on* $[n] \setminus \{1\}$*;*

(ii) $(SA_{n,K})_{-n} = SA_{n-1,K\setminus\{n-1\}}$, *where the latter domain is defined on* $[n] \setminus \{n\}$*.*

Proof. (i) Let $\{i, j, k\} \subseteq [n] \setminus \{1\}$ such that $2 \le i < j < k \le n$. The never conditions for these triples for both domains coincide. Hence $(SA_{n,K})_{-1} \subseteq S_{n-1,K\setminus\{2\}}$. To show the equality we must prove that every order v from $SA_{n-1,K\setminus\{2\}}$ can be obtained by removal of 1 from an order from $SA_{n,K}$. It is enough to show that $1v \in SA_{n,K}$. No matter if 2 is in K or not $1v$ satisfies both $1N_{\{1,2,i\}}3$ and $iN_{\{1,2,i\}}1$. Thus $1v \in SA_{n,K}$ and $v \in (SA_{n,K})_{-1}$. The proof of (ii) is similar. □

Theorem 5.4.3. *For any $n \ge 3$ and $K \subseteq \{2, .., n\}$ the domain SA_K is directly connected.*

Proof. By Theorem 1.3.16 it is enough to prove connectedness. When $n = 3$, we have two choices $K = \emptyset$ and $K = \{2\}$. In both cases the domain $SA_{3,K}$ is known to be connected. Let us assume that for $n = k-1$ and all $K \subseteq \{2, \dots, k-2\}$ the domain $SA_{k-1,K}$ is connected. Let now $n = k$. Then by Lemma 5.4.2 we have $(SA_{k,K})_{-1} = SA_{k-1,K\setminus\{2\}}$ is connected by the induction hypothesis.

Let $u, v \in SA_{k,K}$ be arbitrary. Then by Lemma 5.4.1 u and v are connected to $1u'$ and $1v'$, respectively. But u' and v' are connected in $SA_{k-1,K\setminus\{2\}}$, hence u and v are connected. □

Theorem 5.4.4. *For any $K \subseteq [2, \dots, n-1]$ the domain $SA_{n,K}$ is copious peak-pit maximal Condorcet domain.*

Proof. By its definition $\mathcal{D} = SA_{n,K}$ is peak-pit. Let us prove it is copious. It is easy to check that for $n = 3$ the statement is true. By Lemma 5.4.2 and induction hypothesis we can assume that copiousness has to be proved only for triples $\{1, j, n\}$. As $\mathcal{D}_{-1}$ and $\mathcal{D}_{-n}$ are copious, hence ample, in $\mathcal{D}|_{\{1,j,n\}}$ we have the following suborders $1jn$, $1nj$, $j1n$.

Let $K = \{k_1, \dots, k_s\}$ with $k_1 < \dots < k_s$ and $L = [2, \dots, n-1] \setminus K = \{\ell_1, \dots, \ell_t\}$ with $\ell_1 < \dots < \ell_t$. Let us show that the order

$$v = \ell_1 \dots \ell_t n 1 k_1 \dots k_s$$

in $SA_{n,K}$. For the triple $p < \ell_q < r$ order v must satisfy $rN_{\{p,\ell_q,r\}}1$. Indeed, if $r \in L$, then $r = \ell_u$ with $u > q$ and $rN_{\{p,\ell_q,r\}}1$ is satisfied since ℓ_u is to the right of ℓ_q in v. If $r \in K \cup \{1, n\}$, then this condition is satisfied too as r is always to the right of ℓ_q in v. The argument is similar for the triple $p < k_q < r$ to satisfy $pN_{\{p,k_q,r\}}3$.

Thus we always have four suborders in $\mathcal{D}|_{\{1,j,n\}}$:

- $1jn, 1nj, j1n, n1j$, if $j \in K$;
- $1jn, 1nj, j1n, jn1$, if $j \notin K$;

So $SA_{n,K}$ is copious. By Proposition 1.1.8 it is also maximal. □

5.5 Cambrian acyclic domains

In this section we follow Labbé and Lange [2020] who introduced the domains in the title. We will show that they are nothing else but GF-domains that we introduced in Section 5.1.

Let (W, S) be a *Coxeter system* with the set of generators $S = \{s_1, \ldots, s_{n-1}\}$. For example, the symmetric group S_n as a Coxeter group of type A_{n-1}, is generated by the simple transpositions $s_i = (i, i+1)$, $i \in [n-1]$. The corresponding Dynkin diagram A_{n-1} is

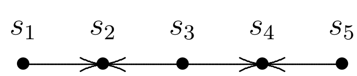
Let $\mathcal{A}$ be an orientation of A_{n-1} converting Dynkin diagram into a Dynkin quiver. Dynkin quivers correspond to Coxeter elements.

Definition 5.5.1. *Let* (W, S) *be a Coxeter system with* $|S| = n$. *A* Coxeter element $\mathbf{c} \in W$ *is the product of* n *distinct generators of* S *in some order and* Cox(W, S) *is the set of all Coxeter elements of* (W, S).

Given a reduced presentation for $\mathbf{c}$ we write $s \to t$, if s and t do not commute and s is to the left of t in $\mathbf{c}$. For example,

$$\mathbf{c} = s_3 s_5 s_1 s_2 s_4 \in S_6$$

correspond to the Dynkin quiver



Following Reading [2006], we distinguish between up and down elements of $\{2, \ldots, n-1\}$. An element $i \in \{2, \ldots, n-1\}$ is up if the edge $\{s_{i-1}, s_i\}$ is directed from s_i to s_{i-1} and down otherwise. As in Reading's work, 1 and n can be chosen to be up or down. Let $D_{\mathbf{c}}$ be the set of down elements (possibly empty) and let $U_{\mathbf{c}}$ be the set of up elements. In this example $U_{\mathbf{c}} = \{1, 3, 5\}$ and $D_{\mathbf{c}} = \{2, 4, 6\}$.

Elements from S satisfy commutation relations of the form $s_i s_j = s_j s_i$ for $|i - j| > 1$. An application of a commutation relation to a product of elements of S is called a *commutation move*. Given a reduced word $\mathbf{u}$ of a permutation, the equivalence class consisting of all words that can be obtained from $\mathbf{u}$ by a sequence of commutation moves is the commutation class of $\mathbf{u}$. We write $\mathbf{u} \sim \mathbf{v}$ if both belong to the same commutation class.

Of particular interest is the unique element $\mathbf{w}_0 \in W$ of maximum length. A reduced expression $\mathbf{w}_0^{\mathbf{c}} = s_1 \ldots s_N$ is called *the longest word*. We note that not all longest words are equivalent. For example, for $\mathbf{w}_0 = 4321 \in S_4$ we have

$$\mathbf{w}_0 = s_1 s_2 s_3 s_1 s_2 s_1 = s_1 s_2 s_3 s_2 s_1 s_2$$

and these two longest words are not equivalent.

A word $s_1 s_2 \dots s_r$ is a prefix up to commutations of a word $\mathbf{w}$ if and only if there is a word $\mathbf{w}' \sim \mathbf{w}$ such that the first r letters of $\mathbf{w}'$ are $s_1 \dots s_r$. Now we can define $\mathbf{c}$-singletons for $\mathbf{c} \in \mathrm{Cox}(W, S)$. When we fix a Coxeter element $\mathbf{c} \in \mathrm{Cox}(\mathbf{W}, \mathbf{S})$ we say that $(W, S, \mathbf{c})$ is a Coxeter triple.

Fix $\mathbf{c}$ and a particular reduced word $s_1 s_2 \dots s_n$ for $\mathbf{c}$ and write a half-infinite word

$$\mathbf{c}^\infty = s_1 s_2 \dots s_n | s_1 s_2 \dots s_n | s_1 s_2 \dots s_n | \dots .$$

The symbols $|$ are inert "dividers" which facilitate the definition of sortable elements but play no other role. When subwords of $\mathbf{c}^\infty$ are interpreted as expressions for elements of W, the dividers are ignored.

Definition 5.5.2. *The* $\mathbf{c}$-sorting word *for* $\mathbf{w} \in W$ *is the lexicographically first (as a sequence of positions in* $\mathbf{c}^\infty$*) subword of* $\mathbf{c}^\infty$ *which is a reduced word for* $\mathbf{w}$*.*

The $\mathbf{c}$-sorting word can be interpreted as a sequence of subsets of S: Each subset in the sequence is the set of letters of the $\mathbf{c}$-sorting word which occur between two adjacent dividers.

Definition 5.5.3. *An element* $\mathbf{w} \in W$ *is* $\mathbf{c}$*-sortable if its* $\mathbf{c}$*-sorting word defines a sequence of subsets which is weakly decreasing under inclusion.*

Example 5.5.1. *For the Coxeter element* $\mathbf{c} = s_1 s_2 s_3$ *of* S_4 *the reverse permutation* $\mathbf{w}_0 = [n+1, n, \dots, 1]$ *(given in one-line notation) is* $\mathbf{c}$*-sortable since for the presentation*

$$\mathbf{w}_0 = s_1 s_2 s_3 | s_1 s_2 | s_1$$

the corresponding sequence of subsets is $\{1, 2, 3\} \supset \{1, 2\} \supset \{1\}$*. We also have*

$$\mathbf{w}_0 = s_1 s_2 s_3 | s_2 | s_1 s_2$$

but $s_1 s_2 s_3 | s_2 | s_1 s_2$ *is not a* $\mathbf{c}$*-sorting word for* $\mathbf{w}_0$ *as it is not lexicographically first to represent* $\mathbf{w}_0$*.*

By $\mathbf{w}_0^{\mathbf{c}}$ we always mean a $\mathbf{c}$-sorting word for $\mathbf{w}_0$.

Since any two equivalent reduced words for $\mathbf{c}$ are related by commutation of letters, the $\mathbf{c}$-sorting words for $\mathbf{w}$ arising from different reduced words for $\mathbf{c}$ are related by commutations of letters, with no commutations across dividers. In particular, the set of $\mathbf{c}$-sortable elements does not depend on the choice of reduced word for $\mathbf{c}$.

Definition 5.5.4 ($\mathbf{c}$-singletons)**.** *Let* (W, S) *be a Coxeter system. An element* $\mathbf{w} \in W$ *is a* $\mathbf{c}$-singleton *if and only if some reduced expression of* $\mathbf{w}$ *is a prefix of* $\mathbf{w}_0^{\mathbf{c}}$ *up to commutations.*

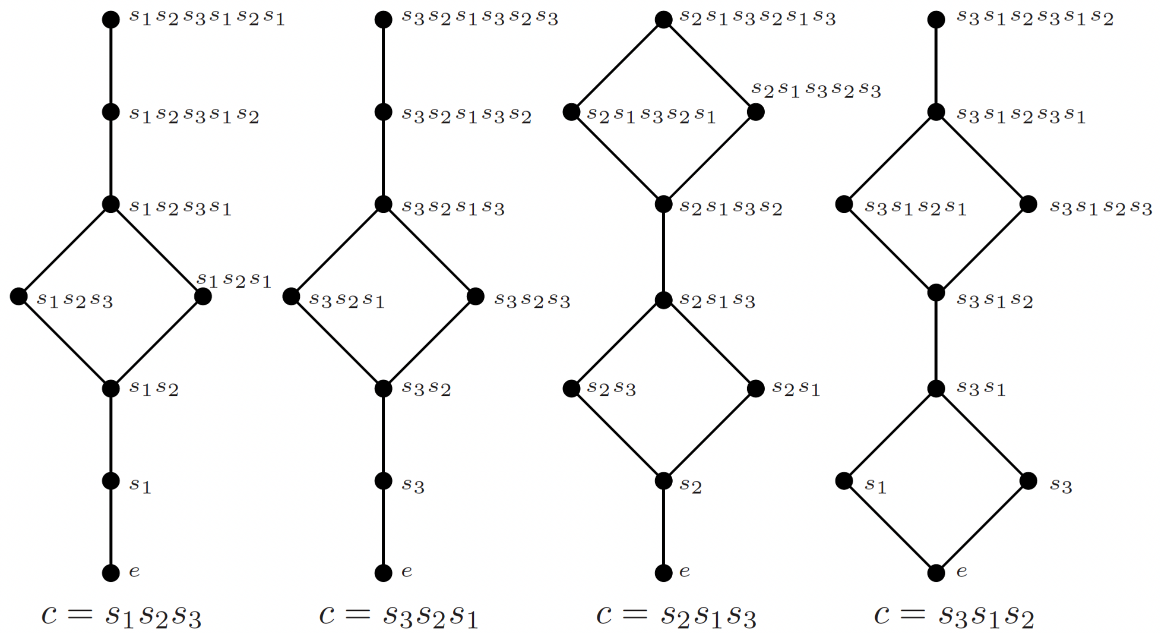


Figure 5.5: There are four Coxeter elements in S_4. Each yields a distributive lattice of singletons.

The **c**-singletons constitute a distributive sublattice of the (right) weak order on W [Reading, 2006].

[Reading, 2006, Proposition 5.7] characterised **c**-sortable and **c**-singleton permutations using pattern-avoidance. We say that a permutation $\mathbf{w}$ avoids the pattern $31\underline{2}$ if w contains no 312-pattern such that the last entry 2 in the pattern is a lower-barred number (i.e., in $U_\mathbf{c}$). This is the same to say that if $j \in U_\mathbf{c}$, then for any triple $\{i, j, k\}$ with $i < j < k$ the restriction of w onto $\{i, j, k\}$ is different from kij. Also, avoidance of the pattern $13\underline{2}$ means that the restriction of w onto $\{i, j, k\}$ is different from ikj. Altogether avoidance of both patterns means that j is never last among i, j, k which is the condition $jN_{\{i,j,k\}}3$. Similarly, avoidance of the patterns $\overline{2}31$ and $\overline{2}13$ is equivalent to never condition $jN_{\{i,j,k\}}1$.

Proposition 5.5.1. *A permutation $\mathbf{w} \in S_n$ is $\mathbf{c}$-sortable if and only if it avoids the patterns $31\underline{2}$ and $\overline{2}31$. Furthermore, a $\mathbf{c}$-sortable permutation $\mathbf{w}$ is a $\mathbf{c}$-singleton if and only if $\mathbf{w}$ avoids the patterns $13\underline{2}$ and $\overline{2}13$.*

In particular, the reverse permutation $\mathbf{w}_0$ is always **c**-sortable.

Theorem 5.5.2. *Let $(W, S, \mathbf{c})$ be a Coxeter triple, where $\mathbf{c}$ is a Coxeter element. The set of $\mathbf{c}$-singletons is a generalised Fishburn domain F_K with $K = U_\mathbf{c}$.*

Hohlweg et al. [2011] showed that the **c**-singletons constitute a distributive sublattice of the (right) weak order on W. The polytopal realisation $\mathrm{Asso}_\mathbf{c}(W)$ of this lattice for any finite Coxeter group W and any Coxeter element $\mathbf{c}$ was obtained [Hohlweg et al., 2011, Theorem 3.4] as well the analogue $\mathrm{Perm}_{\mathbf{c}}(W)$.

It was proved that

Theorem 5.5.3. *The set $Acyc_{\mathbf{c}}$ of $\mathbf{c}$-singleton is the set of common vertices of $Perm_{\mathbf{c}}(A_{n-1})$ and $Asso_{\mathbf{c}}(A_{n-1})$.*

One would want to compare this theorem with Theorem 7.5.2.

Chapter 6

Compositions and decompositions of Condorcet domains

Some Condorcet domains can be composed from smaller Condorcet domains by means of a certain construction, called *composition*. We will introduce several such constructions. Some of them are more suitable for never-middle domains and some for peak-pit domains. Here we consider a composition that is 'friendly' for never-middle domains. Another domains of interest are *symmetric domains* which are domains that together with a linear order x contain its completely reversed (flipped) order $\bar{x}$.

6.1 A composition of Condorcet domains

Definition 6.1.1. *Let $\mathcal{E}$ be a Condorcet domain on an m-element set of alternatives $A = \{a_1, \ldots, a_m\}$. Let also $\mathcal{D}_1, \ldots, \mathcal{D}_m$ be Condorcet domains on disjoint sets $C_1, \ldots, C_m$ of alternatives. Then we define the domain on $C_1 \cup \ldots \cup C_m$ as*

$$\mathcal{E}(a_1 \to \mathcal{D}_1, \ldots, a_m \to \mathcal{D}_m) := \{\mathbf{u}_1 \ldots \mathbf{u}_m \mid \mathbf{u}_j \in \mathcal{D}_{i_j} \text{ and } a_{i_1} \ldots a_{i_m} \in \mathcal{E}\}.$$

When it can cause no confusion, we will denote this domain as $\mathcal{E}(\mathcal{D}_1, \ldots, \mathcal{D}_m)$. We call $\mathcal{E}$ the top level domain *and $\mathcal{D}_1, \ldots, \mathcal{D}_m$* ground level domains*.*

This definition is similar, in spirit, to the definition of the wreath product of permutations introduced in Atkinson and Stitt [2002].

Definition 6.1.2. *A domain $\mathcal{D} \subseteq \mathcal{L}(A)$ is called* decomposable*, if it is isomorphic to $\mathcal{E}(\mathcal{D}_1, \ldots, \mathcal{D}_m)$, where $|C_i| > 1$ for at least one $\mathcal{D}_i$ where $i \in [m]$.*

Proposition 6.1.1. *Let $|A| = m$ and $\mathcal{E}, \mathcal{D}_1, \ldots, \mathcal{D}_m$ be Condorcet domains on disjoint sets of alternatives $A, C_1, \ldots, C_m$, respectively. Then $\mathcal{D} = \mathcal{E}(\mathcal{D}_1, \ldots, \mathcal{D}_m)$ is again a Condorcet domain with*

$$|\mathcal{E}(\mathcal{D}_1, \ldots, \mathcal{D}_m)| = |\mathcal{E}| \prod_{i=1}^{m} |\mathcal{D}_i|. \tag{6.1.1}$$

Moreover, $\mathcal{D}$ is a symmetric domain if and only if all domains $\mathcal{E}, \mathcal{D}_1, \ldots, \mathcal{D}_m$ are symmetric.

Proof. We show that for any triple of elements a, b, c from $C_1 \cup \ldots \cup C_m$ one of the never-conditions is satisfied. Indeed, if $\{a, b, c\} \subseteq C_i$ for some i, or $a \in C_i$, $b \in C_j$ and $c \in C_k$ for distinct i, j, k, this follows from the fact that domains $\mathcal{D}_i$ and $\mathcal{E}$ are Condorcet domains, respectively. Each of these triples satisfies the same never condition in $\mathcal{D}$ as in $\mathcal{D}_i$ or $\mathcal{E}$, respectively. If, however, $a, b \in C_i$ and $c \in C_j$ with $i \neq j$, then we can have only orders abc, bac, cab, cba in the restriction $\mathcal{D}|_{\{a,b,c\}}$ which means that this triple satisfies $cN_{\{a,b,c\}}2$. If $\mathcal{E}$ and all $\mathcal{D}_1, \ldots, \mathcal{D}_m$ are symmetric, then it is straightforward to see that $\mathcal{D}$ is symmetric as well. The formula (6.1.1) follows directly from the definition. □

As we will see later, if $\mathcal{E}, \mathcal{D}_1, \ldots, \mathcal{D}_m$ are symmetric maximal Condorcet domains on their respective sets of alternatives, then $\mathcal{D} = \mathcal{E}(\mathcal{D}_1, \ldots, \mathcal{D}_m)$ is again a symmetric maximal Condorcet domain.

When $|A| = 2$, $\mathcal{E} = \{a_1a_2, a_2a_1\}$ and $\mathcal{D}_1, \mathcal{D}_2$ be two Condorcet domains the operation

$$\mathcal{D}_1 \star \mathcal{D}_2 \ := \ \mathcal{E}(\mathcal{D}_1, \mathcal{D}_2) \ = \ \{\mathbf{u}_1\mathbf{u}_2 \mid \mathbf{u}_1 \in \mathcal{D}_1,\ \mathbf{u}_2 \in \mathcal{D}_2\} \cup \{\mathbf{u}_2\mathbf{u}_1 \mid \mathbf{u}_1 \in \mathcal{D}_1,\ \mathbf{u}_2 \in \mathcal{D}_2\}$$

was used in Danilov and Koshevoy [2013]. We note that bipartition of a domain $\mathcal{D}$ in the sense of Raynaud [1981] is a representation of $\mathcal{D}$ as a composition of two domains $\mathcal{D} = \mathcal{D}_1 \star \mathcal{D}_2$. This composition allowed them to construct a series of never-middle maximal Condorcet domains, namely,

$$a_1 \star a_2 \star a_3 \star \ldots \star a_n$$

with some parenthesisation, where a_i is identified with the trivial domain on a single alternative a_i. In full generality it appeared in Karpov and Slinko [2023a].

We note that this composition $\star$ is commutative (which follows directly from the definition) but not associative. For example,

$$\begin{aligned} a \star (b \star c) &= \{abc, acb, bca, cba\} = \mathcal{D}_{3,2}(a, b, c), \\ (a \star b) \star c &= \{abc, bac, cab, cba\} = \mathcal{D}_{3,2}(c, b, a). \end{aligned}$$

These domains are isomorphic under the mapping $a \to c$, $b \to b$ and $c \to a$. However, this isomorphism does not extends to monomials of length four as the following example shows.

Example 6.1.1. *Consider the following two domains:*

$$\begin{aligned} ((a \star b) \star c) \star d &= \{abcd, bacd, cabd, cbad, dabc, dbac, dcab, dcba\}, \\ (a \star b) \star (c \star d) &= \{abcd, bacd, abdc, badc, cdab, cdba, dcab, dcba\} \end{aligned}$$

are obviously not isomorphic (one alternative occupy the last position in the first domain four times while there is no such alternative in the second domain). However, as shown on Figure 6.1, their graphs are nevertheless identical.

As we have seen $\mathcal{D}_{3,2}$ is decomposable. However, the other domains listed in (1.1.2) are not.

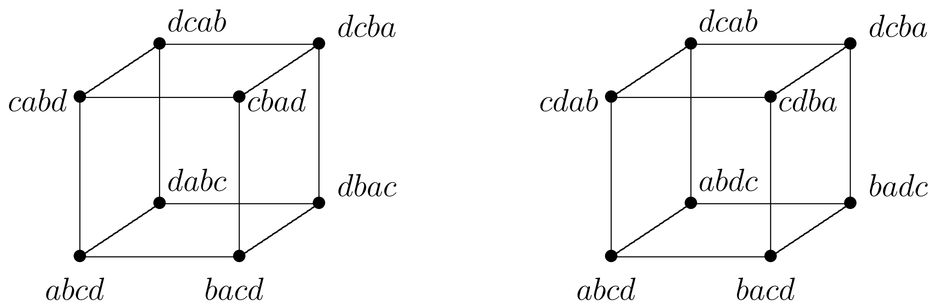


Figure 6.1: The graphs of two non-isomorphic maximal decomposable Condorcet domain are isomorphic.

Proposition 6.1.2. *$\mathcal{D}_{3,1}$ and $\mathcal{D}_{3,3}$ are indecomposable.*

Proof. Since both $(a \star b) \star c$ and $a \star (b \star c)$ lead to $\mathcal{D}_{3,2}$ we conclude that $\mathcal{D}_{3,1}$ and $\mathcal{D}_{3,3}$ are indecomposable. □

The following decomposable example is interesting because the top level domain now is $\mathcal{D}_{3,3}$ which is indecomposable. It cannot be obtained by the Danolov-Koshevoy construction.

Example 6.1.2. *Let us consider the following maximal Condorcet domain for $m = 4$ alternatives:*

$$\mathcal{D}_{3,3}(a \star b, c, d) = \{abcd, abdc, bacd, badc, dbac, dabc, cabd, cbad\}.$$

which satisfies never-conditions

$$cN_{\{a,b,c\}}2, \quad dN_{\{a,b,d\}}2, \quad aN_{\{a,c,d\}}3, \quad bN_{\{b,c,d\}}3.$$

The graph of this domain is shown on Figure 6.2.

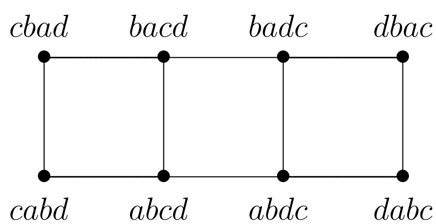


Figure 6.2: Graph of a decomposable maximal Condorcet domain on four alternatives

Theorem 6.1.3. *Every decomposable maximal Condorcet domain on the set of four alternatives has cardinality 8.*

Proof. Indeed, it can be isomorphic to one of the following $(a \star b) \star (c \star d)$, $((a \star b) \star c) \star d$, $\mathcal{D}_{3,3}(a, b \star c, d)$, $\mathcal{D}_{3,3}(a \star b, c, d)$. Each of these contains eight orders. □

We note that this phenomenon can be observed only for $m = 4$. For $m = 5$ we will have already decomposable maximal domains of two different cardinalities 8 and 16.

As we will see in Chapter 7 for any number of alternatives there are indecomposable symmetric maximal Condorcet domains of cardinality 4.

6.2 Clones and decomposability of domains

Let $\mathcal{E}$ be a Condorcet domain on the m-element set of alternatives $A = \{a_1, \ldots, a_m\}$. Let also $\mathcal{D}_1, \ldots, \mathcal{D}_m$ be Condorcet domains on disjoint sets $C_1, \ldots, C_m$ of alternatives.

Definition 6.2.1 (Tideman [1987]). *Let $P = (\succ_1, \ldots, \succ_n)$ be a profile on A. We say that a non-empty subset $C \subseteq A$ is a* clone set *for P if for every $a \in A \setminus C$ and every $c, c' \in C$*

$$c \succ_i a \Longrightarrow c' \succ_i a \quad \text{and} \quad a \succ_i c \Longrightarrow a \succ_i c'$$

for every $i = 1, 2, \ldots, n$. A clone set C is proper, *if $1 < |C| < |A|$. This definition can be also applied to domains in the obvious way.*

Proposition 6.2.1. *Let $\mathcal{E}$ be a Condorcet domain on the m-element set of alternatives $A = \{a_1, \ldots, a_m\}$. Let also $\mathcal{D}_1, \ldots, \mathcal{D}_m$ be Condorcet domains on disjoint sets $C_1, \ldots, C_m$ of alternatives. If $m \geq 2$ and $|C_i| \geq 2$ for some i, then C_i is a proper clone set of the domain $\mathcal{E}(\mathcal{D}_1, \ldots, \mathcal{D}_n)$.*

Proof. Follows from the definitions of the composition and the clone set. □

We note that $\{a, b\}$ and $\{a, b, c\}$ are the only non-trivial clone sets of the left domain in Figure 6.1 while the domain on the right had non-trivial clone sets $\{a, b\}$ and $\{c, d\}$.

Lemma 6.2.2. *Suppose $\mathcal{D}$ is a Condorcet domain on the set of alternatives A and $C \subseteq A$ is a clone set in $\mathcal{D}$. Then*

(a) the restriction $\mathcal{D}|_C$ of $\mathcal{D}$ onto C is a Condorcet domain in $\mathcal{L}(C)$;

(b) the contraction $\mathcal{D}(C \to x)$, obtained by replacing in every order of $\mathcal{D}$ the block of occurrences of elements of C (no matter how the elements of C are ordered in those orders) by a single alternative $x \notin A$, is a Condorcet domain on $(A \setminus C) \cup \{x\}$.

Proof. (a) follows from the fact that $\mathcal{D}|_C$ satisfies a complete set of never conditions of $\mathcal{D}$ in which only elements of C are involved. To prove (b) we notice that if $\mathcal{D}$ satisfies one of the never conditions for a triple $\{a, a', c\}$, where $a, a' \in A \setminus C$, then, since C is a clone, it will satisfy the same never condition for any triple $\{a, a', c'\}$, where $c' \in C$. Hence $\mathcal{D}(C \to x)$ will satisfy the same never condition for the triple $\{a, a', x\}$. □

Theorem 6.2.3. *If a Condorcet domain $\mathcal{D} \subset \mathcal{L}(A)$ has a non-trivial clone set, then $\mathcal{D}$ is contained in some decomposable Condorcet domain. In particular, if $\mathcal{D}$ is maximal, it is decomposable.*

Proof. Suppose C is a clone set of $\mathcal{D}$. Let us introduce a new alternative $x \notin A$ and consider the contraction $\mathcal{E} = \mathcal{D}(C \to x)$ and also $\mathcal{D}' = \mathcal{D}|_C$ which is the restriction of $\mathcal{D}$ on C. By Lemma 6.2.2 both are Condorcet domains. Then $\mathcal{D} \subseteq \mathcal{E}(x \to \mathcal{D}')$ and the latter is decomposable. If $\mathcal{D}$ is maximal, then $\mathcal{D} = \mathcal{E}(x \to \mathcal{D}')$ and it is itself decomposable. □

We see that the existence of a non-trivial clone set is almost equivalent to decomposability. For maximal Condorcet domains it is exactly equivalent.

Corollary 6.2.4. *A maximal Condorcet domain $\mathcal{D} \subset \mathcal{L}(A)$ is indecomposable if and only if it has no non-trivial clone sets.*

As we will see symmetric maximal Condorcet domains tend to be decomposable. This is very different from the peak-pit ones. To see that we prove the following result.

Theorem 6.2.5. *Any connected maximal Condorcet domain $\mathcal{D}$ has no proper clone sets and, in particular, is indecomposable.*

Proof. Suppose on the contrary that there is a proper clone set $C \subset A$ in $\mathcal{D}$ and $C = \{c_1, \ldots, c_\ell\}$. In all linear orders of $\mathcal{D}$ the alternatives from C are standing 'together'. Let $u \in \mathcal{D}$. Then $u = u_1u_2u_3$, where $u_1 = a_1, \ldots, a_s$, $u_2 = c_{i_1} \ldots c_{i_\ell}$, $u_3 = b_1, \ldots, b_t$ for some $a_i, b_j \in A$ and $c_{i_j} \in C$. Let $C' = \{a_1, \ldots, a_s\}$ and $C'' = \{b_1, \ldots, b_t\}$ (one of these sets may be empty). Due to connectedness, in all linear orders of $\mathcal{D}$ the alternatives from C' and also from C'' are also standing 'together' as they cannot jump over C so C' and C'', if non-empty, are also proper clone sets. One clone set cannot split the other one. It also cannot 'jump' over another clone set since there must be a sequence of transformations from one linear order to any other where only two elements of A are swapped at a time.

We note that $|C| > 1$. Suppose $C' \neq \emptyset$ and $C'' \neq \emptyset$. For the triple $\{a_i, b_j, c_k\}$ with $a_i \in C'$, $b_j \in C$, $c_k \in C''$, the restriction of $\mathcal{D}$ onto this triple will contain a single linear order $a_ic_kb_j$. Consider the linear order v which is not in $\mathcal{D}$. Then the restriction of $\mathcal{D} \cup \{v\}$ onto the triple, $\{a_i, b_j, c_k\}$ will be enlarged by a single linear order. However the set of two linear orders always satisfy one of the never conditions.

Consider a triple $\{a_i, c_j, c_k\}$, where $a \in C'$ and $c_i, c_j \in C$. In this case we may have $C'' = \emptyset$. Then the restriction of $\mathcal{D}$ contains at most two linear orders $a_ic_jc_k$ and $a_ic_kc_j$. If $v \notin \mathcal{D}$, then for $\mathcal{D} \cup \{v\}$ there will, therefore, be at most three triples on $\{a_i, c_j, c_k\}$. However, adding an additional linear order to $a_ic_jc_k$ and $a_ic_kc_j$ one cannot obtain a cyclic set of linear orders since a_i will not occupy either the middle or the last positions. The case of a triple $\{c_j, c_k, b_\ell\}$ is similar. Thus adding an arbitrary $v \notin \mathcal{D}$ we obtain a larger domain, a contradiction. Thus any proper clone set cannot exist in $\mathcal{D}$. □

The question may be asked: Are all domains which are not connected decomposable? This can be answered in the negative as several examples show from Dittrich [2018] classification. One can, for example, take the domain $\mathcal{D}_{4,11}$ from Example 6.4.5 which is indecomposable relative to $\star$ but decomposable relative to composition $\diamond$ which will be introduced later.

It is important that maximality is preserved under a composition.

Lemma 6.2.6. *Let $\mathcal{E}$ be a Condorcet domain on a set of alternatives $A = \{a_1, \ldots, a_m\}$ and $\mathcal{D}$ be a Condorcet domain on a set of alternatives $B = \{b_1, \ldots, b_k\}$ and $\mathcal{F} = \mathcal{E}(a_1, \ldots, a_{m-1}, a_m \to \mathcal{D})$. Suppose $\mathcal{E}$ and $\mathcal{D}$ are ample. Then $\mathcal{F}$ is ample maximal Condorcet domain on the set $C = (A \setminus \{a_m\}) \cup B$ if and only if $\mathcal{E}$ and $\mathcal{D}$ are maximal Condorcet domains.*

Proof. Firstly, we note that by Proposition 6.1.1 $\mathcal{F}$ is a Condorcet domain. Suppose $\mathcal{E}$ and $\mathcal{D}$ are maximal. Since they are ample, obviously, $\mathcal{F}$ is ample as well, let us prove its maximality. If $k = 1$, then $\mathcal{F}$ and $\mathcal{E}$ are isomorphic and the result is trivial. Suppose $k > 1$. Let $a \in A \setminus \{a_m\}$ and $b, b' \in B$. We note that, due to the fact that both domains are ample, the restriction of $\mathcal{F}$ onto $\{a, b, b'\}$ contains four orders: $abb', ab'b, bb'a, b'ba$. This means that any linear order $u \in \mathcal{L}(C)$ such that $\mathcal{F}' = \mathcal{F} \cup \{u\}$ is a Condorcet domain must satisfy $aN_{\{a,b,b'\}}2$ as well. Hence, b and b' in u cannot be split by any $a \in A \setminus \{a_m\}$ in the middle. Thus, if we collapse u by glueing all elements of B together to obtain $u' = u(u|_B \to a_m)$, then $u' \in \mathcal{E}$ will satisfy all never conditions of $\mathcal{E}$ and will belong to $\mathcal{E}$ due to maximality of the latter.

The restriction $u|_B$ of u onto B must satisfy the same never conditions as $\mathcal{D}$, hence, due to maximality of $\mathcal{D}$ it must be that $u|_B \in \mathcal{D}$. In this case $u = u'(a_m \to u|_B) \in \mathcal{F}$ and $\mathcal{F}$ is maximal. The converse is straightforward. □

This lemma implies the following theorem.

Theorem 6.2.7. *$\mathcal{E}, \mathcal{D}_1, \ldots, \mathcal{D}_n$ are ample maximal Condorcet domains, if and only if $\mathcal{D} = \mathcal{E}(\mathcal{D}_1, \ldots, \mathcal{D}_n)$ is also a maximal Condorcet domain.*

Proof. To deduce this theorem from Lemma 6.2.6 it is sufficient to note that every composition can be obtained through a succession of elementary compositions in which only one element of the top-level domain is replaced at a time. □

Corollary 6.2.8. *If $\mathcal{E}, \mathcal{D}_1, \ldots, \mathcal{D}_n$ are symmetric maximal Condorcet domains, then $\mathcal{D} = \mathcal{E}(\mathcal{D}_1, \ldots, \mathcal{D}_n)$ is also a symmetric maximal Condorcet domain.*

Proof. It suffices to note that any symmetric domain is ample. □

A partial case of Theorem 6.2.7 was proved in Proposition 4 of Danilov and Koshevoy [2013].

6.3 Group separable maximal Condorcet domains

Inada [1964] was the first to note that the condition which he called group separability is sufficient for acyclicity of the majority relation.

Definition 6.3.1. *A domain $\mathcal{D}$ on the set A of alternatives is called* group separable *if every subset $B \subseteq A$ with at least two elements can be partitioned into two non-empty subsets B' and B'' such that all orders in $\mathcal{D}$ either (i) rank all elements of B' above all elements in B'', or (ii) rank all elements of B'' above all elements in B'.*

The acyclicity follows from the following

Proposition 6.3.1. *A group separable domain $\mathcal{D}$ is a never-middle domain, that is, for any triple $a, b, c \in \mathcal{D}$ a never condition $xN_{\{a,b,c\}}2$ for $x \in \{a, b, c\}$ is satisfied.*

Proof. Let $B = \{a, b, c\}$ with $B' = \{a, b\}$ and $B'' = \{c\}$. Then in $\mathcal{D}_{\{a,b,c\}}$ we can have orders *abc*, *bac*, *cab*, *cba* which means $\mathcal{D}$ satisfies $cN_{\{a,b,c\}}2$. The other splitting give us a similar never condition. □

However, as we will see later the class of never-middle domains is larger than the class of group separable domains.

Definition 6.3.2. *A Condorcet domain $\mathcal{D}$ is said to be* completely decomposable *if it is isomorphic to a product of trivial domains which are domains of linear orders, each on a set of a single alternative.*

It appears that complete decomposability and group separability are equivalent for maximal Condorcet domains.

Theorem 6.3.2. *A maximal Condorcet domain $\mathcal{D} \subseteq \mathcal{L}([n])$ is group separable if it is completely decomposable, that is*

$$\mathcal{D} = i_1 \star i_2 \star \cdots \star i_m$$

with some parenthesis. In such a case it is symmetric and has cardinality 2^{n-1}.

Proof. By the definition of group separability we can split $[n] = A \cup B$ in such a way that A and B are clone sets. By Theorem 6.2.3 we have $\mathcal{D} = \mathcal{D}_1 \star \mathcal{D}_2$, where the restrictions $\mathcal{D}_1 = \mathcal{D}|_A$ and $\mathcal{D}_2 = \mathcal{D}|_B$ are Condorcet domains on A and B, respectively. By the induction hypothesis we may assume that $\mathcal{D}_1$ and $\mathcal{D}_2$ are completely decomposable which proves the statement. The converse is clear. □

6.4 Never-last composition

6.4.1 The composition $\diamond$

Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two domains of linear orders on sets $\{1, \ldots, n-1\}$ and $\{2, \ldots, n\}$ of alternatives, respectively. Then we define the *nl*-composition (*never-last composition*) of these domains as

$$\mathcal{D}_1 \diamond \mathcal{D}_2 = \{un \mid u \in \mathcal{D}_1\} \cup \{v1 \mid v \in \mathcal{D}_2\}.$$

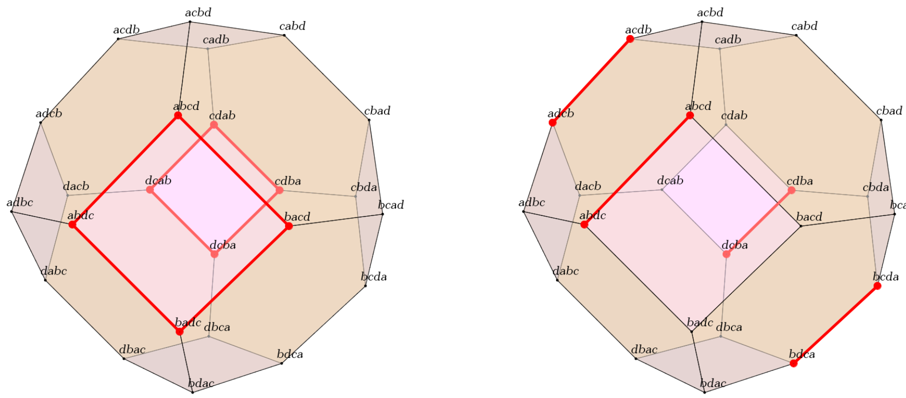


Figure 6.3: The two group separable maximal Condercet domains

Graphically this domain can be displayed as

$$\mathcal{D}_1 \diamond \mathcal{D}_2 = \left[\begin{array}{cccc|cccc} \multicolumn{4}{c|}{\mathcal{D}_1(1,\ldots,n-1)} & \multicolumn{4}{c}{\mathcal{D}_2(2,\ldots,n)} \\ \hline n & n & \cdots & n \quad n & 1 & 1 & \cdots & 1 \quad 1 \end{array} \right],$$

where linear orders of the domain are displayed as columns.

We immediately see that some small domains are nl-compositions. For example, the unique domain $(1) \star (2) = \{12, 21\}$ (notation from Danilov and Koshevoy [2013]) can also be written as $(1) \diamond (2)$. There are three maximal domains on three alternatives

$$\mathcal{D}_{3,1} = \{123, 312, 132, 321\},\ \mathcal{D}_{3,2} = \{123, 231, 132, 321\},\ \mathcal{D}_{3,3} = \{123, 213, 231, 321\}.$$

among which one is completely reducible using never-last joins, namely

$$\mathcal{D}_{3,3}(1,2,3) = ((1) \diamond (2)) \diamond ((2) \diamond (3)),$$

where (i) is a trivial linear order on the set of a single alternative i.

As was demonstrated in Slinko [2019] Arrow's single-peaked domain is always an nl-decomposition of two smaller Arrow's single-peaked domains. But as we will see the use of this construction is not restricted to Arrow's single-peaked domains.

Here we try to answer the following question: Can this composition help us to construct new maximal Condorcet domains and which domains it is possible to combine to obtain a new maximal Condorcet domain? To answer we need to introduce the following concept.

Definition 6.4.1. *Let $\mathcal{D}$ be a Condorcet domain. We say that in a linear order $w = \ldots bc \ldots a \ldots \in \mathcal{D}$ alternative a is a* right obstruction *to the swap $bc \to cb$, if cba cannot be potentially in the restriction $\mathcal{D}|_{\{a,b,c\}}$ of $\mathcal{D}$ onto $\{a,b,c\}$, that is the domain $\mathcal{D}|_{\{a,b,c\}} \cup \{cba\}$ is not Condorcet.*

Proposition 6.4.1. *Let $\mathcal{D} \subseteq \mathcal{L}(A)$ be a Condorcet domain and $a, b, c \in A$. An alternative a is a right obstruction to the swap $bc \to cb$ if and only if $\mathcal{D}|_{\{a,b,c\}}$ satisfies either $cN_{\{a,b,c\}}1$ or $bN_{\{a,b,c\}}2$ (or both) and no other never condition.*

Proof. If we allow bca but not cba in $\mathcal{D}|_{\{a,b,c\}}$ overall we might have four more linear orders listed as columns of the following matrix

$$\begin{bmatrix} b & b & a & a & c \\ c & a & b & c & a \\ a & c & c & b & b \end{bmatrix}$$

Only the third and the fifth column being removed give us a Condorcet domain. These domains satisfy $cN_{\{a,b,c\}}1$ or $bN_{\{a,b,c\}}2$, respectively. Either of the two preclude order cba. □

Theorem 6.4.2. *If $\mathcal{D}_1$ and $\mathcal{D}_2$ are Condorcet domains on $[n-1]$ and $[n]\setminus\{1\}$, respectively, such that $\mathcal{E} = (\mathcal{D}_1)_{-(n-1)} \cup (\mathcal{D}_2)_{-1}$ is also Condorcet. Suppose n is not a right obstruction in $\mathcal{D}_2$ and 1 is not a right obstruction in $\mathcal{D}_1$. Then $\mathcal{D} = \mathcal{D}_1 \diamond \mathcal{D}_2$ is also a Condorcet domain. Moreover, if $\mathcal{D}_1$ and $\mathcal{D}_2$ are maximal and ample, then $\mathcal{D}$ is also maximal and ample. If $\mathcal{D}_1$ and $\mathcal{D}_2$ are copious, then $\mathcal{D}$ is copious as well.*

Proof. Let us prove $\mathcal{D}$ is Condorcet. Let $i, j, k \in [n]\setminus\{1, n\}$. Then $\mathcal{D}|_{\{i,j,k\}}$ is Condorcet as $\mathcal{E}$ is. If $i, j \in [n] \setminus \{1, n\}$, consider the triple $\{i, j, n\}$. Since n is not a right obstruction to both swaps $ij \to ji$ and $ji \to ij$ in $\mathcal{D}_2$ we have ijn and jin compatible with $\mathcal{D}_2|_{\{i,j,n\}}$, that is $\mathcal{D}_2|_{\{i,j,n\}} \cup \{ijn, jin\}$ is Condorcet. However, only these two orders can be additionally added from $\mathcal{D}_1$ with n at the bottom, that is $\mathcal{D}|_{\{i,j,n\}} \subseteq \mathcal{D}_2|_{\{i,j,n\}} \cup \{ijn, jin\}$ is Condorcet. If $i \in [n] \setminus \{1, n\}$, then the triple $\{1, i, n\}$ is Condorcet since it satisfies $iN_{\{1,i,n\}}3$.

If $\mathcal{D}_1$ and $\mathcal{D}_2$ are maximal, we cannot add to $\mathcal{D}$ an order ending with 1 or n. Let $1 < j < n$. Then due to ampleness of $\mathcal{D}_1$ orders $1jn$, $j1n$ are contained in $\mathcal{D}$ and due to ampleness of $\mathcal{D}_2$ orders $jn1$, $nj1$ are contained in $\mathcal{D}$, hence $\mathcal{D}$ is copious and satisfies $jN_{\{1,j,n\}}3$. Hence adding to $\mathcal{D}$ an order with j as its last alternative, would violate $iN_{\{1,i,n\}}3$.

Suppose $\mathcal{D}_1$ and $\mathcal{D}_2$ are copious. If $\{i, j, k\} \subset [n-1]$, then $|\mathcal{D}|_{\{i,j,k\}}| = 4$ as $\mathcal{D}_1$ is copious. Similarly, $|\mathcal{D}|_{\{i,j,k\}}| = 4$ if $\{i, j, k\} \subset [n] \setminus \{1\}$. If $\{i, j, k\} \supset \{1, n\}$, then $|\mathcal{D}|_{\{i,j,k\}}| = 4$ as we have seen already. This proves the theorem. □

This theorem gives us only a sufficient set of conditions. As we will see the *nl*-composition can be maximal without $\mathcal{D}_1$ or $\mathcal{D}_2$ being maximal. This, in particular, means that *nl*-joins construction does not always produce a Condorcet domain.

6.4.2 Domains on four alternatives obtained by never-last joins

We illustrate the result using the classification by Dittrich [2018], this paper will be referred as TD. He determined all 18 maximal Condorcet domains on four alternatives up to an isomorphism and flip-isomorphism. Using the order given by Table 5.4 on p. 94 in TD we denote those maximal domains as $\mathcal{D}_{4,1}, \ldots, \mathcal{D}_{4,18}$. We will show that nine out of 18 domains in that list are obtained by using *nl*-composition. These domains are: $\mathcal{D}_{4,4}$ (The Single Peaked); $\mathcal{D}_{4,5}$ (The Crab); $\mathcal{D}_{4,6}$ (The Sun); $\mathcal{D}_{4,7}$ (The Half-Crab-Half-Sun); $\mathcal{D}_{4,11}$ (Miscellaneous I), $\mathcal{D}_{4,16}$ (Miscellaneous VI), $\mathcal{D}_{4,17}$ (Miscellaneous VII); $\mathcal{D}_{4,2}$ (The Snake); $\mathcal{D}_{4,3}$ (The Broken Snake).

Example 6.4.1 (The Single-peaked in TD)**.** *Let us consider a unique maximal single-peaked domain $\mathcal{SP}(\triangleleft, A)$ on four alternatives with $A = \{1,2,3,4\}$ relative to spectrum $1 \triangleleft 2 \triangleleft 3 \triangleleft 4$:*

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 2 & 3 & 2 & 3 & 3 & 4 \\ 2 & 1 & 3 & 2 & 3 & 2 & 4 & 3 \\ 3 & 3 & 1 & 1 & 4 & 4 & 2 & 2 \\ \hline 4 & 4 & 4 & 4 & 1 & 1 & 1 & 1 \end{array}\right]$$

whose graph is presented on Figure 6.4 (cf. Fig. 2.2).

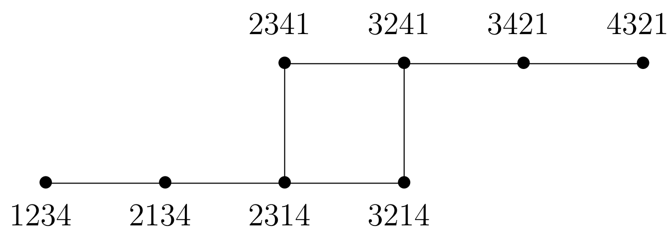


Figure 6.4: Graph of the single-peaked domain $\mathcal{SP}(\triangleleft, A)$ on four alternatives

We have $\mathcal{SP}(\triangleleft, A) = \mathcal{D}(\mathcal{N})$, where

$$\mathcal{N} = \{bN_{\{a,b,c\}}3,\ bN_{\{a,b,d\}}3,\ cN_{\{a,c,d\}}3,\ cN_{\{b,c,d\}}3\}.$$

This domain will also be denoted as $\mathcal{D}_{4,4}$. We have

$$\mathcal{D}_{4,4} = \mathcal{D}_{3,3}(1,2,3) \diamond \mathcal{D}_{3,3}(2,3,4).$$

Example 6.4.2 (The Crab in TD)**.** *Let us consider an Arrow's single-peaked maximal Condorcet domain $\mathcal{D}_{4,5}$ on four alternatives:*

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 2 & 3 & 2 & 3 & 2 & 4 \\ 2 & 1 & 3 & 2 & 3 & 2 & 4 & 2 \\ 3 & 3 & 1 & 1 & 4 & 4 & 3 & 3 \\ \hline 4 & 4 & 4 & 4 & 1 & 1 & 1 & 1 \end{array}\right]$$

$$\mathcal{D}_{4,5} = \mathcal{D}_{3,3}(1,2,3) \diamond \mathcal{D}_{3,3}(3,2,4).$$

whose graph is presented on Figure 2.4. We have $\mathcal{D}_{4,2} = \mathcal{D}(\mathcal{N})$, where

$$\mathcal{N} = \{2N_{\{1,2,3\}}3,\ 2N_{\{1,2,4\}}3,\ 3N_{\{1,3,4\}}3,\ 2N_{\{2,3,4\}}3\}$$

and $\mathcal{D}_{4,2}$ is copious. This domain is not single-peaked (for example, because it does not have two completely reversed orders).

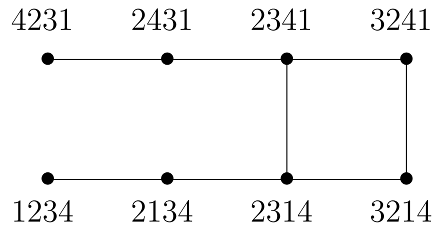


Figure 6.5: Graph of an Arrow's single-peaked domain $\mathcal{D}_{4,5}$.

Example 6.4.3 (The Sun in TD)**.** *Let us consider the following maximal Condorcet domain on four alternatives:*

$$\left[\begin{array}{cccc|cccc} 3 & 3 & 2 & 2 & 3 & 3 & 2 & 2 \\ 1 & 2 & 3 & 1 & 4 & 2 & 3 & 4 \\ 2 & 1 & 1 & 3 & 2 & 4 & 4 & 3 \\ \hline 4 & 4 & 4 & 4 & 1 & 1 & 1 & 1 \end{array}\right]$$

It is defined by the following complete set of never-conditions:

$$1N_{\{1,2,3\}}1, \quad 2N_{\{1,2,4\}}3, \quad 3N_{\{1,3,4\}}3, \quad 4N_{\{2,3,4\}}1.$$

It is copious but does not have maximal width. The median graph of this domain is here:

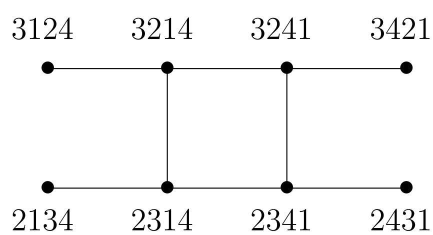


Figure 6.6: Graph of $\mathcal{D}_{4,6}$.

In terms of nl-joins

$$\mathcal{D}_{4,6} = \mathcal{D}_{3,1}(2,1,3) \diamond \mathcal{D}_{3,1}(2,4,3).$$

Example 6.4.4 (Half Crab half Sun domain in TD)**.** *Let us consider the following maximal Condorcet domain on four alternatives:*

$$\left[\begin{array}{cccc|cccc} 3 & 3 & 2 & 2 & 4 & 3 & 3 & 2 \\ 1 & 2 & 3 & 1 & 3 & 4 & 2 & 3 \\ 2 & 1 & 1 & 3 & 2 & 2 & 4 & 4 \\ \hline 4 & 4 & 4 & 4 & 1 & 1 & 1 & 1 \end{array}\right]$$

It is defined by the following complete set of never-conditions:

$$3N_{\{1,2,3\}}1, \quad 1N_{\{1,2,4\}}3, \quad 1N_{\{1,3,4\}}3, \quad 2N_{\{2,3,4\}}3.$$

It is copious but does not have maximal width.

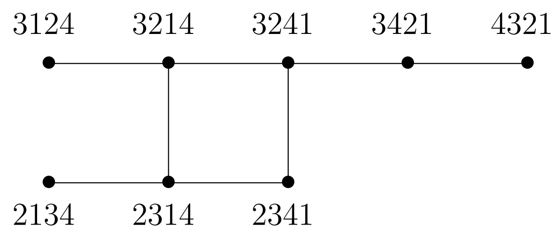


Figure 6.7: Graph of $\mathcal{D}_{4,7}$.

In terms of nl-joins

$$\mathcal{D}_{4,7} = \mathcal{D}_{3,1}(2,1,3) \diamond \mathcal{D}_{3,1}(2,3,4).$$

Example 6.4.5 (Miscellaneous I in TD)**.** *Let us consider the following maximal Condorcet domain for $m = 4$ alternatives:*

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 2 & 3 & 2 & 3 & 4 & 4 \\ 2 & 1 & 3 & 2 & 3 & 2 & 2 & 3 \\ 3 & 3 & 1 & 1 & 4 & 4 & 3 & 2 \\ \hline 4 & 4 & 4 & 4 & 1 & 1 & 1 & 1 \end{array}\right]$$

which satisfies never-conditions

$$2N_{\{1,2,3\}}3, \quad 2N_{\{1,2,4\}}3, \quad 3N_{\{1,3,4\}}3, \quad 4N_{\{2,3,4\}}2.$$

In terms of nl-joins

$$\mathcal{D}_{4,11} = \mathcal{D}_{3,3}(1,2,3) \diamond \mathcal{D}_{3,2}(4,2,3).$$

Figure 6.8 shows the graph of this domain.

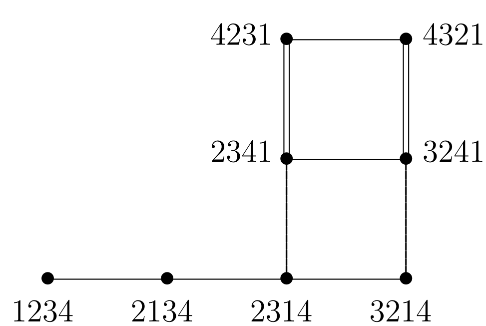


Figure 6.8: Graph of $\mathcal{D}_{4,11}$.

Example 6.4.6 (Miscellaneous VI in TD). *One of the most interesting domains for $m = 4$ alternatives. We encountered it already in Example 6.1.2. As we will see, It allows two distinct decompositions.*

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 2 & 3 & 3 & 2 & 4 & 4 \\ 2 & 3 & 3 & 2 & 2 & 3 & 3 & 2 \\ 3 & 2 & 1 & 1 & 4 & 4 & 2 & 3 \\ \hline 4 & 4 & 4 & 4 & 1 & 1 & 1 & 1 \end{array}\right]$$

which satisfies never-conditions

$$1N_{\{1,2,3\}}2, \quad 2N_{\{1,2,4\}}3, \quad 3N_{\{1,3,4\}}3, \quad 4N_{\{2,3,4\}}2.$$

Figure 6.9 shows the median graph of the domain. It has decompositions

$$\mathcal{D}_{4,16} = \mathcal{D}_{3,3}(1,2,3) \diamond \mathcal{D}_{3,2}(4,2,3) = \mathcal{D}_{3,3}(1, 2 \star 3, 4).$$

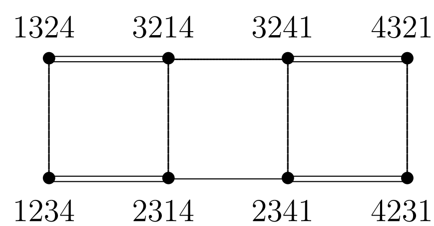


Figure 6.9: Graph of $\mathcal{D}_{4,16}$.

Example 6.4.7 (Miscellaneous VII in TD). *Let us consider the following maximal Condorcet domain for $m = 4$ alternatives:*

$$\left[\begin{array}{cccc|cccc} 2 & 3 & 2 & 3 & 2 & 3 & 4 & 4 \\ 1 & 1 & 3 & 2 & 3 & 2 & 2 & 3 \\ 3 & 2 & 1 & 1 & 4 & 4 & 3 & 2 \\ \hline 4 & 4 & 4 & 4 & 1 & 1 & 1 & 1 \end{array}\right]$$

which satisfies never-conditions

$$1N_{\{1,2,3\}}1, \quad 2N_{\{1,2,4\}}3, \quad 3N_{\{1,3,4\}}3, \quad 4N_{\{2,3,4\}}2.$$

Figure 6.10 shows the median graph of the domain. In terms of nl-joins

$$\mathcal{D}_{4,17} = \mathcal{D}_{3,1}(2,1,3) \diamond \mathcal{D}_{3,2}(4,2,3).$$

Figure 6.10 shows the median graph of the domain. In terms of nl-joins
However, the converse of Theorem 6.4.2 is not correct as the following example shows.

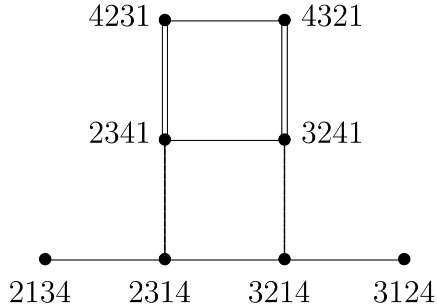


Figure 6.10: Graph of $\mathcal{D}_{4,17}$.

Example 6.4.8 (The snake in TD)**.** *Let us consider a single-crossing maximal domain $\mathcal{D}_{4,2}$ whose orders are represented as columns of the following matrix*

$$\left[\begin{array}{ccc|cccc} 1 & 2 & 2 & 2 & 2 & 4 & 4 \\ 2 & 1 & 3 & 3 & 4 & 2 & 3 \\ 3 & 3 & 1 & 4 & 3 & 3 & 2 \\ \hline 4 & 4 & 4 & 1 & 1 & 1 & 1 \end{array}\right]$$

We see that

$$\mathcal{D}_{4,2} = \mathcal{E} \diamond \mathcal{D}_{3,1}(2,3,4),$$

where $\mathcal{E} = \{123, 213, 231\} \subseteq \mathcal{D}_{3,3}(1,2,3) \cap \mathcal{D}_{3,1}(2,3,1)$. *and $\mathcal{D}_1$ is not maximal. This happens due to* 4 *being an obstruction to swap* $23 \to 32$ *in* $\mathcal{D}_{3,1}(2,3,4)$ *hence we cannot add order* 321 *or* 132 *to $\mathcal{D}_1$ to make it maximal.*

The graph of the single-crossing domain is a line graph shown on a Figure 6.11.

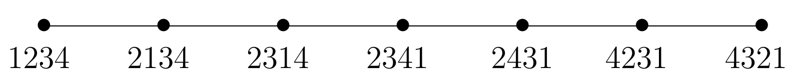


Figure 6.11: Graph of single-crossing domain on four alternatives

One other such example.

Example 6.4.9 (The broken snake in TD)**.** *Let us consider the maximal domain $\mathcal{D}_{4,3}$ whose orders are represented as columns of the following matrix*

$$\left[\begin{array}{ccc|cccc} 1 & 3 & 3 & 3 & 3 & 2 & 4 \\ 3 & 1 & 2 & 2 & 4 & 4 & 2 \\ 2 & 2 & 1 & 4 & 2 & 3 & 3 \\ \hline 4 & 4 & 4 & 1 & 1 & 1 & 1 \end{array}\right]$$

We see that

$$\mathcal{D}_{4,3} = \mathcal{E} \diamond \mathcal{D}_{3,2}(3,2,4),$$

where

$$\mathcal{E} = \{132, 312, 321\} \subseteq \mathcal{D}_2 = \{123, 132, 231, 321\}.$$

and $\mathcal{E}$ *is not maximal. This happens due to* 4 *being an obstruction to swap* $32 \to 23$ *in* $\mathcal{D}_{3,2}(3, 2, 4)$ *hence we cannot add order* 123 *or* 231 *to* $\mathcal{E}$ *to make it maximal.*

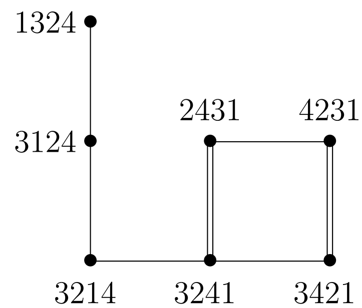


Figure 6.12: Graph of $\mathcal{D}_{4,3}$.

Chapter 7

Symmetric Condorcet domains

The main goal of this section is to characterise the class of never-middle maximal Condorcet domains which, as we noticed in Proposition 1.1.9, coincides with the class of symmetric maximal Condorcet domains. Naturally, it is expedient to classify those domains that are not compositions of smaller domains in the sense of Chapter 6. In this respect it is important that Danilov and Koshevoy [2013] discovered a series of symmetric maximal Condorcet domains that for any number of alternatives m have the size of just 4, we call them *Raynaud domains* as Raynaud [1981] was the first who discovered such a domain in case of four alternatives. In this chapter we characterise Raynaud domains by means of simple permutations which is a well-known object in combinatorics. We hypothesise that Raynaud domains are the only non-trivial symmetric indecomposable maximal domains. Finally we show how a symmetric Condorcet domain can emerge in the object called *associahedron*

7.1 The structure of symmetric Condorcet domains

In this section we will pay a special attention to the weak Bruhat poset $(\mathcal{L}([n]), \ll)$ introduced in Definition 1.4.3. We will also exploit the one-to-one correspondence $v \mapsto \mathrm{Inv}(v)$ between linear orders over $[n]$ and balanced subsets of the set of pairs $\Omega_n = \{(i,j), 1 \le i < j \le n\}$ introduced in Section 1.4.1. We will often identify a linear order v with the set of its inversions $\mathrm{Inv}(v)$. We also remind the reader that

$$\mathrm{Inv}(\bar{v}) = \Omega_n \setminus \mathrm{Inv}(v),$$

where $\bar{v}$ is the flipped v.

In this section $\mathcal{D}$ is a symmetric Condorcet domain of maximal width. In Section 1.4 we saw that $\mathcal{D}$ can naturally be identified with a distributive sublattice of the Bruhat lattice $(\mathcal{L}([n]), \ll)$ with $0 = e$ and $1 = \bar{e}$. Now we have an additional structure on $\mathcal{D}$ as, due to symmetry, it has complements. Indeed, if $u \in \mathcal{D}$, then $\bar{u} \in \mathcal{D}$ and $u \cup \bar{u} = 1$.

A distributive lattice in which every element has a complement is known as a Boolean lattice or a Boolean algebra. We note that any finite Boolean algebra contains 2^k elements, where k is the number of its atoms. The key observation in the proof of the upcoming Theorem 7.1.2 is the following one.

Proposition 7.1.1. *Any non-empty co-transitive subset $S \subseteq \Omega_n$ contains a pair $(i, i+1)$ for some $i = 1, \dots, n-1$.*

Proof. Suppose $(k, m) \in \Omega_n$ such that $m - k$ is the smallest. We claim $m = k + 1$. If not, then there exists a number ℓ such that $k \le \ell \le m$. By co-transitivity we have either $(k, \ell) \in S$ or $(\ell, m) \in S$ which contradicts to the minimality of $m - k$. □

Theorem 7.1.2 (Danilov and Koshevoy [2013])**.** *Let $\mathcal{D} \subseteq \mathcal{L}([n])$ be a symmetric maximal Condorcet domain. Then $\mathcal{D}$ is a Boolean sublattice in the Bruhat lattice $(\mathcal{L}([n]), \ll)$. Its cardinality is a power of 2 and at most 2^{n-1}.*

Proof. By Theorem 1.4.9 $\mathcal{D}$ is a sublattice of the Bruhat lattice ans since it is symmetric $\mathcal{D}$ is a Boolean sublattice. Let $v_1, \dots, v_k$ be atoms of this Boolean lattice, and $S_i = \mathrm{Inv}(v_i)$, $i = 1, \dots, k$, be the corresponding inversion sets. Theorem 1.4.9 the S_i's do not intersect. Thus $|\mathcal{D}| = 2^k$. Let us show that $k \le n-1$. For this we note that S_i are co-transitive and therefore each contains a pair $(i, i+1)$ with two neighbouring numbers in it. But there are at most $n-1$ such pairs. Hence $k \le n-1$. □

We note that this maximum value 2^{n-1} can be achieved. By Theorem 6.3.2 any completely decomposable maximal Condorcet domain is symmetric and has cardinality 2^{n-1}.

Corollary 7.1.3. *The graph $G_{\mathcal{D}}$ of a symmetric maximal Condorcet domain $\mathcal{D} \subseteq \mathcal{L}([n])$ is a k-dimensional cube where $k \le n-1$.*

Example 7.1.1. *In the symmetric domain $\mathcal{D} = ((1 \star 2) \star 3) \star 4$ the atoms will be $v_1 = 2134$, $v_2 = 3124$ and $v_3 = 4123$ with the corresponding inversion sets*

$$S_1 = \{(1,2)\}, \quad S_2 = \{(1,3),(2,3)\}, \quad \{(1,4),(2,4),(3,4)\}.$$

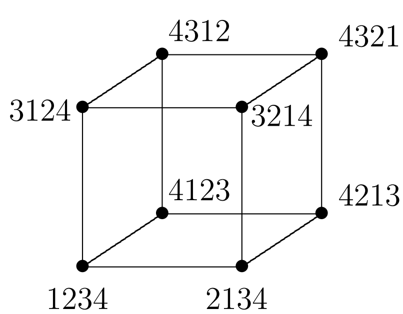


Figure 7.1: A symmetric domain for $n = 4$ with maximal number of linear orders

7.2 Raynaud domains

In Figure 1.2 only domain $\mathcal{D}_{3,2}$ is symmetric. Let us give a series of more sophisticated examples which will be important later.

Example 7.2.1. *Let* $|A| = m$ *and*

$$x = a_{i_1}a_{i_2}\dots a_{i_m}, \qquad y = a_{j_1}a_{j_2}\dots a_{j_m}$$

be two sequences without repetitions interpreted as linear orders from $\mathcal{L}(A)$*. Then*

$$K(x,y) = \{x, y, \bar{x}, \bar{y}\}$$

is a symmetric Condorcet domain. Up to an isomorphism we can think that $y = e = 123\dots m$ *and we denote* $K(x,e)$ *as* $K_m(x)$ *or* $K(x)$*. It is immediate to see that* $K(x)$ *coincides with* $K(\bar{x})$*.*

Raynaud [1981] discovered a remarkable domain of this sort

$$K = \{1234, 4321, 2413, 3142\}$$

and called it *configuration* K. Up to an isomorphism it is $K(2413) = K(3142)$. As a corollary of Theorem 6.1.3 we obtain

Theorem 7.2.1 ([Raynaud, 1981])**.** *Configuration* K *is an indecomposable Condorcet domain (does not have a bipartition in Raynaud terminology*[1]*).*

Danilov and Koshevoy [2013] showed that domain K is a maximal Condorcet domain and generalised it to an arbitrary size of the set A of alternatives. They showed that for any number of alternatives there are indecomposable maximal symmetric Condorcet domains of cardinality four. We will call maximal Condorcet domains of type $K_m(x)$ for $x \in \mathcal{L}(A)$ *Raynaud domains.*

Example 7.2.2 (Danilov and Koshevoy [2013])**.** *Let* $A = [m]$ *and define*

$$d_m = 24\cdots(2k)1(2k \pm 1)\cdots 53, \tag{7.2.1}$$

where $2k \pm 1$ *is equal to* $2k + 1 = m$*, if* m *is odd, and* $2k - 1 = m - 1$ *if* m *is even. For example,* $d_4 = 2413$*,* $d_5 = 24153$*,* $d_6 = 246153$ *and* $d_7 = 2461753$*.*

From a more general result of Section 7.4 it will follow that $K(d_m)$ is a symmetric maximal Condorcet domain for every $m \geq 3$. In the meantime we note the following.

Proposition 7.2.2. $K(d_4) = K(2413)$ *is a copious maximal Condorcet domain consisting of all orders that satisfy the following set of never-middle conditions*

$$1N_{\{1,2,4\}}2, \quad 2N_{\{2,3,4\}}2, \quad 3N_{\{1,2,3\}}2, \quad 4N_{\{1,3,4\}}2. \tag{7.2.2}$$

[1]A *bipartition* of a domain $\mathcal{D}$ in the sense of Raynaud [1981] is a representation of $\mathcal{D}$ as a composition of two domains $\mathcal{D} = \mathcal{D}_1 \star \mathcal{D}_2$.

Proof. It is easy to see that $K(d_4)$ is a copious never-middle domain. Let us prove that $K(2413)$ is maximal. Suppose $u \in \mathcal{L}([4]) \setminus K(d_4)$ can be added to it so that $\mathcal{D} = K(2413) \cup \{u\}$ is a Condorcet domain. This means u must satisfy the same never conditions (7.2.2) as $K(2413)$. Then, since $\mathcal{D}$ satisfies $3N_{\{1,2,3\}}2$, we have $u' = u|_{\{1,2,3\}} \in \{123, 321, 213, 312\}$. Suppose $u' = 123$. Then $u \neq 4123$ as 1 cannot be between 2 and 4, $u \neq 1423$ and $u \neq 1243$ as 4 cannot be between 1 and 3. Thus $u = 1234$, a contradiction.

Suppose $u' = 213$. Then $u \neq 4213$ as 2 cannot be between 3 and 4, $u \neq 2413 = d_4$, $u \neq 2143$ as 4 cannot be between 1 and 3, $u \neq 2134$ as 1 cannot be between 2 and 4. The other cases are flipped to the considered ones. □

Proposition 7.2.3. $K(d_5)$*, however, is a maximal Condorcet domain which is not copious.*

Proof. Indeed, the projection of $K(d_5)$ onto $\{2, 4, 5\}$ is only $\{245, 542\}$. □

7.3 Simple sequences

Our classification of Raynaud domains will rely on the following combinatorial theory. We identified a linear order $i_1 > i_2 > \ldots > i_m$ on $[m]$ with a sequence $i_1 i_2 \ldots i_m$ but it can also be identified with the permutation $\sigma\colon k \to i_k$. The concept of a *simple permutation* from [Albert and Atkinson, 2005] can then be applied to sequences.

Definition 7.3.1. *Let* $i_1 i_2 \ldots i_m$ *be a sequence of distinct elements of* $[m]$*. We say that a subsequence* $i_k i_{k+1} \ldots i_\ell$ *is an* interval *of length* ℓ *in the sequence* $i_1 i_2 \ldots i_m$ *if the set* $\{i_k, i_{k+1}, \ldots, i_{k+\ell-1}\} = \{a, a+1, \ldots, a+\ell-1\}$ *for some* $a \in [m]$*. This interval is* trivial *if this subsequence has length 1 or* m*. A sequence without non-trivial intervals is called* simple.

Example 7.3.1.

- 21 *is the only non-trivial interval in* 521463*;*
- 52463 *is an interval in* 152463*;*
- 2413*,* 41352*,* 24153*,* 2475316*, and* 24683157 *are simple;*
- *series* (d_m) *of sequences in Example 7.2.2 are also simple.*

A survey on simple sequences (permutations) can be found in Brignall [2010]. In particular, the number s_n of simple permutations of length n for vaious n is known to be

$$s_n = 1, 2, 0, 2, 6, 46, 338, 2926, 28146, \ldots,$$

which is the sequence $A111111$ of Sloane et al. [2003]. The number of simple permutations of length n is asymptotically $\frac{n!}{e^2}$ [Albert et al., 2003].

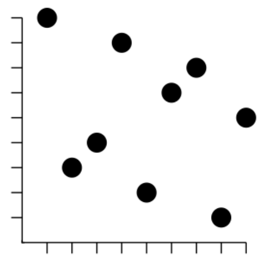


Figure 7.2: The plot of the permutation 934826715.

Geometrically the sequences can be represented by plots. For example, the plot in Figure 7.2 represents permutation 934826715.

Simple permutations remain simple under the following transformations which in the theory of simple permutations are called *symmetries*:

Reversal: If $x = x_1x_2\dots x_m$, then $\bar{x} = x_m \dots x_2x_1$;

Flip: If $x = x_1x_2\dots x_m$, then $x^\circ = z_1z_2\dots z_m$, where $z_i = m+1-x_i$;

Inversion: If $x = x_1x_2\dots x_m$, then $x^{-1} = y_1y_2\dots y_m$, where

$$\begin{pmatrix} 1 & 2 & \dots & m \\ y_1 & y_2 & \dots & y_m \end{pmatrix} = \begin{pmatrix} 1 & 2 & \dots & m \\ x_1 & x_2 & \dots & x_m \end{pmatrix}^{-1}.$$

For example, $(24153)^{-1} = 31524$.

On Figure 7.2 these transformations correspond to the reflections about vertical line $x = 5$, horizontal line $y = 5$ and the bisector of the right angle $y = x$. Simple permutations are usually described up to symmetries.

The importance of simple sequences to us is clarified in the following

Proposition 7.3.1. *The sequence x is simple if and only if the domain $K(x)$ has no non-trivial clone sets.*

Proof. Let $x = x_1x_2\dots x_m$. If $x_kx_{k+1}\dots x_\ell$ is an interval, then $C = \{x_kx_{k+1}\dots x_\ell\}$ is contiguous in $12\dots m$, hence it is also an interval in $e = 12\dots m$, hence C is a clone of $K(x)$. The converse is also obvious. □

Firstly we show that symmetries of permutations lead to a number of important isomorphisms.

Proposition 7.3.2. *For an arbitrary $x \in \mathcal{L}([m])$*

1. $K(x) = K(\bar{x})$;
2. $K(x) \cong K(x^\circ)$;

3. $K(x) \cong K(x^{-1})$.

Proof. The first two are evident. The third isomorphism is $\sigma\colon i \mapsto x_i$. This will map x to e and e to x^{-1}. □

Theorem 7.3.3 (Raynaud [1981]). *$K = K(2413)$ is the unique, up to an isomorphism, indecomposable symmetric maximal Condorcet domain for four alternatives.*

Proof. By Proposition 7.2.2 and Theorem 7.2.1 K is maximal and indecomposable. We need to establish uniqueness.

Let $\mathcal{D}$ be an indecomposable symmetric maximal Condorcet domain on $A = [4]$. If $|\mathcal{D}| = 4$, then, up to an isomorphism, $\mathcal{D} = K(x)$ for some sequence $x = a_1a_2a_3a_4$ with $x \neq e$ and $x \neq \bar{e}$. By Corollary 6.2.4 $K(x)$ has no clone sets and by Proposition 7.3.1 x is simple. This implies $\{a_1, a_4\} \cap \{1, 4\} = \emptyset$, hence $\{a_1, a_4\} = \{2, 3\}$. Then $x = 2413$ or $x = 3142$, so $\mathcal{D} = K(2413)$.

Suppose $|\mathcal{D}| > 4$, and $\mathcal{D}$ has no clone sets. Then 4 is not the first or the last alternative in one of the orders and so is 1.

Assume, first, that there exists $x \in \mathcal{D}$ which has both 1 and 4 in the middle and without loss of generality assume that 1 precedes 4 in x. Then $x \neq 3142$, as 3142 is simple, and $K(3142)$ would be maximal, hence $\mathcal{D} = K(3142) = K(2413)$. This leaves us with $x = 2143$ and $\mathcal{D}|_{\{1,2,3\}} = \{123, 321, 213, 312\}$ which shows that there cannot be a $y \in \mathcal{D}$ where 1 and 2 are split by 3. Similarly, they cannot be split by 4. In such a case $\{1, 2\}$ is a clone set.

Now let us assume that there exists $x \in \mathcal{D}$ which has both 1 and 4 are outside. Then without loss of generality $x = 1324$. We then have $\mathcal{D}|_{\{1,2,3\}} = \{123, 321, 132, 231\}$ which shows that 1 can never split $\{2, 3\}$. Same with 4 so $\{2, 3\}$ is a clone set.

The remaining case we can assume that $x \in \mathcal{D}$ exists with 4 outside and 1 inside. In such a case $x = 2134$ or $x = 3124$. Arguing as above we can show that in both cases $\{1, 2\}$ is a clone set. □

Corollary 7.3.4. *Up to an isomorphism there are only three symmetric maximal Condorcet domains on four alternatives:* $((1 \star 2) \star 3) \star 4$, $(1 \star 2) \star (3 \star 4)$, $K(2413)$.

The graph $G_{\mathcal{D}}$ of each Raynaud domain is a 4-cycle.

As Danilov and Koshevoy [2013] show, for any n, it is possible to construct a symmetric maximal Condorcet domains of size 2^m for any positive integer m such that $2 \leq m \leq n$. For this we can, for example, consider

$$K(d_{n-m+2}) \star (n-m+3) \star (n-m+2) \star \cdots \star n,$$

where d_{n-m+2} is given in (7.2.1).

7.4 Raynaud domains and simple sequences

The goal of this Section is to establish that Raynaud domains are in one-to-one correspondence with simple sequences and the following theorem describes them.

Theorem 7.4.1. *$K_m(x)$ is a maximal Condorcet domain if and only if $m \geq 4$ and x is a simple sequence of length m.*

Let x represents a linear order from $\mathcal{L}(A)$. If x is not simple, then $K_m(x)$ has a non-trivial clone set and by Theorem 6.2.3 it is decomposable and by Theorem 6.1.3 cannot be maximal.

Thus we need to prove that if x is simple of length m, then $K_m(x)$ is maximal. We will prove this by induction on m. By Theorem 7.3.3 the induction hypothesis is true for $m = 4$. For the induction step we need some facts from the theory of simple sequences. By a *subsequence* of a sequence x we call any sequence that can be obtained by removal of certain elements from x without changing the order of those which are left. One of the most important properties of simple sequences can be derived from Schmerl and Trotter [1993]. Namely, it follows from that paper that any simple sequence x of length n contains a simple subsequence of length either $n-1$ or, if no such subsequence of length $n-1$ exists, of length $n-2$. In the latter case the simple sequence x is said to be *exceptional*. Exceptional sequences have been classified in Albert and Atkinson [2005] and they are

$$246\ldots(2k)135\ldots(2k-1), \quad (k \geq 2) \tag{7.4.1}$$

and all symmetries of these. We note that Raynaud's configuration K is one of them. We note that all exceptional sequences have even length.

We will now act under the assumption that $m \geq 5$ and that for all numbers n such that $4 \leq n < m$ Theorem 7.4.1 is true. Further proof of this theorem will be split into two lemmas.

Lemma 7.4.2. *If $x \in \mathcal{L}([m])$ is an exceptional simple sequence, then $K_m(x)$ is a maximal Condorcet domain.*

Proof. Due to Proposition 7.3.2 we may consider that x is as in (7.4.1) with $k \geq 3$. By Theorem 7.3.3 the result is true for $k = 3$ as we get configuration K. Suppose $k > 3$.

By removing $2k$ and $2k-1$ from x we get the sequence

$$x' = 246\ldots(2k-2)135\ldots(2k-3),$$

which is exceptional simple and for which $K(x')$ is maximal by the induction hypothesis. Let also $e' = 12\ldots(2k-2)$. If $K(x)$ is not a maximal domain, then there is a sequence $u \notin K(x)$ such that $K(x) \cup \{u\}$ is also a Condorcet domain. Firstly, since $K(x')$ is maximal, the restriction u' of u onto $[2k-2]$, coincides, without loss of generality, with either x' or with e'. We assume it is x' (otherwise we replace x with x^{-1} using Proposition 7.3.2). So, we have

$$u' = 246\ldots(2k-2)135\ldots(2k-3).$$

Similarly, we can remove 1 and 2 and obtain a simple exceptional sequence again, hence the restriction u'' of u onto $\{3, 4, \ldots, 2k\}$ will be

$$u'' = 46\ldots(2k-2)(2k)35\ldots(2k-3)(2k-1).$$

Hence 2 is in the beginning of u and $2k-1$ is the last symbol of it. If u is different from x, the only option for it is

$$246\dots(2k-2)1(2k)35\dots(2k-3)(2k-1),$$

where 1 and $2k$ in x are switched. However, 1 cannot be between $2k-2$ and $2k$ since in $e, \bar{e}, x, \bar{x}$ we already have four subsequences

$$1(2k-2)(2k),\ (2k)(2k-2)1,\ (2k-2)(2k)1,\ 1(2k)(2k-2).$$

Thus $u = x$, a contradiction. □

Now we will look at the case when x is simple but not exceptional. Then it has the so-called "one point deletion", i.e., there exists an element $a \in A$ such that $x_{-a} = x|_{A\setminus\{a\}}$ is also simple [Brignall, 2010]. In this case a is said to be an *inessential alternative*, otherwise it is said to be *essential*. It is important for us to know how many inessential alternatives x might have. On the one hand, asymptotically, almost all of them are inessential [Brignall and Vincent, 2014], on the other, there exist permutations, like, for example, 48352617 with only one inessential alternative 3. As a result we cannot use the idea used in the previous lemma where we compared two projections of the additional order u and deduced that $u = x$.

Lemma 7.4.3. *If $x \in \mathcal{L}([m])$ is a simple sequence, different from exceptional, then $K_m(x)$ is a symmetric maximal Condorcet domain.*

Proof. Suppose $K_m(x)$ is not maximal and there exists $u \in \mathcal{L}([m])$ such that $\mathcal{D} = K_m(x) \cup \{u\}$ is also a Condorcet domain. Since x is not exceptional, there is an alternative $a \in [m]$, whose deletion leads to a simple sequence $x' = x_{-a}$ of length $m-1$ and by the induction hypothesis Condorcet domain $K_{m-1}(x')$ on $m-1$ alternatives is maximal. As in the previous lemma $u' = u_{-a}$ must coincide either with $e' = e_{-a}$ or $x' = x_{-a}$ and without loss of generality we may assume that $u' = e'$ (otherwise we use Proposition 7.3.2 and replace x with x^{-1} and consider $K_m(x^{-1})$). There may be two options:

$$u = 12\dots ka(k+1)\dots(a-1)(a+1)\dots m,$$

or

$$u = 12\dots(a-1)(a+1)\dots ka(k+1)\dots m.$$

where in both cases $k \in \{0, 1, \dots, m\}$ (this, in particular, means that a can be the first or the last symbol of u). We consider the first case; for the second case the proof is similar.

Because of simplicity of x alternatives $a-1$ and a are not consecutive in x. We note that, if there is an alternative $b \notin \{k+1, \dots, a-2\}$, which is between a and $a-1$ in x, then we can show that the triple $a-1, a, b$ does not satisfy a never-middle condition in $\mathcal{D}$. Indeed, suppose $b \le k$. Then, depending on relative position of $a-1$ and a, we have triples

$$b(a-1)a,\ ba(a-1),\ \text{and}\ (a-1)ba\ \text{or}\ ab(a-1)$$

in e, u, and x, respectively, which contradicts to the never-middle condition for this triple.

If, on the other hand, $b \geq a+1$, then we have triples

$$(a-1)ab,\ a(a-1)b,\ \text{and } (a-1)ba \text{ or } ab(a-1)$$

in e, u, and x, respectively, which also gives a contradiction.

Now we assume that all alternatives in x that are between $a-1$ and a lie in the set $\{k+1,\dots,a-2\}$. Since x is simple, this set is not empty. Hence the cardinality of the set $X = \{k+1,\dots,a-1,a\}$ is at least 3 and at most $m-1$ (except the case $k=0$ and $a=m$). Let b_1 be the leftmost alternative from X in x and b_2 be the rightmost one. Due to the simplicity of x there exists an alternative $c \notin X$ which prevents the alternatives of X from forming an interval in x. As discussed above c cannot be between $a-1$ and a but it is between b_1 and b_2. Without loss of generality (otherwise we consider $\bar{x}$) we can assume that c in x is to the right of b_1 and to the left of both $a-1$ and a. Thus we have

$$x = \dots b_1 \dots c \dots (a-1) \dots a \dots b_2 \dots$$

or

$$x = \dots b_1 \dots c \dots a \dots (a-1) \dots b_2 \dots,$$

where we can have $b_2 = a$ in the first case and $b_2 = a-1$ in the second.

In both cases, if $c < k+1$, then we have $c < b_1 < a$ and the triples cb_1a, b_1ca, cab_1, in e, x and u, respectively, which contradicts to any never-middle condition. If $c > a$, then we $b_1 < a < c$ and therefore we have triples b_1ac, b_1ca, and ab_1c in e, x and u, respectively, which also gives a contradiction.

In the extreme case, when $k = 0$ and $a = m$, due to simplicity of x, there are alternatives c_1, c_2, such that alternative m is between alternatives c_1, c_2 in linear order x. Triple c_1, c_2, m then does not satisfy a never-middle condition in $\mathcal{D}$. □

End of proof of Theorem 7.4.1. Combining Lemmas 7.4.2 and 7.4.3 we see that the induction step can be done in both cases. □

Classification of indecomposable symmetric maximal Condorcet domains is of great interest. We believe the following conjecture is true.

Conjecture 3. *Any non-trivial indecomposable symmetric maximal Condorcet domain is a Raynaud domain.*

We checked it for $m = 5$ and $m = 6$ by brute force.

7.5 A symmetric domain in the associahedron

As we have seen, Condorcet domains viewed as a set of permutations have deep relations to various branches of combinatorics [Galambos and Reiner, 2008, Danilov et al., 2012, Labbé and Lange, 2020, Slinko, 2024a]. In this note we present one more combinatorial connection of Condorcet domains.

7.5.1 Permutations and binary trees.

The group of permutations S_n of order n is in a bijection with the set of full binary trees with levels on leaves labelled $0, 1, \dots, n$. Every internal vertex is assigned a level as shown on Figure 7.3. A full binary tree with levels corresponds to the permutation

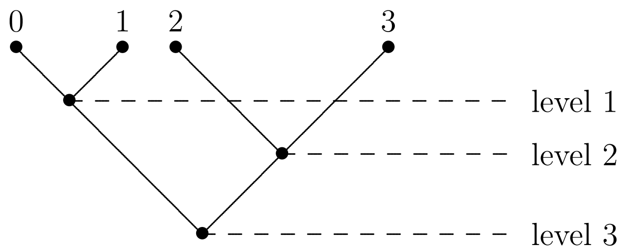
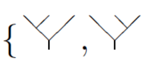


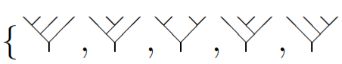
Figure 7.3: Binary tree with levels

$$\sigma(i) = \text{the lowest level on the unique path from } i-1 \text{ to } i.$$

For example, the tree on Figure 7.3 corresponds to the transposition $(2, 3)$.

Let Y_n be the set of full binary trees with $n+1$ leaves $0, 1, \dots, n$:

$Y_0 = \{ \mid \}$, $Y_1 = \{$  $\}$, $Y_2 = \{$  $\}$,

$Y_3 = \{$  $\}$.

Using bijection of S_n with trees with levels and forgetting the levels we obtain a surjective mapping

$$\psi\colon S_n \to Y_n.$$

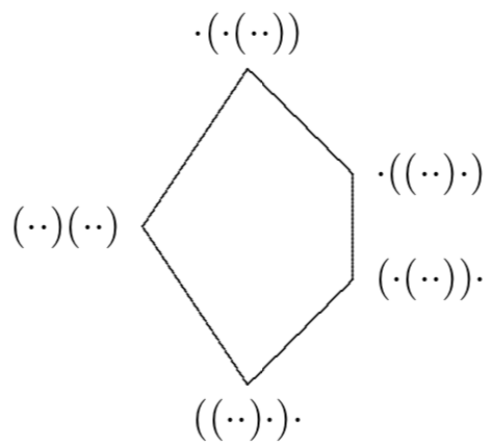
For example,

$\psi((1, 3, 2)) =$  .

An nth Tamari lattice T_n [Tamari, 1962], is a partially ordered set in which the elements consist of different ways of parenthesization of a product of n elements in an associative algebra. For $n = 4$ we have five ways to calculate the product $abcd$, namely,

$$((ab)c)d = (ab)(cd) = (a(bc))d = a((bc)d) = a(b(cd)).$$

One parenthesization is ordered before another if the second parenthesization may be obtained from the first by only rightward applications of the associative law replacing $(xy)z$ with $x(yz)$. For example, the fourth Tamari lattice T_4 is



which can be identified with Y_3.

So Y_{n+1} can be obviously identified with the elements of the Tamari lattice T_n. Indeed, suppose we have a bijections $\mu_i \colon Y_i \to T_{i-1}$ for $i = 1, \dots, n-1$. Then given a tree $t \in Y_n$ we consider its left and right branches t_ℓ and t_r, with k and $n+1-k$ leaves, respectively, and put in correspondence to t the parenthesization $(\mu_k(t_\ell))(\mu_{n-k}(t_r)) \in T_{n-1}$. This establishes a bijection between Y_n and T_{n-1} which is clearly an isomorphism of the lattices.

7.5.2 Permutohedron and associahedron

We view a permutation $u \in S_n$ as a vector $[u(1), u(2), \dots, u(n)] \in \mathbb{R}^n$ and sometimes as a sequence $u(1)u(2)\dots u(n)$. Geometrically, the permutations of $[n]$ form a set of $n!$ points in $\mathbb{N}^n$. According to Ziegler [2012] the *permutohedron* of order n is an $(n-1)$-dimensional polytope embedded in an n-dimensional space. Its vertex coordinates (labels) are the permutations of $[n]$, it is denoted as Perm_n.

The skeleton of a polytope is the graph structure given by the vertices and edges of the polytope. Given a graph G, its *polytopal realisation* is a polytope P whose skeleton is isomorphic to G.

Associahedron is a (convex) polytopal realisation of the Tamari lattice. Associahedra are also called Stasheff polytopes [Stasheff, 1963] after the work of Jim Stasheff who was the first to give their polytopal realisations. The associahedron can be also viewed as a convex polytope whose vertices are in correspondence with triangulations of a convex polygon and whose edges are flips among them [Ziegler, 2012].

The following beautiful polytopal realisation is due to Loday [2004]. As an element of the Tamari lattice T_{n+1} can be identified with a full binary tree in Y_n with leaves $0, 1, \dots, n$, we label the internal vertices of that tree so that the ith vertex is the one which falls in between the leaves $i-1$ and i. Then we denote by a_i, respectively, b_i, the number of leaf descendants of the left child, respectively, right child, of the ith vertex. The product $w(i) = a_i b_i$ is the *weight* of the ith vertex. To the tree t in Y_n we associate the point $M_n(t) \in \mathbb{R}^n$ whose ith coordinate is the weight of the ith vertex, that is, $M_n \colon Y_n \to \mathbb{R}^n$ such that

$$M_n(t) = (a_1 b_1, a_2 b_2, \dots, a_n b_n) \in \mathbb{R}^n.$$

Theorem 7.5.1 (Loday, 2004)**.** *The polytope with vertices* $\{M_n(t) \mid t \in Y_n\}$ *is a polytopal realisation of the Tamari lattice* T_{n+1}*, i.e., an associahedron, denoted* $Asso_n$*.*

Example 7.5.1. *For the tree* t *on Figure 7.3 we have* $M_3(t) = (1,4,1)$ *and* $M_3($$) =$ $(3,1,2)$*. Figure 7.4 shows the coordinates of* $Asso_3$*.*

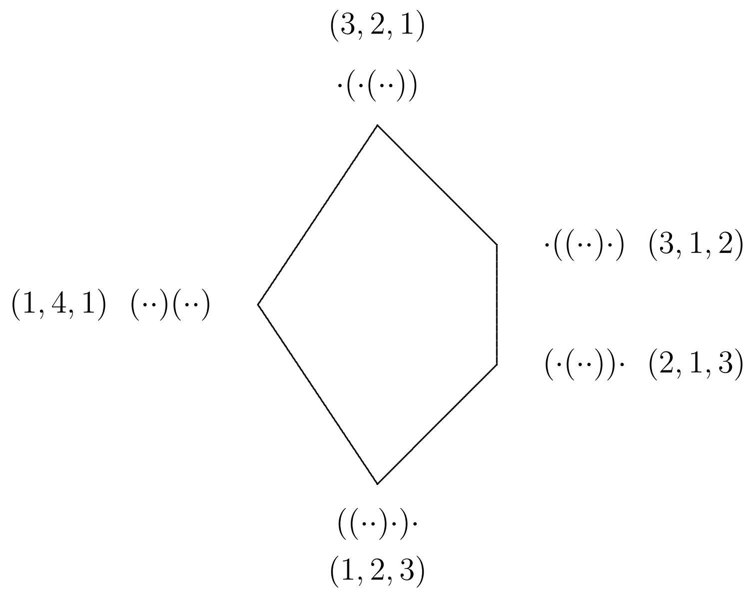


Figure 7.4: Loday's coordinates of the associahedron Asso_3.

The Tamari lattices T_n can be described in many ways via the known bijections between families of Catalan objects.

7.5.3 The Condorcet domain

Example 7.5.1 shows that some vertices of Perm_n and Asso_n are in common. We are interested in the intersection of these two polytopes.

Theorem 7.5.2. $Perm_n \cap Asso_n = (\ldots((1 \star 2) \star 3) \star \ldots \star n-1) \star n$ *is the maximal never-middle Condorcet domain of size* 2^{n-1}*.*

Proof. As seen in Example 7.5.1, for $n = 3$ the common vertices of the two polytopes are the points corresponding to the permutations belonging to maximal Condorcet domain

$$\mathcal{D}_{3,2} = \{123, 213, 312, 321\} = (1 \star 2) \star 3.$$

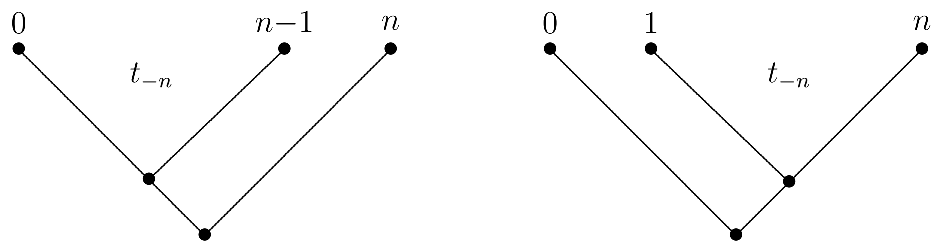
We see that 3 can never be in the middle, hence the condition $3N_{\{1,2,3\}}3$ is satisfied. Inductively, suppose that

$$\text{Perm}_{n-1} \cap \text{Asso}_{n-1} = (\ldots(1 \star 2) \star 3) \star \ldots) \star (n-1). \tag{7.5.1}$$

Let $t \in Y_n$ and consider $M_n(t) \in \mathrm{Asso}_n$. Consider the root v of this tree and its weight $w(v)$. We have $w(v) = (i+1)(n-i)$ for some $i \in \{0, \ldots, n-1\}$. If $M_n(t) \in \mathrm{Perm}_n$, we must have $w(v) \leq n$. Thus,

$$w(v) = (i+1)(n-i) \leq n,$$

from which $i = 0$ or $i = n-1$. Thus t is equal to one of the following trees:



where $t_{-n} \in Y_{n-1}$ and $M_{n-1}(t_{-n}) \in \mathrm{Perm}_{n-1} \cap \mathrm{Asso}_{n-1}$. This means that

$$M_n(t) = n \ldots \quad \text{or} \quad M_n(t) = \ldots n,$$

respectively. We note that the weights for the vertices of t_{n-1} are exactly the same as they were in t_n, Hence the induction hypothesis (7.5.1) can be used and

$$\mathrm{Perm}_n \cap \mathrm{Asso}_n = ((\ldots(1 \star 2) \star 3) \star \ldots) \star (n-1)) \star n.$$

It is known [Karpov and Slinko, 2023c] that such domain is maximal and satisfies for each triple $i < j < k$ the never condition $kN_{\{i,j,k\}}2$ (never-middle one) and has cardinality 2^{n-1}. This proves the theorem. □

Note 1. *The never condition $kN_{\{i,j,k\}}2$ (never-middle one) can be reformulated in terms of pattern avoidance. This will mean that $Perm_n \cap Asso_n$ consists of all permutations of S_n that avoid patterns* 132 *and* 231*.*

Example 7.5.2. *For $n = 4$ we get the domain*

$$\mathcal{D} = ((1 \star 2) \star 3) \star 4 = \{1234, 2134, 3124, 3214, 4123, 4213, 4312, 4321\}.$$

Chapter 8

Constructions of large Condorcet domains

Maximal Condorcet domains historically have attracted a special attention since they represent a compromise which allows a society to always have transitive collective preferences and, under this constraint, provide voters with as much individual freedom as possible. However, maximal Condorcet domains may have different cardinalities which can be as low as four [Danilov and Koshevoy, 2013]. Thus the question: "How large a Condorcet domain can be?" has attracted even more attention (see the survey of Monjardet [2009] for a fascinating account of historical developments). Kim et al. [1992] identified this problem as a major unsolved problem in the mathematical social sciences.

Fishburn [1996a] addressing this question introduced the function

$$f(n) = \max\{|\mathcal{D}| : \mathcal{D} \text{ is a Condorcet domain on a set of } n \text{ alternatives}\}.$$

and put this problem in the mathematical perspective asking for the maximal values of this function to be found or at least estimated.

Abello [1991] and Fishburn [1996a, 2002] managed to construct some "large" Condorcet domains based on different ideas. Fishburn, in particular, taking a clue from Monjardet example (sent to him in a private communication), came up with the so-called alternating scheme domains, later called Fishburn's domains, considered in Section 5.1. This scheme produces Condorcet domains with some nice properties, which, in particular, are connected and have maximal width.

Fishburn [1996a] conjectured (Conjecture 2) that among Condorcet domains that, for any triple of alternatives, do not satisfy a never-middle condition (these were later called peak-pit domains), the alternating scheme provides domains of maximum cardinality. Labbé and Lange [2020] showed that Fishburn's domain has the largest size among domains associated with the weak order of a finite Coxeter group. The focus of attention on peak-pit maximal domains was justified by their important connections to such classical combinatorial objects as rhombus tilings, arrangements of pseudolines and maximal separated set systems [Leclerc and Zelevinsky, 1998, Galambos and Reiner, 2008, Danilov et al., 2012].

In early work, Condorcet domains on the set of n alternatives were usually required to have maximal width which (up to an isomorphism) means that they must contain two completely reversed linear orders $12\dots n$ and $n\dots 21$. Monjardet [2006] introduced (in slightly different terms) the function

$$g(n) = \max\{|\mathcal{D}| : \mathcal{D} \text{ is a a peak-pit Condorcet domain of maximal width on a set of } n \text{ alternatives}\}.$$

It was hypothesised [Fishburn, 1996a] that $g(n) = |F_n|$, which means that Fishburn's domains are the largest in the class of peak-pit Condorcet domain of maximal width. This was disproved by Danilov et al. [2012].

Recently, the necessity of the requirement of maximal width was questioned for a number of reasons. In particular, Arrow's single-peaked domain (which is a Condorcet domain whose restriction on each triple of alternatives is single-peaked is a very natural class of Condorcet domains without maximal width requirement. Also, not every society is liberal enough to allow completely opposite views. Thus, we introduce another function

$$h(n) = \max\{|\mathcal{D}| : \mathcal{D} \text{ is a peak-pit domain on a set of } n \text{ alternatives}\}.$$

It is known that $f(n) = g(n) = h(n)$ for $n \leq 7$ [Fishburn, 1996a, Galambos and Reiner, 2008] and it was believed that $g(16) < f(16)$ [Monjardet, 2009]. This is because Fishburn [1996a] showed that $f(16) > |F_{16}|$ — we will prove this in Section 8.1. Thus, if Fishburn's hypothesis was true, we would get $f(n) > g(n)$ for $n \geq 16$. However, this hypothesis appeared to be false.

Danilov et al. [2012] introduced the class of tiling domains which are peak-pit domains of maximal width and defined an operation of concatenation on tiling domains that allowed them to show that $g(42) > |F_{42}|$ refuting the aforementioned Fishburn's conjecture. In this section we give an algebraic definition, a generalisation, and an extension of the Danilov-Karzanov-Koshevoy construction (abbreviated DKK-construction), which we call concatenation+shuffle scheme, and investigate its properties.

In our interpretation the DKK-construction $\otimes_1$ involves two peak-pit Condorcet domains $\mathcal{D}_1$ and $\mathcal{D}_2$ on sets of n and m alternatives, respectively, and two linear orders $u \in \mathcal{D}_1$ and $v \in \mathcal{D}_2$; the result is denoted as $(\mathcal{D}_1 \otimes_1 \mathcal{D}_2)(u, v)$. It is again a peak-pit Condorcet domain on $n + m$ alternatives whose exact cardinality we can calculate. A drawback of the DKK-construction is that, when $\mathcal{D}_1$ and $\mathcal{D}_2$ are maximal Condorcet domains, $(\mathcal{D}_1 \otimes_1 \mathcal{D}_2)(u, v)$ may not be maximal. We will fix this shortcoming by introducing a new construction $(\mathcal{D}_1 \otimes_2 \mathcal{D}_2)(u, v)$ that always contains $(\mathcal{D}_1 \otimes_1 \mathcal{D}_2)(u, v)$ and is always maximal. It allows us to prove the inequality $h(13) > |F_{13}|$. Using the new construction we can also show that $g(34) > |F_{34}|$ further improving the result of Danilov et al. [2012].

We also prove that for large n we have lower bounds $g(n) \geq 2.0767^n$, $h(n) \geq 2.1045^n$ and $f(n) \geq 2.1890^n$, where the first confirms the unpublished result of Ondřej Bilka announced in Felsner and Valtr [2011] and the third improves Fishburn's (1996a) bound while the second result is completely new. We also get an upper bound $g(n) < 2.4870^n$ for function g. For other functions, it is only known that they are bounded by c^n for some constant c.

Concatenation+shuffle scheme reveals the structural properties and generation algorithm for Arrow's single-peaked domains. We show that all Arrow's single-peaked domains can be constructed by concatenation+shuffle scheme starting from the trivial domain.

The chapter is organised as follows. Section 8.2 gives the algebraic description and properties of the original Danilov-Karzanov-Koshevoy construction and proves the lower bound for function g. Section 8.3 introduces the concatenation+shuffle scheme, studies its properties and proves that all Arrow's single-peaked domains can be obtained iteratively using the concatenation+shuffle scheme from the trivial domain. Section 8.4 is devoted to construction of Condorcet domains of cardinality larger than Fishburn's domains. It also contains a new bound for function h and improved bound for function f. A short Section 8.5 outlines what is known in relation to upper bounds, most notably, an upper bound for function g.

8.1 Fishburn's replacement scheme

Fishburn's replacement scheme is a partial case of the construction given in Definition 6.1.1. Given a domain $\mathcal{E}$ on the set A of $m+1$ alternatives $\{a_0, a_1, \ldots, a_m\}$ and domain $\mathcal{D}$ on the set of n alternatives $B = \{b_1, \ldots, b_n\}$ with $A \cap B = \emptyset$ we define the domain

$$\mathcal{E}(a_0 \to \mathcal{D}, a_1, \ldots, a_n).$$

For function f, the Fishburn's replacement scheme allowed him to prove the following inequality.

Proposition 8.1.1 (Lemma 1 in Fishburn [1996b])**.**

$$f(n+m) \geq f(n)f(m+1). \tag{8.1.1}$$

Proof. Let $\mathcal{E}$ be a Condorcet domain on the set of $m+1$ alternatives of size $f(m+1)$ and $\mathcal{D}$ be a Condorcet domain on the set of n alternatives of size $f(n)$. Then

$$f(n+m) \geq |\mathcal{E}(a_0 \to \mathcal{D}, a_1, \ldots, a_n)| = |\mathcal{E}||\mathcal{D}| = f(n)f(m+1).$$ □

Corollary 8.1.2. $f(16) > |F_{16}|$.

Proof. We have due to (8.1.1)

$$f(16) > f(8)f(9) \geq |F_8||F_9| = 222 \cdot 488 = 108336 > 105884 = |F_{16}|.$$ □

8.2 Danilov-Karzanov-Koshevoy construction

Danilov et al. [2012] define the "concatenation" of two tiling domains by the picture shown in Figure 8.1. Let us now start describing this construction algebraically. In fact, this will be a generalisation of their construction since in our construction two arbitrary linear orders are additionally involved. Firstly, we describe the "pure" concatenation.

Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two Condorcet domains on disjoint sets of alternatives A and B, respectively. We define a *concatenation* of these domains as the domain

$$\mathcal{D}_1 \odot \mathcal{D}_2 = \{xy \mid x \in \mathcal{D}_1 \text{ and } y \in \mathcal{D}_2\}$$

on $A \cup B$. It is immediately clear that $\mathcal{D}_1 \odot \mathcal{D}_2$ is also a Condorcet domain of cardinality $|\mathcal{D}_1 \odot \mathcal{D}_2| = |\mathcal{D}_1||\mathcal{D}_2|$. We have only to check that one of the never-conditions is satisfied for each triple $\{a_1, a_2, b\}$ where $a_1, a_2 \in A$ and $b \in B$ (for triples $\{a, b_1, b_2\}$ the argument will be similar). The restriction $(\mathcal{D}_1 \odot \mathcal{D}_2)|_{\{a_1,a_2,b\}}$ will contain at most two linear orders a_1a_2b and a_2a_1b, which is consistent with both never-top and never-bottom conditions. This domain corresponds to the snakes lying entirely within T and T' on Figure 8.1.

Definition 8.2.1. *Let A and B be two disjoint sets of alternatives, $u \in \mathcal{L}(A)$ and $v \in \mathcal{L}(B)$. An order $w \in \mathcal{L}(A \cup B)$ is said to be a* shuffle *of u and v if $w|_A = u$ and $w|_B = v$, i.e., the restriction of w onto A is equal to u and the restriction of w onto B is equal to v.*

For example, 516723849 is a shuffle of 1234 and 56789.

Given two linear orders u and v, we define domain $u \oplus v$ on $A \cup B$ as the set of all shuffles of u and v. It is clear from the definition that $u \oplus v = v \oplus u$. The cardinality of this domain is $|u \oplus v| = \binom{n+m}{m}$. We believe this domain corresponds to what is depicted in Figure 8.1 outside of T and T' for some particular u and v.

Now we combine the two domains we have just introduced together.

Theorem 8.2.1. *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two Condorcet domains on disjoint sets of alternatives A and B. Let $u \in \mathcal{D}_1$ and $v \in \mathcal{D}_2$ be arbitrary linear orders. Then*

$$(\mathcal{D}_1 \otimes_1 \mathcal{D}_2)(u, v) := (\mathcal{D}_1 \odot \mathcal{D}_2) \cup (u \oplus v)$$

is a Condorcet domain. If $\mathcal{D}_1$ and $\mathcal{D}_2$ are peak-pit domains, so is $(\mathcal{D}_1 \otimes_1 \mathcal{D}_2)(u, v)$.

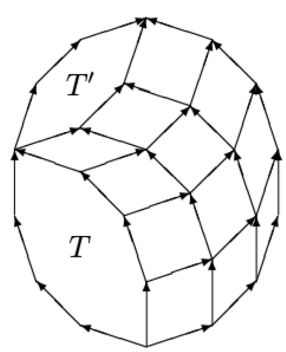


Figure 8.1: Concatenation of tilings T and T'.

Proof. Let us fix u and v in this construction and denote $(\mathcal{D}_1 \otimes_1 \mathcal{D}_2)(u,v)$ as simply $\mathcal{D}_1 \otimes_1 \mathcal{D}_2$. If $a,b,c \in A$, then $(\mathcal{D}_1 \otimes_1 \mathcal{D}_2)|_{\{a,b,c\}} = \mathcal{D}_1|_{\{a,b,c\}}$, i.e., the restriction of $\mathcal{D}_1 \otimes_1 \mathcal{D}_2$ onto $\{a,b,c\}$ is the same as the restriction of $\mathcal{D}_1$. Hence $\mathcal{D}_1 \otimes_1 \mathcal{D}_2$ satisfies the same never condition for $\{a,b,c\}$ as $\mathcal{D}_1$. For $x,y,z \in B$ the same thing happens.

Suppose now $a,b \in A$ and $x \in B$. Then $(\mathcal{D}_1 \odot \mathcal{D}_2)|_{\{a,b,x\}} \subseteq \{abx, bax\}$. Let us also assume (without loss of generality) that $u|_{\{a,b\}} = \{ab\}$. Then $(u \oplus v)|_{\{a,b,x\}} = \{abx, axb, xab\}$, hence

$$(\mathcal{D}_1 \otimes_1 \mathcal{D}_2)_{\{a,b,x\}} \subseteq \{abx, bax, axb, xab\}, \tag{8.2.1}$$

thus $\mathcal{D}_1 \otimes_1 \mathcal{D}_2$ satisfies $aN_{\{a,b,x\}}3$. For $a \in A$ and $x,y \in B$ we have $(\mathcal{D}_1 \odot \mathcal{D}_2)_{\{a,x,y\}} \subseteq \{axy, ayx\}$. Let also $v|_{\{x,y\}} = \{xy\}$. Then $(u \oplus v)|_{\{a,x,y\}} = \{axy, xay, xya\}$, hence

$$(\mathcal{D}_1 \otimes_1 \mathcal{D}_2)|_{\{a,x,y\}} \subseteq \{axy, ayx, xay, xya\}, \tag{8.2.2}$$

thus $\mathcal{D}_1 \otimes_1 \mathcal{D}_2$ satisfies $yN_{\{a,x,y\}}1$. □

From the proof of Theorem 8.2.1 we can extract the following additional information.

Corollary 8.2.2. *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two Condorcet domains. Then*

(i) If $\mathcal{D}_1$ and $\mathcal{D}_2$ are ample, then $(\mathcal{D}_1 \otimes_1 \mathcal{D}_2)(u,v)$ is copious.

(ii) If for any $a,b \in A$ and $x,y \in B$ with $a \succ_u b$ and $x \succ_v y$, then domain $(\mathcal{D}_1 \otimes_1 \mathcal{D}_2)(u,v)$ satisfies $aN_{\{a,b,x\}}3$ and $yN_{\{a,x,y\}}1$.

Proof. (i) The inequalities (8.2.1) and (8.2.2) become equalities if $\mathcal{D}_1$ and $\mathcal{D}_2$ are ample. (ii) also follows from (8.2.1) and (8.2.2). □

If $\mathcal{D}_1$ is a domain of maximal width on $[m]$ and $\mathcal{D}_2$ is a domain of maximal width on $[m+n] \setminus [m]$, then to obtain the original DKK-construction we should choose $u = m(m-1)\ldots 1$ and $v = (m+n)(m+n-1)\ldots(m+1)$.

Proposition 8.2.3. *If $|A| = m$ and $|B| = n$, then for any $u \in \mathcal{D}_1$ and $v \in \mathcal{D}_2$*

$$|(\mathcal{D}_1 \otimes_1 \mathcal{D}_2)(u,v)| = |\mathcal{D}_1||\mathcal{D}_2| + \binom{n+m}{m} - 1. \tag{8.2.3}$$

Proof. We have $|\mathcal{D}_1 \otimes_1 \mathcal{D}_2| = |\mathcal{D}_1||\mathcal{D}_2|$ and $|u \oplus v| = \binom{n+m}{m}$. These two sets have only one linear order in common which is uv. This proves (8.2.3). □

Proposition 8.2.4. *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be of maximal width with $u, \bar{u} \in \mathcal{D}_1$ and $v, \bar{v} \in \mathcal{D}_2$. Then $(\mathcal{D}_1 \otimes_1 \mathcal{D}_2)(u,v)$ also has maximal width with two completely reversed orders $\bar{u}\bar{v}$ and vu.*

Proof. We note that $\bar{u}\bar{v} \in \mathcal{D}_1 \odot \mathcal{D}_2$. We also have $\overline{\bar{u}\bar{v}} = vu \in u \oplus v$, hence $(\mathcal{D}_1 \otimes_1 \mathcal{D}_2)(u,v)$ has maximal width. □

The following corollary will be important during the construction of large domains of maximal width.

Corollary 8.2.5. *Let $\mathcal{D}_1$ be a domain of maximal width on $[m]$ with $e = 12\dots m$ and $\bar{e} = m\dots 21$ and $\mathcal{D}_2$ be a domain of maximal width on $[n]\setminus[m]$ with $f = (m+1)(m+2)\dots n$ and $\bar{f} = n\dots(m+2)(m+1)$. Then for the product $(\mathcal{D}_1 \otimes_1 \mathcal{D}_2)(u,v)$ to have maximal width with two completely reversed orders $12\dots n$ and $n\dots 21$ we need to choose $u = \bar{e}$ and $v = \bar{f}$. In particular, if $\mathcal{D}_1$ and $\mathcal{D}_2$ are semi-connected, then so is $(\mathcal{D}_1 \otimes_1 \mathcal{D}_2)(\bar{e},\bar{f})$.*

Proof. The first statement follows directly from Proposition 8.2.4. Suppose now that $\mathcal{D}_1$ and $\mathcal{D}_2$ are semi-connected. By Proposition 8.2.4 the two completely reversed orders in $(\mathcal{D}_1 \otimes_1 \mathcal{D}_2)(\bar{e},\bar{f})$ will be ef and $\bar{f}\bar{e}$. Then we note that ef can be connected to $\bar{e}\bar{f}$ (which belongs both to $\mathcal{D}_1 \odot \mathcal{D}_2$ and to $\bar{e} \oplus \bar{f}$) by a geodesic path within $\mathcal{D}_1 \odot \mathcal{D}_2$ and $\bar{e}\bar{f}$, in turn, can be connected to $\bar{f}\bar{e}$ by a geodesic path within $\bar{e} \oplus \bar{f}$. □

Theorem 8.2.6. *Let $e \in \{g,h\}$ be one of the functions defined above. Then*

$$e(n+m) > e(n)e(m). \tag{8.2.4}$$

Proof. If $\mathcal{D}_1$ is one of the largest Condorcet domains on the set of n alternatives of size $e(n)$ and $\mathcal{D}_2$ is the largest Condorcet domain on the set of m alternatives of size $e(m)$, then, by Theorem 8.2.1, $(\mathcal{D}_1 \otimes_1 \mathcal{D}_2)(u,v)$ is a Condorcet domain on the set of $n+m$ alternatives that has the size greater than $e(n)e(m)$ but smaller than or equal to $e(n+m)$. If both were peak-pit domains, then the product is also peak-pit and, moreover, by Proposition 8.2.4 u and v could be chosen so that the product has maximum width if $\mathcal{D}_1$ and $\mathcal{D}_2$ had one. This proves the equation (8.2.4) for both functions. □

For function f, the Fishburn's replacement scheme allows to prove a slightly stronger inequality (8.1.1). However, the replacement scheme used to obtain this inequality, given two Condorcet domains $\mathcal{D}_1$ and $\mathcal{D}_2$, produces a Condorcet domain for which, unlike our composition, some triples of alternatives satisfy the never-middle conditions.

If both $\mathcal{D}_1$ and $\mathcal{D}_2$ have maximal width, it is not true, however, that $(\mathcal{D}_1 \otimes_1 \mathcal{D}_2)(u,v)$ will have maximal width for any $u \in \mathcal{D}_1$ and $v \in \mathcal{D}_2$. Let us take, for example, $\mathcal{D}_1 = \{x = ab, \bar{x} = ba\}$ and $\mathcal{D}_2 = \{u = cde, v = dec, w = dce, \bar{u} = edc\}$. Then $(\mathcal{D}_1 \otimes_1 \mathcal{D}_2)(\bar{x},\bar{u})$ has maximal width with *abcde* and *edcba* belonging to it, while $(\mathcal{D}_1 \otimes_1 \mathcal{D}_2)(\bar{x},v)$ does not since $\bar{v} \notin \mathcal{D}_2$. In particular,

$$(\mathcal{D}_1 \otimes_1 \mathcal{D}_2)(\bar{x},\bar{u}) \ncong (\mathcal{D}_1 \otimes_1 \mathcal{D}_2)(\bar{x},v).$$

This indicates that the construction $\otimes_1$ may be useful in description and construction of Condorcet domains which do not satisfy the requirement of maximal width and we will see exactly that in characterisation of Arrow's single-peaked domains.

Proposition 8.2.7. *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two connected Condorcet domains on disjoint sets of alternatives A and B, $u \in \mathcal{D}_1$ and $v \in \mathcal{D}_2$. Then $\mathcal{D} = (\mathcal{D}_1 \otimes_1 \mathcal{D}_2)(u,v)$ is also connected.*

Proof. Suppose that $\mathcal{D}_1$ and $\mathcal{D}_2$ are both connected, and consider two neighbours $w, w' \in \mathcal{D}$ in $G_{\mathcal{D}}$. Due to connectedness of $\mathcal{D}_1$ and $\mathcal{D}_2$, if $w, w' \in \mathcal{D}_1 \odot \mathcal{D}_2$ they are connected by a series of swaps and since they are neighbours, they are connected by a single swap. If $w, w' \in u \oplus v$ the situation is similar. So it is enough to consider the case when $w \in \mathcal{D}_1 \odot \mathcal{D}_2$ and $w' \in u \oplus v$. Then w and w' are connected by a series of swaps and any such path contains $uv \in (\mathcal{D}_1 \odot \mathcal{D}_2) \cap (u \oplus v)$. Thus $uv \in [w, w']_{\mathcal{D}}$. Hence either $w = uv$ or $w' = uv$ from which the statement follows. □

Proposition 8.2.8. *The following isomorphism holds*

$$(F_2(a,b) \otimes_1 F_2(c,d))(ba, dc) \cong F_4. \tag{8.2.5}$$

Proof. We list orders of this domain as columns of the following matrix

$$[F_2(a,b) \odot F_2(c,d) \mid ba \oplus dc] = \left[\begin{array}{cccc|ccccc} a & a & b & b & b & b & d & d & d \\ b & b & a & a & d & d & b & b & c \\ c & d & c & d & a & c & c & a & b \\ d & c & d & c & c & a & a & c & a \end{array}\right].$$

We see that the following never conditions are satisfied: $bN_{\{a,b,c\}}3$, $bN_{\{a,b,d\}}3$, $cN_{\{a,c,d\}}1$, $cN_{\{b,c,d\}}1$. Hence the mapping $1 \to a$, $2 \to b$, $3 \to c$ and $4 \to d$ is an isomorphism of F_4 onto $(F_2(a,b) \otimes_1 F_2(c,d))(ba, dc)$. □

The isomorphism (8.2.5) is very nice but unfortunately, as we will see in example that follows, for larger m, n we have no such isomorphisms. Moreover, it appears that for two maximal Condorcet domains $\mathcal{D}_1$ and $\mathcal{D}_2$ on sets A and B, respectively, $(\mathcal{D}_1 \otimes_1 \mathcal{D}_2)(u, v)$ may not be maximal on $A \cup B$. Here is an example.

Example 8.2.1. *Let us calculate* $\mathcal{E} := F_3(1,2,3) \otimes_1 F_2(4,5)(321, 54)$*:*

$$\left[\begin{array}{cccccccc|ccccccccc} 1 & 2 & 2 & 3 & 1 & 2 & 2 & 3 & 3 & 3 & 5 & 3 & 3 & 3 & 5 & 5 & 5 \\ 2 & 1 & 3 & 2 & 2 & 1 & 3 & 2 & 2 & 5 & 3 & 2 & 5 & 5 & 3 & 3 & 4 \\ 3 & 3 & 1 & 1 & 3 & 3 & 1 & 1 & 5 & 2 & 2 & 5 & 2 & 4 & 2 & 4 & 3 \\ 4 & 4 & 4 & 4 & 5 & 5 & 5 & 5 & 1 & 1 & 1 & 4 & 4 & 2 & 4 & 2 & 2 \\ 5 & 5 & 5 & 5 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right].$$

There are 17 linear orders in this domain. It is known, however, that F_5 *has 20 but this fact alone does not mean non-maximality of* $\mathcal{E}$*. By Corollary 8.2.2(i) this domain is copious. By its construction (see Corollary 8.2.2(ii)) it satisfies just three inversion triples:*

$$[1,4,5], \quad [2,4,5], \quad [3,4,5].$$

Now we see that there are two more linear orders 23514 *and* 23541 *that satisfy these conditions. Hence* $\mathcal{E}$ *is not maximal. In particular,* $\mathcal{E}$ *is not isomorphic to* F_5*. So the isomorphism* (8.2.5) *is just one of a kind.*

Despite not providing maximal Condorcet domains DKK-construction allowed Danilov et al. [2012] to refute a long standing conjecture [Fishburn, 1996a, Galambos and Reiner, 2008] that $g(n) = |F_n|$, where F_n is the nth Fishburn's domain. In other words, they showed that for large n it is not true that F_n is the largest peak-pit Condorcet domain of maximal width on n alternatives.

Below by $F_n \otimes_1 F_m$ we will mean the product

$$(\mathcal{D}_1 \otimes_1 \mathcal{D}_2)(\bar{e}, \bar{f}),$$

where $\mathcal{D}_1$ is the nth Fishburn's domain on the set of alternatives $[n]$, $\mathcal{D}_2$ is the mth Fishburn's domain on the set of alternatives $[n+m] \setminus [n]$, and $e = 12\ldots n$, $f = (n+1)(n+2)\ldots(n+m)$. (As we previously noted such a choice of u and v in $(\mathcal{D}_1 \otimes_1 \mathcal{D}_2)(u,v)$ secures the maximal width of the product.)

To disprove the aforementioned conjecture Danilov et al. [2012] showed that $|F_{21} \otimes_1 F_{21}| > |F_{42}|$ which implies that $g(42) > |F_{42}|$. Our calculations, using exact formulas (1.2.2) and (8.2.3) show that

$$|F_n \otimes_1 F_n| < |F_{2n}|$$

for $2 < n \leq 19$ but $4611858343415 = |F_{20} \otimes_1 F_{20}| > |F_{40}| = 4549082342996$ which shows that $g(40) > |F_{40}|$. Later in the chapter we will further improve this inequality.

The fact that the Fishburn's hypothesis is not true also follows from the unpublished result of Ondřej Bilka announced in Felsner and Valtr [2011]. It states that for large n it is true that $g(n) > 2.0767^n$. Since this lower bound asymptotically grows faster than the cardinality of Fishburn's domain (1.2.2), the result follows. Below we give a proof of Bilka's bound.

Another use of DKK-construction, due to Proposition 8.2.3, allows us to note, for example, that $g(x+y) > g(x)g(y)$ and $g(ax) > g(x)^a$ for any positive integer a, and a similar inequalities hold for function h. This helps us to find lower bounds for functions g, h with the help of the following theorem that is analogous (but slightly weaker) than Fishburn's result in Fishburn [1996b] which was proved for function f and was based on the replacement scheme. We remind the reader the statement of Fishburn's theorem.

Theorem 8.2.9 (Fishburn [1996b]). *For any natural k, positive ϵ, and for sufficiently large n we have*

$$f(n) > \left(f(k)^{\frac{1}{k-1}-\epsilon}\right)^n.$$

For two other functions we can now prove a similar theorem.

Theorem 8.2.10. *For any natural k, positive ϵ, and for sufficiently large n we have*

$$e(n) > \left(e(k)^{\frac{1}{k}-\epsilon}\right)^n,$$

where $e \in \{g, h\}$.

Proof. We note that by Theorem 8.2.6 for any positive integer a we have $e(ax) > e(x)^a$. Let us divide n by k with remainder: $n = ak + b$, where $0 \le b < k$. Then we will have

$$e(n) > e(k)^a e(b) = e(k)^{(n-b)/k} e(b) \ge e(k)^{(n-b)/k} = \left(e(k)^{\frac{1}{k}-\epsilon}\right)^n,$$

for $\epsilon = b/kn \to 0$ for $n \to \infty$. □

With the help of Theorem 8.2.10 we can now get a lower bound for g.

Theorem 8.2.11. *$g(n) > 2.0767^n$ for all large n.*

Proof. Using Proposition 8.2.3 we can calculate the size of domain $\mathcal{D} = (F_{27} \otimes_1 F_{27}) \otimes_1 (F_{27} \otimes_1 F_{27})$ which is a connected Condorcet domain of maximal width on 108 alternatives with $|\mathcal{D}| = 1.8917 \cdot 10^{34}$ linear orders. We note that $|\mathcal{D}|^{1/108} = 2.07672\ldots$. We can choose a small ϵ such that $|\mathcal{D}|^{\frac{1}{108}-\epsilon} = 2.0767$. We set $k = 108$. Then from Theorem 8.2.10 we get the result.

$$g(n) > \left(g(108)^{\frac{1}{108}-\epsilon}\right)^n \ge \left(|\mathcal{D}|^{\frac{1}{108}-\epsilon}\right)^n = 2.0767^n.$$ □

This confirms the result announced in Felsner and Valtr [2011].

8.3 Karpov-Slinko construction

As we have seen in Example 8.2.1 the DKK-construction does not guarantee the maximality of the product of two maximal domains, and we could add two more linear orders without violating all the never conditions. Let us extend their construction following Karpov and Slinko [2023b]. We need to introduce a necessary notation first.

Given a linear order $x = x_1 \ldots x_n$ in $\mathcal{L}(A)$ we can view it as a concatenation $x = x^1 x^2$ of two suborders, where for some $s \in \{0, 1, \ldots, n\}$ we have $x^1 = x_1 \ldots x_s$ the top part of x and by $x^2 = x_{s+1} \ldots x_n$ the bottom part of it. We allow for a trivial splitting when $s = 0$ and x^1 is empty or $s = n$ and x^2 is empty.

Given a domain of linear orders $\mathcal{D} \subseteq \mathcal{L}(A)$ and a linear order $x = x_{i_1} \ldots x_{i_k}$ defined on a subset $X = \{x_{i_1} \ldots x_{i_k}\} \subseteq A$, we define the upper and lower contour sets of x as

$$U_{\mathcal{D}}(x) = \{z \in \mathcal{L}(A \setminus X) \mid zx \in \mathcal{D}\},$$
$$L_{\mathcal{D}}(x) = \{y \in \mathcal{L}(A \setminus X) \mid xy \in \mathcal{D}\}.$$

These sets can be empty sometimes. Also, if x is empty, $U_{\mathcal{D}}(x) = L_{\mathcal{D}}(x) = \mathcal{D}$.

For example, if $\mathcal{D} = F_4$, given in (1.2.1), then $U_{\mathcal{D}}(34) = \{12, 21\}$ and $L_{\mathcal{D}}(12) = \{34, 43\}$.

Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two Condorcet domains on disjoint sets of alternatives A and B. Let $u \in \mathcal{D}_1$ and $v \in \mathcal{D}_2$ be arbitrary linear orders. Let also $u = u^1 u^2$ and $v = v^1 v^2$ be any splittings of u and v. Let us define the domain

$$u \boxplus v := \bigcup (U_{\mathcal{D}_1}(u^2) \odot (u^2 \oplus v^1) \odot L_{\mathcal{D}_2}(v^1)),$$

where the union is over all splittings of u and v (including the trivial ones).

Proposition 8.3.1. *$u \boxplus v \supseteq u \oplus v$.*

Proof. Under the trivial splittings of u and v we have $v^1 = v$, $u^2 = u$ and $L_{\mathcal{D}_2}(v^1) = U_{\mathcal{D}_1}(u^2) = \emptyset$ so $U_{\mathcal{D}_1}(u^2) \odot (u^2 \oplus v^1) \odot L_{\mathcal{D}_2}(v^1) = u \oplus v$. □

Lemma 8.3.2. *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two Condorcet domains on disjoint sets of alternatives A and B. Let $u \in \mathcal{D}_1$ and $v \in \mathcal{D}_2$ be arbitrary linear orders. Then the domain $u \boxplus v$ satisfies all the never conditions that $(\mathcal{D}_1 \otimes_1 \mathcal{D}_2)(u, v)$ satisfies.*

Proof. Let us consider an arbitrary element $w \in U_{\mathcal{D}_1}(u^2) \odot (u^2 \oplus v^1) \odot L_{\mathcal{D}_2}(v^1)$ for $u = u^1u^2 \in \mathcal{D}_1$ and $v = v^1v^2 \in \mathcal{D}_2$, where $u^1 \in \mathcal{L}(A_1)$ and $u^2 \in \mathcal{L}(A_2)$ for a partition $A = A_1 \cup A_2$ and where $v^1 \in \mathcal{L}(B_1)$ and $v^2 \in \mathcal{L}(B_2)$ for a partition $B = B_1 \cup B_2$. Suppose $w = \tilde{u} \odot (u^2 \oplus v^1) \odot \tilde{v}$, where $\tilde{u} \in U_{\mathcal{D}_1}(u^2)$ and $\tilde{v} \in L_{\mathcal{D}_2}(v_1)$. We note that, by the definition, $\tilde{u}u^2 \in \mathcal{D}_1$ and $v^1\tilde{v} \in \mathcal{D}_2$. Hence for any triple $\{a, b, c\} \subseteq A$ or $\{x, y, z\} \subseteq B$, the order w will satisfy the same never condition as in $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively.

Suppose, for example, that $a \in A_2$, $x, y \in B$ with $x \succ_v y$. Then by Corollary 8.2.2(ii) $(\mathcal{D}_1 \otimes_1 \mathcal{D}_2)(u, v)$ satisfies $yN_{\{a,x,y\}}1$. It is easy to see that w satisfies the same never condition as y cannot come ahead of both a and x. □

Example 8.3.1. *As we saw in Example 8.2.1 the DKK-construction does not capture two linear orders* 23514 *and* 23541 *which results in the product being not maximal. However, if we define $u^1 = 32$, $u^2 = 1$, $v^1 = 54$ and $v^2 = \emptyset$, then $U_{\mathcal{D}_1}(u^2) = U_{\mathcal{D}_1}(1) = \{23, 32\}$, $L_{\mathcal{D}_2}(v^1) = \emptyset$, $u^2 \oplus v^1 = 1 \oplus 54 = \{154, 514, 541\}$ and thus* 23514 *and* 23541 *will be then captured by*

$$U_{\mathcal{D}_1}(u_2) \odot (u^2 \oplus v^1) \odot L_{\mathcal{D}_2}(v^1) \in u \boxplus v.$$

Now, given two Condorcet domains $\mathcal{D}_1$ and $\mathcal{D}_2$ we define a new domain

$$(\mathcal{D}_1 \otimes_2 \mathcal{D}_2)(u, v) := (\mathcal{D}_1 \odot \mathcal{D}_2) \cup (u \boxplus v).$$

By Corollary 8.2.2 and Lemma 8.3.2 this domain is a Condorcet domain which is copious if $\mathcal{D}_1$ and $\mathcal{D}_2$ are ample and is a peak-pit domain if $\mathcal{D}_1$ and $\mathcal{D}_2$ were peak-pit.

We saw in Example 8.3.1 that the new construction - *concatenation+shuffle scheme* - yields a domain which is strictly bigger than the DKK-construction does. Moreover, the concatenation+shuffle scheme always yields a maximal Condorcet domain which will be proved in the following theorem.

Theorem 8.3.3. *For any two ample maximal Condorcet domains $\mathcal{D}_1$ and $\mathcal{D}_2$ and any $u \in \mathcal{D}_1$, $v \in \mathcal{D}_2$, the domain*

$$\mathcal{D} = (\mathcal{D}_1 \otimes_2 \mathcal{D}_2)(u, v)$$

is also a maximal Condorcet domain.

Proof. Suppose that $\mathcal{D}$ is not a maximal Condorcet domain, then there is a linear order w, not belonging to this set, which addition, however, does not violate all the never conditions that $(\mathcal{D}_1 \otimes_2 \mathcal{D}_2)(u, v)$ satisfies. Let $w_{\mathcal{D}_1} = \tilde{u}$ and $w_{\mathcal{D}_2} = \tilde{v}$. Because domains $\mathcal{D}_1$

and $\mathcal{D}_2$ are maximal, these restrictions belong to $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively, and we have $w \in \tilde{u} \oplus \tilde{v}$. If $w = \tilde{u}\tilde{v}$, then $w \in \mathcal{D}_1 \odot \mathcal{D}_2$. Thus we may assume that $w \neq \tilde{u}\tilde{v}$.

If there are splittings $\tilde{u} = \tilde{u}^1\tilde{u}^2$ and $\tilde{v} = \tilde{v}^1\tilde{v}^2$, $u = u^1u^2$, and $v = v^1v^2$ such that $\tilde{u}^2 = u^2$ and $\tilde{v}^1 = v^1$, then $w \in U_{\mathcal{D}_1}(u^2) \odot (u^2 \oplus v^1) \odot L_{\mathcal{D}_2}(v^1)$ and hence $w \in \mathcal{D}$. Thus we may assume that such splittings do not exist. In particular, we may assume that either the minimal elements of $\tilde{u}^2$ and u^2 are different or the maximal elements of $\tilde{v}^1$ and v^1 are different.

Suppose that the minimal elements a and b in linear orders $\tilde{u}, u$, respectively, are different and suppose without loss of generality that $a \succ_u b$. Let x be the maximal element in linear order $\tilde{v} = v$. Then for the triple of alternatives $\{a, b, x\}$, by Corollary 8.2.2 the condition $aN_{\{a,b,x\}}3$ is satisfied and also $\mathcal{D}_{\{a,b,x\}} = \{abx, bax, axb, xab\}$ as by Corollary 8.2.2 $\mathcal{D}$ is copious. However, since $w \neq \tilde{u}\tilde{v}$ we will have $x \succ_w a$ which implies that either bxa or xba is in $\mathcal{D}_{\{a,b,x\}}$, a contradiction. The case when the maximal elements x and y in linear orders $\tilde{v}, v$, respectively, are different is similar. □

The new definition enables recursive construction of domains.

Theorem 8.3.4. *Let $\mathcal{D}_1$ be an Arrow's single-peaked domain on the set of alternatives A and $\mathcal{D}_2$ be a trivial domain on a one-element set $\{b\}$, where $b \notin A$. Let $u \in \mathcal{D}_1$ and $v = b \in \mathcal{D}_2$. Then $\mathcal{D} = (\mathcal{D}_1 \otimes_2 \mathcal{D}_2)(u, v)$ is also Arrow's single-peaked. If $\mathcal{D}_1$ were maximal, then $\mathcal{D}$ would be also maximal.*

Proof. By Corollary 8.2.2 we add to the set of never conditions governing $\mathcal{D}_1$ the set

$$\{aN_{\{a,a',b\}}3, \quad a, a' \in A, \quad a \succ_u a'\},$$

hence the obtained domain $\mathcal{D}$ is also defined by the never-bottom conditions, hence it is Arrow's single-peaked. If $\mathcal{D}_1$ were maximal, then $\mathcal{D}$ would be also maximal by Theorem 8.3.3. □

We recall that an alternative of an Arrow's single-peaked domain $\mathcal{D}$ is called terminal if it is a bottom alternative of at least one linear order in $\mathcal{D}$. In each maximal Arrow's single-peaked domain there are exactly two terminal alternatives. A pair of orders in Arrow's single-peaked domain is extremal if top and bottom alternatives are reversed in these orders. Each Arrow's single-peaked domain has exactly one pair of extremal linear orders (see Section 2.2).

Lemma 8.3.5. *If two maximal Arrow's single-peaked domains on the set of n alternatives have identical extremal orders with top/bottom alternatives a, b and these domains have identical restrictions on the set $A \setminus \{a, b\}$, then these domains are identical.*

Proof. Let us prove by induction. The statement is true for $n = 3$.

Let us consider two maximal Arrow's single-peaked domains $\mathcal{D}_1$ and $\mathcal{D}_2$ on the set A of n alternatives with the same extremal orders e_1, e_2, which have the following structure $e_1 = av_1b$, $e_2 = bv_2a$ so a and b are terminal. Both domains have the same restrictions on the set $A \setminus \{a, b\}$ which is domain $\mathcal{D}_3$. Let us define the following domains $\mathcal{D}_{1a} = U_{\mathcal{D}_1}(a)$,

$\mathcal{D}_{1b} = U_{\mathcal{D}_1}(b)$, $\mathcal{D}_{2a} = U_{\mathcal{D}_2}(a)$, $\mathcal{D}_{2b} = U_{\mathcal{D}_2}(b)$. These domains are restrictions of respective domains $\mathcal{D}_1$ and $\mathcal{D}_2$ onto the sets $A \setminus \{a\}$ and $A \setminus \{b\}$. According to Lemma 4.3 in Slinko [2019] all of these are maximal Arrow's single-peaked domains on the respective sets of $n-1$ alternatives.

Let us consider domain $\mathcal{D}_{1b}$. We claim that a is a terminal alternative in this domain and $e_{1b} = av_1$ is one of its extremal orders. Assume for a moment that a is not terminal and x and y are terminal for $\mathcal{D}_{1b}$ instead. Then for the triple $\{a, x, y\}$ in $\mathcal{D}$ no never-bottom condition is satisfied. Another extremal linear order starts from the last alternative in order v_1, let us call it c, and has structure $e_{2b} = cv_3a$, where $cv_3 \in \mathcal{D}_3$. All of these arguments are applicable also to $\mathcal{D}_{2b}$ so their extremal orders coincide. Thus, the two domains $\mathcal{D}_{1b}$ and $\mathcal{D}_{2b}$ are maximal Arrow's single-peaked, have identical extremal linear orders, and have identical restrictions on the set $A \setminus \{a, b, c\}$. By the induction hypothesis we have $\mathcal{D}_{1b} = \mathcal{D}_{2b}$. Similarly, we have $\mathcal{D}_{1a} = \mathcal{D}_{2a}$. This implies $\mathcal{D}_1 = \mathcal{D}_2$. □

Theorem 8.3.6. *Each Arrow's single-peaked domain $\mathcal{D}$ on the set A of n alternatives is a composition $(\mathcal{D}_1 \otimes_2 \mathcal{D}_2)(u, v)$ of an Arrow's single-peaked domain $\mathcal{D}_1$ on the set of $n-1$ alternatives $A \setminus \{a\}$ and the trivial domain $\mathcal{D}_2$ on $\{a\}$, where $u \in \mathcal{D}_1$ and $v = a$.*

Proof. Let us consider extremal orders e_1, e_2 of $\mathcal{D}$ which are $e_1 = aw_1b$, $e_2 = bw_2a$ for terminal alternatives a and b and some orders $w_1, w_2 \in \mathcal{L}(A \setminus \{a, b\})$.

Let $\mathcal{D}_1 = U_{\mathcal{D}}(a)$, $\mathcal{D}_2 = \{a\}$, $u = w_1b$ and $v = a$, then by Theorem 8.3.4 domain $\mathcal{E} = (\mathcal{D}_1 \otimes_2 \mathcal{D}_2)(u, v)$ is a maximal Arrow's single-peaked domain. Also we have $e_2 = bw_2a \in \mathcal{D}_1 \odot \mathcal{D}_2 \subseteq \mathcal{E}$.

As in the proof of Theorem 8.3.4 b is a terminal alternative of $\mathcal{D}_1$. Let us consider $\mathcal{D}_3 = U_{\mathcal{D}_1}(b)$. Again by Theorem 8.3.4 it is a maximal Arrow's single-peaked domain on $A \setminus \{a, b\}$. Due to the maximality of the latter $w_1 \in \mathcal{D}_3$, hence $u = w_1b \in \mathcal{D}_1$. Now $e_1 = aw_1b \in u \oplus a \subseteq u \boxplus v \subseteq \mathcal{E}$.

Thus $\mathcal{E}$ contains the same extremal orders e_1 and e_2, and has the same restriction on the set $A \setminus \{a, b\}$ as domain $\mathcal{D}$. From Lemma 8.3.5, $\mathcal{D} = \mathcal{E}$. □

Corollary 8.3.7. *Each Arrow's single-peaked domain can be obtained by iterated application of concatenation+shuffle construction to the trivial domain.*

One additional case is noteworthy. We remind the reader that a domain of linear orders is called (Black's) single-peaked if there exists a left-to-right arrangement (axis) of alternatives such that an upper contour set of any linear order in this domain is convex subset of the axis. Up to an isomorphism we can assume that the axis is $12\ldots n$.

Corollary 8.3.8. *Let $\mathcal{D}_1$ be Black's single-peaked domain on the set of alternatives $[n]$ with the axis $12\ldots n$ and $\mathcal{D}_2$ be the trivial domain on a one-element set $\{n+1\}$. Let $u = n(n-1)\ldots 21$ and $v = n+1$. Then $(\mathcal{D}_1 \otimes_2 \mathcal{D}_2)(u, v)$ is Black's single-peaked domain with axis $12\ldots(n+1)$.*

Zhan [2019] observed this recursive property of the single-peaked domains considering their tilling representations.

Not all peak-pit domains are reducible in the sense of Corollary 8.3.7. As we established in Section 4.8.1, up to an isomorphism, there are 18 peak-pit domains of maximal width with five alternatives. Ten of them can be obtained from smaller domains by the concatenation+shuffle scheme.

8.4 Constructing large peak-pit Condorcet domains

We again take two Fishburn's domains F_n and $\bar{F}_n$ (which are not isomorphic but flip-isomorphic) as building blocks for constructing new large domains. However, now we use the new construction to combine them. Firstly, we will look at the highest possible value T_n of the cardinality of the product

$$(\mathcal{D} \otimes_2 \mathcal{E})(u, v),$$

where $\mathcal{D} \in \{F_{\lfloor \frac{n}{2} \rfloor}, \bar{F}_{\lfloor \frac{n}{2} \rfloor}\}$ and $\mathcal{E} \in \{F_{\lceil \frac{n}{2} \rceil}, \bar{F}_{\lceil \frac{n}{2} \rceil}\}$, with arbitrary $u \in \mathcal{D}$ and $v \in \mathcal{E}$.

Secondly, if we restrict ourselves with domains of maximal width we have to choose u and v as shown in Corollary 8.2.5. The highest possible value of the cardinality of the product under these conditions will be denoted as S_n. As we will see, Table 8.1 shows the importance of choosing linear orders u and v for concatenation+shuffle scheme since the resulting domains may have different cardinality depending on the selected orders. This is why S_n is generally smaller than T_n. Our goal is to see for which n we have $S_n > |F_n|$ and for which n it is true that $T_n > |F_n|$. In the following table we present the results of our calculations.

n	$\lvert F_n \rvert$	$\lvert F_{\lfloor \frac{n}{2} \rfloor} \otimes_1 F_{\lceil \frac{n}{2} \rceil} \rvert$	S_n	T_n
2	2	2	2	2
3	4	4	4	4
4	9	9	9	9
5	20	17	19	19
6	45	35	42	42
7	100	70	85	93
8	222	150	183	211
9	488	305	392	462
10	1069	651	860	1028
11	2324	1361	1791	2233
12	5034	2948	3843	4916
13	10840	6215	8214	10668
14	23266	13431	17828	23360

Table 8.1: Products of Fishburn's domains under $\otimes_1$, under $\otimes_2$ with u and v chosen to secure maximal width (S_n), and under $\otimes_2$ where u and v are chosen to get the size as large as possible (T_n).

We see that neither S_n, nor T_n exceeds $|F_n|$ for $n < 14$. But for $n = 14$

$$T_{14} = 23360 > 23266 = |F_{14}|,$$

which means that $h(14) > |F_{14}|$. However, as we will see later, we can do even better.

When we restrict ourselves with Condorcet domains of maximal width (the classical case) we have much less freedom. As a result S_n reaches $|F_n|$ much later. We have $S_n < |F_n|$ for $n = 5, \ldots, 33$ but

$$S_{34} = |(F_{17} \otimes_2 F_{17})(\bar{e}, \bar{f})| = 60034795696 > |F_{34}| = 59921782301.$$

This gives us

Theorem 8.4.1 (Karpov and Slinko [2023b])**.** $g(34) > |F_{34}|$.

For the general case the choice of linear orders u and v for concatenation+shuffle scheme influences the structure of resulting domains even when they have equal cardinality. Let us consider the following example.

Example 8.4.1. *Let us find* $\mathcal{E}_1 := (F_3 \otimes_2 F_2)(321, 54)$*:*

$$\begin{bmatrix} 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & \boldsymbol{3} & \boldsymbol{3} & \boldsymbol{3} & \boldsymbol{3} & \boldsymbol{3} & \boldsymbol{3} & \boldsymbol{3} & 5 & 5 & 5 & 5 \\ 2 & 2 & 1 & 1 & 3 & 3 & 3 & 3 & \boldsymbol{2} & \boldsymbol{2} & \boldsymbol{2} & \boldsymbol{2} & 5 & 5 & 5 & 3 & 3 & 3 & 4 \\ 3 & 3 & 3 & 3 & 1 & 1 & 5 & 5 & \boldsymbol{1} & \boldsymbol{1} & 5 & 5 & 2 & 2 & 4 & 2 & 2 & 4 & 3 \\ 4 & 5 & 4 & 5 & 4 & 5 & 1 & 4 & 4 & \boldsymbol{5} & 1 & 4 & 1 & 4 & 2 & 1 & 4 & 2 & 2 \\ 5 & 4 & 5 & 4 & 5 & 4 & 4 & 1 & 5 & \boldsymbol{4} & 4 & 1 & 4 & 1 & 1 & 4 & 1 & 1 & 1 \end{bmatrix}$$

and $\mathcal{E}_2 := (F_3 \otimes_2 F_2)(213, 45)$*:*

$$\begin{bmatrix} 1 & 1 & 1 & 1 & \boldsymbol{2} & \boldsymbol{2} & \boldsymbol{2} & \boldsymbol{2} & \boldsymbol{2} & \boldsymbol{2} & \boldsymbol{2} & \boldsymbol{2} & \boldsymbol{2} & 3 & 3 & 4 & 4 & 4 & 4 \\ 2 & 2 & 2 & 2 & \boldsymbol{1} & \boldsymbol{1} & \boldsymbol{1} & \boldsymbol{1} & 3 & 3 & 4 & 4 & 4 & 2 & 2 & 2 & 2 & 2 & 5 \\ 3 & 3 & 4 & 4 & \boldsymbol{3} & \boldsymbol{3} & 4 & 4 & 1 & 1 & 1 & 1 & 5 & 1 & 1 & 1 & 1 & 5 & 2 \\ 4 & 5 & 3 & 5 & \boldsymbol{4} & 5 & 3 & 5 & 4 & 5 & 3 & 5 & 1 & 4 & 5 & 3 & 5 & 1 & 1 \\ 5 & 4 & 5 & 3 & \boldsymbol{5} & 4 & 5 & 3 & 5 & 4 & 5 & 3 & 3 & 5 & 4 & 5 & 3 & 3 & 3 \end{bmatrix}.$$

Both domains have equal cardinality, but they are neither isomorphic, nor flip-isomorphic. Domain $\mathcal{E}_2$ *has alternative* 2 *with the lower contour set of cardinality 9 while the highest possible cardinality of a lower contour set of domain* $\mathcal{E}_1$ *is 7 (of alternative* 3*).*

To explain why $(\mathcal{D}_1 \otimes_2 \mathcal{D}_2)(u, v)$ *may have different cardinalities depending on* u *and* v *let us consider* $F_3 = \{123, 213, 231, 321\}$*. In* F_3 *alternative* 1 *has the lower contour set of cardinality* 1*, while alternative* 2 *has its lower contour set of cardinality* 2*. Hence the domain* $(\mathcal{D} \otimes_2 F_3)(u, v)$ *with* $v = 213$ *has higher cardinality of* $(u \oplus v^1) \odot L_{F_3}(v^1)$ *for* $v^1 = 2$ *than such domain with* $v = 123$ *and* $v^1 = 1$*.*

Given domains $\mathcal{D}_1$ and $\mathcal{D}_2$ on sets of alternatives A and B, respectively, for generating a large concatenation+shuffle domain $(\mathcal{D}_1 \otimes_2 \mathcal{D}_2)(u, v)$ with large contour sets (for further use in the concatenation+shuffle construction) we use the following heuristic greedy algorithm to find the "best" orders u and v.

Algorithm 1 (Heuristically selecting u and v in $\oplus_2$). *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two Condorcet domains on sets of alternatives A and B of cardinalities m and n, respectively. Firstly, to find $v \in \mathcal{D}_2$ we find an alternative $v_1 \in B$ with the largest $L_{\mathcal{D}_2}(v_1)$, (if two alternatives have the same number of first placements, we take any of them), then we find an alternative $v_2 \in B$ with the largest $L_{\mathcal{D}_2}(v_1 v_2)$, etc. Then we set $v = v_1 \ldots v_n$. To find $u \in \mathcal{D}_1$ with upper contour sets of the highest cardinality we find alternative $u_m \in A$, with the largest $U_{\mathcal{D}_1}(u_m)$, then we find an alternative $u_{m-1} \in A$ with the largest $U_{\mathcal{D}_1}(u_{m-1}u_m)$, etc. Then we set $u = u_1 \ldots u_m$.*

When using this heuristics for choosing u and v in the product $(F_m \otimes_2 F_n)(u, v)$, we get a domain that have usually larger lower and upper contour sets than Fishburn's domain F_{m+n}. Due to this we obtain

$$\max_{u,v} |((F_4 \otimes_2 F_3) \otimes_2 (F_4 \otimes_2 F_3))(u, v)| \geq 24481 > 23360 = \max_{u,v} |(F_7 \otimes_2 F_7)(u, v)|$$

despite $\max_{u,v} |(F_4 \otimes_2 F_3)(u, v)| < |F_7|$. We also discovered that

$$\max_{u,v} |((F_4 \otimes_2 F_3) \otimes_2 (F_3 \otimes_2 F_3))(u, v)| \geq 10940 > 10840 = |F_{13}|,$$

which proves one of the main results of this paper.

Theorem 8.4.2 (Karpov and Slinko [2023b]). $h(13) > |F_{13}|$.

The cardinality and the structure of the biggest domains that we found are summarised in Table 8.2 in which for better readability we removed the subscript from $\otimes_2$ and did not mention the orders u, v used in the construction. These orders are found using Algorithm 1.

Having $F_4 = F_2 \otimes_2 F_2$, $F_3 = F_2 \otimes_2 F_1$ and $F_2 = F_1 \otimes_2 F_1$ we note that all presented domains are constructed via iterative mixing of trivial domains. For example, we have

$$\begin{aligned}(F_3 \otimes F_3) \otimes (F_3 \otimes F_2) = ((F_2 \otimes F_1) \otimes (F_2 \otimes F_1)) \otimes ((F_2 \otimes F_1) \otimes (F_1 \otimes F_1)) = \\ (((F_1 \otimes F_1) \otimes F_1) \otimes ((F_1 \otimes F_1) \otimes F_1)) \otimes (((F_1 \otimes F_1) \otimes F_1) \otimes (F_1 \otimes F_1)),\end{aligned}$$

where we omitted a subscript 2 in $\otimes$ and also u and v that maximise the cardinalities of both intermediate and the top level product.

For calculation of the lower bound for function h using Theorem 8.2.10 we need to calculate the values $h(k)^{1/k}$ for some positive integer k which is as large as possible. Observing the Table 8.2 we note that the numbers $h(k)^{1/k}$ are strictly increasing with k reaching 2.1045 for $k = 20$.

Theorem 8.4.3. *For large n we have $h(n) > (2.1045)^n$.*

Using Fishburn's Theorem 8.2.9 we can improve his lower bound for f

Theorem 8.4.4. *$f(n) > 2.1890^n$ for all large n.*

Proof. Observing the Table 8.2 we note that the numbers $f(k)^{1/k-1}$ are strictly increasing with k reaching 2.18909 for $k = 20$. □

Recently the lower bound in this theorem was improved to 2.1973 [Karpov et al., 2025].

It is interesting to note that, as Fishburn showed, the numbers $|F_k|^{1/k-1}$ steadily decline starting from $k = 12$. This is where the Fishburn's domains become not optimal.

n	size	the structure of the largest domain found
2	2	F_2
3	4	F_3
4	9	F_4
5	20	F_5
6	45	F_6
7	100	F_7
8	222	F_8
9	488	F_9
10	1069	F_{10}
11	2324	F_{11}
12	5034	F_{12}
13	10940	$(F_4 \otimes F_3) \otimes (F_3 \otimes F_3)$
14	24481	$(F_4 \otimes F_3) \otimes (F_4 \otimes F_3)$
15	54752	$(F_4 \otimes F_4) \otimes (F_4 \otimes F_3)$
16	123004	$(F_4 \otimes F_4) \otimes (F_4 \otimes F_4)$
17	271758	$((F_4 \otimes F_4)) \otimes ((F_3 \otimes F_2) \otimes F_4)$
18	602299	$((F_3 \otimes F_2) \otimes F_4)) \otimes ((F_3 \otimes F_2) \otimes F_4)$
19	1323862	$((F_3 \otimes F_2) \otimes (F_3 \otimes F_2)) \otimes ((F_3 \otimes F_2) \otimes F_4)$
20	2917604	$((F_3 \otimes F_2) \otimes (F_3 \otimes F_2)) \otimes ((F_3 \otimes F_2) \otimes (F_3 \otimes F_2))$

Table 8.2: The largest domains found for different n using Algorithm 1.

8.5 An upper bound

We have much less information about upper bounds for functions f, g, h. It is known from Raz [2000] that there exists a constant $c > 0$ such that $f(n) < c^n$. However, no such particular constant c is known. Fishburn [1996a] conjectured (Conjecture 3) that the following inequality holds:

$$f(n+m) \leq f(n+1)f(m+1).$$

This conjecture, if proved, would imply $f(n) \leq 2.591^{n-2}$ [Fishburn, 2002].

As was shown in Theorem 4.6.1 there is a bijection between the set of peak-pit maximal Condorcet domains of maximal width and arrangements of pseudolines. The number of flags of an arrangement of pseudolines is exactly the number of linear orders in a corresponding peak-pit domain of maximal width. Felsner and Valtr [2011] gave an upper bound for the number of cutpaths (flags in our terminology) in an arrangements of pseudolines, which leads to the following theorem.

Theorem 8.5.1 (Felsner and Valtr [2011]). *For a sufficiently large n, we have $g(n) < 2.4870^n$.*

Chapter 9

Dittrich's classification of maximal Condorcet domains on four alternatives

Here we provide an original Dittrich [2018] classification where all 18 maximal Condorcet domains on four alternatives were determined up to an isomorphism and flip-isomorphism. We note that we have met many of these domains in previous sections. In particular, in Section 6.4.2 we described nine out of 18 of the domains that appear as never-last compositions. As before the Dittrich's paper will be referred as TD. Using the order given by Table 5.4 on p. 94 in his dissertation, we denote those maximal domains as $\mathcal{D}_{4,1}, \ldots, \mathcal{D}_{4,18}$. Here we go.

- $\mathcal{D}_{4,1}$ is the familiar Raynaud's configuration K (Section 7.2);
- $\mathcal{D}_{4,2}$ is the single-crossing domain (see e.g., Example 6.4.8);
- $\mathcal{D}_{4,3}$ is described in Example 6.4.9;
- $\mathcal{D}_{4,4}$ is the single-peaked domain;
- $\mathcal{D}_{4,5}$ is the only maximal Arrow's single-peaked domain that is not single-peaked (Example 2.2.2);
- $\mathcal{D}_{4,6}$ is the Ladder domain (Example 4.8.1);
- $\mathcal{D}_{4,7}$ is the Broken Ladder domain (Example 4.8.2);
- $\mathcal{D}_{4,8} = ((1) \star (2)) \star ((3) \star (4))$ - the first domain from Example 6.1.1;
- $\mathcal{D}_{4,9}$. This domain has not appeared previously. The set of eight linear orders is

1	1	1	1	2	2	4	4
2	2	4	4	3	4	2	3
3	4	2	3	4	3	3	2
4	3	3	2	1	1	1	1

It is defined by a mixture of never conditions

$$1N_{\{1,2,3\}}2, \quad 1N_{\{1,2,4\}}2, \quad 1N_{\{1,3,4\}}2, \quad 3N_{\{2,3,4\}}1.$$

Obviously

$$\mathcal{D}_{4,9} = \mathcal{D}_{3,3}(2,3,4) \star (1).$$

The graph of the domain is

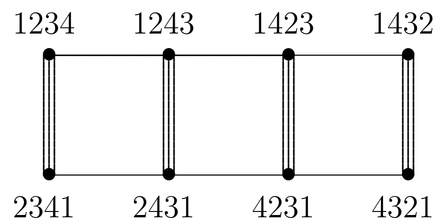


Figure 9.1: Graph of $\mathcal{D}_{4,9}$.

- $\mathcal{D}_{4,10} = (1) \star ((2)) \star ((3) \star (4))$ - the second domain from Example 6.1.1;
- $\mathcal{D}_{4,11}$ is described in Example 6.4.5;
- $\mathcal{D}_{4,12}$. This domain has not appeared previously. The set of eight linear orders is

1	1	2	2	2	3	3	3
2	2	1	1	4	1	1	2
3	4	3	4	1	4	2	1
4	3	4	3	3	2	4	4

It is defined by a mixture of never conditions

$$3N_{\{1,2,3\}}2, \quad 4N_{\{1,2,4\}}1, \quad 1N_{\{1,3,4\}}3, \quad 4N_{\{2,3,4\}}1$$

The graph of the domain is

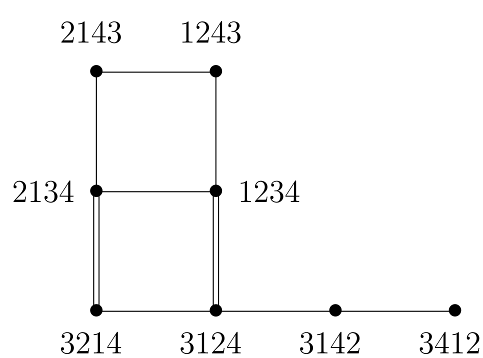


Figure 9.2: Graph of domain $\mathcal{D}_{4,12}$.

- $\mathcal{D}_{4,13}$: This domain has not appeared previously. The set of eight linear orders is

1	1	2	2	3	3	3	3
2	2	1	1	4	1	1	2
3	4	3	4	1	4	2	1
4	3	4	3	2	2	4	4

It is defined by a mixture of never conditions

$$3N_{\{1,2,3\}}2, \quad 1N_{\{1,2,4\}}3, \quad 4N_{\{1,3,4\}}1, \quad 4N_{\{2,3,4\}}1$$

The graph of the domain is

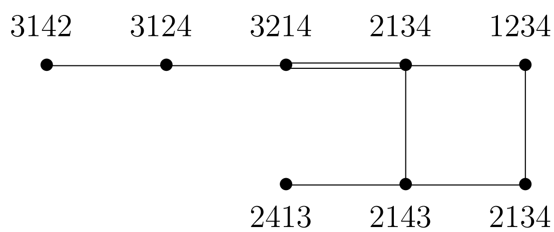


Figure 9.3: Graph of domain $\mathcal{D}_{4,13}$.

- $\mathcal{D}_{4,14}$: This domain has not appeared previously. The set of eight linear orders is

1	1	2	2	3	3	3	3
2	2	1	1	1	1	2	2
3	4	3	4	4	2	1	4
4	3	4	3	2	4	4	1

It is defined by a mixture of never conditions

$$3N_{\{1,2,3\}}2, \quad 4N_{\{1,2,4\}}1, \quad 4N_{\{1,3,4\}}1, \quad 4N_{\{2,3,4\}}1$$

The graph of the domain is shown on Figure 9.4.

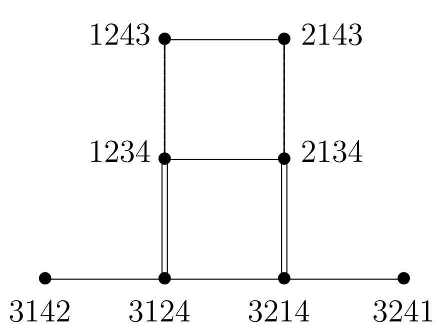


Figure 9.4: Graph of $\mathcal{D}_{4,14}$.

Interesting that the graph is the same as for $\mathcal{D}_{4,17}$ but domains are not isomorphic.

- $\mathcal{D}_{4,15}$: This domain has not appeared previously. The set of eight linear orders is

$$\begin{matrix} 1 & 2 & 3 & 3 & 1 & 2 & 3 & 3 \\ 2 & 1 & 1 & 2 & 2 & 1 & 4 & 4 \\ 3 & 3 & 2 & 1 & 4 & 4 & 1 & 2 \\ 4 & 4 & 4 & 4 & 3 & 3 & 2 & 1 \end{matrix}$$

It is defined by a mixture of never conditions

$$3N_{\{1,2,3\}}2, \quad 4N_{\{1,2,4\}}2, \quad 4N_{\{1,3,4\}}1, \quad 4N_{\{2,3,4\}}1$$

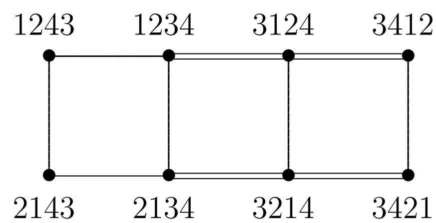


Figure 9.5: Graph of $\mathcal{D}_{4,15}$.

- $\mathcal{D}_{4,16}$ is described in Example 6.4.6;
- $\mathcal{D}_{4,17}$ is described in Example 6.4.7;
- $\mathcal{D}_{4,18}$ - Fishburn's domain

Part II

Applications and Extensions

Chapter 10

Arrovian aggregation and strategy-proof social choice

Closed Condorcet domains not only preclude intransitivities in pairwise majority voting, they are also endowed with a large class of further aggregation rules satisfying Arrow's independence condition. This follows from the analysis of Nehring and Puppe [2007, 2010]. Indeed, their main result entails a characterisation of all Arrovian aggregators on such domains under an additional monotonicity condition. In the first section of this chapter, we apply their result to describe the class of all monotone Arrovian aggregators on closed Condorcet domains. In the second section the monotonicity condition plays a crucial role in the construction of strategy-proof social choice functions on such domains.

10.1 Characterisation of all Arrovian aggregators

10.1.1 General properties of Arrovian aggregators

Let $[n] = \{1, 2, \ldots, n\}$ be the set of voters. An *aggregator* on a domain $\mathcal{D} \subseteq \mathcal{L}(A)$ is a mapping $F\colon \mathcal{D}^n \to \mathcal{D}$ that assigns an order in $\mathcal{D}$ to each profile of n individual orders in $\mathcal{D}$. The following conditions on aggregators have been extensively studied in the literature.

Full Range. For all $R \in \mathcal{D}$, there exist $R_1, \ldots, R_n$ such that $F(R_1, \ldots, R_n) = R$.

Unanimity. For all $R \in \mathcal{D}$, $F(R, \ldots, R) = R$.

Independence. For all $R_1, \ldots, R_n, R'_1, \ldots, R'_n \in \mathcal{D}$ with $R = F(R_1, \ldots, R_n)$ and $R' = F(R'_1, \ldots, R'_n)$, and all pairs of distinct alternatives $a, b \in A$, if aRb and

$$\forall_{i\in[n]}(aR_ib \Longleftrightarrow aR'_ib),$$

then $aR'b$.

An aggregator F is called *Arrovian* if it satisfies unanimity and independence. Due to independence, every Arrovian aggregator can be defined by a system of subsets

$$\mathcal{W}_F = \{\mathcal{W}_{ab} \subseteq 2^{[n]} \mid (a, b) \in A \times A\},$$

where $\mathcal{W}_{ab}$ consists of those subsets $X \subseteq [n]$ such that for every profile $R = (R_1, \ldots, R_n) \in \mathcal{D}^n$

$$[\forall_{i \in X}(aR_i b) \text{ and } \forall_{j \in X^c}(bR_j a)] \Longrightarrow aF(R)b,$$

where $X^c = [n] \setminus W$ denotes the complement of the coalition X in $[n]$.

We call $\mathcal{W}_F$ the *structure of winning coalitions* for F and we observe that

$$X \in \mathcal{W}_{ab} \iff X^c \notin \mathcal{W}_{ba}. \tag{10.1.1}$$

Example 10.1.1. *A special case of the structure of winning coalitions that leads to F being the* pairwise majority voting *requires that n is odd and for every pair $(a, b) \in A \times A$ of distinct alternatives*

$$X \in \mathcal{W}_{ab} \iff |X| > n/2.$$

In what follows, we will be concerned with Arrovian aggregators that satisfy the following monotonicity condition.

Monotonicity For all $R_1, \ldots, R_n, R'_i \in \mathcal{D}$ with $R = F(R_1, \ldots, R_i, \ldots, R_n)$ and $R' = F(R_1, \ldots, R'_i, \ldots, R_n)$, and all pairs of distinct alternatives $a, b \in A$, if $aR'b$ and $aR_i b$, then aRb. In such a case the aggregator F is called *monotone*.

We now have to recap the following basic mathematical structure.

Definition 10.1.1 (Von Neumann and Morgenstern [2007])**.** *A pair $G = ([n], W)$, where $W \subseteq 2^{[n]}$, is called a* simple game *if W satisfies the following monotone property:*

$$\text{If } X \in W \text{ and } Y \supseteq X, \text{ then } Y \in W. \tag{10.1.2}$$

The game is said to be a constant-sum game *if*

$$X \in W \iff X^c \notin W. \tag{10.1.3}$$

There are numerous examples of simple games. The simplest are the *dictatorships* which arise whenever there exists an individual i such that a coalition X is winning in $G = ([n], W)$ if and only if $i \in X$. On the other side of the spectrum are *uniform* games in which case W consists of all coalitions of size at least q for some positive integer q, called the threshold. In such a case membership of a coalition X in W depends only on the number of individuals in X.

Proposition 10.1.1. *If F is a monotone Arrovian aggregator, then for every two alternatives $a, b \in A$ the pair $([n], \mathcal{W}_{ab})$ is a constant-sum simple game.*

Proof. Let us prove (10.1.2). Suppose $X \in \mathcal{W}_{ab}$. Then let $R = (R_1, \ldots, R_n)$ be a profile for which

$$\forall_{i \in X}(aR_i b) \text{ and } \forall_{j \in X^c}(bR_j a)$$

and $aF(R)b$. Let $j \notin X$ and $aR'_j b$ for some linear order R'_j. Let $R' = (R_1, \ldots, R'_j, \ldots, R_n)$. Monotonicity implies that $aF(R')b$ which means $X \cup \{j\} \in \mathcal{W}_{ab}$. Moreover, (10.1.1) implies (10.1.3) so $\mathcal{W}_{ab}$ is constant-sum. ☐

The monotonicity condition has the following interpretation. Consider any pair of distinct alternatives a and b, and suppose that aR_ib according to agent i's true order R_i. Then, if agent i can force the social order to rank a above b by submitting some order R_i', the social order would also rank a above b if agent i submitted his true preference R_i. In other words, no agent can benefit in a pairwise comparison from any misrepresentation. The monotonicity condition thus has a clear 'non-manipulability' flavour which we will further exploit.

The monotonicity condition can be rephrased as follows. An aggregator F is monotone if and only if, for all $R = (R_1, \dots, R_n) \in \mathcal{D}^n$ and $R_i' \in \mathcal{D}$ and all $R_{-i} \in \mathcal{D}^{n-1}$,

$$F(R) \in [R_i, F(R_{-i}, R_i')] \ . \tag{10.1.4}$$

The conjunction of independence and monotonicity is equivalent to the following single condition.

Monotone Independence For all $R_1, \dots, R_n, R_i' \in \mathcal{D}$ with $R = F(R_1, \dots, R_i, \dots, R_n)$ and $R' = F(R_1, \dots, R_i', \dots, R_n)$, and for all pairs of distinct alternatives $a, b \in A$, if aRb and

$$\forall_{i\in[n]}(aR_ib \Longrightarrow aR_i'b),$$

then $aR'b$.

Note also that under monotonicity, unanimity can be deduced from the full range condition, i.e., from the assumption that the aggregator is onto.

Given a domain $\mathcal{D} \in \mathcal{L}(A)$, a structure of winning coalitions is said to be *compatible* with $\mathcal{D}$ if, for every pair of distinct alternatives $(a, b) \in A \times A$ and every pair of distinct alternatives $(c, d) \in A \times A$,

$$\mathcal{V}_{ab}^{\mathcal{D}} \subseteq \mathcal{V}_{cd}^{\mathcal{D}} \implies \mathcal{W}_{ab} \subseteq \mathcal{W}_{cd}, \tag{10.1.5}$$

where the sets $\mathcal{V}_{xy}^{\mathcal{D}}$ for $x, y \in A$ are defined in (1.3.1). Observe that pairwise majority voting always defines a compatible structure of winning coalitions (since in the case of pairwise majority voting $\mathcal{W}_{ab} = \mathcal{W}_{cd}$ for all distinct pairs $(a, b) \neq (c, d)$.

Theorem 10.1.2 (Nehring and Puppe [2007]). *Let $\mathcal{D}$ be a closed Condorcet domain and let $\mathcal{W} = \{\mathcal{W}_{ab} \mid a, b \in A\}$ be a compatible structure of winning coalitions on $\mathcal{D}$. For every preference profile $R = (R_1, \dots, R_n) \in \mathcal{D}^n$ there exists a unique linear order $R^* \in \mathcal{D}$ such that, for every pair of distinct alternatives $(a, b) \in A \times A$*

$$aR^*b \iff \{i \in [n] \mid aR_ib\} \in \mathcal{W}_{ab}. \tag{10.1.6}$$

The aggregator $F_{\mathcal{W}}\colon R \mapsto R^*$ defined by (10.1.6) *is a monotone Arrovian aggregator. Conversely, every monotone Arrovian aggregator on $\mathcal{D}$ takes this form for some compatible structure of winning coalitions $\mathcal{W}$.*

Proof. Let $R = (R_1, \dots, R_n) \in \mathcal{D}^n$ be a profile and consider the binary relation R^* defined by (10.1.6).

Firstly, we note that R^* is complete. Let $X = \{i \in [n] \mid aR_i b\}$. If $X \in \mathcal{W}_{ab}$, then aR^*b. If $X \notin \mathcal{W}_{ab}$, then $X^c \in \mathcal{W}_{ba}$ by (10.1.1) and since $X^c = \{i \in [n] \mid bR_i a\}$, we have bR^*a. Also R^* is antisymmetric since both aR^*b and bR^*a means by (10.1.6) that both $X \in \mathcal{W}_{ab}$ and $X^c \in \mathcal{W}_{ba}$, which contradicts to (10.1.1).

Secondly, we must show that R^* defined in (10.1.6) is a linear order belonging to $\mathcal{D}$. Consider two distinct pairs of alternatives $(a, b) \neq (c, d)$ such that aR^*b and cR^*d; we will show that $\mathcal{V}_{ab}^{\mathcal{D}} \cap \mathcal{V}_{cd}^{\mathcal{D}} \neq \emptyset$. By contradiction, suppose that this does not hold, then $\mathcal{V}_{ab}^{\mathcal{D}}$ will be contained in the complement of $\mathcal{V}_{cd}^{\mathcal{D}}$, that is, $\mathcal{V}_{ab}^{\mathcal{D}} \subseteq \mathcal{V}_{dc}^{\mathcal{D}}$. By compatibility we would have $\mathcal{W}_{ab} \subseteq \mathcal{W}_{dc}$. Since $\mathcal{V}_{ab}^{\mathcal{D}} \subseteq \mathcal{V}_{dc}^{\mathcal{D}}$, we have $\{i \in [n] \mid aR_i b\} \subseteq \{i \in [n] \mid dR_i c\}$ and by monotonicity of $\mathcal{W}_{ab}$, we have $\{i \in [n] \mid dR_i c\} \in \mathcal{W}_{ab}$, hence by compatibility $\{i \in [n] \mid dR_i c\} \in \mathcal{W}_{dc}$ and dR^*c. But the latter contradicts cR^*d as R^* is antisymmetric.

By Proposition 1.3.5 and Theorem 1.3.6 $\mathcal{D}$ has the Helly property for convex sets. Thus, the collection of convex sets $\{\mathcal{V}_{xy}^{\mathcal{D}} \mid xR^*y\}$ which has non-empty pairwise intersections must satisfy

$$\bigcap_{xR^*y} \mathcal{V}_{xy}^{\mathcal{D}} \neq \emptyset. \tag{10.1.7}$$

This means that R^* coincides with the linear order in the intersection on the right-hand-side of (10.1.7). This means that R^* is an element of $\mathcal{D}$. Thus, (10.1.6) indeed defines a mapping from $\mathcal{D}^n$ to $\mathcal{D}$. It is easily verified that $F_{\mathcal{W}}$ is monotone and independent, i.e., an Arrovian aggregator.

Conversely, let F be a monotone Arrovian aggregator and $\mathcal{W}_F$ be the corresponding structure of winning coalitions. We thus have only to verify that $\mathcal{W}_F$ is compatible with $\mathcal{D}$. To this end, assume that, for distinct $(a, b) \in A \times A$ and distinct $(c, d) \in A \times A$, we have $\mathcal{V}_{ab}^{\mathcal{D}} \subseteq \mathcal{V}_{cd}^{\mathcal{D}}$, and $W \in \mathcal{W}_{ab}$. Then, if the profile $R = (R_1, \ldots, R_n)$ is such that all voters in W prefer a to b, we must have $aF(R)b$. As $F(R) \in \mathcal{D}$ and, by assumption, all orders in $\mathcal{D}$ which rank a above b must also rank c above d, we have $cF(R)d$. By independence, this holds for *all* profiles in which the agents in W rank c above d, hence the coalition of agents W is also winning for c versus d, i.e., $W \in \mathcal{W}_{cd}$. □

A few words on the anonymous aggregators.

Definition 10.1.2. *An aggregator $F\colon \mathcal{D}^n \to \mathcal{D}$ is* anonymous *if for every profile $(R_1, \ldots, R_n)$ and any permutation $\sigma \in S_n$ we have*

$$F(R_1, \ldots, R_n) = F(R_{\sigma(1)}, \ldots, R_{\sigma(n)}).$$

In such a case membership of a coalition X in $\mathcal{W}_{xy}$ depends only on the number of individuals in W. We can formulate it in the following proposition.

Proposition 10.1.3. *The aggregator $F\colon \mathcal{D}^n \to \mathcal{D}$ is anonymous if and only if all the games in $\mathcal{W}_F$ are uniform.*

Let us denote by q_{ab} the threshold for the uniform game $G_{ab} = ([n], \mathcal{W}_{ab}) \in \mathcal{W}_F$ of an anonymous aggregator F.

Proposition 10.1.4. $q_{ab} + q_{ba} = n + 1.$

Proof. Let $X \in \mathcal{W}_{ab}$ such that $|X| = q_{ab}$. By (10.1.1) $X^c \notin \mathcal{W}_{ba}$, hence $|X^c| = n - q_{ab} < q_{ba}$. Let us consider $U \subseteq X$ with $|U| = q_{ab} - 1$. Then $U \notin \mathcal{W}_{ab}$, hence by (10.1.1) we have $U^c \in \mathcal{W}_{ba}$. Since $|U^c| = n - q_{ab} + 1$ we have $q_{ba} = n - q_{ab} + 1$. □

10.1.2 Examples of Arrovian aggregators on closed Condorcet domains

Let $\mathcal{D}$ be a Condorcet domain. An aggregator $F\colon \mathcal{D}^n \to \mathcal{D}$ is called a *generalised median function* if, there exists a fixed profile $\mu = (\mu_1, \dots, \mu_{n-1}) \in \mathcal{D}^{n-1}$ of $n-1$ 'phantom' voters such that for any $v = (v_1, \dots, v_n) \in \mathcal{D}^n$

$$F(v)\colon = \operatorname{Med}(v_1, \dots, v_n, \mu_1, \dots, \mu_{n-1}), \tag{10.1.8}$$

where the function Med is defined as in (1.3.4). We denote this function also as $F_\mu(v)$.

Theorem 10.1.5. *Let $\mathcal{D}$ be a closed Condorcet domain over the set of alternatives A, and $\mu = (\mu_1, \dots, \mu_{n-1}) \in \mathcal{D}^{n-1}$ be arbitrary. Then F_μ is compatible with $\mathcal{D}$ and anonymous.*

Proof. Let $|A| = m$. Let $v = (v_1, \dots, v_n) \in \mathcal{D}^n$ and $\mu = (\mu_1, \dots, \mu_{n-1})$. By Proposition 1.3.23

$$F_\mu(v) = \operatorname{Med}(v_1, \dots, v_n, \mu_1, \dots, \mu_{n-1})$$

is the majority relation of (v, μ) which is a linear order from $\mathcal{D}$. Thus, F_μ is anonymous.

For $x = (x_1, \dots, x_n) \in \mathcal{D}^n$ let $N(x) = (n_{ij}(x))$ be the matrix of advantages of the profile x, i.e.,

$$n_{ab}(x) = \#\{i \in [n] \mid a \succ_{x_i} b\}.$$

Obviously, $N(u) = N(v) + N(\mu)$. By Proposition 1.3.23

$$a \succ_u b \iff n_{ab}(u) \geq n \iff n_{ab}(v) \geq n - n_{ab}(\mu).$$

We set

$$\mathcal{W}_{ab} := \{X \subseteq [n] \mid |X| \geq n - n_{ab}(\mu)\}.$$

that is, we make the game $\mathcal{W}_{ab}$ uniform with the threshold $q_{ab} = n - n_{ab}(\mu)$. Suppose now that $\mathcal{V}^{\mathcal{D}}_{ab} \subseteq \mathcal{V}^{\mathcal{D}}_{cd}$. Then

$$n_{ab}(v) \leq n_{cd}(v) \quad \text{and} \quad n_{ab}(\mu) \leq n_{cd}(\mu),$$

and

$$n_{ab}(u) = n_{ab}(v) + n_{ab}(\mu) \leq n_{cd}(v) + n_{cd}(\mu) = n_{cd}(u),$$

then $q_{ab} = n - n_{ab}(\mu) \geq q_{cd} = n - n_{cd}(\mu)$, hence $\mathcal{W}_{ab} \subseteq \mathcal{W}_{cd}$ and F_μ is compatible. □

If one takes $\mu_1 = \mu_2 = \dots = \mu_{n-1} =: \bar{\mu}$ in Theorem 10.1.5, we obtain the so-called *unanimity rule with default $\bar{\mu}$*; for each pair $a, b \in A$, it ranks the pair (a, b) in the same way as $\bar{\mu}$ unless there is unanimous consent to rank it in the opposite direction.

10.2 Strategy-Proof Social Choice

It is well-known that on domains on which pairwise majority voting with an odd number of voters is transitive, choosing the Condorcet winner yields a strategy-proof social choice function (see, e.g., Lemma 10.3 in Moulin [1988]). Here we use Theorem 10.1.2 and property (10.1.4) which is entailed by monotonicity to construct a rich class of strategy-proof social choice functions on any closed Condorcet domain.

A social choice function f that maps every profile $R = (R_1, \ldots, R_n) \in \mathcal{D}^n$ to an element $f(R) \in A$ is *strategy-proof* if, for all $i \in [n]$, all $R_1, \ldots, R_n, R'_i \in \mathcal{D}$

$$f(R)\, R_i\, f(R_{-i}, R'_i),$$

i.e., if no voter can benefit by misrepresenting her true preferences R_i by submitting R'_i.

For each order $R \in \mathcal{L}(A)$ denote by $\tau(R) \in A$ the top-ranked element of R. Let $\mathcal{D} \subseteq \mathcal{L}(A)$ be any closed Condorcet domain, and consider any structure of winning coalitions $\mathcal{W}$ that is compatible with $\mathcal{D}$. For every profile $R \in \mathcal{D}^n$ let $F_{\mathcal{W}}(R) \in \mathcal{D}$ be the unique order determined by (10.1.6). Define a social choice function $f_{\mathcal{W}} \colon \mathcal{D}^n \to A$ by

$$f_{\mathcal{W}}(R) = \tau(F_{\mathcal{W}}(R)). \tag{10.2.1}$$

Theorem 10.2.1. *Let $\mathcal{D} \subseteq \mathcal{L}(A)$ be any closed Condorcet domain. For every structure of winning of coalitions $\mathcal{W}$ compatible with $\mathcal{D}$, the social choice function $f_{\mathcal{W}}$ defined by* (10.2.1) *is strategy-proof.*

Proof. By Theorem 10.1.2, the aggregator $F_{\mathcal{W}} \colon \mathcal{D}^n \to \mathcal{D}$ that maps any profile $R \in \mathcal{D}^n$ to the social order $F_{\mathcal{W}}(R)$ is a monotone Arrovian aggregator; in particular, it satisfies (10.1.4). In other words, if we denote $R_{\mathcal{W}} = F_{\mathcal{W}}(R)$ and $R'_{\mathcal{W}} = F_{\mathcal{W}}(R_{-i}, R'_i)$, we have for all distinct pairs of alternatives $(x, y) \in A \times A$,

$$(xR_iy \text{ and } xR'_{\mathcal{W}}y) \implies xR_{\mathcal{W}}y. \tag{10.2.2}$$

This implies at once the strategy-proofness of $f_{\mathcal{W}}$, by contraposition. Indeed, suppose that agent i could benefit by misreporting R'_i, i.e., suppose that $\tau(R'_{\mathcal{W}})R_i\tau(R_{\mathcal{W}})$, where R_i is agent i's true preference order. Then, since $\tau(R'_{\mathcal{W}})$ is the top element of the order $R'_{\mathcal{W}}$ we obtain from (10.2.2) , $\tau(R'_{\mathcal{W}})R_{\mathcal{W}}\tau(R_{\mathcal{W}})$. Since $\tau(R_{\mathcal{W}})$ is the top element of $R_{\mathcal{W}}$ this implies $\tau(R'_{\mathcal{W}}) = \tau(R_{\mathcal{W}})$, i.e., the misrepresentation does not change the chosen alternative. □

In case of the domain of all single-peaked preferences and the domain of all single-crossing preferences, the class of anonymous social choice functions defined by (10.2.1) exhaust the class of all anonymous strategy-proof social choice functions (see Moulin [1980], Saporiti [2009]). This observation naturally leads to the question if the same is true of all closed Condorcet domains. The answer is negative as the following simple example shows.

On $A = \{a, b, c\}$ consider the closed (but not maximal) Condorcet domain $\mathcal{D} = \{abc, cba\}$. For $0 < l < k \leq n$, define a social choice function $f_{l,k}$ as follows. For all profiles $R = (R_1, \ldots, R_n) \in \mathcal{D}^n$ denote by n_a the number of voters i with $R_i = abc$, and let

$$f_{l,k}(R) \ := \ \begin{cases} a & \text{if } k \leq n_a, \\ b & \text{if } l \leq n_a < k, \\ c & \text{if } n_a < l. \end{cases}$$

As is easily verified, $f_{l,k}$ is anonymous, strategy-proof and onto (i.e., every alternative is in the range of $f_{l,k}$). But evidently, $f_{l,k}$ is not of the form (10.2.1) for any structure of winning coalitions $\mathcal{W}$, because the range of any (10.2.1) equals the set of tops in $\mathcal{D}$, i.e., $\{a, b\}$ in the present case.

The characterisation of the class of all strategy-proof social functions for any given closed (or maximal) Condorcet domain appears to be interesting open problem. The fact that the set of top alternatives does not coincide with the set of all alternatives in case of domains that are not minimally rich complicates matters, however, significantly. Moreover, while different general characterisations of the class of all strategy-proof and onto social choice functions on the domain of all single-peaked preferences are available in the literature [Moulin, 1980, Nehring and Puppe, 2007, Jennings et al., 2024], the characterisation of Saporiti [2009] for maximal single-crossing domains hinges on the additional anonymity condition. (Observe that the definition $f_{\mathcal{W}}$ does not require anonymity.)

A detailed analysis of the class of all strategy-proof social choice functions on closed Condorcet domains is beyond the scope of the present inquiry and left to future work.

Chapter 11

Condorcet domains of weak and partial orders

Many of the representations of Condorcet domains presented so far crucially hinge on the fact that we are considering strict linear orders (i.e. permutations) of the alternatives. However, economically meaningful domain restrictions such a single-peakedness and single-crossingness, are naturally generalisable to weak orders and even partial orders. In this chapter, we present some basic facts and results about Condorcet domains of weak and partial orders. We do not aim at a comprehensive treatment but rather highlight the main differences and similarities to the case of linear orders in the hope to stimulate further research along this, so far unexplored route. As a first indication of some of the new phenomena that arise, note that any pair of Kemeny neighbours differs in the ranking of exactly one pair of (adjacent) alternatives both in the case of linear and in the case of partial orders, but not in the case of weak orders. Indeed, if there is indifference between three alternatives a, b, c, breaking the indifference for one pair among them forces one to break the indifference for at least one other pair in order to preserve transitivity of the indifference relation.

11.1 Weak order Condorcet domains

Consider now the set $\mathcal{R}(A)$ of all *weak orders* (i.e., complete and transitive binary relations) on A, and subdomains $\mathcal{D} \subseteq \mathcal{R}(A)$. Individual weak preferences are denoted by R, R' etc., and profiles by $\rho = (R_1, \ldots, R_n) \in \mathcal{R}(A)^n$. Frequently, we will denote weak orders by listing the alternatives in descending order and putting indifferent alternatives in brackets, e.g. the weak order that ranks a first and b and c indifferently as second best is denoted by $a(bc)\ldots$; similarly, the weak order that has a and b indifferently at the top and c at the following rank is denoted by $(ab)c\ldots$; finally, the weak order that has a, b and c indifferently on top is denoted by $(abc)\ldots$, etc.

The *majority relation* associated with a profile ρ is the binary relation R^{maj}_ρ on A such that $xR^{\text{maj}}_\rho y$ if and only if xR_iy for more than half of the voters.[1] Note that, according

[1]It is well-known that this is not the only possible definition of majority rule with weak preferences

to this definition, the majority relation is complete for every odd profile ρ. As before, the domains $\mathcal{D} \subseteq \mathcal{R}(A)$ such that, for all odd n, the majority relation associated with any profile $\rho \in \mathcal{D}^n$ is transitive are referred to as *Condorcet domains.* Condorcet domains of weak orders have been studied much less than their counterparts with linear orders, see the monograph of Gaertner [2001] for some examples. A domain $\mathcal{D} \subseteq \mathcal{R}(A)$ is called a *maximal Condorcet domain (of weak orders)* if every Condorcet domain (on the same set of alternatives) that contains $\mathcal{D}$ as a subset must in fact coincide with $\mathcal{D}$.

The generalisation of even the most basic results to the case of weak orders requires careful adaption of the employed concepts since some additional complications arise. For instance, the equivalence stated in Proposition 1.1.1 does not carry over to the domain of weak orders. As a simple example, consider the domain on $A = \{a, b, c\}$ consisting of the three weak orders $(ab)c$, $a(bc)$ and cba. If ρ is the profile in which three voters have each one of these preferences, respectively, the majority relation is acyclic but not transitive, since a majority strictly prefers a to c while $[aR^{\text{maj}}_\rho b$ and $bR^{\text{maj}}_\rho a]$ as well as $[bR^{\text{maj}}_\rho c$ and $cR^{\text{maj}}_\rho b]$, i.e. both pairs of alternatives (a, b) and (b, c) are deemed indifferent, respectively, according to the majority relation. Thus, the domain of these three weak orders is not a Condorcet domain in our sense.

A domain $\mathcal{D} \subseteq \mathcal{R}(A)$ is called *weakly single-peaked with respect to the (linear) spectrum* $\lhd$ on A if, for all $R \in \mathcal{D}$ and all $w \in A$, the upper contour sets $U_R(w) := \{y \in A : yRw\}$ are connected ('convex') in the spectrum $\lhd$, i.e. $\{x, z\} \subseteq U_R(w)$ and $x \lhd y \lhd z$ jointly imply $y \in U_R(w)$. A domain $\mathcal{D} \subseteq \mathcal{R}(A)$ is called *weakly single-peaked* if there exists *some* spectrum $\lhd$ such that $\mathcal{D}$ is weakly single-peaked with respect to $\lhd$. Similarly, a domain $\mathcal{D} \subseteq \mathcal{R}(A)$ is called *weakly single-dipped* if there exists a spectrum $\lhd$ on X such that, for all $R \in \mathcal{D}$, the lower contour sets $L_R(w) := \{y \in A : wRy\}$ are connected with respect to $\lhd$.

Remark. Several generalisations of the concept of single-peakedness in order to accommodate indifference have been discussed in the literature. [Moulin, 1988, p. 264] notes that for many purposes indifferences *across* the (unique) peak could be allowed without difficulty. On the other hand, the concept of 'single-plateaued' preferences allows for multiple best alternatives and has proven useful in some contexts ([Gaertner, 2001, p. 68]); however, this concept still assumes strict monotonicity below the optimum (Moulin [1984]). The above notion of 'weakly' single-peaked preferences has also been employed by Duggan [2016] (under this name) and is weaker than single-plateauedness as it allows for multiple 'plateaus' and, in particular, not only at the top. From an abstract perspective, it represents a natural generalisation of the usual notion of single-peakedness, as it corresponds to convexity of all upper contour sets of the weak relations, just as standard single-peakedness corresponds to convexity of the upper contour sets of all strict relations. However, we do by no means want to argue that the adopted definitions are the only reasonable extensions of the concepts of single-peakedness and Condorcet domain to the case of weak orders. For instance, one could require only acyclicity of the majority relation for

(see, e.g., [Gaertner, 2001, Ch. 3]); for our purpose it turns out to be the appropriate one. Moreover, it represents the natural notion of majority rule inherited from the general judgement aggregation model, see List and Puppe [2009], Nehring et al. [2014, 2016].

a domain to be called 'Condorcet domain,' or base the notion of Condorcet domain on the so-called 'strict' majority relation ([Gaertner, 2001, p. 27]), or require uniqueness of the top alternative in the definition of single-peakedness. However, as our analysis will show, the adopted definitions are *precisely* the appropriate notions for which the robustness of some of our characterisation result can be demonstrated.

A further important difference to the case of linear orders is that not all weakly single-peaked domains are Condorcet domains; in particular, the domain of *all* weak orders that are weakly single-peaked with respect to some order $\triangleleft$ is not a Condorcet domain. The same example as above can be used to demonstrate this. Indeed, we have already argued above that the domain $\mathcal{D} = \{(ab)c, a(bc), cba\}$ is not a Condorcet domain. On the other hand, it is clearly weakly single-peaked with respect to the spectrum $a \triangleleft b \triangleleft c$.

We also have to generalise the notion of 'connectedness,' as follows. For $R \in \mathcal{R}(A)$, denote by $\neg R$ the negation of R, i.e., for all $x, y \in X$, $x \neg R y \Leftrightarrow$ not xRy. For any two orders $R, R' \in \mathcal{R}(A)$, denote by

$$[R, R'] := \{Q \in \mathcal{R}(A) \mid Q \supseteq R \cap R' \text{ and } \neg Q \supseteq \neg R \cap \neg R' \}. \tag{11.1.1}$$

the *interval* spanned by R and R', and to its elements as the orders *between* R and R'. Thus, an order Q is between R and R' if and only if, for all pairs $x, y \in A$, (i) Q ranks x weakly above y whenever both R and R' do so, and (ii) Q ranks x strictly above y whenever both R and R' do so. Two distinct orders R and R' are called $\mathcal{R}(A)$-*neighbours* if $[R, R'] = \{R, R'\}$. Figure 11.1 depicts the 'neighbourhood structure' of $\mathcal{R}(\{a, b, c\})$, connecting all neighbours by an edge. Note that, due to transitivity of the indifference relation, the six neighbours of the weak order (abc) on the set $X = \{a, b, c\}$ each differ from (abc) in the ranking of two pairs of alternatives, respectively.[2]

A *path* in $\mathcal{R}(A)$ is a subset $\{R_1, \ldots, R_m\} \subseteq \mathcal{R}(A)$ with $m \geq 2$ such that for all $j = 1, \ldots, m-1$, the two consecutive orders R_j and R_{j+1} are $\mathcal{R}(A)$-neighbours. A domain $\mathcal{D} \subseteq \mathcal{R}(A)$ will be called *connected* if, for every pair $R, R' \in \mathcal{D}$ of distinct orders in $\mathcal{D}$, there exists a path $\{R_1, \ldots, R_m\}$ that connects R and R' (i.e. $R_1 = R$ and $R_m = R'$) and that lies entirely in $\mathcal{D}$ (i.e. $R_j \in \mathcal{D}$ for all $j = 1, \ldots, m$).

Two orders R and R^{inv} are called *totally reversed* if, for all distinct x and y, $xRy \Leftrightarrow$ not$(xR^{\text{inv}}y)$. Note that by the completeness assumption, two totally reversed weak orders must both in fact be linear orders, i.e. neither of the two can contain any non-trivial indifference. A domain $\mathcal{D} \subseteq \mathcal{R}(A)$ is said to have *maximal width* if $\mathcal{D}$ contains at least one pair of totally reversed orders. Finally, a domain $\mathcal{D} \subseteq \mathcal{R}(A)$ is called *semi-connected* if $\mathcal{D}$ contains at least two totally reversed orders and an entire path connecting them.

Our goal is to show that the (weakly) single-peaked domain is characterised by a few simple properties, similar to the case of linear orders (cf. Theorem 2.1.7).

[2] The neighbourhood relation can be defined analogously on the space $\mathcal{P}(A)$ of all linear orders; in this case, definition (11.1.1) simplifies to $[P, P'] = \{Q \in \mathcal{P}(A) \mid Q \supseteq P \cap P'\}$ due to the antisymmetry condition.

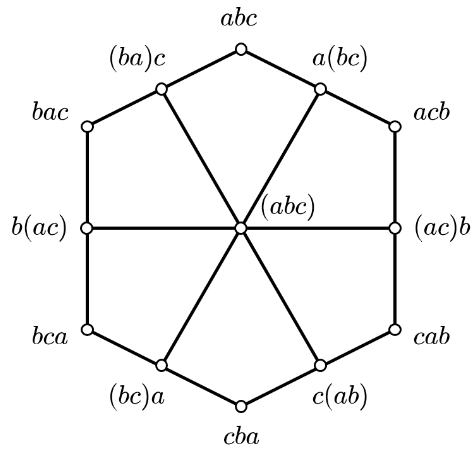


Figure 11.1: The neighbourhood structure of $\mathcal{R}(\{a, b, c\})$.

11.1.1 All semi-connected Condorcet domains on triples

As in the case of linear orders, a crucial step in the characterisation of the single-peaked (and single-dipped) domain will be the complete classification of the semi-connected Condorcet domains on triples. This is provided by the next result (see Figure 11.2 for illustration with the domains $\mathcal{D}_1 - \mathcal{D}_6$ depicted clockwise from top left in the first row to bottom left in the second row).

Proposition 11.1.1. *Let $x, y, z \in A$ be pairwise distinct. The following are the semi-connected Condorcet subdomains of $\mathcal{R}(\{x, y, z\})$ that contain the totally reversed orders xyz and zyx. All of them are maximal and in fact connected.*

$$\begin{aligned}
\mathcal{D}_1(x, y, z) &= \{xyz, (xy)z, yxz, y(xz), yzx, (yz)x, zyx\}, \\
\mathcal{D}_2(x, y, z) &= \{xyz, (xy)z, (xyz), z(xy), zyx\}, \\
\mathcal{D}_3(x, y, z) &= \{xyz, (xy)z, (xyz), (yz)x, zyx\}, \\
\mathcal{D}_4(x, y, z) &= \{xyz, x(yz), (xyz), z(xy), zyx\}, \\
\mathcal{D}_5(x, y, z) &= \{xyz, x(yz), (xyz), (yz)x, zyx\}, \\
\mathcal{D}_6(x, y, z) &= \{xyz, x(yz), xzy, (xz)y, zxy, z(xy), zyx\}.
\end{aligned}$$

Proof. By the semi-connectedness of $\mathcal{D}$, there exists a path in $\mathcal{D}$ connecting xyz and zyx. Thus, $\mathcal{D}$ contains at least one neighbour of xyz, i.e. either $(xy)z$ or $x(yz)$. On the other hand, $\mathcal{D}$ cannot contain both of these neighbours because the triple $\{(xy)z, x(yz), zyx\}$ is 'forbidden:' if three voters each have one of these preference orders, respectively, we obtain a non-transitive majority relation since x and y, as well as y and z are deemed indifferent by the majority relation while x is strictly superior to z.

Case 1. Suppose first that $\mathcal{D}$ contains $(xy)z$. By the semi-connectedness, $\mathcal{D}$ must contain either the neighbour yxz, or the neighbour (xyz) of $(xy)z$ 'in direction of' zyx. Again, $\mathcal{D}$

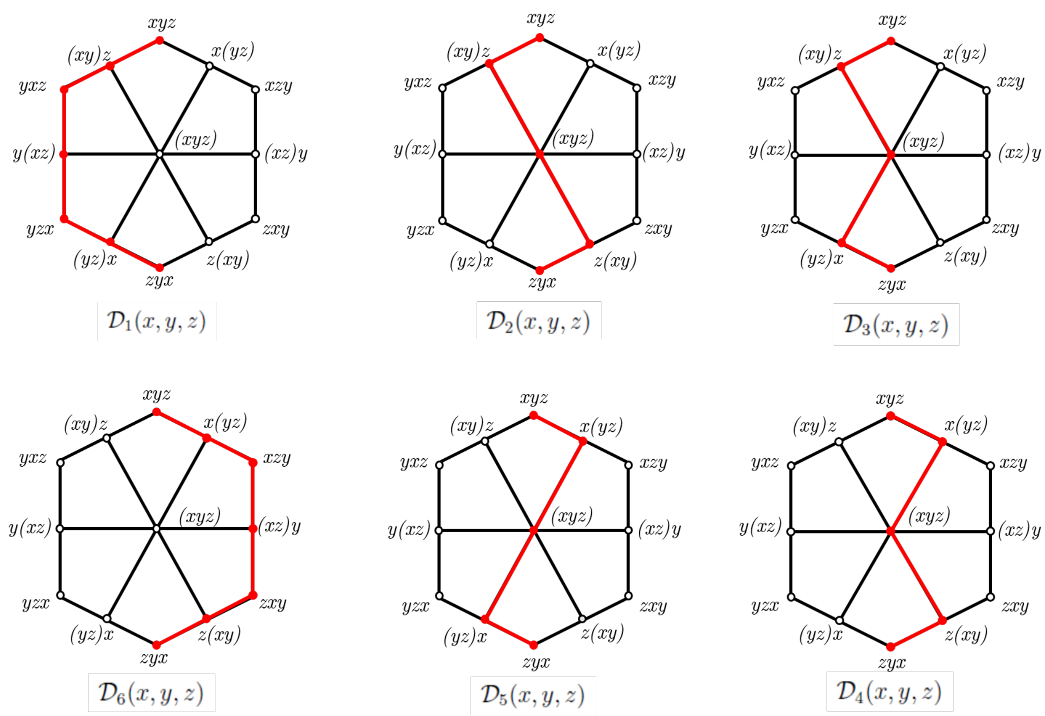


Figure 11.2: The (maximal) (semi-)connected Condorcet subdomains of $\mathcal{R}(\{x,y,z\})$.

cannot contain both of them since the triple $\{yxz, (xyz), zyx\}$ is forbidden (in the same sense as above).

Case 1.1. Thus, suppose $\mathcal{D}$ contains yxz in addition to $(xy)z$. By the semi-connectedness it must contain $\mathcal{D}_1(x,y,z)$. On the other hand, $\mathcal{D}_1(x,y,z)$ is a maximal Condorcet domain. For this it remains to show that it cannot contain any order from the set $\{xzy, (xz)y, zxy, z(xy)\}$. For the two linear orders xzy and zxy this follows immediately from the existence of appropriate Condorcet cycles with two elements from $\mathcal{D}_1(x,y,z)$, respectively; for the order $z(xy)$ it follows since the triple $\{xyz, (yz)x, z(xy)\}$ is forbidden; and for $(xz)y$ it follows since the triple $\{(xy)z, yzx, (xz)y\}$ is forbidden. Thus, in Case 1.1 we obtain $\mathcal{D} = \mathcal{D}_1(x,y,z)$.

Case 1.2. Now suppose $\mathcal{D}$ contains (xyz) in addition to $(xy)z$. Then, $\mathcal{D}$ can neither contain $y(xz)$ nor $(xz)y$ since both triples $\{(xy)z, (xyz), y(xz)\}$ and $\{xyz, (xyz), (xz)y\}$ are forbidden.[3] Thus, by the semi-connectedness $\mathcal{D}$ must contain either $z(xy)$ or $(yz)x$, but as already argued, it cannot contain both. In the first case, $\mathcal{D}$ thus contains $\mathcal{D}_2(x,y,z)$, and in the second case $\mathcal{D}$ contains $\mathcal{D}_3(x,y,z)$. But by arguments completely symmetric to those given so far it follows that either of these domains is maximal.

Case 2. If $\mathcal{D}$ contains $x(yz)$, a completely symmetric analysis yields the possible cases $\mathcal{D} =$

[3] That these two triples are forbidden hinges on the adopted definition of the majority relation. For instance, if in a profile ρ, each of the orders $(xy)z$, (xyz), $y(xz)$ receives one third of the votes, respectively, we obtain both $xR^{\mathrm{maj}}_{\rho}y$ and $yR^{\mathrm{maj}}_{\rho}x$; moreover, $xR^{\mathrm{maj}}_{\rho}z$ and $zR^{\mathrm{maj}}_{\rho}x$ but not $zR^{\mathrm{maj}}_{\rho}y$.

$\mathcal{D}_4(x, y, z)$, $\mathcal{D} = \mathcal{D}_5(x, y, z)$, or $\mathcal{D} = \mathcal{D}_6(x, y, z)$. This completes the proof of Proposition 11.1.1. □

A crucial observation for the following is that the domains $\mathcal{D}_1 - \mathcal{D}_5$ are weakly single-peaked, while the domains $\mathcal{D}_2 - \mathcal{D}_6$ are weakly single-dipped with respect to the spectrum $x \lhd y \lhd z$ (in particular, the five-element domains $\mathcal{D}_2 - \mathcal{D}_5$ are simultaneously weakly single-peaked and weakly single-dipped with respect to $x \lhd y \lhd z$).

11.1.2 Characterisation of the Single-Peaked Domain

Due to the inclusion of indifferences, there are two natural formulations of the minimal richness condition. Let us say that a domain $\mathcal{D} \subseteq \mathcal{R}(A)$ is *strongly minimally rich* if, for each alternative $x \in A$, there is at least one weak order $R \in \mathcal{D}$ that has x has the unique top alternative. Analogously, say that a domain $\mathcal{D} \subseteq \mathcal{R}(A)$ is *weakly minimally rich* if, for each alternative $x \in A$, there is at least one weak order $R \in \mathcal{D}$ such that x is among the top alternatives of R.

The following result extends the main findings of Section 3.1 to the case of weak orders.

Theorem 11.1.2. **a)** *For every spectrum $\lhd$ on A, there exists a unique maximal Condorcet domain $\hat{\mathcal{R}}_\lhd \subseteq \mathcal{R}(A)$ that contains the domain $\mathcal{SP}(\lhd, A)$ of all single-peaked linear orders with respect to $\lhd$. The domain $\hat{\mathcal{R}}_\lhd$ is weakly single-peaked and connected. Evidently, $\hat{\mathcal{R}}_\lhd$ has maximal width and is strongly minimally rich.*

b) *Conversely, let $\mathcal{D} \subseteq \mathcal{R}(A)$ be a semi-connected and weakly minimally rich Condorcet domain. Then, $\mathcal{D}$ is weakly single-peaked.*

c) *Moreover, if $\mathcal{D} \subseteq \mathcal{R}(A)$ is a semi-connected and strongly minimally rich Condorcet domain, then $\mathcal{D} \subseteq \hat{\mathcal{R}}_\lhd$ for some spectrum $\lhd$ on A. (In particular, $\mathcal{D}$ is weakly single-peaked.)*

The domain $\hat{\mathcal{R}}_\lhd$ admits the following explicit characterisation. First, each indifference class of every weak order in $\hat{\mathcal{R}}_\lhd$ has at most two elements, i.e. any indifference prevails over at most two distinct alternatives. Moreover, whenever all indifferences are 'resolved' for a weak order in $\hat{\mathcal{R}}_\lhd$ by transforming all indifferent pairs into strictly ranked adjacent alternatives, one obtains an order in $\mathcal{SP}(\lhd, A)$, no matter which combination of strict rankings for the indifferent pairs is chosen. For illustration, consider Figure 11.3 which depicts the domain $\hat{\mathcal{R}}_\lhd$ on $A = \{a, b, c, d\}$ for the spectrum $a \lhd b \lhd c \lhd d$. As an example, consider the weak order $(bc)(ad)$ in the middle of Fig. 11.3, i.e. the order that declares b and c as indifferent on the top and a and d as indifferent at the bottom. There are exactly four ways to transform these two indifferent pairs into adjacent strictly ranked pairs resulting in the linear orders *bcad*, *cbad*, *bcda*, and *cbda*, all of which belong to $\mathcal{SP}(\lhd, \{a, b, c, d\})$ for the spectrum $a \lhd b \lhd c \lhd d$. By contrast, the order $a(bc)d$ does not belong to the domain $\hat{\mathcal{R}}_\lhd$ for the spectrum $a \lhd b \lhd c \lhd d$, since the linear order *acbd* is not single-peaked. Note that the weak order $a(bc)d$ is nevertheless weakly single-peaked.

Proof of Theorem 11.1.2. **a)** Let $\mathcal{D} \subseteq \mathcal{R}(A)$ be a maximal Condorcet domain with $\mathcal{D} \supseteq \mathcal{SP}(\lhd, A)$ for a given spectrum $\lhd$ on A. As is easily verified, for every triple of distinct

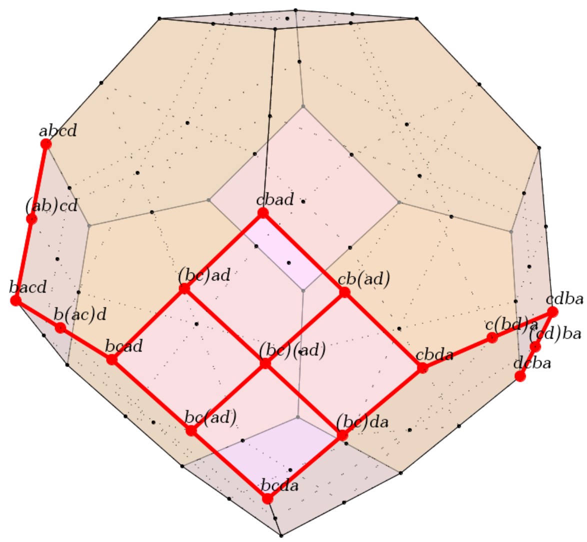


Figure 11.3: The domain $\hat{\mathcal{R}}_{\triangleleft}$ on $X = \{a, b, c, d\}$ with spectrum $a > b > c > d$.

alternatives $x, y, z \in A$ with $x \triangleleft y \triangleleft z$, the restriction $\mathcal{D}_{\{x,y,z\}}$ of $\mathcal{D}$ to $\{x, y, z\}$ is a maximal Condorcet domain of weak orders on $\{x, y, z\}$ that contains the domain $\mathcal{SP}(\triangleleft, \{x, y, z\})$ (strictly speaking, in the latter term '$\triangleleft$' denotes the restriction of $\triangleleft$ to $\{x, y, z\}$). By acyclicity of the majority relation, $\mathcal{D}_{\{x,y,z\}}$ can neither contain the linear order xzy nor the linear order zxy. Moreover, $\mathcal{D}_{\{x,y,z\}}$ can contain neither of the weak orders $x(yz)$, $(xz)y$, $z(xy)$, nor the complete indifference relation (xyz); indeed, the following are forbidden triples of weak orders: $\{x(yz), yxz, zyx\}$, $\{(xz)y, yxz, zyx\}$, $\{z(xy), xyz, yzx\}$, $\{(xyz), xyz, yzx\}$. This implies that $\mathcal{D}_{\{x,y,z\}} \subseteq \mathcal{D}_1(x, y, z)$, where $\mathcal{D}_1(x, y, z)$ is defined as in Proposition 11.1.1 above. By the maximality of both $\mathcal{D}_{\{x,y,z\}}$ and $\mathcal{D}_1(x, y, z)$, we obtain that in fact $\mathcal{D}_{\{x,y,z\}} = \mathcal{D}_1(x, y, z)$.[4] This implies the desired conclusion $\mathcal{D} = \hat{\mathcal{R}}_{\triangleleft}$ since evidently no indifference class in any order in $\mathcal{D}$ can have more than two elements; moreover, for no weak order in $\mathcal{D}$, one can obtain a linear order outside $\mathcal{SP}(\triangleleft, A)$ by strictly ranking all indifferent pairs and keeping the relative position of all other pairs of alternatives fixed (indeed, if this was possible there would exist a triple $x \triangleleft y \triangleleft z$ such that $\mathcal{D}_{\{x,y,z\}}$ would contain at least one of the weak orders $x(yz)$, $(xz)y$, or $z(xy)$). From the above it is immediate that the domain $\hat{\mathcal{R}}_{\triangleleft}$ is weakly single-peaked. It is easily seen that $\hat{\mathcal{R}}_{\triangleleft}$ is semi-connected: indeed, any two single-peaked orders in $\mathcal{SP}(\triangleleft, A)$ that differ in the ranking of exactly one pair are connected to each other in $\hat{\mathcal{R}}_{\triangleleft}$ by the weak order that

[4]Note that we cannot invoke Proposition 11.1.1 to directly conclude this, since we do not yet know whether or not $\mathcal{D}$ is semi-connected. In fact, there exist several maximal Condorcet domains of weak orders on a triple $\{x, y, z\}$ different from the ones listed in Proposition 11.1.1; these domains are either not connected or do not have maximal width (cf. Dittrich [2018]). The argument just given in the main text shows that none of them contains the single-peaked domain $\mathcal{SP}_{>}(\{x, y, z\})$ as a subdomain.

declares this pair of alternatives as indifferent (keeping the position of all other alternatives fixed). This implies the full connectedness of $\hat{\mathcal{R}}_\triangleleft$ recursively: each weak order R in $\hat{\mathcal{R}}_\triangleleft \setminus \mathcal{SP}(\triangleleft, A)$ has a neighbour in $\hat{\mathcal{R}}_\triangleleft$ that displays one indifference less by resolving one binary indifference to a strict preference (in any direction). By recursion, any element of $\hat{\mathcal{R}}_\triangleleft \setminus \mathcal{SP}(\triangleleft, A)$ is thus connected by a path in $\hat{\mathcal{R}}_\triangleleft$ to some element in $\mathcal{SP}(\triangleleft, A) \subsetneq \hat{\mathcal{R}}_\triangleleft$. Since every two elements in $\mathcal{SP}(\triangleleft, A)$ are connected by a path in $\hat{\mathcal{R}}_\triangleleft$, the domain $\hat{\mathcal{R}}_\triangleleft$ is connected. This completes the proof of part a).

b) Suppose that $\mathcal{D} \subseteq \mathcal{R}(A)$ is a semi-connected and weakly minimally rich Condorcet domain. Denote by $\overline{P}$ one of the pair of orders in $\mathcal{D}$ that are complete inverses of each other and are connected by a path in $\mathcal{D}$ (recall that each one of a pair of completely reversed orders in $\mathcal{R}(A)$ must be a linear order). For each triple of distinct alternatives $x, y, z \in A$ with $x\overline{P}y\overline{P}z$, the restriction $\mathcal{D}_{\{x,y,z\}}$ is a semi-connected and weakly minimally rich Condorcet domain that contains the completely reversed orders xyz and zyx. By Proposition 11.1.1, $\mathcal{D}_{\{x,y,z\}}$ coincides with one of the domains $\mathcal{D}_1(x,y,z) - \mathcal{D}_6(x,y,z)$, but by the weak minimal richness in fact with one of the domains $\mathcal{D}_1(x,y,z) - \mathcal{D}_5(x,y,z)$ (observe that no weak order in $\mathcal{D}_6(x,y,z)$ has y among its top alternatives). But we have already observed that the domains $\mathcal{D}_1(x,y,z) - \mathcal{D}_5(x,y,z)$ are all weakly single-peaked. As in Section 2 above, the weak single-peakedness of $\mathcal{D}$ on all triples with respect to the same spectrum $\triangleleft := \overline{P}$ implies its weak single-peakedness globally with respect to $\triangleleft$.

c) Finally, suppose that $\mathcal{D} \subseteq \mathcal{R}(A)$ is a semi-connected and even strongly minimally rich Condorcet domain. Then, for all triples $x, y, z \in A$ with $x\overline{P}y\overline{P}z$ the restriction $\mathcal{D}_{\{x,y,z\}}$ coincides with $\mathcal{D}_1(x,y,z)$ (all other semi-connected Condorcet domains on $\{x,y,z\}$ are not strongly minimally rich). By the arguments given in part a) this implies that $\mathcal{D} \subseteq \hat{\mathcal{R}}_\triangleleft$. This completes the proof of Theorem 3. □

While, for a given spectrum $\triangleleft$, the Condorcet domain $\hat{\mathcal{R}}_\triangleleft$ is uniquely determined by the conditions in Theorem 11.1.2 c) and maximality, there are several non-isomorphic maximal single-peaked domains that satisfy the weaker richness property in part b). Figure 11.4 depicts four of them. The reason for the multiplicity of weakly single-peaked Condorcet domains that satisfy the weak minimal richness condition is that the restrictions to different triples of alternatives may correspond to *different* types of Condorcet domains on the triples (for each triple $x, y, z \in X$, one of the domains $\mathcal{D}_1(x,y,z) - \mathcal{D}_5(x,y,z)$). By contrast, under the strong minimal richness conditions, the restrictions to all triples x, y, z coincide uniformly with $\mathcal{D}_1(x,y,z)$.

11.2 Condorcet domains of partial orders

We now turn to the even more general case in which each voter is characterised by a (strict) partial order. Note that the case of linear orders corresponds to the special case in which all orders are complete, and the case of weak orders to the special case in which all orders are negatively transitive (in which case the associated incomparability relation is transitive and can therefore be interpreted as proper indifference).

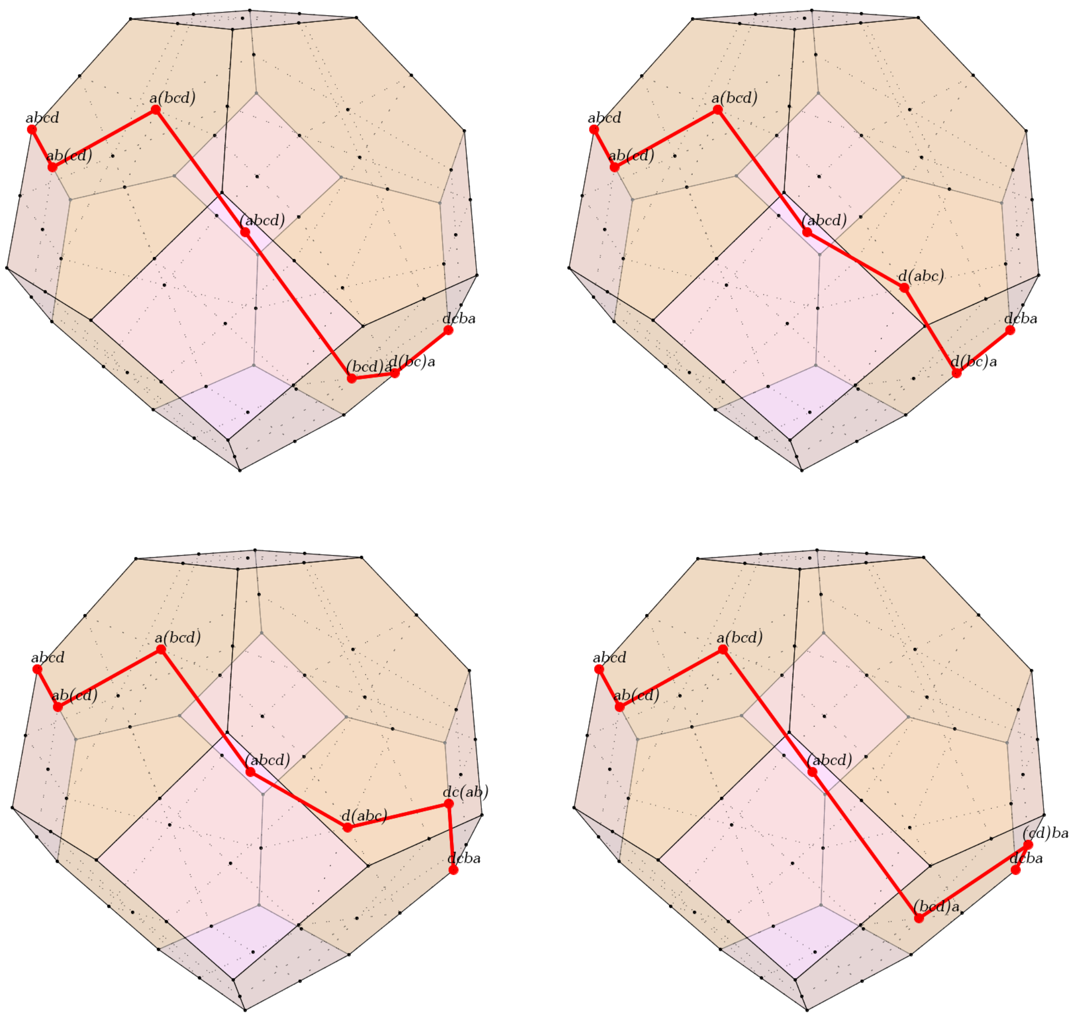


Figure 11.4: Four maximal weakly single-peaked domains satisfying weak minimal richness.

Our first finding is that there exist 'large' Condorcet domains of partial orders that in fact do not contain any linear order. In the case of three alternatives, a case of special interest as detailed below, the Condorcet domain containing the maximal number of orders is indeed of that sort. Specifically, the domain of all partial orders over three alternatives that are *not* linear orders is a Condorcet domain attaining this maximal number (namely 13 out of the 19 possible partial orders over three alternatives). This is in sharp contrast to the case of weak orders in which the maximal Condorcet domain over three alternatives that does not contain any linear order is the Condorcet domain with the *minimal* number of elements (containing 4 out of the 13 possible weak orders over three alternatives); up to isomorphism or flip isomorphism it is given by $\{a(bc), b(ac), c(ab), (abc)\}$, see Dittrich [2018].

We then prove that, for each spectrum $\lhd$, there exists a *unique* maximal Condorcet domain of partial orders that contains the domain $\mathcal{SP}(\lhd, A)$ of all *linear* orders on X that are single-peaked with respect to the spectrum $\lhd$. This domain is given by the set of all intersections of elements of $\mathcal{SP}(\lhd, A)$. As will be shown, this domain is much smaller than the set of all partial orders that are single peaked with respect to $\lhd$ in the sense that they can be extended to an element of $\mathcal{SP}(\lhd, A)$.

As in the case of weak orders, our analysis relies on a close study of the case of three alternatives, in which case a complete description of all Condorcet domains of partial orders has been given by Dittrich [2018].

Consider again a finite set of alternatives A, and the set $\mathcal{Q}(A)$ of all *(strict) partial orders* (i.e., transitive and asymmetric binary relations) on A. A subset $\mathcal{D} \subseteq \mathcal{Q}(A)$ will be called a *domain*. A *profile* $\pi = (P_1, \ldots, P_n)$ on $\mathcal{D}$ is an element of the Cartesian product $\mathcal{D}^n$ for some number $n \in \mathbb{N}$ of 'voters,' where the partial order P_i represents the (possibly incomplete) preference relation of the ith voter over the alternatives from A. A profile with an odd number of voters will simply be referred to as an *odd profile*. As above, we will denote linear orders simply by listing the alternatives in descending order, e.g. the linear order that ranks a first, b second, c third, etc., is denoted by $abc\ldots$. We will use a similar notation for partial orders, e.g. the partial order that ranks a above b and leaves all other pairs incomparable is denoted by ab ; moreover, the partial order that ranks both a and b above c, respectively, and leaves all other pairs incomparable is denoted by $\langle ac, bc\rangle$; as a final example, the partial order that ranks a above b, b above c (and hence by transitivity also a above c), and a above d, leaving all other pairs incomparable, is denoted by $\langle abc, ad\rangle$, etc.

The *majority relation* associated with a profile π is the binary relation P_π^{maj} on A such that $xP_\pi^{\mathrm{maj}}y$ if and only if more than half of the voters rank x above y. Note that, according to this definition, the majority relation is always asymmetric. A domain $\mathcal{D} \subseteq \mathcal{P}(A)$ is called a *Condorcet domain (of partial orders)* if the majority relation associated to every odd profile is transitive.[5] While Condorcet domains of *linear* orders have received significant attention in the literature, to the best of our knowledge, Condorcet domains

[5] As in the case of weak orders, there are different ways to extend the various concepts from the case of linear orders to the case of partial orders. For instance, one could base the majority relation corresponding to a profile on the *net majorities* and say that x is majority preferred if and only if more voters prefer x to y than vice versa. We do not exclude that such approach could give rise to interesting insights as well.

of partial orders have not been studied in the literature so far.

A domain $\mathcal{D}$ is called a *maximal Condorcet domain* if every Condorcet domain (on the same set of alternatives) that contains $\mathcal{D}$ as a subset must in fact coincide with $\mathcal{D}$. It is easily verified that every maximal Condorcet domain $\mathcal{D}$ is *closed* in the sense that the majority relation of any odd profile from $\mathcal{D}$ is again an element of $\mathcal{D}$ (and not only of $\mathcal{Q}(A)$).

Two orders P and P^{inv} are called *completely reversed* if $xPy \Leftrightarrow yP^{\text{inv}}x$. Note that two completely reversed partial orders need not necessarily be linear (in contrast to two *totally* reversed weak orders, cf. Subsection 12.1). Recall that a domain is *symmetric* if it contains with every order also its completely reversed order. Finally, a domain $\mathcal{D}$ will be called *minimally rich at the top (resp. at the bottom)* if, for every alternative $x \in A$, there exists an order $P \in \mathcal{D}$ such that xPw (resp. wPx) for all $w \in A$.

Our analysis of Condorcet domains of partial orders is based on the following classification of all maximal Condorcet domains of partial orders on three alternatives.

Lemma 11.2.1 (Dittrich [2018]). *Let $a, b, c \in A$ be pairwise distinct. There are, up to isomorphism or flip isomorphism, thirteen different maximal Condorcet domains on the set $\{a, b, c\}$; Figure 11.5 depicts them.*

Proof. The proof of Lemma 11.2.1 is based on an analysis of the 'forbidden' triples of orders, i.e. the profiles of three orders that give rise to a non-transitive majority relation. Up to relabeling of the alternatives, these are the following nine triples:

$$\begin{array}{lcccr}
\{ & xyz, & yzx, & zxy & \}, \\
\{ & xyz, & yzx, & xy & \}, \\
\{ & xyz, & zxy, & yz & \}, \\
\{ & xyz, & yzx, & \langle xy, zy\rangle & \}, \\
\{ & xyz, & zxy, & \langle yx, yz\rangle & \}, \\
\{ & xyz, & xy, & yz & \}, \\
\{ & xyz, & xy, & \langle yx, yz\rangle & \}, \\
\{ & xyz, & \langle xy, zy\rangle, & \langle yx, yz\rangle & \}, \\
\{ & xyz, & yz, & \langle xy, zy\rangle & \}.
\end{array}$$

The systematic exclusion of all triples of orders of one of these nine types leads to the domains depicted in Fig. 11.5, which proves Lemma 11.2.1. □

A number of comments and observations are in order. First, the 13 maximal Condorcet domains depicted in Fig. 11.5 have different sizes (the first picture in the first row does not represent a domain but just shows the neighbourhood structure of $\mathcal{Q}(\{a, b, c\})$). The uniquely largest (with 13 elements) is the domain to the left in the third row consisting of all partial orders on $\{a, b, c\}$ that are *not* linear. The uniquely smallest (with 9 elements) is the one to the right in the third row which extends the group separable domain with the two 'groups' $\{\{a, b\}, \{c\}\}$. The first domain (second picture from left in the first row) is the unique maximal Condorcet domain that contains the domain of all single-peaked linear orders with respect to the spectrum $a \lhd b \lhd c$.

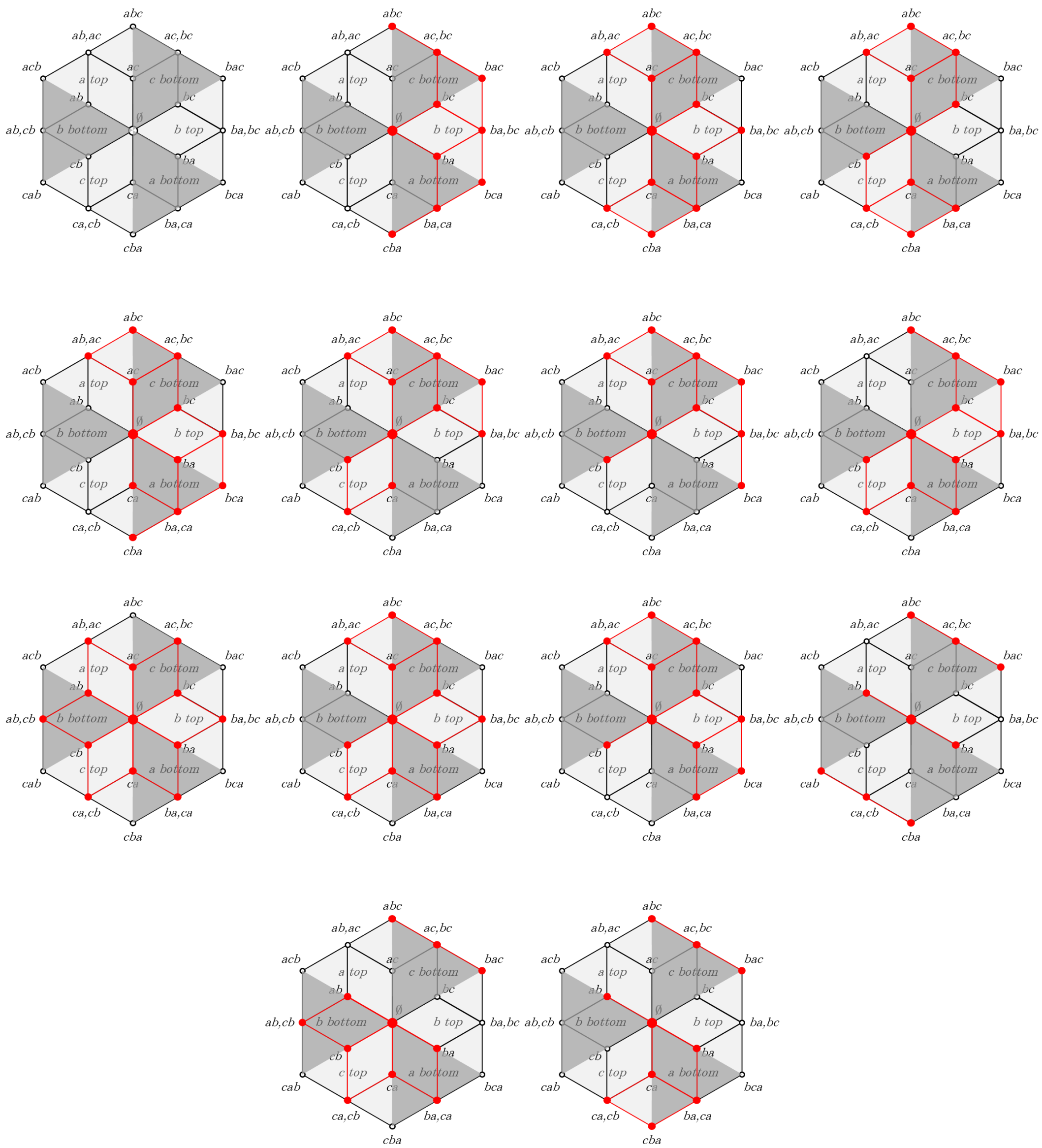


Figure 11.5: All maximal Condorcet subdomains of $\mathcal{Q}(\{a,b,c\})$.

11.2.1 A connected, symmetric and minimally rich Condorcet domain

Denote by $\mathcal{D}^{\leq 2}(A)$ the domain of all partial orders that contain no 'string' of three or more ordered alternatives. I.e. $P \in \mathcal{D}^{\leq 2}(A)$ if and only if, for no triple $x, y, z \in A$ of pairwise distinct alternatives, xPy and yPz (and therefore by transitivity also xPz). In other words, $P \in \mathcal{D}^{\leq 2}(A)$ if and only if, for no triple $x, y, z \in A$ of pairwise distinct alternatives, $xyz \subseteq P$. In the following result, by a *sub-order* of a partial order P we mean any partial order Q such that $Q \subseteq P$ (allowing for $Q = \emptyset$).

Theorem 11.2.2. *For all A, the domain $\mathcal{D}^{\leq 2}(A)$ is a maximal Condorcet domain of partial orders that is connected, symmetric, minimally rich (both at the top and at the bottom), and closed under taking sub-orders.*

Proof. That $\mathcal{D}^{\leq 2}(A)$ is a maximal Condorcet domain follows at once from the fact that all 'forbidden' triples include at least one string of three ordered alternatives (see the proof of Lemma 11.2.1 above); by definition, no restriction of any order in $\mathcal{D}^{\leq 2}(A)$ to any triple contains such a string.

The domain $\mathcal{D}^{\leq 2}(A)$ is connected since every of its element is connected by a path to the empty partial order; this path is obtained by sequentially removing all pairwise comparisons entailed by a given element of $\mathcal{D}^{\leq 2}(A)$. Observe that pairwise comparisons can indeed be sequentially removed in any order because there are no transitivity implications: for all $P \in \mathcal{D}^{\leq 2}(A)$ and all $(x, y) \in P$, we have $P \setminus \{(x, y)\} \in \mathcal{D}^{\leq 2}(A)$ (in particular, $P \setminus \{(x, y)\}$ is in fact a partial order). This also shows that $\mathcal{D}^{\leq 2}(A)$ is closed under taking sub-orders.

The domain is symmetric since, evidently, an order does not contain a string of three ordered alternatives if and only if its complete reverse does not.

Finally, the domain $\mathcal{D}^{\leq 2}(A)$ is minimally rich at the top because, for all $x \in A$, it contains the partial order $\langle xy_1, ..., xy_m \rangle$ where $A \setminus \{x\} = \{y_1, ..., y_m\}$; similarly, it is minimally rich at the bottom because it contains the order $\langle y_1x, ..., y_mx \rangle$. □

Observe that in the case of linear orders, there does not exist a maximal Condorcet domain that is connected, symmetric and minimally rich (neither at the top nor at the bottom) whenever $|A| \geq 3$.

An inspection of the maximal Condorcet domains in Fig. 11.5 reveals that, on a domain of three alternatives, there are only two symmetric domains other than $\mathcal{D}^{\leq 2}(A)$, the domain to the right of the first row and the domain to the right of the third row. The latter is not connected, and the former is not minimally rich (neither on the top nor on the bottom). Moreover, the domain $\mathcal{D}^{\leq 2}(A)$ is the only domain among the domains shown in Fig. 11.5 that is closed under taking sub-orders. These observations motivate the following conjectures.

Conjecture 4. *The domain $\mathcal{D}^{\leq 2}(A)$ is the only maximal Condorcet domain of partial orders that is connected, symmetric and minimally rich (either at the top or at the bottom).*

Conjecture 5. *The domain $\mathcal{D}^{\leq 2}(A)$ is the only maximal Condorcet domain of partial orders on X that is closed under taking sub-orders.*

11.2.2 The unique maximal Condorcet domain containing $\mathcal{SP}(\lhd, A)$

Next, we establish a result akin to Theorem 11.1.2a) in the case of partial orders. Indeed there exists, for every spectrum $\lhd$, a unique maximal Condorcet domain of partial orders that contains the domain of all single-peaked linear orders $\mathcal{SP}(\lhd, A)$. On the other hand, results similar to Theorems 11.1.2b) and c) cannot be obtained in the case of partial orders; this follows from the analysis and the properties of the domain $\mathcal{D}^{\leq 2}(A)$ in the previous subsection.

Denote by $\hat{\mathcal{Q}}_{\lhd}$ the domain of all intersections of orders in $\mathcal{SP}(\lhd, A)$, i.e.

$$\hat{\mathcal{Q}}_{\lhd} \;:=\; \Big\{ P \in \mathcal{Q} \mid P = \bigcap \mathcal{S} \text{ for some non-empty subset } \mathcal{S} \subseteq \mathcal{SP}(\lhd, A) \Big\}.$$

Theorem 11.2.3. *For every spectrum $\lhd$ on X, the domain $\hat{\mathcal{Q}}_{\lhd}$ is the unique maximal Condorcet domain of partial orders that contains $\mathcal{SP}(\lhd, A)$.*

Proof. First, we show that no triple of orders from $\hat{\mathcal{Q}}_{\lhd}$ when restricted to a triple $\{x, y, z\}$ induces any of the nine forbidden constellations listed in Lemma 11.2.1.

Case 1. It is evident that no triple of orders from $\hat{\mathcal{Q}}_{\lhd}$ can induce the cyclic triple $\{xyz, yzx, zxy\}$ on $\{x, y, z\}$.

Case 2. Now suppose, by contradiction, that the triple Q_1, Q_2, Q_3 if restricted to $\{x, y, z\}$ induces the constellation $\{xyz, yzx, xy\}$; concretely, suppose that $Q_1|_{\{x,y,z\}} = xyz$, $Q_2|_{\{x,y,z\}} = yzx$ and $Q_3|_{\{x,y,z\}} = xy$. Also, let $Q_3 = \bigcap \mathcal{S}_3$ where $\mathcal{S}_3 \subseteq \mathcal{SP}(\lhd, A)$. Every element $P \in \mathcal{S}_3$ ranks x above y, and by the single-peakedness, no element of $\mathcal{S}_3$ contains the string zxy. Hence, all elements of $\mathcal{S}_3$ when restricted to $\{x, y, z\}$ either induce xyz or xzy, and in fact some of them must induce xyz and some of them must induce xzy (otherwise, their intersection would not result in the partial order xy). In other words, the restrictions of $\mathcal{SP}(\lhd, A)$ to $\{x, y, z\}$ include the set $\{xyz, yzx, xzy\}$; but there is no spectrum $\lhd$ such that all elements of $\{xyz, yzx, xzy\}$ are single-peaked on $\{x, y, z\}$ with respect to $\lhd$ (since all three alternatives are sometimes at the bottom).

Case 3. Suppose, again by contradiction, that the triple Q_1, Q_2, Q_3 if restricted to $\{x, y, z\}$ induces the constellation $\{xyz, zxy, yz\}$; concretely, suppose that $Q_1|_{\{x,y,z\}} = xyz$, $Q_2|_{\{x,y,z\}} = zxy$ and $Q_3|_{\{x,y,z\}} = yz$. Also, let $Q_3 = \bigcap \mathcal{S}_3$ where $\mathcal{S}_3 \subseteq \mathcal{SP}(\lhd, A)$. Every element $P \in \mathcal{S}_3$ ranks y above z, and by the single-peakedness no element of $\mathcal{S}_3$ contains the string yzx. Hence, all elements of $\mathcal{S}_3$ when restricted to $\{x, y, z\}$ either induce xyz or yxz; but in this case their intersection contains the partial order $\langle xz, yz \rangle$, a contradiction.

Case 4. Again by contradiction, suppose that the triple Q_1, Q_2, Q_3 if restricted to $\{x, y, z\}$ induces the constellation $\{xyz, yzx, \langle xy, zy \rangle\}$; concretely, suppose that $Q_1|_{\{x,y,z\}} = xyz$, $Q_2|_{\{x,y,z\}} = yzx$ and $Q_3|_{\{x,y,z\}} = \langle xy, zy \rangle$. Also, let $Q_3 = \bigcap \mathcal{S}_3$ where $\mathcal{S}_3 \subseteq \mathcal{SP}(\lhd, A)$. Since $Q_3|_{\{x,y,z\}} = \langle xy, zy \rangle$, all elements of $\mathcal{S}_3$ when restricted to $\{x, y, z\}$ either induce xyz or zxy. However, no single-peaked linear order can induce zxy, hence all elements of $\mathcal{S}_3$ when restricted to $\{x, y, z\}$ must induce xyz; but this contradicts the fact that x and z are incomparable according to Q_3.

Case 5. Again by contradiction, suppose that the triple Q_1, Q_2, Q_3 if restricted to $\{x, y, z\}$ induces the constellation $\{xyz, zxy, \langle yx, yz \rangle\}$; concretely, suppose that $Q_1|_{\{x,y,z\}} = xyz$, $Q_2|_{\{x,y,z\}} = zxy$ and $Q_3|_{\{x,y,z\}} = \langle yx, yz \rangle$. Also, let $Q_3 = \bigcap \mathcal{S}_3$ where $\mathcal{S}_3 \subseteq \mathcal{SP}(\lhd, A)$. Since $Q_3|_{\{x,y,z\}} = \langle yx, yz \rangle$, all elements of $\mathcal{S}_3$ when restricted to $\{x, y, z\}$ either induce yxz or yzx, and in fact both strings must occur in the restrictions of the elements of $\mathcal{S}_3$ to $\{x, y, z\}$ (because x and z are incomparable with respect to $Q_3|_{\{x,y,z\}}$). Summarizing, the strings induced by $\mathcal{SP}(\lhd, A)$ on the triple $\{x, y, z\}$ must include $\{xyz, zxy, yxz, yzx\}$. As in Case 2, there is no spectrum $\lhd$ such that these four orders are single-peaked on $\{x, y, z\}$ with respect to $\lhd$.

Case 6. Next suppose, again by contradiction, that the triple Q_1, Q_2, Q_3 if restricted to $\{x, y, z\}$ induces the constellation $\{xyz, xy, yz\}$; concretely, suppose that $Q_1|_{\{x,y,z\}} = xyz$, $Q_2|_{\{x,y,z\}} = xy$ and $Q_3|_{\{x,y,z\}} = yz$. Also, let $Q_2 = \bigcap \mathcal{S}_2$ and $Q_3 = \bigcap \mathcal{S}_3$, respectively, where $\mathcal{S}_2, \mathcal{S}_3 \subseteq \mathcal{SP}(\lhd, A)$. Since $Q_2|_{\{x,y,z\}} = xy$, all elements of $\mathcal{S}_2$ induce one of the following three strings on $\{x, y, z\}$: xyz, xzy, zxy; in fact, since Q_2 entails no other comparisons, at least the strings xyz and zxy must occur. Similarly, since $Q_3|_{\{x,y,z\}} = yz$, all elements of $\mathcal{S}_3$ induce one of the following three strings on $\{x, y, z\}$: xyz, yxz, yzx; and in fact, since Q_3 entails no other comparisons, at least the strings xyz and yzx must occur. Thus, summarizing the strings induced by $\mathcal{SP}(\lhd, A)$ on the triple $\{x, y, z\}$ must include $\{xyz, yzx, zxy\}$; but as in Case 2, there is no spectrum $\lhd$ such that these three orders are single-peaked on $\{x, y, z\}$ with respect to $\lhd$.

Case 7. Suppose, again by contradiction, that the triple Q_1, Q_2, Q_3 if restricted to $\{x, y, z\}$ induces the constellation $\{xyz, xy, \langle yx, yz \rangle\}$; concretely, suppose that $Q_1|_{\{x,y,z\}} = xyz$, $Q_2|_{\{x,y,z\}} = xy$ and $Q_3|_{\{x,y,z\}} = \langle yx, yz \rangle$. Also, let $Q_2 = \bigcap \mathcal{S}_2$ and $Q_3 = \bigcap \mathcal{S}_3$, respectively, where $\mathcal{S}_2, \mathcal{S}_3 \subseteq \mathcal{SP}(\lhd, A)$. Since $Q_2|_{\{x,y,z\}} = xy$, all elements of $\mathcal{S}_2$ induce one of the following three strings on $\{x, y, z\}$: xyz, xzy, zxy; in fact, since Q_2 entails no other comparisons, at least the strings xyz and zxy must occur. Similarly, since $Q_3|_{\{x,y,z\}} = \langle yx, yz \rangle$, all elements of $\mathcal{S}_3$ induce one of the following two strings on $\{x, y, z\}$: yxz, yzx, and since Q_3 entails no other comparisons both strings must in fact occur. Thus, summarizing the strings induced by $\mathcal{SP}(\lhd, A)$ on the triple $\{x, y, z\}$ must include $\{xyz, yxz, yzx, zxy\}$; but again, there is no spectrum $\lhd$ such that these four orders are single-peaked on $\{x, y, z\}$ with respect to $\lhd$.

Case 8. Suppose, again by contradiction, that the triple Q_1, Q_2, Q_3 if restricted to $\{x, y, z\}$ induces the constellation $\{xyz, \langle xy, zy \rangle, \langle yx, yz \rangle\}$; concretely, let us suppose that $Q_1|_{\{x,y,z\}} = xyz$, $Q_2|_{\{x,y,z\}} = \langle xy, zy \rangle$ and $Q_3|_{\{x,y,z\}} = \langle yx, yz \rangle$. Also, let $Q_2 = \bigcap \mathcal{S}_2$ and $Q_3 = \bigcap \mathcal{S}_3$, respectively, where $\mathcal{S}_2, \mathcal{S}_3 \subseteq \mathcal{SP}(\lhd, A)$. Since $Q_2|_{\{x,y,z\}} = \langle xy, zy \rangle$, all elements of $\mathcal{S}_2$ induce one of the two strings xzy or zxy on $\{x, y, z\}$, and since Q_2 entails no other comparisons both strings must in fact occur. Similarly, since $Q_3|_{\{x,y,z\}} = \langle yx, yz \rangle$, all elements of $\mathcal{S}_3$ induce one of the two strings yxz or yzx on $\{x, y, z\}$, and since Q_3 entails no other comparisons both strings must in fact occur. Thus, summarizing the strings induced by $\mathcal{SP}(\lhd, A)$ on the triple $\{x, y, z\}$ must include $\{xyz, xzy, yxz, yzx, zxy\}$; but there is no spectrum $\lhd$ such that these five orders are single-peaked on $\{x, y, z\}$ with respect to $\lhd$.

Case 9. Finally, suppose by contradiction, that the triple Q_1, Q_2, Q_3 if restricted to

$\{x, y, z\}$ induces the constellation $\{xyz, yz, \langle xy, zy\rangle\}$; concretely, suppose that $Q_1|_{\{x,y,z\}} = xyz$, $Q_2|_{\{x,y,z\}} = yz$ and $Q_3|_{\{x,y,z\}} = \langle xy, zy\rangle$. Also, let $Q_2 = \bigcap \mathcal{S}_2$ and $Q_3 = \bigcap \mathcal{S}_3$, respectively, where $\mathcal{S}_2, \mathcal{S}_3 \subseteq \mathcal{SP}(\lhd, A)$. Since $Q_2|_{\{x,y,z\}} = yz$, all elements of $\mathcal{S}_2$ induce one of the following three strings on $\{x, y, z\}$: xyz, yxz, yzx; in fact, since Q_2 entails no other comparisons, at least the strings xyz and yzx must occur. Similarly, since $Q_3|_{\{x,y,z\}} = \langle xy, zy\rangle$, all elements of $\mathcal{S}_3$ induce one of the following two strings on $\{x, y, z\}$: xzy, zxy, and since Q_3 entails no other comparisons both strings must in fact occur. Thus, summarizing the strings induced by $\mathcal{SP}(\lhd, A)$ on the triple $\{x, y, z\}$ must include $\{xyz, xzy, yzx, zxy\}$; but again, there is no spectrum $\lhd$ such that these four orders are single-peaked on $\{x, y, z\}$ with respect to $\lhd$.

So far we have shown that $\hat{\mathcal{Q}}_\lhd$ is indeed a Condorcet domain. Now we demonstrate that it is maximal. To do so, we show that for every partial order Q that cannot be represented as an intersection of orders in $\mathcal{SP}(\lhd, A)$ one can find two elements $P_1, P_2 \in \mathcal{SP}(\lhd, A)$ such that the triple Q, P_1, P_2 induces a forbidden constellation on some triple $\{x, y, z\}$. If Q cannot be represented as an intersection of single-peaked orders, then there exists a triple $\{x, y, z\}$ such that $Q|_{\{x,y,z\}}$ cannot be represented as the intersection of single-peaked orders on $\{x, y, z\}$. Without loss of generality assume that $x \lhd y \lhd z$. Evidently, if $Q|_{\{x,y,z\}}$ is complete on $\{x, y, z\}$ but not single-peaked with respect to $\lhd$, Q cannot be added to $\mathcal{SP}(\lhd, A)$ without inducing a cyclic majority relation. Thus, we may assume that $Q|_{\{x,y,z\}}$ is incomplete. As is easily verified, the incomplete partial orders on $\{x, y, z\}$ that cannot be obtained as intersections of elements of $\mathcal{SP}(\lhd, A)$ are: xy, xz, zx, zy, $\langle xy, xz\rangle$, $\langle xy, zy\rangle$ and $\langle zx, zy\rangle$. For each of these we can choose two orders from $\mathcal{SP}(\lhd, A)$ such that a forbidden constellation results, respectively. For the order xy, this follows at once from the second forbidden constellation in the proof of Lemma 11.2.1; for the order xz it follows from a re-labelling of the third forbidden constellation (exchange x and y); for the order zx it follows again from a suitable re-labelling of the third forbidden constellation ($x \to y$, $y \to z$, $z \to x$); for the order zy it follows from a suitable re-labelling of the second forbidden constellation (exchange x and z); for the order $\langle xy, xz\rangle$ it follows from a re-labelling of the fifth forbidden constellation (exchange x and y); for the order $\langle xy, zy\rangle$ it follows directly from the fourth forbidden constellation; and finally, for the order $\langle zx, zy\rangle$ it follows again from the fifth forbidden constellation (by relabelling $x \to y$, $y \to z$ and $z \to x$).

This shows that no Condorcet domain that contains $\mathcal{SP}(\lhd, A)$ can contain any partial order outside $\hat{\mathcal{Q}}_\lhd$. In other words, $\hat{\mathcal{Q}}_\lhd$ is the maximal Condorcet domain that contains $\mathcal{SP}(\lhd, A)$. □

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