Efficient and Strategy-Proof Voting over Connected Coalitions: A Possibility Result

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July 2009

Abstract

We consider preferences over connected coalitions that are single-peaked with respect to an appropriate betweenness relation. We show that on this preference domain there exist non-dictatorial, strategy-proof and efficient social choice functions.

1 Motivation

By the Gibbard-Satterthwaite-Theorem ([2] and [6]) the only strategy-proof voting rule on an unrestricted preference domain over at least three alternatives is the dictatorship of one individual. For possibility results restrictions of the preference domain are necessary. Well-known examples are the domain of all single-peaked preferences on a line (see Moulin [3]) and the domain of all separable preferences on the hypercube (see Barberá, Sonnenschein, Zhou [1]).

In this paper a novel example of a possibility domain is presented. As the two preference domains mentioned above (and a number of other possibility domains as well) it belongs to the large class of generalized single-peaked domains considered in Nehring and Puppe [5].

To motivate our preference domain, consider a finite set of political parties ordered from left to right on the political spectrum. The space of alternatives is the family of all *connected* coalitions, i.e. the family of all non-empty coalitions that contain with any two parties all parties that are between them in the political spectrum. The family of connected coalitions can be endowed

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with a natural betweenness relation as follows: a connected coalition C is *between* two connected coalitions C_1 and C_2 if (i) the leftmost element of C is between the leftmost elements of C_1 and C_2, respectively, and (ii) the rightmost element of C is between the rightmost elements of C_1 and C_2, respectively.

A preference ordering on the family of all non-empty connected coalitions is called *generalized single-peaked* if it admits a unique most preferred coalition (the "peak"), say C^*, such that a coalition C is strictly preferred to another coalition C' whenever C lies between C' and C^*.

We show that on the domain of all generalized single-peaked preferences over connected coalitions there exist anonymous and strategy-proof social choice functions. One example is the social choice function that selects the connected coalition which has as leftmost element the median of the leftmost elements of the individually most preferred coalitions and as rightmost element the median of the rightmost elements of the individually most preferred coalitions.

The existence of anonymous and strategy-proof social choice functions on the domain of all generalized single-peaked preferences over connected coalitions follows from general results derived in Nehring and Puppe [5] since the underlying betweenness relation gives rise to a *median space*. In fact it follows from the analysis in [5] that the social choice function described above is the *only* anonymous and strategy-proof voting rule that is neutral in an appropriate sense. Moreover, using the main result of [4] one can show that the above voting rule is *efficient*. In this paper, we provide elementary proofs of its strategy-proofness and efficiency.

2 Generalized Single-Peaked Preferences over Connected Coalitions

Let $A = \{a_1, \ldots, a_m\}$ be a finite set containing $m \geq 2$ objects. We consider the case in which individuals have preferences over a subset of the power set $\mathcal{P}(A)$. Specifically, we consider the following domain restriction. Let $<$ be a linear ordering of A, w.l.o.g. $a_1 < \cdots < a_m$. As a specific example one may think of A as representing a set of political parties which can be ordered from left to right on a political spectrum. In this case the power set $\mathcal{P}(A)$ represents the class of possible coalitions. While other interpretations may be applicable as well, in the remainder will refer to the elements of A as political parties and to the elements of $\mathcal{P}(A)$ as coalitions. For notational convenience we identify parties with their indices and simply write (ijk) for $\{a_i, a_j, a_k\}$. A non-empty coalition C is called **connected**, if for all i, j, k,

$$i, j \in C \text{ and } i < k < j \Rightarrow k \in C.$$

We denote by $\mathcal{C}_< \subset \mathcal{P}(A)$ the set of all connected coalitions. For every connected coalition C we call

$$l_C \in C \textbf{ leftmost in C} \text{ if for all } k < l_C \Rightarrow k \notin C$$

and

$$r_C \in C \textbf{ rightmost in C} \text{ if for all } k > r_C \Rightarrow k \notin C.$$

Evidently one has $C = (l_C ... r_C)$ for every connected coalition C.

Example: The coalition $C = (234)$ consisting of the parties a_2, a_3 and a_4 is connected with $l_C = 2$ and $r_C = 4$. $C' = (24)$ is not connected and therefore not an element of $\mathcal{C}_<$.

We define the following betweenness relation on $\mathcal{C}_<$. A coalition C is **between** C_1 and C_2 if

- its leftmost party l_C is between l_{C_1} and l_{C_2}, and
- its rightmost party r_C is between r_{C_1} and r_{C_2}.

Formally, C is between C_1 and C_2 if

$$l_C \in [\min\{l_{C_1}, l_{C_2}\}, \max\{l_{C_1}, l_{C_2}\}] \text{ and } r_C \in [\min\{r_{C_1}, r_{C_2}\}, \max\{r_{C_1}, r_{C_2}\}].$$

For graphical illustration of the betweenness relation consider Figure 1 with $m = 6$ parties. A coalition C is between C_1 and C_2 if and only if it lies on a shortest path connecting C_1 and C_2 on the graph. Note that shortest paths need not be unique.

Obviously the betweenness relation respects the subset ordering, i.e. a coalition is between any of its subsets and any of its supersets (see Figure 2).

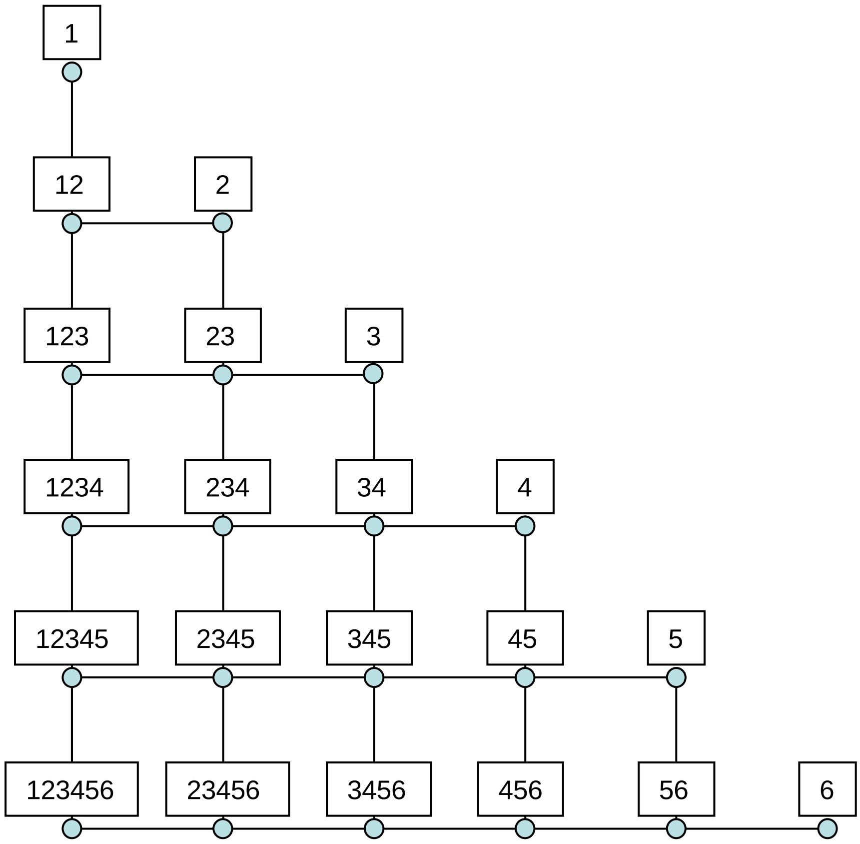

Figure 1: Graphical illustration for six parties

The neighbors of a coalition $(l_C \dots r_C)$ are all coalitions which are connected and which consist either of exactly one party more $((l_C - 1 \dots r_C)$ or $(l_C \dots r_C + 1))$ or one party less $((l_C \dots r_C - 1)$ or $(l_C + 1 \dots r_C))$.
We are now able to define the preference structure over connected coalitions. Let $N = \{1, \dots, n\}$ denote the set of voters[1]. Suppose that every individual i has a unique favorite coalition $C_i^* = (l_{C_i}^* \dots r_{C_i}^*)$ which is called **peak of i**, i.e. for all $C \in \mathcal{C}_<$

$$C \neq C_i^* \Rightarrow C_i^* \succ_i C.$$

The preference relation $(\succsim_i)$ of individual i is **generalized single-peaked** if for all connected coalitions C and $C' \neq C_i^*$ we have that

$$C \text{ is between } C_i^* \text{ and } C' \Rightarrow C \succ_i C'.$$

[1]For simplicity we assume throughout that the number of voters is odd.

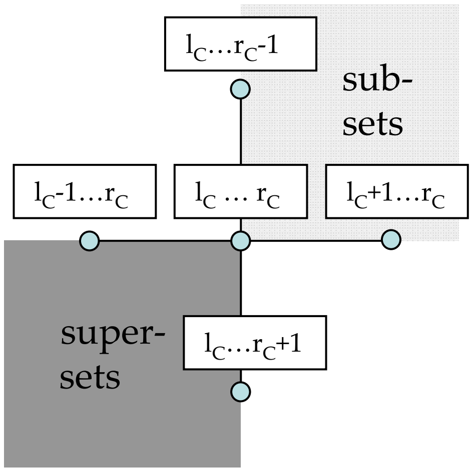

Figure 2: The coalition C with its neighbors

Denote by $\mathcal{S}(\mathcal{C}_<)$ the set of all generalized single-peaked preferences on $\mathcal{C}_<$.

Example: Suppose that $\succ$ is generalized single-peaked with peak $C_i^* = (234)$. Then, for instance, $(234) \succ_i (23) \succ_i (123)$, but there is no restriction on the preference over (23) and (1234).

Remark: The concept of generalized single-peakedness over connected coalitions cannot be reduced to single-peakedness in the classical sense, i.e. there does not exist a linear ordering of the set of connected coalition such that all elements of $\mathcal{S}(\mathcal{C}_<)$ are single-peaked with respect to the given linear ordering in the classical sense.[2]

A social choice function is a mapping

$$F := \begin{cases} \mathcal{S}(\mathcal{C}_<)^n & \longrightarrow \quad \mathcal{C}_< \\ (\succsim_1, \ldots, \succsim_n) & \longmapsto \quad C \end{cases}$$

F is called **strategy-proof** if for all $i \in N$ and $\succsim_i, \succsim_i' \in \mathcal{S}(\mathcal{C}_<)$:

$$F(\succsim_1, \ldots, \succsim_i, \ldots, \succsim_n) \succsim_i F(\succsim_1, \ldots, \succsim_i', \ldots, \succsim_n)$$

[2]To see this, consider three generalized single-peaked preferences $\succsim_i, \succsim_j, \succsim_k$ which have the same peak but pairwise different second-best coalitions. Evidently, it is not possible to arrange all three second-best coalitions as direct neighbors of the peak in one dimension.

3 Results

Consider the social choice function

$$F(\succsim_1, \ldots, \succsim_n) = (med\{l_{C_1^*}, \ldots, l_{C_n^*}\} \ldots med\{r_{C_1^*}, \ldots, r_{C_n^*}\}),$$

where med denotes the median-operator, i.e. $medX$ satisfies
$\#\{x \in X : x \leq medX\} = \#\{x \in X : x \geq medX\}$.

Theorem 1.
$F(\cdot)$ is strategy-proof and anonymous.

Proof:
F is anonymous:
The median operator is anonymous, hence F is anonymous as well.

F is strategy-proof:
Let $F(\cdot) = C^* = (l^* \ldots r^*)$ be the social choice.
Suppose that F is not strategy-proof. Then, there exists a misrepresentation $\succsim_i' \in \mathcal{S}(C_<)$ such that $F(\succsim_1, \ldots, \succsim_i', \ldots, \succsim_n) \succ_i C^*$ for an individual i with peak $C_i^* = (l_{C_i}^* \ldots r_{C_i}^*)$. As F depends only on the peak profile it follows that $F(\succsim_1, \ldots, \succsim_i', \ldots, \succsim_n) = C^*$ for all $\succsim_i'$ with peak C_i^*. So the peak of $\succsim_i'$ has to be different from C_i^*. Let $C' = (l' \ldots r')$ be this misrepresented peak and $F(\succsim_1, \cdots \succsim_i', \cdots \succsim_n) = C_M' = (l_M' \ldots r_M')$ the resulting (manipulated) social choice.

For the relative position of the leftmost elements $l^*, l_{C_i}^*, l'$ there are the following three possible cases:

Case 1) l' is between l^* and $l_{C_i}^*$ $\Rightarrow$ the median of the leftmost elements does not change.

Case 2) $l_{C_i}^*$ is between l' and l^* $\Rightarrow$ the median of the leftmost elements does not change.

Case 3) l^* is between $l_{C_i}^*$ and l' $\Rightarrow$ the median of the leftmost elements is between l^* and l'. This implies that l^* is between $l_{C_i}^*$ and l_M'.

Notice that since betweenness is always understood in the weak sense we have that l^* is between $l_{C_i}^*$ and l_M' also if the median of the leftmost elements does not change, i.e. if $l^* = l_M'$.

Analogously, one easily shows that r^* is between $r^*_{C_i}$ and r'_M. This implies that C^* is between C^*_i and C'_M, hence by generalized single-peakedness, $C^* \succsim_i C'_M$. Thus, i has no incentive to misrepresent. $\square$

Example: There are $m = 6$ parties and $n = 5$ individuals with peaks on the coalitions $C^*_1 = (2), C^*_2 = (123), C^*_3 = (34), C^*_4 = (45), C^*_5 = (23456)$ respectively. The median of the leftmost parties is $med\{2, 1, 3, 4, 2\} = 2$ and the median of the rightmost parties is $med\{2, 3, 4, 5, 6\} = 4$. Therefore $F(\succsim_1, \ldots, \succsim_n) = (234)$.

Remark: It follows from the analysis of Nehring and Puppe [5] that the social choice function $F(\cdot)$ given above is the only anonymous and strategy-proof voting rule on $\mathcal{S}(\mathcal{C}_<)^n$ that is *neutral* (in an appropriate sense). As shown in Nehring and Puppe [4] the property of neutrality is closely related to efficiency, to which we turn now.

Proposition 2.
$F(\cdot)$ is efficient, i.e. for all $(\succsim_1, \ldots, \succsim_n)$ there exists no $C \in \mathcal{C}_<$ such that $C \succsim_i F(\succsim_1, \ldots, \succsim_n)$ with at least one strict preference.

Proof: Consider a situation where $F(\succsim_1, \ldots, \succsim_n) = C^* = (l^* \ldots r^*)$ and let $O_1 := \{C \in \mathcal{C}_< | l_C \geq l^* \text{ and } r_C \leq r^*\}$, $O_2 := \{C \in \mathcal{C}_< | l_C \leq l^* \text{ and } r_C \leq r^*\}$, $O_3 := \{C \in \mathcal{C}_< | l_C \leq l^* \text{ and } r_C \geq r^*\}$, $O_4 := \{C \in \mathcal{C}_< | l_C \geq l^* \text{ and } r_C \geq r^*\}$ (see Figure 3).

Step 1: We prove that there exists a peak in every orthant $O_1, \ldots, O_4$ by contradiction.

By symmetry we assume w.l.o.g. that there is no peak in O_1, i.e. for all $i = 1, \ldots n$, $l^*_{C_i} < l^*$ or $r^*_{C_i} > r^*$.

As l^* is the median of the leftmost elements, we have $\#\{i |\ l^*_{C_i} < l^*\} < \frac{n}{2}$ (remember that n is odd). By the same argument, $\#\{i |\ r^*_{C_i} > r^*\} < \frac{n}{2}$.

Summing up both inequalities, we obtain:
$\#\{i |\ l^*_{C_i} < l^* \text{ or } r^*_{C_i} > r^*\} < n$, a contradiction; thus O_1 contains a peak.

Step 2: If there is a peak in every orthant, then C^* is efficient.

Case a) Evidently, C^* is efficient if $C^*_i = C^*$ for some i.

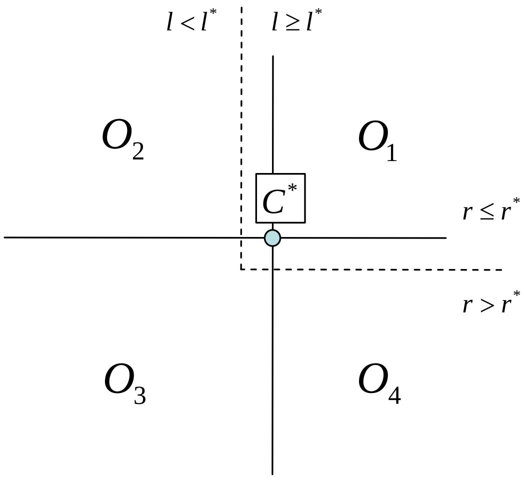

Figure 3: The social choice C^* and the orthants $O_1, \ldots, O_4$

Case b) No individual has his peak on C^*.
Let $\hat{C} \neq C^*$ be an arbitrary connected coalition, say in orthant O_a. By Step 1 we know that there exists a peak $C_j^* \in O_{a+2(mod4)}$ in the opposite orthant. As C^* is between $\hat{C}$ and C_j^*, it follows by generalized single-peakedness that $C^* \succ_j \hat{C}$. Thus, C^* is efficient.

□

References

[1] S. Barberà, H. Sonnenschein, and L. Zhou. Voting by committees. *Econometrica*, pages 595–609, 1991.

[2] A. Gibbard. Manipulation of voting schemes: A general result. *Econometrica*, pages 587–601, 1973.

[3] H. Moulin. On strategy-proofness and single peakedness. *Public Choice*, 35(4):437–455, 1980.

[4] K. Nehring and C. Puppe. Efficient and strategy-proof voting rules: A characterization. *Games and Economic Behavior*, 59(1):132–153, 2007.

[5] K. Nehring and C. Puppe. The structure of strategy-proof social choice-Part I: General characterization and possibility results on median spaces. *Journal of Economic Theory*, 135(1):269–305, 2007.

[6] M.A. Satterthwaite. Strategy-proofness and Arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions. *Journal of Economic Theory*, 10(2):187–217, 1975.