Essential Alternatives and Freedom Rankings[1]

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Abstract. We study the problem of ranking opportunity sets in terms of freedom of choice. The analysis is based on the notion of essential alternatives introduced in Puppe (1996). An alternative in an opportunity set is called essential if by deleting it, the reduced opportunity set offers less freedom than the original set. We provide an axiomatic characterization of the ranking according to which an opportunity set offers more freedom than another opportunity set if its share of essential elements in their union is larger.

1 Introduction

The paper addresses the problem of assessing an individual's freedom as reflected in his opportunity set.[2] Our analysis is based on the notion of essential alternatives as introduced in Puppe (1996). An alternative in an opportunity set is called *essential* (in that set) if by deleting it, the reduced set of opportunities offers an individual less freedom than the original opportunity set.[3] Of course, this raises the question of when the deletion of an alternative reduces freedom, i.e. when is an alternative essential in a given set of opportunities? This problem of determining the informational basis for the assessment of freedom has indeed crystallized as one of the central issues in the literature. In an earlier paper, Pattanaik and Xu (1990) introduced a set of axioms which characterized the ranking of opportunity sets according to their cardinalities. Subsequently, many writers have argued that the informational basis for the evaluation of opportunity sets in terms of freedom should be enlarged to include information about preference over alternatives. Some of these writers have emphasized the role of the agent's own preferences (see e.g. Sen 1991) whereas others have argued in favor of considering *sets* of preference orderings over alternatives (see e.g. Jones and Sugden 1982, Foster 1992, Pattanaik and Xu 1998, Nehring and Puppe 1999). We refer to the latter approach as the multi-preference approach. In this paper, we show that the notion of essentiality can be given a natural interpretation in terms of multiple preferences, and we investigate the precise relation between essentiality and multi-preference representations.

By construction, the essentiality information is contained in the restriction of the freedom ranking to the comparisons of a set with all of its subsets containing exactly one alternative less. In this paper, we explore the implications of various internal consistency conditions imposed on this notion of essentiality. For instance, we will require that if an alternative x is essential in an opportunity set A then it must be essential in any subset of A that contains x (Property α, sometimes also called "contraction consistency"). Our main novel contribution is an axiomatic characterization of the following rule of comparing opportunity sets: A is ranked above B if and only if A contains more alternatives that are essential in $A \cup B$ than B. In the special

[2]For a recent overview of the literature on freedom of choice, see Dowding and van Hees (2009).

[3]The connection to the notion of "eligibility" used by Jones and Sugden (1982) and others is discussed in van Hees (2008).

case in which *every* alternative is essential in *every* set that contains it this rule reduces to the cardinality ranking characterized by Pattanaik and Xu (1990).

By restricting the informational basis of set comparisons to the endogenously defined notion of essentiality in the way just described our approach entails clear limitations. For instance, our ranking neglects both, information on the *(dis)similarity* of alternatives and *cardinal* preference information.[4] Moreover, we show that our ranking of opportunity sets according to their respective share of essential elements can be transitive only under very restrictive circumstances. This gives our results an "impossibility flavor" suggesting that the informational basis for ranking opportunity sets in terms of freedom has to be expanded. In a closely related contribution, van Hees (2008) discusses ways to do this.

The remainder of this paper is organized as follows. In the next section, Section 2, we lay down our notation, introduce our axioms and discuss several examples. We also examine the relation between the notion of essentiality and representations in terms of multiple preferences. Section 3 provides our main characterization result. Section 4 analyzes the transitive case, and Section 5 concludes the paper.

2 Axioms and Examples

2.1 Basic Conditions

Let X be the universal set of alternatives, assumed to be non empty and finite. Let Z be the set of all non-empty subsets of X, i.e. $Z = 2^X - \{\emptyset\}$. The elements of Z are the alternative feasible sets with which the agent may be faced and will be referred to as *opportunity sets*. Let $\succeq$ be a binary relation defined over Z. For all $A, B \in Z$, $A \succeq B$ will be interpreted as "A offers at least as much freedom as B". For all $A, B \in Z$, [$A \succ B$ iff $A \succeq B$ and not($B \succeq A$)] and [$A \sim B$ iff $A \succeq B$ and $B \succeq A$].

The first condition for a ranking of opportunity sets in terms of freedom of choice is the following.

[4] See Bervoets and Gravel (2007) and Pattanaik and Xu (2000, 2008) for models with ordinal dissimilarity information, and Nehring and Puppe (2010a,b) for a model with cardinal dissimilarity and preference information.

Axiom M (Monotonicity) For all $A, B \in Z, B \subseteq A \Rightarrow A \succeq B$.

Thus, expanding an agent's opportunities can never reduce the freedom to choose. This seems to be an uncontroversial condition and we expect that any sensible freedom-ranking should satisfy this condition.

The next condition, due to Puppe (1996), requires a minimal form of strict monotonicity of the freedom ranking.

Axiom F (Freedom of Choice) For all $A \in Z$, there exists $x \in A$ such that $A \succ A - \{x\}$.

For notational convenience, in Axiom F, we have extended the binary relation $\succeq$ over Z to $Z \cup \{\emptyset\}$ by defining $A \succ \emptyset$ for all $A \in Z$. The intuition behind Axiom F may be explained in terms of the notion of *availability*. Indeed, what F requires is that in every opportunity set there should exist at least one alternative such that its availability contributes to the freedom associated with that set. Viewed in this way, Axiom F seems to be a rather mild condition (for a more detailed discussion of Axiom F, see Puppe 1995, 1996). For any non-empty opportunity set A, define its subset of "essential alternatives" $E(A)$ as follows:

$$E(A) := \{x \in A : A \succ A - \{x\}\}.$$

Obviously, $E(A) \subseteq A$ for all $A \in Z$. Axiom F implies $E(A) \neq \emptyset$ for all $A \in Z$. Thus, given Axiom F, E can be thought of as a mapping $E : Z \to Z$ with $E(A) \subseteq A$. An element x is called *essential* in A iff $x \in E(A)$. Otherwise, x is *non-essential* in A. Notice that since $E : Z \to Z$ is implicitly defined by the ranking $\succeq$ the notion of "essentiality" is endogenously defined by the individual's freedom ranking. In order to give the abstract notion of essentiality a content consider the following examples. The first example is the multi-preference approach of Jones and Sugden (1982), Foster (1992), Pattanaik and Xu (1998), and Nehring and Puppe (1999).

Example 2.1 Let $\mathcal{R} = \{R_1, ..., R_n\}$ be a set of preference orderings (transitive and complete binary relations) on X, where each R_i corresponds to a different way of evaluating the alternatives in X. Now suppose that each $R_i \in \mathcal{R}$ plays a certain role in the agent's assessment of the alternatives. For instance, the set $\mathcal{R}$ could represent different points of view from which the agent may evaluate the alternatives. In a sense, one may thus think of the R_i's as corresponding to an agent's different *selves*. Alternatively, the set $\mathcal{R}$ could represent the set of preference orderings of all individuals in the society to which the agent belongs. More generally, $\mathcal{R}$ could be the set of all preferences which a *reasonable person* may conceivably have. The latter interpretation is the one favored by Pattanaik and Xu (1998) who also provide further possible ways of interpreting the multi-preference approach.

Given such a *reference set* $\mathcal{R}$ of preference orderings one might argue that for each $A \in Z$, an alternative x is essential in A, i.e. A offers strictly more freedom than $A - \{x\}$, iff x is a best alternative in A with respect to *some* $R_i \in \mathcal{R}$. Hence,

$$E(A) = \bigcup_{i \in \{1,\dots,n\}} \max_{R_i} A. \tag{2.1}$$

Here, $\max_R A$ denotes the set of best elements of A with respect to the weak order R. Obviously, in this case the set of essential alternatives is never empty so that Axiom F is satisfied.

One might object that optimality of x in A is not enough for x to be considered essential in A. For instance, if x and y are indifferent with respect to *all* orderings in $\mathcal{R}$, and optimal with respect to some ordering, one may argue that A, $A - \{x\}$, and $A - \{y\}$ are all *equivalent* in terms of entailed freedom. This problem can be avoided by considering sets of *linear* orderings, i.e. antisymmetric relations, only. Note that from the perspective of the present paper this would entail no loss of generality. Indeed, it can be shown that given a correspondence $E : Z \to Z$, the existence of a set $\mathcal{R}$ of weak orderings satisfying (2.1) is equivalent to the existence of a set $\mathcal{R}'$ of linear orderings satisfying (2.1). This is because, for any weak ordering R over X, we can construct a set of linear orderings $\{P_1, \cdots, P_q\}$ such that $P_i \subset R$ for $i = 1, ..., q$ and $\max_R A = \cup_{i \in \{1,\dots,q\}} \max_{P_i} A$ for any $A \subseteq X$.

Example 2.2 Consider an agent who is unable to make up his mind about his preferences over the set of alternatives. However, suppose that the agent can identify a kind of "status quo" point $x_0 \in X$ and may always tell whether a given alternative $y \in X$ is at least as good as, or worse than, x_0. No other comparisons of alternatives are possible. Denote by $G \subseteq X$ the set of all alternatives which are at least as good as x_0, and by $W \subseteq X$ the set of alternatives worse than x_0, so that $W = X - G$. Then for any $A \in Z$ a reasonable specification of the subset $E(A)$ of essential alternatives could be

$$E(A) = \begin{cases} A \cap G & \text{if } A \cap G \neq \emptyset \\ A & \text{if } A \subseteq W \end{cases} .$$

Hence, an alternative $y \in A$ is essential in A if, either y is at least as good as the status quo x_0, or if there are no such alternatives at all in A. Note that this example, although motivated quite differently, may formally be also subsumed under the multi-preference approach. This is easily verified by defining the reference set of preference orderings as

$$\mathcal{R} = \{R \subseteq X \times X : xPy \text{ whenever } x \in G \text{ and } y \in W\},$$

where P denotes the asymmetric part of R.

Example 2.3 In the literature, there is some controversy whether preference information should play a role for the assessment of individual freedom at all. Some writers have substantiated the intuition that the intrinsic value of freedom may be described by the property of *strict* monotonicity in the sense that $A \succ B$ whenever B is a proper subset of A, regardless of the value of the alternatives in $(A - B)$ with respect to any preference ordering (see, e.g. Gravel 1994). This property of strict monotonicity would imply that for all $A \in Z$, $E(A) = A$.

Given the notion of essentiality one may ask whether the mapping $E : Z \to Z$ can be expected to satisfy some *internal consistency* properties. Our intuition is that in our context a very basic consistency requirement is the familiar contraction property known in the literature as property α. In terms of essential alternatives, it states that if x is essential in A then x must also be essential in every subset B of A which contains x (cf. Puppe 1996).

Axiom α (Property α) For all $A, B \in Z$, if $x \in B \subseteq A$ and $x \in E(A)$ then $x \in E(B)$.

To illustrate the intuition behind Axiom α imagine the case where an alternative $x \in B$ is not essential in B. Thus, there is no potential point of view from which x may contribute to the "value" of B. How could x then possibly contribute to the value of an enlarged set A? It seems that expanding an agent's opportunities can only make essential alternatives non-essential in the enlarged set but it cannot transform non-essential alternatives into essential alternatives. Note that all the mappings $E : Z \to Z$ specified in Examples 2.1 – 2.3 satisfy Axiom α. Indeed, property α is a necessary condition for $E : Z \to Z$ to be representable by a reference set of weak orders in the manner of (2.1). On the other hand, property α alone is not sufficient for the existence of the representation (2.1). The following result is due to

Aizerman and Malishevski (1981) (see also Moulin 1985 and Suzumura and Xu 2003). Let $E : Z \to Z$ be a mapping with $E(A) \subseteq A$ for all $A \in Z$. Then there exist a natural number n and a set $\mathcal{R} = \{R_1, ..., R_n\}$ such that (2.1) holds iff E satisfies α and the following property, known in the literature as the *Aizerman Condition*. For all $A, B \in Z$,

$$E(A) \subseteq B \subseteq A \Rightarrow E(B) \subseteq E(A). \tag{2.2}$$

2.2 Further Conditions

The conditions considered so far are *basic* in the sense that they put restrictions only on comparisons of pairs (A, B) such that $B \subseteq A$. Indeed, the mapping $E : Z \to Z$ is completely determined by comparing for each $A \in Z$ and each $x \in A$ the sets A and $A - \{x\}$. In order to further specify a ranking $\succeq$ of opportunity sets one clearly needs additional restrictions. The common idea underlying the formulation of the following axioms is as follows. Suppose that $A, B \in Z$ are such that neither set is contained in the other. In this case knowledge of $E(A)$ and $E(B)$ may not help determining the ranking between A and B. However, one can still try to limit the informational basis for the comparison of A and B to the endogenously defined notion of essentiality by using the information derived from the set $E(A \cup B)$.

First, consider two opportunity sets A and B and the case in which none of the elements in B is essential in $A \cup B$. It seems highly reasonable, in this case, to require that the freedom offered by the opportunity set A be strictly greater than the freedom offered by the opportunity set B. This idea is captured by the following axiom.

Axiom D (Dominance) For all $A, B \in Z, E(A \cup B) \cap B = \emptyset \Rightarrow A \succ B$.

The next condition provides a particularly clear illustration of our goal to limit the informational basis of freedom comparisons to the notion of essentiality. In order to formulate the condition some additional notation is needed. Let $A, B \in Z$ with $\#A = \#B$ and $A \cap B = \emptyset$, and let $f : A \to B$ be a bijection between A and B. For each $x \in X$, let

$$x_f = \begin{cases} f(x) & \text{if } x \in A \\ f^{-1}(x) & \text{if } x \in B \\ x & \text{if } x \in X - (A \cup B) \end{cases},$$

and for each $C \in Z$, let C_f denote the set $\{x_f : x \in C\}$. Intuitively, C_f is the *reflection* of C with respect to f.
Definition Say that $A \in Z$ and $B \in Z$ are *essentially equivalent*, denoted by $A \approx_E B$, if there exists a bijection $f : (A - B) \to (B - A)$ such that, for all $C \subseteq A \cup B$, $[x \in E(C) \Leftrightarrow x_f \in E(C_f)$ for all $x \in C]$.

Axiom SYM (Symmetry) For all $A, B \in Z$, $A \approx_E B \Rightarrow A \sim B$.

Note that $A \approx_E B$ in particular implies $a \in E(A) \Leftrightarrow f(a) \in E(B)$ and $a \in E(A \cup B) \Leftrightarrow f(a) \in E(A \cup B)$. However, in general, essential equivalence of A and B requires a lot more, namely that for each subset C of $A \cup B$ an alternative is essential in C iff the corresponding alternative is essential in the corresponding set C_f. Hence, if $A \approx_E B$ then with respect to essentiality the sets A and B are completely symmetric within the set $A \cup B$. Consequently, if the notion of essentiality is to be the only informational basis for comparing A and B one must have $A \sim B$ as required by Axiom SYM.

Note that as a consequence of SYM one has the following property which is a restricted version of the axiom of Indifference between No-choice Situations proposed by Pattanaik and Xu (1990). For all $x, y \in X$,

$$E(\{x, y\}) = \{x, y\} \Rightarrow \{x\} \sim \{y\}.$$

Hence, instead of requiring all singleton sets offering the same degree of freedom of choice, the condition of Indifference between No-choice situations is now restricted to the case in which both, x and y, are essential in $\{x, y\}$. Indeed, Axiom SYM is perfectly consistent with the assumption that preference information may enter a ranking of opportunity sets *via* the notion of essentiality.

Next, consider a case where $A \succeq B$ and x is an alternative in $X - A$ such that x is not only essential in $\{x\} \cup A$ but is even essential in $\{x\} \cup A \cup B$. More specifically, assume that any element of $\{x\} \cup A \cup B$ is essential in that set, i.e. $\{x\} \cup A \cup B = E(\{x\} \cup A \cup B)$. In such a case, it seems highly reasonable to assume that adding the alternative x to the opportunity set A would strictly increase an individual's freedom in the comparison with B. Formally, we will consider the following condition.

Axiom ED (Expansion Dominance) For all $A, B \in Z$, and all $x \in X - A$, if $\{x\} \cup A \cup B = E(\{x\} \cup A \cup B)$ then $A \succeq B \Rightarrow \{x\} \cup A \succ B$.

Note that, given property α, Axiom ED would be immediately implied by transitivity of $\succeq$.

The last axiom is a composition-type axiom and originates from Sen (1991) (see also Pattanaik and Xu 1998).

Axiom C (Composition) For all $A, B \in Z$ and all $C, D \in 2^X$, if $(A \cup B) \cap (C \cup D) = \emptyset$, and $A \cup B = E(A \cup B \cup C \cup D)$ then $A \succeq B \Leftrightarrow A \cup C \succeq B \cup D$.

In Axiom C, the set $A \cup B$ is exactly the set of essential alternatives in $A \cup B \cup C \cup D$ whereas all alternatives in $C \cup D$ are non-essential. Intuitively, Axiom C thus requires that the ranking between sets should not depend on non-essential alternatives.[5] Note that in Axiom C the sets C and D are allowed to be empty.

3 A Characterization Result

The axioms considered so far uniquely characterize a rule of ranking opportunity sets, as follows.

Proposition 3.1 *Let $\succeq$ be a binary relation on Z satisfying F and α. Then $\succeq$ satisfies D, SYM, ED and C iff for all $A, B \in Z$,*

$$A \succeq B \Leftrightarrow \#[E(A \cup B) \cap A] \geq \#[E(A \cup B) \cap B]. \tag{3.1}$$

Proof: Necessity can be checked easily. We prove only sufficiency. First, we show that for all $A, B \in Z$,

$$\#[E(A \cup B) \cap A] = \#[E(A \cup B) \cap B] \Rightarrow A \sim B. \tag{3.2}$$

Suppose $\#[E(A \cup B) \cap A] = \#[E(A \cup B) \cap B] = t$. Given Axiom F, it can be checked that $t \neq 0$. Let $E(A \cup B) \cap A = \{a_1, ..., a_t\}$ and $E(A \cup B) \cap B = \{b_1, ..., b_t\}$. Clearly, $\{a_1, ..., a_t\} \cup \{b_1, ..., b_t\} = E(A \cup B)$. By Axiom α, $E(C) = C$ for all subsets $C \subseteq E(A \cup B)$. Hence, the sets $\{a_1, ..., a_t\}$ and $\{b_1, ..., b_t\}$ are essentially equivalent. Consequently, by Axiom SYM, $\{a_1, ..., a_t\} \sim \{b_1, ..., b_t\}$. Now apply Axiom C to obtain $A \sim B$. This completes the proof of (3.2).

[5] Innocuous as this may sound, it is this condition that is mainly responsible for the fact that no information about the similarity/diversity of alternatives can enter the ranking of opportunity sets.

Now we prove that, for all $A, B \in Z$,

$$\#[E(A \cup B) \cap A] > \#[E(A \cup B) \cap B] \Rightarrow A \succ B. \tag{3.3}$$

Suppose $\#[E(A \cup B) \cap A] > \#[E(A \cup B) \cap B]$. First, consider the case where $\#[E(A \cup B) \cap B] = 0$, that is, $E(A \cup B) \cap B = \emptyset$. Then Axiom D implies $A \succ B$ immediately.

Next consider the case where $\#[E(A \cup B) \cap A] > \#[E(A \cup B) \cap B] > 0$. Hence, suppose $E(A \cup B) \cap B = \{b_1, ..., b_g\}$ and $E(A \cup B) \cap A = \{a_1, ..., a_g, a_{g+1}, ..., a_{g+h}\}$. By Axioms α, SYM and the argument from above one has $\{a_1, ..., a_g\} \sim \{b_1, ..., b_g\}$. Repeated use of Axiom ED together with Axiom α then implies

$$\{a_1, ..., a_g, a_{g+1}, ..., a_{g+h}\} \succ \{b_1, ..., b_g\}.$$

Finally, from this one obtains $A \succ B$ by a straightforward application of Axiom C. This completes the proof of (3.3) and the proof of sufficiency.

Remark 3.2 It may be noted that all rules of the form (3.1) satisfy Axiom M. Hence, Axiom M is implied by the conjunction of the other conditions stated in Proposition 3.1.

Remark 3.3 Suppose for all $A \in Z$ and any $a \in A, A \succ A - \{a\}$. It can be checked that, in this special case, the binary relation $\succeq$ defined by (3.1) takes the following simple form. For all $A, B \in Z$,

$$A \succeq B \Leftrightarrow \#A \geq \#B. \tag{3.4}$$

This form was originally characterized by Pattanaik and Xu (1990). Intuitively, it corresponds to the case in which, for all $A \in Z$, *every* alternative $a \in A$ is essential in A (cf. Example 2.3).

Remark 3.4 Suppose that for all $A \in X$ there exists a unique $a \in A$ such that $A \succ A - \{a\}$ and $A \sim A - \{a'\}$ for any $a \in A - \{a'\}$, and assume that α is satisfied. Then, in this special case, the binary relation $\succeq$ defined by (3.1) takes the following simple form. There exists a linear ordering R over X such that, for all $A, B \in Z$, $A \succeq B \Leftrightarrow a^* R b^*$, where $\max_R A = \{a^*\}$ and $\max_R B = \{b^*\}$. If R is to be interpreted as a preference relation of an individual, then this rule is the simple indirect-utility rule.

Remark 3.5 In the characterization result, completeness and transitivity of $\succeq$ are not assumed. Indeed, in general, the relation $\succeq$ defined by (3.1) may violate even acyclicity of $\succ$. This can be demonstrated by means of the following example.

Let $X = \{x, y, z, u, v, w\}$. Furthermore, let $E(A) = A$ for all $A \subseteq X$ such that $x \notin A$ and $E(A) = A - \{y, z, w\}$ if $x \in A$. It can be checked that the E defined satisfies property α. In fact, it also satisfies the Aizerman Condition (2.2) so that there exists a representation of E by a reference set of weak orders in the manner of (2.1). Nevertheless, according to the binary relation defined by (3.1), we have: $\{y, z, w\} \succ \{u, v\}, \{u, v\} \succ \{x\}$, and $\{x\} \succ \{y, z, w\}$. Therefore, $\succ$ violates acyclicity.

Remark 3.6 The rule (3.1) for comparing opportunity sets is different from the rules considered in Pattanaik and Xu (1998). Let $\mathcal{R}$ be a reference set of preference orderings. One of the rules characterized by Pattanaik and Xu (1998) is as follows. For all $A, B \in Z$,

$$A \succeq B \Leftrightarrow \#(max(A) - A^B) \geq \#(max(B) - B^A), \tag{3.5}$$

where for $G, H \in Z$, $max(G) := \{x \in G : \exists R_i \in \mathcal{R} \text{ such that } x \in \max_{R_i} G\}$ and $G^H := \{y \in G : y \notin max(H \cup \{y\})\}$. In general, the rule given by (3.1) is not equivalent to the rule given above in (3.5). Even in the case in which essentiality is defined by (2.1), the two rules yield different rankings of opportunity sets. This can be shown by the following example. Consider $A = \{x, y, z\}$ and $B = \{a, b\}$, and $\mathcal{R} = \{R_1, \cdots, R_4\}$, where $xP_1aP_1bP_1yP_1z$, $xP_2yP_2zP_2bP_2a$, $aP_3zP_3yP_3bP_3x$, $bP_4yP_4xP_4zP_4a$. Then, $E(A \cup B) = \{x, a, b\}$, $max(A) = \{x, y, z\}$, $max(B) = \{a, b\}$, and $A^B = B^A = \emptyset$. Consequently, according to (3.1), we have $B \succ A$, while (3.5) gives $A \succ B$.

Remark 3.7 (Independence of the Axioms) We now show that the axioms used in Proposition 3.1 are logically independent. For each of the Axioms D, SYM, ED and C we provide an example that violates it but satisfies all other axioms stated in the proposition.

(i) Fix $x_0 \in X$ and define a binary relation $\succeq$ as follows. If $x_0 \in A$, then $A \sim B$ for all $B \subseteq A$ with $x_0 \in B$, and $A \succ B$ for all subsets $B \subseteq A - \{x_0\}$; if $x_0 \notin A$, then $A \succ B$ for all proper subsets B of A. In particular, $\succeq$ induces $E(A) = \{x_0\}$ if $x_0 \in A$ and $E(A) = A$ if $x_0 \notin A$, thus satisfying Axioms F and α. If neither A is a subset of B, nor B a subset of A, let

$$A \succeq B \Leftrightarrow \#E(A) \geq \#E(B).$$

It is easily verified that $\succeq$ satisfies SYM, ED and C. But if y and z are distinct alternatives different from x_0, we obtain $\{y, z\} \succ \{x_0\}$ in violation of Axiom D.

(ii) Let $v : X \to \{1, 2, ..., \#X\}$ be one-to-one, and define a binary relation $\succeq$ on X by

$$A \succeq B \Leftrightarrow \sum_{x \in A} v(x) \geq \sum_{x \in B} v(x).$$

Evidently, one has $E(A) = A$ for all A, thus $\succeq$ satisfies Axioms F and α. Moreover, $\succeq$ is easily seen to violate SYM while satisfying D, ED and C.

(iii) Define a binary relation $\succeq$ as follows: $A \succ B$ if B is a proper subset of A, and $A \sim B$ otherwise. Evidently, one then has $E(A) = A$ for all A; in particular, $\succeq$ satisfies Axioms F and α. Moreover, it is easily verified that satisfies D, SYM and C, but violates ED.

(iv) Let $X = \{x, y, z\}$, and construct a binary relation $\succeq$ such that the following holds: $E(\{x, y, z\}) = \{x, y\}$, $E(\{x, y\}) = \{x, y\}$, $E(\{x, z\}) = \{x, z\}$, $E(\{y, z\}) = \{y\}$, and $E(\{w\}) = \{w\}$ for all $w \in X$. Then, $\succeq$ satisfies Axioms F and α. If, in addition, $\{x\} \sim \{y\}$ and $\{y, z\} \succ \{x, z\}$, then $\succeq$ violates Axiom C. It is easily verified that $\succeq$ can be extended to a complete binary relation on X satisfying D, SYM and ED.

4 The Transitive Case

In this section we investigate the case in which the freedom ranking $\succeq$ is assumed to be transitive. First, it is shown that under this assumption the characterization result obtained in the previous section can be substantially simplified. Secondly, we show that in the transitive case condition (2.2) in the result of Aizerman and Malishevski may be replaced by a very simple condition of "Independence of Non-essential Alternatives". Finally, we point out that transitivity of the ranking (3.1) considered in the previous section imposes strong restrictions on the structure of the mapping $E : Z \to Z$. The proofs of all results of this section are found in an appendix.

Proposition 4.1 *Let $\succeq$ be a transitive relation on Z satisfying F. Then $\succeq$ satisfies Axioms M, α, SYM and C iff for all $A, B \in Z$,*

$$A \succeq B \Leftrightarrow \#[E(A \cup B) \cap A] \geq \#[E(A \cup B) \cap B].$$

It may be noted that, when $\succeq$ is required to be transitive, ED is an immediate consequence of F and α. To see this, let $A, B \in Z$ and $x \in X - A$ be such that $E(\{x\} \cup A \cup B) = \{x\} \cup A \cup B$ and $A \succeq B$. By F and α, $\{x\} \cup A \succ A$, hence by transitivity $\{x\} \cup A \succ B$.

We also note that in Proposition 4.1 the full strength of Axiom C is not needed. Consider the following two conditions which are both straightforward implications of Axiom C (for the first condition see also Puppe 1996).

Axiom INE (Independence of Non-essential Alternatives) For all $A \in Z$, $A \sim E(A)$.

Axiom EC (Expansion Consistency) For all $A, B \in Z$ and all $x \notin A \cup B$ such that $A \cup B = E(A \cup B \cup \{x\})$, $A \sim B \Rightarrow \{x\} \cup A \sim B$.

Proposition 4.2 *Let $\succeq$ be a transitive relation on Z satisfying F. Then $\succeq$ satisfies Axioms M, α, SYM, INE and EC iff for all $A, B \in Z$,*

$$A \succeq B \Leftrightarrow \#[E(A \cup B) \cap A] \geq \#[E(A \cup B) \cap B].$$

Note that Proposition 4.1 is an immediate corollary of Proposition 4.2.

Remark 4.3 In the transitive case, Axiom INE is equivalent to the Aizerman Condition (2.2). To see this, let $\succeq$ be transitive and suppose that Axiom M holds. First, assume that $E(A) \subseteq B \subseteq A$. If there would exist $y \in E(B)$ such that $y \notin E(A)$ one would obtain $B \succ B - \{y\}$ and by Axiom M, $B - \{y\} \succeq E(A)$. This implies by transitivity of $\succeq$ and by Axiom M, $A \succeq B \succ E(A)$, hence $A \succ E(A)$. But this contradicts Axiom INE. Consequently, under INE one must have $E(B) \subseteq E(A)$ as required by the Aizerman Condition.

Conversely, let $A - E(A) = \{x_1, ..., x_n\}$. In particular, $A \sim A - \{x_1\}$. Now apply the Aizerman Condition (2.2) with $B = A - \{x_1\}$ to obtain $E(A - \{x_1\}) \subseteq E(A)$. In particular, $A - \{x_1\} \sim A - \{x_1, x_2\}$. Hence, by transitivity $A \sim A - \{x_1, x_2\}$. By induction, one can thus show that $A \sim A - \{x_1, ..., x_n\}$, i.e. $A \sim E(A)$ as required by INE. Hence, using Aizerman's and Malishevski's Theorem we have the following corollary.

Corollary 4.4 *Let $\succeq$ be a transitive relation on Z satisfying Axioms F and M. Then the corresponding mapping $E : Z \to Z$ is rationalizable by a reference set $\mathcal{R} = \{R_1, ..., R_n\}$ of weak orders in the sense that for all $A \in Z$,*

$$E(A) = \bigcup_{i \in \{1,...,n\}} \max_{R_i} A$$

iff $\succeq$ *satisfies Axioms* α *and INE.*

The following result gives necessary and sufficient conditions for the ranking (3.1) to be transitive.

Proposition 4.5 *The relation* $\succeq$ *defined by (3.1) is transitive iff* $E : Z \to Z$ *satisfies property* α *and the following property (property* β*, see e.g. Sen 1970). For all* $A, B \in Z$*,*

$$B \subseteq A \text{ and } \{x, y\} \subseteq E(B) \Rightarrow [x \in E(A) \Leftrightarrow y \in E(A)].$$

The result of Proposition 4.5 may seem slightly discouraging. By a well-known result of Sen (1970), it implies that the ranking $\succeq$ defined in (3.1) can be transitive only in the case where the endogenously defined notion of essentiality can be rationalized by means of a *single* weak order R. However, one does not necessarily have to interpret this weak order as the agent's preferences over alternatives. Suppose for example that an agent has a definite linear preference ordering over the alternatives. At the same time he might take into account all preference orderings which a reasonable person could have. If these include *all* possible linear orderings over the alternatives the induced ranking over opportunity sets would be of the form (3.4). Clearly, the mapping $E : Z \to Z$ corresponding to the ranking (3.4) can be rationalized by the universal indifference relation which, however, may be very "far" from the agent's actual preferences.[6] It might also be the case that the agent does in fact *not have* a definite preference ordering over the alternatives but considers different orderings which he might regard as "approximations" to his imprecise judgements. Again, if the set of such approximations is rich enough it could turn out that the agent behaves as if to maximize the number of best elements with respect to *one single* weak preference ordering. Indeed, by Proposition 4.5 this will be the case if and only if his ranking over opportunity sets is transitive.

Hence, if one allows for the possibility that actual preference over alternatives is different from what is "revealed" by the ranking of opportunities, or that there might be no (single) preference at all which could correspond to "revealed preference," then Proposition 4.5 does not harm the

[6]This example also shows that the weak ordering R that rationalizes the endogenously defined notion of essentiality *cannot* be interpreted as the agent's preference over alternatives in case one restricts the multi-preference approach to *linear* preferences as suggested at the end of Example 2.1 above.

multi-preference approach in the transitive case. It only demonstrates that transitivity of the ranking (3.1) requires a special internal structure of the reference set $\mathcal{R}$.

The following result summarizes our above results.

Corollary 4.6 *A binary relation $\succeq$ is transitive and satisfies F,M, α, SYM, INE and EC iff there exists a weak order R on X such that, for all $A, B \in Z$,*

$$A \succeq B \Leftrightarrow \#[\max_R(A \cup B) \cap A] \geq \#[\max_R(A \cup B) \cap B].$$

We conclude this section by providing an alternative characterization of the ranking defined in (3.1). This characterization is based on property γ instead of Axiom C, or its implications. Property γ is the following condition.

Axiom γ (Property γ) For all $A, B \in Z$, $x \in E(A) \cap E(B) \Rightarrow x \in E(A \cup B)$.

Proposition 4.7 *Let $\succeq$ be a transitive relation on Z satisfying F. Then $\succeq$ satisfies Axioms M, SYM, α and γ iff for all $A, B \in Z$,*

$$A \succeq B \Leftrightarrow \#[E(A \cup B) \cap A] \geq \#[E(A \cup B) \cap B].$$

Note that, by Proposition 4.7, we could also replace the conjunction of conditions INE and EC by Property γ in Corollary 4.6.[7]

5 Concluding Remarks

In this paper, we have explored the possibility of formulating a rule for ranking opportunity sets in terms of freedom of choice based on the notion of essentiality. The rule characterized by our axioms seems to have some intuitive appeal and combines features of the approaches of Puppe (1996) and Pattanaik and Xu (1990). Moreover, we have shown that the rule has a natural interpretation in terms of multiple preferences relating our analysis to the approach of Jones and Sugden (1982), Foster (1992), Pattanaik and Xu (1998) and Nehring and Puppe (1999).

[7] We do not know to which extent the conditions used in each of the results of this section are logically independent. Indeed, under transitivity there is a subtle interplay between conditions imposed directly on $\succeq$ and consistency conditions imposed on the induced mapping $E : Z \to Z$. A more detailed analysis of this issue is left to future work.

It turned out that the rule considered in this paper can be transitive only in the case where $E(A) = \max_R A$ for a *transitive* relation R on the set X. Indeed, this is an inevitable consequence of the conditions considered in this paper. However, as we have argued one does not necessarily have to interpret the revealed ordering R as the decision maker's actual preferences. In this case, transitivity of the ranking of opportunity sets requires a special internal structure of the reference set $\mathcal{R}$ of weak orders.

Appendix

The proof of Proposition 4.2 is given via the following two lemmata.

Lemma A.1 *Let $\succeq$ be transitive. If $\succeq$ satisfies Axioms M, F, α, SYM, INE and EC then for all $A \in Z$,*

$$E(A) = \max_R A,$$

where R is defined by $xRy :\Leftrightarrow \{x\} \succeq \{y\}$.

Proof: First we show that for all $x, y \in X$,

$$x \in E(\{x, y\}) \Leftrightarrow \{x\} \succeq \{y\}. \tag{A.1}$$

Suppose that $x \in E(\{x, y\})$. Then either $E(\{x, y\}) = \{x\}$, or $E(\{x, y\}) = \{x, y\}$. In the first case, Axioms M and F imply $\{x\} \sim \{x, y\} \succ \{y\}$, hence by transitivity of $\succeq$, $\{x\} \succ \{y\}$. In the second case, one obtains $\{x\} \sim \{y\}$ by Axiom SYM. Conversely, $\{x\} \succeq \{y\}$ is only possible if $x \in E(\{x, y\})$ since $E(\{x, y\}) = \{y\}$ would imply $\{y\} \succ \{x\}$ by the argument from above. This proves (A.1).

Now we show $E(A) \subseteq \max_R A$ for all $A \in Z$. Let $x \in E(A)$. By Axiom α, $x \in E(\{x, y\})$ for all $y \in A$. By (A.1), this implies $\{x\} \succeq \{y\}$ for all $y \in A$, i.e. $x \in \max_R A$.

It remains to show that $\max_R A \subseteq E(A)$ for all $A \in Z$. This is proved by induction over $n = \#A$. Obviously, this is true for $n = 1$. Also note that for $n = 2$ the claim immediately follows from (A.1). Hence, suppose the claim is true for all B with $\#B = n \geq 2$ and consider a set $A = \{x_1, ..., x_{n+1}\}$. We distinguish two cases.

Case (i) $\max_R A$ consists of exactly one element, say $\max_R A = \{x_1\}$. Then by Axiom F and the fact that $E(A) \subseteq \max_R A$ one must have $E(A) = \{x_1\}$.

Case (ii) $\max_R A$ contains at least two elements, say $\{x_1, x_2\} \subseteq \max_R A$. First consider the case where $A \neq \max_R A$, say $x_{n+1} \notin \max_R A$. By the first part of this proof $x_{n+1} \notin E(A)$. Hence, $A \sim A - \{x_{n+1}\}$. Consequently, by the induction hypothesis one obtains for each $x_i \in \max_R A$,

$$A \sim A - \{x_{n+1}\} \succ A - \{x_i, x_{n+1}\} \sim A - \{x_i\},$$

and therefore $x_i \in E(A)$.

Thus, in the remainder we may assume that $A = \max_R A$. First assume that there are two different elements of A which are not contained in $E(A)$. Without loss of generality, assume $\{x_1, x_2\} \cap E(A) = \emptyset$. By Axioms M and INE, one has $A \sim A - \{x_1, x_2\}$. However, this leads to a contradiction since by induction hypothesis $A - \{x_1\} \succ A - \{x_1, x_2\}$. Hence, there can exist at most one element of $\max_R A$ which is not in $E(A)$. Without loss of generality, assume $x_1 \notin E(A)$. By the induction hypothesis, the sets $\{x_2, x_4, ..., x_{n+1}\}$ and $\{x_3, x_4, ..., x_{n+1}\}$ are essentially equivalent, hence by Axiom SYM

$$\{x_2, x_4, ..., x_{n+1}\} \sim \{x_3, x_4, ..., x_{n+1}\}$$

(in the case $n + 1 = 3$ this should, of course, be read as $\{x_2\} \sim \{x_3\}$). Now Axiom EC implies $\{x_1, x_2, x_4, ..., x_{n+1}\} \sim \{x_3, x_4, ..., x_{n+1}\}$. However, this again contradicts the induction hypothesis. Hence, one must have $\max_R A \subseteq E(A)$. This completes the proof of Lemma A.1.

Lemma A.2 *Let $\succeq$ be transitive. Then Axioms M, F and INE imply Axiom D.*

Proof: Suppose A and B are such that $E(A \cup B) \cap B = \emptyset$. By INE, $A \cup B \sim E(A \cup B)$. By assumption $E(A \cup B) \subseteq A$. Hence, by Axiom M and transitivity, $A \sim A \cup B$. By Axiom F, let $x \in E(A \cup B)$. Since x must be in $A - B$ one obtains

$$A \sim A \cup B \succ (A \cup B) - \{x\} \succeq B,$$

hence by transitivity, $A \succ B$, which completes the proof of Lemma A.2.

Proof of Proposition 4.2: Necessity of Axioms M, F, SYM, INE and EC is easily verified. Necessity of α follows from Proposition 4.5 which is proved below. Thus, we only need to prove sufficiency of the conditions for the representation (3.1). This is shown by induction over $n = \max\{\#A, \#B\}$. It is easily verified that for $n = 1$ Axioms M, F and SYM imply the desired representation. Hence, suppose that (3.1) holds for all A, B such that $\#A, \#B \leq n$. We distinguish three cases.

Case (i) $\#A = n + 1$ and $\#B \leq n$. First, suppose that $E(A) = A$ so that every alternative in A is essential. By Lemma A.1, $E(C) = \max_R C$ for all $C \in Z$ if R is defined by $xRy :\Leftrightarrow \{x\} \succeq \{y\}$. By assumption, R is transitive. Consequently, if one element of A is essential in $A \cup B$ then *every* element of A must be essential in $A \cup B$. That is, $\#[E(A \cup B) \cap A]$ is either $n+1$ or 0. If

$\#[E(A\cup B)\cap A] = n+1$ then for any $x \in A$, $A \succ A - \{x\}$. By the induction hypothesis one has $A - \{x\} \succeq B$ since $\#[E((A-\{x\})\cup B)\cap(A-\{x\})] = n$. Therefore, by transitivity $A \succ B$. If, on the other hand, $\#[E(A\cup B)\cap A] = 0$ then $E(A\cup B) \subseteq B$, which implies $B \succ A$ by Lemma A.2.

Next, suppose that $E(A) \neq A$, say $x \in A - E(A)$, that is, $x \notin \max_R A$. It is easily verified that in this case,

$$\begin{array}{lcl} \#[E((A-\{x\})\cup B)\cap(A-\{x\})] & = & \#[E(A\cup B)\cap A] \\ \#[E((A-\{x\})\cup B)\cap B] & = & \#[E(A\cup B)\cap B] \end{array}. \tag{A.2}$$

Also, since $A \sim A - \{x\}$ one has $A \succeq B \Leftrightarrow A - \{x\} \succeq B$, so the claim follows from the induction hypothesis.

Case (ii) $\#A \leq n$ and $\#B = n+1$. This case is completely symmetric to case (i) with the roles of A and B interchanged.

Case (iii) $\#A = n+1$ and $\#B = n+1$. If there exists $x \in A - E(A)$ one has $A \sim A - \{x\}$ and by (A.2) the claim then follows from Case (ii). Similarly, if there exists $y \in B - E(B)$. Hence, we may assume without loss of generality that $A = E(A) = \max_R A$ and $B = E(B) = \max_R B$. Then, either xIy for all $x \in A$ and all $y \in B$, or xPy for all $x \in A$ and all $y \in B$, or yPx for all $x \in A$ and all $y \in B$ (where I and P are the symmetric and asymmetric part of R, respectively). In the first case, one obtains $A \sim B$ by Axiom SYM. In the second case, $A \succ B$ by Lemma A.2, and in the third case $B \succ A$ again by Lemma A.2. This completes the proof of Proposition 4.2

Proof of Proposition 4.5: Sufficiency of α and β for transitivity of the ranking $\succeq$ defined in (3.1) is easily verified. Necessity is proved by verifying that for the ranking $\succeq$ defined in (3.1) one has for all $A \in Z$,

$$E(A) = \max_R A, \tag{A.3}$$

where $xRy :\Leftrightarrow \{x\} \succeq \{y\} \Leftrightarrow x \in E(\{x,y\})$. Transitivity of the ranking (3.1) of course implies transitivity of the induced relation R. By a well-known result, properties α and β are together necessary and sufficient for the existence of a transitive relation R such that (A.3) holds (see Sen 1970).

We now prove (A.3) by induction over $n = \#A$. Obviously, (A.3) holds for $n = 1$. Since the ranking (3.1) satisfies the equivalence (A.1), the claim is also true for $n = 2$. Thus, suppose that (A.3) holds for all B with $\#B = n \geq 2$ and consider a set A with $n+1$ elements. First, we show $\max_R A \subseteq E(A)$ by a contradiction argument. Thus, suppose that $x \in \max_R A$ but $x \notin E(A)$.

Then there must exist $y \in E(A)$ with $y \neq x$. We distinguish two cases.
Case (i) $E(A) = \{y\}$. In this case, one obtains for the ranking $\succeq$ defined by (3.1), $\{y\} \sim A \sim \{x, y\}$. Hence, by transitivity $\{y\} \sim \{x, y\}$. But this implies $\{y\} \succ \{x\}$, which is a contradiction.
Case (ii) $\{y, z\} \subseteq E(A)$ for some $z \in A$ with $z \neq y$. In this case, one obtains $A - \{y\} \sim A - \{z\}$ and $A - \{z\} \sim A - \{x, y\}$. Hence, by transitivity $A - \{y\} \sim A - \{x, y\}$ which implies $x \notin E(A - \{y\})$. But this contradicts our induction hypothesis.

Next, we show $E(A) \subseteq \max_R A$, again by contradiction. Thus, suppose $y \in E(A)$ but $y \notin \max_R A$. Let $x \in \max_R A$. By the first part of this proof one must have $x \in E(A)$ and therefore, $A - \{y\} \sim A - \{x\}$. Again, we distinguish two cases.
Case (i) $\max_R A = \{x\}$. In this case, $E(A - \{y\}) = \{x\}$ by the induction hypothesis. Hence for the ranking (3.1) one obtains $\{x\} \sim A - \{y\}$ and by transitivity also $\{x\} \sim A - \{x\}$. Let $z \in A$ be different from x and y. Again, by the induction hypothesis, $\{x\} \sim A - \{z\}$. Hence, by transitivity, $A - \{y\} \sim A - \{z\}$. But this implies $z \in E(A)$ which contradicts $\{x\} \sim A - \{x\}$.
Case (ii) $\max_R A \neq \{x\}$. In this case, one has $A - \{x\} \sim A - \{x, y\}$ by the induction hypothesis. Hence, by transitivity $A - \{y\} \sim A - \{x, y\}$ which would imply $x \notin E(A - \{y\})$. But this contradicts the induction hypothesis. Thus, the proof of Proposition 4.5 is complete.

Proof of Corollary 4.6 Suppose that $\succeq$ satisfies the stated conditions. By Proposition 4.2, $\succeq$ has the form (3.1) for some mapping $E : Z \to Z$ such that $E(A) \subseteq A$ for all $A \in Z$. By transitivity and Proposition 4.5, the mapping E satisfies both α and β, hence it can be rationalized by a weak order R on X.

Conversely, if E can be rationalized by a weak order, the corresponding ranking defined by (3.1) is transitive by Proposition 4.5. All other stated conditions on $\succeq$ follow as in Proposition 4.2.

Proof of Proposition 4.7: Necessity of γ follows immediately from Proposition 4.5 once it is noted that property β implies property γ. Sufficiency of the conditions can be verified along the following lines. First, it can be shown that in the transitive case, property γ implies Axiom INE (for a proof see Lemma 2 in Puppe 1996). Consequently, by Lemma A.2 one can deduce Axiom D. An inspection of the proof of Proposition 4.2 shows that it suffices to verify that Axioms M, F, SYM, α and γ imply

$$E(A) = \max_R A \text{ for all } A \in Z, \tag{A.4}$$

where R is defined by $xRy \Leftrightarrow \{x\} \succeq \{y\}$. By properties α and γ, in order to verify (A.4) it suffices to show that the R defined is the base relation of E. However, this is a consequence of Axioms M, F and SYM as has been shown in the proof of Lemma A.1.

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