Efficient and Strategy-Proof Voting Rules: A Characterization

Klaus Nehring

Department of Economics, University of California at Davis

Davis, CA 95616, U.S.A.

kdnehring@ucdavis.edu

and

Clemens Puppe

Department of Economics, University of Karlsruhe

D – 76128 Karlsruhe, Germany

puppe@wior.uni-karlsruhe.de

Abstract *The paper provides a characterization of all efficient and strategy-proof voting mechanisms on a large class of preference domains, the class of all generalized single-peaked domains. It is shown that a strategy-proof voting mechanism on such a domain is efficient if and only if it satisfies a weak neutrality condition and is either almost dictatorial, or defined on a median space of dimension less than or equal to two. In more than two dimensions, weakly neutral voting mechanisms are still "locally" efficient.*

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1 Introduction

A main conclusion from the literature on dominant strategy implementation is that strategy-proofness and efficiency are compatible in a significant way only on very special preference domains. For instance, in quasi-linear environments ex-post efficiency is, in general, incompatible with strategy-proofness (Green and Laffont (1979)). In a voting context, it has been shown that even on those domains that enable non-dictatorial and strategy-proof social choice rules, often none of these guarantee efficiency. For instance, Kim and Roush (1984) provide an impossibility result, later generalized by Peters, van der Stel and Storcken (1992), for Euclidean preferences in at least three-dimensional spaces. Similarly, Border and Jordan (1983) show that among all social choice functions on the domain of "separable quadratic" preferences in a Euclidean space of more than two dimensions only the dictatorial ones are strategy-proof and efficient.[1] These and other negative results in the literature seem to have led to the general view that there is possibly only one "natural" preference domain on which there exists a significant range of non-dictatorial social choice functions that are both strategy-proof and efficient: the set of all single-peaked preferences on a line (Moulin (1980)), and more generally, the set of of all single-peaked preferences on a tree (Demange (1982)).

The goal of this paper is to exactly determine the set of all strategy-proof and efficient social choice functions on a large class of preference domains, the class of all generalized single-peaked domains introduced in Nehring and Puppe (2002) and further analyzed in Nehring and Puppe (2004), henceforth NP (2004). This class includes the domain of all single-peaked preferences on a line, the domain of all separable preferences on the hypercube (Barberá, Sonnenschein and Zhou (1991)), the domain of all "multi-dimensionally" single-peaked preferences on the Cartesian product of lines (Barberá, Gul and Stacchetti (1993)), the unrestricted preference domain of the Gibbard-Satterthwaite theorem, and many others. On generalized single-peaked domains, strategy-proof social choice can be described in a unified manner as "voting by issues." We prove that, within this class of preference domains, strategy-proof social choice is efficient if and only if it satisfies a weak neutrality condition and is either almost dictatorial, or defined on an at most two-dimensional median space.[2]

Our characterization is based on three insights in particular. First, efficiency requires a "weak" neutrality condition on the structure of winning coalitions. In one-dimensional spaces, this condition is vacuously satisfied; in multi-dimensional spaces such as the product of lines, by contrast, weak neutrality requires that all issues are decided using the *same* family of winning coalitions. Secondly, voting by issues can be weakly neutral only if it is "almost-dictatorial," or defined on a median space; examples of median spaces are lines, trees, products of these, and appropriate subspaces of such products. Finally, weakly neutral rules are indeed efficient if and only if the underlying (median) space is at most two-dimensional. A key step in our characterization is the definition of an appropriate notion of dimension which is based on taking trees rather than lines to be the fundamental one-dimensional structure. We present a natural location example of a median space that can be embedded in the Cartesian product of two trees, but not of two lines. While efficiency is unattainable in more than two

[1] In fact, Border and Jordan (1983) claim this also for two dimensions in which case the statement is, however, not correct; see Section 3 below for further discussion.

[2] A space is a median space if any triple of elements admits a fourth element (their "median") that is "between" any pair of the triple, see Section 2 below.

dimensions, it is shown that weakly neutral rules still enjoy superior partial efficiency properties.

Our main result has the corollary that there is one and only one *anonymous* and efficient strategy-proof social choice rule on the Cartesian product of two lines, namely to choose the coordinate-wise median of the voters' peaks ("issue-by-issue majority voting"). This generalizes corresponding results in Kim and Roush (1984) and Peters, van der Stel and Storcken (1992) who consider the (much smaller) domain of preferences that can be represented by the negative Euclidean distance to some ideal point. Peters, van der Stel and Storcken (1993) generalize the analysis to non-Euclidean metrics, but still retain the restrictive assumption that the relevant metric is common to all agents. On the necessity side of our result, all previous contributions have assumed that the underlying space is a product of lines. Besides the already mentioned ones, these include Barberá, Sonnenschein and Zhou (1991) who assume the underlying space to be a hypercube (product of two-element sets). Our analysis is far broader by considering domains of preferences that are single-peaked with respect to very general betweenness geometries and by deriving the necessity of a median space structure for efficiency in the absence of almost-dictatorship.

The body of the paper is organized as follows. In the following Section 2, we briefly state the basic definitions and results of NP (2004). In particular, we review the characterization of strategy-proof social choice on generalized single-peaked domains as voting by issues satisfying the Intersection Property. Section 3 contains our main result, the characterization of all strategy-proof and efficient social choice rules on these domains. To illustrate the scope of our analysis, we discuss examples dealing with location problems, public goods and judgement aggregation, respectively. In Section 4, we show that voting by issues always satisfies a weak efficiency condition which admits a natural interpretation as renegotiation proofness. We also show that on median spaces of any dimension, the crucial weak neutrality condition is equivalent to an appropriate notion of "local" efficiency. Section 5 concludes, and all proofs are collected in an appendix.

2 Background: Strategy-Proof Social Choice on Generalized Single-Peaked Domains

In this section, we briefly summarize the basic concepts and results from Nehring and Puppe (2002) and NP (2004) needed for the later analysis.

Property space A *property space* is a pair $(X, \mathcal{H})$, where X is a finite universe of social states or social alternatives, and $\mathcal{H}$ is a collection of subsets of X satisfying
H1 $\emptyset \notin \mathcal{H}$,
H2 $H \in \mathcal{H} \Rightarrow H^c \in \mathcal{H}$,
H3 for all $x \neq y$ there exists $H \in \mathcal{H}$ such that $x \in H$ and $y \notin H$,
where, for any $S \subseteq X$, S^c denotes the complement of S in X. The elements $H \in \mathcal{H}$ are referred to as the *basic properties* (with the understanding that a property is extensionally identified with the subset of all social states possessing that property), and a pair (H, H^c) is referred to as an *issue*.

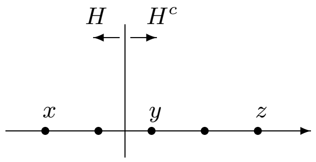
Betweenness A property space $(X, \mathcal{H})$ induces a ternary *betweenness relation* $T \subseteq X^3$

according to

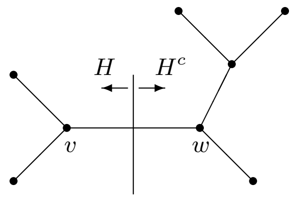
$$(x, y, z) \in T :\Leftrightarrow [\text{ for all } H \in \mathcal{H} : \{x, z\} \subseteq H \Rightarrow y \in H] \quad (1)$$

(see Nehring (1999)). If $(x, y, z) \in T$, we say that y is *between* x and z.

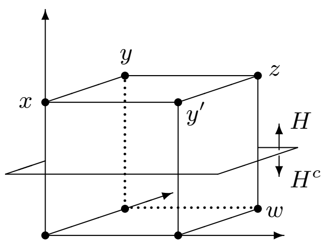
The following figure shows some basic examples of property spaces.



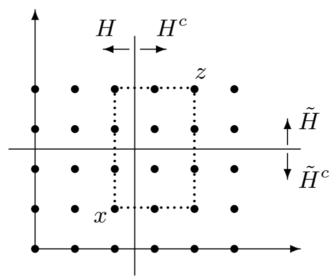
1a: Lines



1b: Trees



1c: The 3*-hypercube*



1d: The product of two lines

Figure 1: Examples of property spaces

In the case of a line, the basic properties are of the form "lying to the left (resp. to the right)" of some given element (see Fig. 1a). More generally, in a tree any edge (v, w) defines two basic properties of the form "lying in direction of v" (the set of all x for which v lies on the shortest path from x to w) and "lying in direction of w" (all x for which w lies on the shortest path from x to v), respectively (see Fig. 1b). An element y is between x and z if it lies on the shortest path that connects x and z. Note that in a tree a shortest path is uniquely determined.

The K-dimensional hypercube is the set $\{0, 1\}^K$ of all binary sequences of length K. The basic properties are, for all $k = 1, ..., K$, the sets H_0^k (resp. H_1^k) of all elements that have a zero (resp. a one) in coordinate k; the 3-hypercube is depicted in Fig. 1c. An element y is between x and z if it agrees with x and z in all coordinates in which these two elements agree. In a product $X = \prod X^k$, the basic properties are of the form $H^j \times \prod_{k \neq j} X^k$, where H^j is a basic property in coordinate j. Thus, e.g., in the product of two lines the basic properties are of the form $H^1 \times X^2$ and $X^1 \times H^2$, respectively. An element y is between x and z if it is contained in the rectangle spanned by x and z, i.e. if it is coordinatewise between x and z (see Fig. 1d).

Convexity A subset $A \subseteq X$ is *convex* in $(X, \mathcal{H})$ if $A = \cap \mathcal{H}_A$ for some family $\mathcal{H}_A \subseteq \mathcal{H}$; by convention, X is also convex. If A is convex, then $[\{x, z\} \subseteq A$ and $(x, y, z) \in T] \Rightarrow y \in A$. For any set $S \subseteq X$, let CoS denote the *convex hull* of S, i.e. the smallest convex set containing S. For any pair x, z, the convex hull of $\{x, z\}$ is denoted by $[x, z]$ and referred to as the *segment* between x and z.

In a line, the convex sets are precisely the intervals; more generally, a subset of a tree is convex if and only if it contains with any two elements the entire shortest path connecting them. In the hypercube, the non-empty convex sets are the subcubes of the form $\prod_k A^k$, where $\emptyset \neq A^k \subseteq \{0, 1\}$ for all k. More generally, in a product space a set is convex if and only if it is the product of convex sets.

Generalized Single-Peakedness A linear preference ordering $\succ$ on X is called *single-peaked* on $(X, \mathcal{H})$ if there exists $x^* \in X$ (the "peak") such that for all $y \neq z$,

$$(x^*, y, z) \in T \Rightarrow y \succ z.$$

Single-peaked preferences in this sense have been studied, among others, by Moulin (1980) in the case of a line, by Demange (1982) in the case of trees, by Barberá, Sonnenschein and Zhou (1991) in the hypercube (under the name of "separable preferences"), and by Barberá, Gul and Stacchetti (1993) in the product of lines (under the name of "multidimensionally single-peaked preferences").

It is important to realize that also the unrestricted preference domain is a single-peaked domain. To see this, consider, for any $x \in X$, the complementary basic properties $H = \{x\}$ ("being equal to x") and $H^c = X \setminus \{x\}$ ("being different from x"). Evidently, the collection of all such properties defines a property space. The corresponding betweenness relation according to (1) is vacuous in the sense that no element x is between two other elements y and z (since y and z share the basic property "being different from x," a property not shared by x). By consequence, *any* linear preference ordering is single-peaked with respect to this betweenness relation.

A large class of generalized single-peaked domains can be described in terms of "graphic" betweenness relations, as follows (cf. NP (2004)).[3] Given a graph, say that y is between x and z if y lies on some shortest path that connects x and z. Often this betweenness relation can be derived from an underlying property space via (1); for instance, the betweenness relations corresponding to the examples in Fig. 1 above are all graphic betweenness relations. As another example, consider the cycle in the following figure.

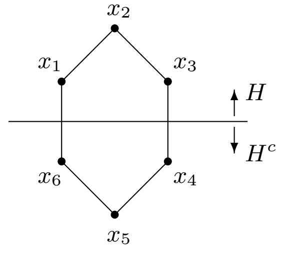


Figure 2: Betweenness and single-peakedness on a cycle

[3]For related domains of single-peaked preferences on graphs, see Schummer and Vohra (2002).

If one takes the family of basic properties to consist of all sets of three consecutive elements in the cycle (all "half-cycles"), the induced betweenness according to (1) coincides with the graphic betweenness just described. For instance, x_j is between x_{j-1} and x_{j+1}, and all elements are between opposite pairs, such as x_1 and x_4.

In the following, we denote by $\mathcal{S}_{(X,\mathcal{H})}$ the set of all single-peaked preference orderings on $(X, \mathcal{H})$ and by $\mathcal{S} \subseteq \mathcal{S}_{(X,\mathcal{H})}$ any subset that is *rich* in the sense of the following two conditions.

R1 For all x, y such that no other element is between x and y there exists a preference ordering in $\mathcal{S}$ that has x as peak and y as the second best element.

R2 For all x, y, z such that y is not between x and z there exists a preference ordering in $\mathcal{S}$ with peak x that ranks z above y.

Any set $\mathcal{S}^n$ such that $\mathcal{S} \subseteq \mathcal{S}_{(X,\mathcal{H})}$ satisfies R1 and R2 is referred to as a *rich single-peaked domain* associated with $(X, \mathcal{H})$. Evidently, the set of *all* single-peaked preference orderings is rich in this sense.

Social Choice Function Let $N = \{1, ..., n\}$ be a set of voters. A *social choice function* on a rich single-peaked domain is a mapping $F : \mathcal{S}^n \to X$ that assigns to each preference profile $(\succ_1, ..., \succ_n) \in \mathcal{S}^n$ a unique social alternative $F(\succ_1, ..., \succ_n) \in X$. The function F satisfies *voter sovereignty* if F is onto, i.e. if any $x \in X$ is in the range of F.

The function F is called a *voting scheme* if there exists a function $f : X^n \to X$ such that $F(\succ_1, ..., \succ_n) = f(x_1^*, ..., x_n^*)$, where x_i^* is voter i's peak. In that case, we say that f *represents* F, and we will also refer to the function $f : X^n \to X$ as a voting scheme.

A social choice function F is *strategy-proof* on $\mathcal{S}$ if for all i and $\succ_i, \succ_i' \in \mathcal{S}$,

$$F(\succ_1, ..., \succ_i, ..., \succ_n) \succeq_i F(\succ_1, ..., \succ_i', ..., \succ_n).$$

Similarly, $f : X^n \to X$ is called strategy-proof if the voting scheme F that it represents is strategy-proof.

Voting by Issues A *family of winning coalitions* is a non-empty family $\mathcal{W}$ of subsets of N satisfying $[W \in \mathcal{W}$ and $W' \supseteq W] \Rightarrow W' \in \mathcal{W}$. A *structure of winning coalitions* on $(X, \mathcal{H})$ is a mapping $\mathcal{W} : H \mapsto \mathcal{W}_H$ that assigns a family of winning coalitions to each basic property $H \in \mathcal{H}$ satisfying the following condition,

$$W \in \mathcal{W}_H \Leftrightarrow W^c \notin \mathcal{W}_{H^c}. \tag{2}$$

Voting by issues is the mapping $f_{\mathcal{W}} : X^n \to 2^X$ defined as follows. For all $\xi \in X^n$,

$$x \in f_{\mathcal{W}}(\xi) :\Leftrightarrow \text{ for all } H \in \mathcal{H} \text{ with } x \in H : \{i : \xi_i \in H\} \in \mathcal{W}_H.$$

Thus, x is the outcome of voting by issues if and only if, for any property H possessed by x, the coalition of those individuals whose peak have property H is winning for H. The induced mapping $F_{\mathcal{W}}(\succ_1, ..., \succ_n) := f_{\mathcal{W}}(x_1^*, ..., x_n^*)$, where x_i^* is the peak of $\succ_i$, is also referred to as *voting by issues*. Condition CS1 implies that,

$$\mathcal{W}_H = \{W \subseteq N : W \cap W' \neq \emptyset \text{ for all } W' \in \mathcal{W}_{H^c}\}. \tag{3}$$

Consistency A structure of winning coalitions $\mathcal{W}$ is called *consistent* if $f_{\mathcal{W}}(\xi) \neq \emptyset$ for all $\xi \in X^n$. If voting by issues is consistent, it is single-valued due to H3; if no confusion can arise, we will therefore identify $F_{\mathcal{W}}$ and $f_{\mathcal{W}}$ with the corresponding functions from $\mathcal{S}^n$ and X^n to X, respectively.

The following theorem extends an important earlier result by Barberá, Massò and Neme (1997).

Theorem A [NP (2004)] *A social choice function $F : \mathcal{S}^n \rightarrow X$ satisfies voter sovereignty and is strategy-proof on a rich single-peaked domain if and only if it is voting by issues with a consistent structure of winning coalitions.*

Critical Family A *critical family* $\mathcal{G} \subseteq \mathcal{H}$ is a minimal set of basic properties with empty intersection, i.e. such that $\cap\mathcal{G} = \emptyset$ and for all $G \in \mathcal{G}$, $\cap(\mathcal{G} \setminus \{G\}) \neq \emptyset$.

A critical family $\mathcal{G} = \{G_1, ..., G_l\}$ thus describes the exclusion of the combination of the corresponding basic properties in the sense that $G_1, ..., G_l$ cannot be jointly realized. "Criticality" (i.e. minimality) means that this exclusion is not implied by a more general exclusion, i.e. the properties in any proper subset of $\mathcal{G}$ are jointly realizable.

Intersection Property Voting by issues $F_{\mathcal{W}}$ satisfies the *Intersection Property* if for any critical family $\mathcal{G} = \{G_1, ..., G_l\}$, and any selection $W_j \in \mathcal{W}_{G_j}$, $\cap_{j=1}^{l} W_j \neq \emptyset$.

Theorem B [NP (2004)] *A social choice function $F : \mathcal{S}^n \rightarrow X$ satisfies voter sovereignty and is strategy-proof on a rich single-peaked domain if and only if it is voting by issues satisfying the Intersection Property.*

Median Space A property space $(X, \mathcal{H})$ is called a *median space* if, for all $x, y, z \in X$, there exists a (unique) element $m = m(x, y, z) \in X$, called the *median* of x, y, z, such that m is between any pair of $\{x, y, z\}$, i.e. such that $\{m\} = [x, y] \cap [x, z] \cap [y, z]$.

A property space $(X, \mathcal{H})$ is a median space if and only if every critical family contains exactly two basic properties (NP (2004, Proposition 4.1)). All property spaces in Fig. 1 are median spaces. Further examples are products of median spaces and median stable subsets of median spaces, i.e. subsets that are closed under taking medians, see Section 3 below for an example. By contrast, the property space underlying the 6-cycle in Fig. 2 is not a median space. For instance, the elements x_1, x_3 and x_5 do not possess a median. Evidently, also the property space underlying the unrestricted domain is not a median space; indeed, in that case *no* triple of distinct elements possesses a median.

3 Efficient Voting by Issues

In this section, we characterize the class of all efficient strategy-proof social choice functions on generalized single-peaked domains. Subsection 3.1 contains the statement of the main result. Subsection 3.2 provides examples, and Subsection 3.3 discusses the literature.

3.1 Main Result

The following definition is fundamental.

Definition (Efficiency) A social choice function $F : \mathcal{S}^n \rightarrow X$ is called *efficient* if, for no preference profile $(\succ_1, ..., \succ_n) \in \mathcal{S}^n$, there exists $y \in X$ such that for all $i \in N$, $y \succeq_i F(\succ_1, ..., \succ_n)$ with strict preference for at least one $i \in N$.

It is well-known that voting by issues is not efficient in general (cf. Barberá, Sonnenschein and Zhou (1991)). As a simple example, consider three voters in the three-dimensional hypercube, and assume that each issue is decided by majority ("issue-by-issue majority voting"), i.e. $\mathcal{W}_H = \{W \subseteq N : \#W > n/2\}$ for all H and $n = 3$. If the peaks are distributed as shown in Figure 3 below, the chosen state is their median m. However, it is entirely possible that the voters unanimously prefer the "antimedian" $\hat{m}$

to the median m since single-peakedness of a preference ordering imposes no restriction on the ranking between m and $\hat{m}$ when the peak is either x_1^*, x_2^* or x_3^* as in Fig. 3.

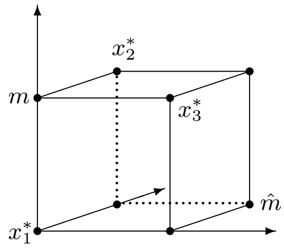


Figure 3: Issue-by-issue majority voting in the three-dimensional hypercube

The possible inefficiency in the three-dimensional hypercube contrasts with the fact that voting by issues is efficient under single-peakedness on a line. Indeed, since the coalition of all voters is winning, voting by issues always selects an element between the left-most and the right-most peak, from which the efficiency is immediate. The argument easily generalizes to the case of all single-peaked preferences on a tree. This suggests that in the analysis of efficiency a crucial role is played by the "dimensionality" of the underlying property space, as follows.

Definition (Independent Properties, Dimension) Say that $H, \tilde{H} \in \mathcal{H}$ are *independent*, denoted by $H \perp \tilde{H}$, if the four sets $H \cap \tilde{H}$, $H \cap \tilde{H}^c$, $H^c \cap \tilde{H}$ and $H^c \cap \tilde{H}^c$ are all non-empty. A set $\mathcal{F} \subseteq \mathcal{H}$ is called *independent* if any pair $H, \tilde{H} \in \mathcal{F}$ with $H \notin \{\tilde{H}, \tilde{H}^c\}$ is independent. The *dimension* $\dim(X, \mathcal{H})$ of a property space $(X, \mathcal{H})$ is the maximal cardinality of an independent subset $\mathcal{F} \subseteq \mathcal{H}$.

Observe that if H and $\tilde{H}$ are not independent and $\tilde{H} \notin \{H, H^c\}$, then *exactly* one of the four sets $H \cap \tilde{H}$, $H \cap \tilde{H}^c$, $H^c \cap \tilde{H}$ and $H^c \cap \tilde{H}^c$ is empty.

It is easily verified that, among all median spaces, trees are characterized by the property that no two basic properties are independent; in other words: trees are precisely the one-dimensional median spaces. Furthermore, the above notion corresponds exactly to the standard notion of dimension in case of the hypercube. More generally, the following observation shows that, for median spaces, the proposed notion of dimension indeed reflects our intuitive understanding of that concept.[4]

Fact 1 *For a median space,* $\dim(X, \mathcal{H}) \leq k$ *if and only if* $(X, \mathcal{H})$ *is embeddable in the product of* k *trees.*[5]

As we shall see, on spaces of dimension ≥ 3 no strategy-proof and non-dictatorial social choice function can be efficient. The case of $\dim(X, \mathcal{H}) \leq 2$ is more complex; the key to the analysis is the following weak neutrality condition.

Definition (Weak Neutrality) Say that voting by issues is *weakly neutral* if $\mathcal{W}_H = \mathcal{W}_{\tilde{H}}$ whenever $H \perp \tilde{H}$.

Note that weak neutrality is vacuously satisfied on all one-dimensional spaces. In the hypercube, and more generally in any product of lines, weak neutrality is easily seen

[4] The terminology may be less appropriate for non-median spaces; van de Vel (1993) uses the term "directional degree" for an analogous concept that coincides in the case of median spaces.

[5] An embedding is a betweenness preserving injection.

to be equivalent to full neutrality in the sense that $\mathcal{W}_H = \mathcal{W}_{\tilde{H}}$ for all $H, \tilde{H} \in \mathcal{H}$.

Definition (Dictators and Almost Dictators) A social choice function $F : \mathcal{S}^n \to X$ is called *dictatorial* if there exists $i \in N$ such that $F(\succ_1, ..., \succ_n) = x_i^*$ whenever x_i^* is the peak of $\succ_i$. Furthermore, F is called *locally dictatorial* if it is dictatorial on some subdomain $\mathcal{D}^n$, where $\mathcal{D} \subseteq \mathcal{S}$ contains at least two preferences with different peaks. Finally, F is called *almost dictatorial* if there exists a local dictator i such that, for any pair $x, y \in X$ with $x \neq y$, voter i can veto x or y if all voters have their peaks in the set $\{x, y\}$, i.e. voter i can ensure that $F(\succ_1, ..., \succ_n) \neq x$ or voter i can ensure that $F(\succ_1, ..., \succ_n) \neq y$ for all preference profiles such that all peaks are in $\{x, y\}$.[6]

Fact 2 *Denote by $F_{\mathcal{W}} : \mathcal{S}^n \to X$ voting by issues with the structure $\mathcal{W}$ of winning coalitions.*

a) *$F_{\mathcal{W}}$ is dictatorial if and only if, for some i, $\{i\} \in \mathcal{W}_H$ for all $H \in \mathcal{H}$.*

b) *$F_{\mathcal{W}}$ is locally dictatorial if and only if, for some i, $\{i\} \in \mathcal{W}_H \cap \mathcal{W}_{H^c}$ for some H.*

c) *$F_{\mathcal{W}}$ is almost dictatorial if and only if, for some i, $\{i\} \in \mathcal{W}_H \cup \mathcal{W}_{H^c}$ for all H, and $\{i\} \in \mathcal{W}_H \cap \mathcal{W}_{H^c}$ for some H.*

Observe that, while there can be multiple local dictators, there can be at most one almost-dictator.

The following is our main result.

Theorem *Let $F : \mathcal{S}^n \to X$ be a strategy-proof social choice function on a rich single-peaked domain associated with $(X, \mathcal{H})$.*

a) *Let F be efficient. Then, F is weakly neutral voting by issues; moreover, F is almost dictatorial or $(X, \mathcal{H})$ is a median space with* $\dim(X, \mathcal{H}) \leq 2$.

b) *Let F be weakly neutral voting by issues. If F is almost dictatorial, or if $(X, \mathcal{H})$ is a median space with* $\dim(X, \mathcal{H}) \leq 2$*, then F is efficient.*

The intuition behind the result is as follows. By strategy-proofness, the social choice function F only depends on the voters' peaks (see NP (2004, Prop. 3.3)). By consequence, the social choice must *guarantee* efficiency simultaneously for *all* profiles of single-peaked preferences with the same profile of peaks. In the hypercube, this is possible only when the social choice in fact coincides with one of the peaks, i.e. if $F(\succ_1, ..., \succ_n) = x_i^*$ for some i. But in three or more dimensions, this is inconsistent with the voting by issues structure and non-dictatorship, as exemplified by Fig. 3 above. On the other hand, in the 2-hypercube the social choice always coincides with a voter's peak under neutrality. For general median spaces, weak neutrality ensures that the social choice is always contained in the smallest median stable subset that contains all voters' peaks (their so-called "median stabilization," see van de Vel (1993, p.139)). If $\dim(X, \mathcal{H}) \leq 2$, this is sufficient for efficiency. Concretely, consider the product of two lines. In this case, weak neutrality is equivalent to $\mathcal{W}_H = \mathcal{W}^*$ for all $H \in \mathcal{H}$, where $\mathcal{W}^*$ is a fixed family $\mathcal{W}^*$ of winning coalitions satisfying $W \cap W' \neq \emptyset$ for all $W, W' \in \mathcal{W}^*$ (by condition (3) above). Let x be the outcome under voting by issues. Neutrality guarantees that any "orthant" $H \cap H'$ containing x must also contain the peak of some voter.[7] In the product of *two* lines, this implies at once the efficiency of x for any

[6] Note that such a voter i may still not be a dictator over the pair $\{x, y\}$ in this situation, since the outcome could lie between x and y.

[7] Indeed, $W := \{j : x_j^* \in H\} \in \mathcal{W}^*$ since $x \in H$, and $W' := \{j : x_j^* \in H'\} \in \mathcal{W}^*$ since $x \in H'$. By condition (3), there exists a voter i with $x_i^* \in H \cap H'$ (see Figure 4).

profile of single-peaked preferences with the given peaks. For instance, in Fig. 4, voter j prefers x to any outcome in the south of x, voter k prefers x to outcomes in the east of x, whereas any voter i with peak in $H \cap H'$ prefers x to outcomes in the northwest of x.

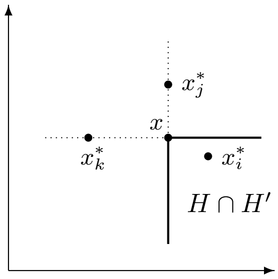


Figure 4: Neutrality implies efficiency in the product of two lines

The following result is immediate from our main theorem and NP (2004, Th. 4).

Corollary 1 *In a product of two trees, there is one and only one anonymous and efficient strategy-proof social choice rule on $\mathcal{S}^n$, namely issue-by-issue majority voting with an odd number n of voters.*

Issue-by-issue majority voting (i.e. choosing the coordinate-wise median of the voters' peaks) with an odd number of voters is strategy-proof also on the domains of separable Euclidean preferences on $\mathbf{R}^m$ considered in Border and Jordan (1983); moreover, for $m = 2$ it is efficient by the same arguments as given here for the larger domain of generalized single-peaked preferences. Thus, Border and Jordan's Corollary 4 (the inconsistency of strategy-proofness, efficiency, and non-dictatorship) is valid only in three or more dimensions, but not in two, contrary to what is claimed there.[8]

3.2 Examples

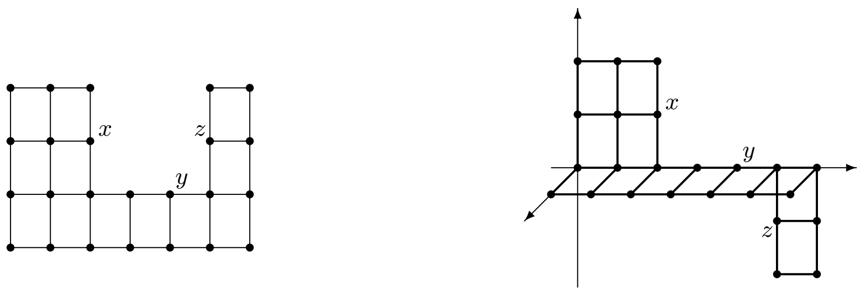
The product of two lines is an important and natural example of a space that enables strategy-proof and efficient social choice via neutral voting by issues. We now discuss three further classes of domains that arise naturally in economic applications. The first two give rise to possibility results, the third to an impossibility result.

3.2.1 A Location Example

The following class of domains is associated with spaces that are two-dimensional in the sense of being embeddable in the product of two trees, but which cannot be embedded in the product of two lines. Concretely, consider the graph in Figure 5a below endowed with the graphic betweenness according to which an alternative y is between x and z if y is on a shortest path connecting x and z. One may think of the edges as the streets of a city with obstructed links (the obstacle that hinders the direct connection of x and

[8]The published version contains a correct proof for the case of two voters; the proof does not generalize, however, to more than two voters. This is due to the fact that *neutral* voting by issues is necessarily dictatorial with only two voters, while it is easily non-dictatorial when there are more than two voters.

z in Fig. 5a may be a lake, for instance). The depicted graph is a median space since it embeds as a median stable subset in the product of a tree and a line (see Fig. 5b). Observe that the same graph does *not* embed in the product of two lines; for instance, the element y is not between x and z if one views the graph in Fig. 5a as a subset of the product of two lines with the induced betweenness.[9] By our above results, the only anonymous and efficient strategy-proof social choice rule on the associated single-peaked domain is issue-by-issue majority voting, which amounts to choosing the unique "local" Condorcet winner, i.e. the location that wins in pairwise comparison against each of its neighbours. Equivalently, the location chosen under issue-by-issue majority voting minimizes the sum of distances to the voters' peaks, where the distance between two points is given by the minimal number of edges needed to connect them.



5a: A city with obstructed links ... *5b: ... and its embedding*

Figure 5: A two-dimensional median space

The example can be naturally generalized as indicated by the following sketch. Take any rectangular grid (product of two lines) and remove a subset of points together with all edges joining them, so that the remaining graph ("induced subgraph") is connected. There are two possible cases. If the induced subgraph has "no holes," then one can show that the associated graphic betweenness gives rise to a median space of dimension ≤ 2. By Fact 1, the space is embeddable in the product of two trees, and by the main theorem, any associated rich single-peaked domain admits efficient strategy-proof social choice functions that are not almost dictatorial (in fact even anonymous ones). On the other hand, if the induced subgraph has "holes," the resulting space is not a median space (and indeed need not even be a property space), hence all efficient strategy-proof social choice functions are almost dictatorial.

3.2.2 Additive Preferences over Public Goods

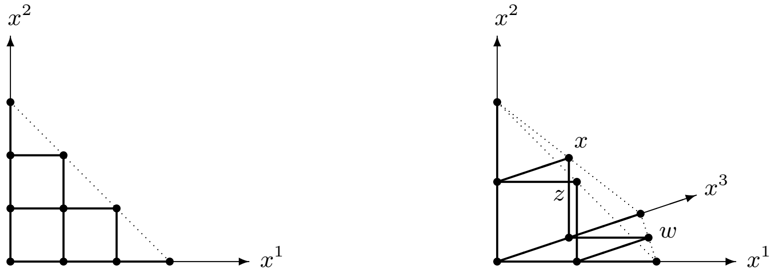
Suppose that there are $K + 1$ public goods, which can be supplied in non-negative discrete quantities. Denote by $x^k \in \mathbf{N}_0$ the quantity of good $k = 0, 1, ..., K$, and suppose that feasibility requires $\sum_k x^k \leq M$ for some fixed amount M. Furthermore, suppose that preferences can be represented by additive utility functions of the form $\sum_k u^k(x^k)$, where each u^k is increasing and concave. By the resulting monotonicity of preferences, the choice will always lie on the budget line $\sum_k x^k = M$. We can therefore

[9] In accordance with an interpretation of the blank area as a lake, the underlying metric may thus be referred to as the "Chicago"- rather than "Manhattan"-metric.

eliminate the coordinate corresponding to good 0, and consider the set $X = \{x \in \mathbf{N}_0^K : \sum_{k=1}^K x^k \leq M\}$ as the universe of alternatives. The utility functions on X can be written as follows,

$$u(x^1, ..., x^K) = \sum_{k=1}^{K} u^k(x^k) + u^0(M - \sum_{k=1}^{K} x^k).$$

Suppose that preferences are *quasi-linear* with good 0 as the numeraire so that $u^0(x^0) = x^0$. Then, preferences can be represented by utility functions of the form $\tilde{u}(x) = \sum_k (u^k(x^k) - x^k)$. Since each summand $u^k(x^k) - x^k$ is concave, any such utility function represents a single-peaked preference with respect to the standard betweenness on X induced by the product of K lines. For $K = 2$ the resulting property space $(X, \mathcal{H})$ is a median space, but not for $K \geq 3$ (see Fig. 6b in which the triple x, z, w has no median).



6a: Median space for $K = 2$ *6b: Non-median space for $K = 3$*

Figure 6: Additive preferences over public goods

It is easily verified that, for any K, the set of all utility functions of the form $\tilde{u}$ gives rise to a rich single-peaked domain. Since for $K = 2$ the associated space is a two-dimensional median space it admits non-degenerate efficient and strategy-proof social choice functions (cf. Corollary 2 below). On the other hand, for $K > 2$ no such social choice functions exist.

3.2.3 Judgement Aggregation

Consider a situation in which a group of individuals has to reach a collective decision on a set of propositions each of which can be affirmed or rejected. Specifically, suppose that there is a set of logically independent "premises" $a_1, ..., a_m$ and a "conclusion" p that depends on them, say p is the logical conjunction of the a_j, i.e. $p = a_1 \wedge ... \wedge a_m$. A concrete example is the so-called "doctrinal paradox" in which a jury in a court has to decide on the liability of a defendant (see Kornhauser and Sager (1986)). It is commonly agreed that the defendant is to be held liable (p) if and only if he broke a contract (a_1) and the contract was legally valid (a_2). Each jury member has an opinion on the premises, i.e. either affirms or rejects each of a_1 and a_2, and a resulting opinion on the conclusion, i.e. affirms p if and only if he/she affirms both a_1 and a_2. A logically consistent opinion on all propositions is called a *judgement* and can be viewed as an element of a property space $(X, \mathcal{H})$, as follows. Each proposition corresponds to an issue with H_{a_j} (resp. H_p) denoting the set of all judgements that entail the affirmation a_j (resp. p). The logical dependence of the conclusion p on the premises a_j is captured by criticality of the families $\{H_1, ..., H_m, H_p^c\}$ ("the affirmation of all premises implies the affirmation of the conclusion") and $\{H_j^c, H_p\}$ for all j ("the rejection of any premise implies the rejection of the conclusion").

The problem of aggregating profiles of individual judgements into a collective judgement has received some attention recently (see, e.g., List and Pettit (2002) for a first formal statement). In order to discuss the strategy-proofness of aggregation rules, following Dietrich and List (2005), it is natural to assume that each individual has a single-peaked preference on the space $(X, \mathcal{H})$ of (logically consistent) judgements with the peak representing the individual's "true" judgement; single-peakedness means that individuals prefer judgements with fewer deviations from their true judgement. By Theorem A above, any strategy-proof judgement aggregation rule on a rich domain of such preferences takes the form of voting by issues, i.e. of propositionwise aggregation.[10]

It is easily shown that the property space $(X, \mathcal{H})$ corresponding to $m \geq 2$ logically independent premises a_j and the conclusion $p = a_1 \wedge ... \wedge a_m$ has dimension m and is never a median space. Thus, by our main result above no strategy-proof judgement aggregation rule can be efficient (unless it is almost dictorial). In fact, in Nehring and Puppe (2005b) we show that, by the Intersection Property, all strategy-proof aggregation rules are oligarchic in this case. For example, in the anonymous case any such rule takes the form of a unanimity rule according to which each premise a_j and the conclusion p are collectively affirmed if and only if *all* individuals affirm it. Even in the two-dimensional case, this rule is inefficient. Suppose, for example, that some voters affirm a_1 but not a_2, while all other voters affirm a_2 but not a_1 (thus all voters reject $p = a_1 \wedge a_2$ by logical consistency). In this case, the unanimity rule prescribes the rejection of all propositions a_1, a_2 and p. However, it is clearly compatible with single-peakedness that all voters prefer the acceptance of all propositions to their rejection. We conclude in noting that the impossibility of non-degenerate strategy-proof and efficient aggregation generalizes to any other logical form of the conclusion p vis-á-vis its premises; this follows from the analysis in Nehring and Puppe (2005b).

3.3 Comparison to the Literature

On both the necessity and the sufficiency side, our results go significantly beyond what is known in the literature. On the sufficiency side, Kim and Roush (1984) and Peters, van der Stel and Storcken (1992) have shown the efficiency of the coordinatewise median in two dimensions under the assumption that voters' preferences are given by the negative Euclidean distance from their ideal point. Under this assumption, any point in the convex hull of voters' ideal points guarantees efficiency. This is no longer true if voters' use different metrics or have more general single-peaked preferences; in such cases, our more general argument based on the median-stabilization of the set of ideal points is needed. Moreover, while these two references assume anonymity, our analysis shows that weak neutrality alone suffices to ensure efficiency in median spaces of dimension less than three.

[10] In contrast to Dietrich and List (2005), we thus need not *assume* that the aggregation rule only depends on the preference peaks, but derive this property from the condition of strategy-proofness. Also, Dietrich and List (2005) do not study the issue of efficiency in this context.

On the necessity side, the literature has so far considered domains associated with the product of lines. It is, however, straightforward that the impossibility of achieving efficiency in the three-dimensional hypercube illustrated in Fig. 3 generalizes to at least three-dimensional median spaces, since any such space contains a 3-hypercube. Here, our contribution is the necessity of the median space structure of the domain itself, as highlighted by the following result.

Corollary 2 *A domain $\mathcal{S}^n$ admits a not-almost dictatorial and weakly neutral strategy-proof social choice function if and only if the underlying space is a median space. Moreover, $\mathcal{S}^n$ admits a not-almost dictatorial and efficient strategy-proof social choice function if and only if the underlying space is a median space of dimension ≤ 2.*

Corollary 2 follows at once from the proof of our main theorem in the appendix together with the observation that issue-by-issue majority voting is (weakly) neutral and well-defined on any median space, by NP (2004, Th. 4).

The two novel insights are thus the necessity of weak neutrality for efficient strategy-proof social choice on any single-peaked domain, and the realization that not-almost dictatorial and weakly neutral strategy-proof social choice functions exist *only* on median spaces.[11]

To illustrate, consider the following graph with the associated graphic betweenness. The underlying basic properties are given by the three four-cycles and their complements. Concretely, suppose that there are three candidates who can be hired by some company or not. The four-cycle through x represents the states in which the first candidate is hired; similarly, the four-cycle through w (resp. z) represents the states in which the second (resp. third) candidate is hired. The point x represents the state in which all but the first candidate are rejected, and similarly for w and z. Observe that since there is no state in which all candidates are rejected, the triple x, z, w does not admit a median. Nevertheless, the associated single-peaked domain admits well-behaved strategy-proof social choice rules. For instance, the rule according to which the social choice belongs to any of the four-cycles if and only if at least one third of the voters' peaks are in that four-cycle can be shown to be consistent and strategy-proof; clearly, it is anonymous, hence in particular not-almost dictatorial. However, efficiency cannot be achieved since no well-behaved social choice function can be weakly neutral, by Corollary 2.

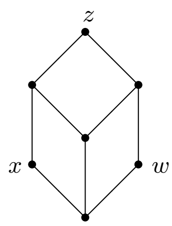


Figure 7: Impossibility of efficiency outside median spaces

[11] Note that the unrestricted domain is associated with a property space of dimension 1 as defined here. By definition, any strategy-proof social choice function on such a property space is weakly neutral. Corollary 2 thus implies that any strategy-proof social choice function on the unrestricted domain must be almost dictatorial – that is, it "almost" implies the Gibbard-Satterthwaite theorem.

The example readily generalizes to the case of K candidates of which at least k *have to* be hired and at most k' *can* be hired (in Fig. 7, we thus have $k = 1$ and $k' = K = 3$). Except for trivial cases, none of these spaces admits non-dictatorial, efficient and strategy-proof social choice functions, since none is a median space. Nevertheless, some of these spaces enables non-degenerate strategy-proof social choice rules, as shown in Nehring and Puppe (2005a).

4 Partial Efficiency

Our main result shows that, under strategy-proofness, full efficiency is attainable only in at most two-dimensional spaces. In this section, we want to determine to what extent strategy-proofness is compatible with weaker notions of efficiency in more than two dimensions.[12]

4.1 Restricted Information Efficiency

For any voter i and any peak x_i^* define a partial order $\geq_i$ by

$$x \geq_i y :\Leftrightarrow x \in [x_i^*, y],$$

i.e. $x \geq_i y$ if and only if x is "closer" than y to the peak x_i^*. The partial order $\geq_i$ reflects the information about i's preference that an outside observer can infer from knowledge of i's peak and the fact that i's preference is single-peaked. Note that, for instance, the voters' preference between m and $\hat{m}$ in Fig. 3 above cannot be determined from $\geq_i$ for $i = 1, 2, 3$.

Definition (Weak Efficiency) Say that a voting scheme $f : X^n \to X$ is *weakly efficient* whenever for no $\xi \in X^n$ there exists $y \in X$ such that for all $i \in N$, $y \geq_i f(\xi)$ with strict inequality for at least one $i \in N$.

Weak efficiency has a natural interpretation as *renegotiation-proofness.* Specifically, suppose that after the voting mechanism has been applied, each voter gets the right to enforce the outcome. Weak efficiency then guarantees that the group of all voters cannot jointly improve upon the outcome *given* the information revealed by the mechanism, namely the voters' peaks.

The following result shows that weak efficiency is closely related to the requirement that the social choice be in the convex hull of the voters' peaks; indeed, on median spaces, these two conditions are equivalent.

Proposition 1 *A voting scheme $f : X^n \to X$ is weakly efficient whenever $f(\xi) \in Co\{\xi_1, ..., \xi_n\}$ for all $\xi \in X^n$. Conversely, if $(X, \mathcal{H})$ is a median space and f is weakly efficient, then $f(\xi) \in Co\{\xi_1, ..., \xi_n\}$ for all $\xi \in X^n$.*

Since the coalition of all individuals is winning for any basic property, voting by issues is clearly onto, i.e. it satisfies voter sovereignty. The weak efficiency of voting by issues is thus an immediate consequence of Proposition 1 together with the following result.

Proposition 2 *A strategy-proof social choice function $F : \mathcal{S}^n \to X$ satisfies voter sovereignty if and only if, for all preference profiles, $F(\succ_1, ..., \succ_n) \in Co\{x_1^*, ..., x_n^*\}$ where x_i^* denotes the peak of $\succ_i$.*

[12]For a different approach to partial efficiency properties in the context of voting by issues ("voting by committees"), see Shimomura (1996).

4.2 Local Efficiency

Even when voting by issues yields an inefficient outcome, finding the Pareto-improvement may be difficult. For instance, under issue-by-issue majority voting in the 3-hypercube with the peaks as in Fig. 3 above, the *only* alternative that is possibly Pareto superior to the outcome m is the opposite point $\hat{m}$, and thus quite far away from m. It thus seems natural to ask whether strategy-proofness is compatible with weaker "local" efficiency criteria requiring that there should not exist Pareto improvements "close" to the actual outcome. For simplicity, we restrict ourselves to median spaces in the remainder of this section.

The following notion of distance between two alternatives intuitively describes the number of dimensions in which they differ. Formally, for any two states x and y, denote by $\mathcal{H}_{x\neg y} := \{H \in \mathcal{H} : x \in H, y \notin H\}$ the basic properties possessed by x but not by y. Define the *distance* $d(x, y)$ between x and y as the maximal cardinality of an independent subset $\mathcal{F} \subseteq \mathcal{H}_{x\neg y}$.

Definition (k-efficiency) Say that a social choice function F is *k-efficient* if for no $x = F(\succ_1, ..., \succ_n)$ there exists $y \in X$ with $d(x, y) \leq k$ such that $y \succeq_i x$ for all i with at least one strict preference.

Clearly, full efficiency corresponds to k-efficiency for all $k \leq \dim(X, \mathcal{H})$. In general, k-efficiency describes "local" optimality in the following sense: if x is the outcome of a k-efficient social choice function, no alternative that differs from x in k or less dimensions can be unanimously preferred to x. In a product of lines, an outcome x is thus k-efficient if it is not Pareto dominated by any alternative that differs from x in no more than k coordinates. Note that in Fig. 3, all alternatives except $\hat{m}$ are 2-efficient at any profile compatible with the given peaks; by contrast, the anti-median $\hat{m}$ need not be 2-efficient since possibly all voters prefer x_3^* to $\hat{m}$.

Part a) of the following result shows that voting by issues always satisfies 1-efficiency. Part b) shows that weak neutrality is necessary and sufficient for "local" efficiency in the sense of 2-efficiency. On the other hand, 3-efficiency is just as demanding as full efficiency, by part c).

Proposition 3 *Let $(X, \mathcal{H})$ be a median space, and let $F : \mathcal{S}^n \to X$ be strategy-proof. Then,*

a) *F is 1-efficient if and only if it satisfies voter sovereignty.*

b) *F is 2-efficient if and only if it is weakly neutral voting by issues.*

c) *F is 3-efficient if and only if it is fully efficient.*

5 Conclusion

We have characterized strategy-proof and efficient social choice on the class of all generalized single-peaked domains. Our analysis has revealed that efficiency obtains under more general circumstances than suggested by the literature. (Weak) neutrality of voting rules has proved to be the key to full efficiency in two dimensions and to superior local efficiency in more than two dimensions. The following table summarizes some noteworthy implications of our main result; all entries pertain to strategy-proof social choice rules.

	Median Spaces	Non-Median Spaces
dim $(X, \mathcal{H}) = 1$	all rules are efficient	all rules are almost dictatorial
dim $(X, \mathcal{H}) = 2$	efficiency $\Leftrightarrow$ weak neutrality	all rules entail veto power
dim $(X, \mathcal{H}) > 2$	all rules are inefficient	non-degenerate rules may exist but all are inefficient

Table 1: Summary of the main implications

The entries in the left column are straightforward from our main theorem above. As to the right column, since on one-dimensional spaces weak neutrality is trivially satisfied by any choice rule, it follows that if these are not median spaces (i.e. trees) any choice function must be almost dictatorial. According to the second entry in the right column, whenever a space admits choice functions without veto power, it must be a median space, and thus admits efficient choice functions as well. This follows from the characterization of the class of all property spaces that admit no veto rules obtained in Nehring (2004). Finally, on non-median spaces with more than two dimensions non-degenerate strategy-proof social choice rules (e.g. anonymous rules without veto power) may or may not exist (see Nehring and Puppe (2005a)), but clearly all such rules must be inefficient.

An interesting topic for future research is to further explore the full extent to which strategy-proofness allows for "approximate" efficiency. One natural way to make the question precise is in terms of the *probability* of efficiency. For example, in the case of three voters in the 3-hypercube, if all profiles of single-peaked linear orderings are equally likely, the probability of obtaining an ex-post inefficient outcome under issue-by-issue majority voting is only 1/18432; this happens because the example of Fig. 3 is, up to isomorphism, the only way in which an ex-post inefficient outcome may be selected.

Moreover, the probability of inefficiency will decrease with the number of voters. One might even conjecture that the theoretical possibility of inefficiency is practically irrelevant based on the following heuristic argument. In real world elections, there are typically many heterogeneous voters, hence almost any alternative will be the peak of *some* voter. Therefore, with high probability the outcome will be the top choice of some voter, hence efficient. However, the argument is flawed since due to the possible high-dimensionality of the underlying space, there may in fact be vastly more alternatives than voters. A well-known case occured in 1990 in Los Angeles county, California, where *no single voter* (among approximately 1.8 million individuals) voted for the winning yes-no-combination on the 28 propositions of the election ballot, a phenomenon known in the literature as the "paradox of multiple elections" (see Brams, Kilgour and Zwicker (1998)).

Appendix: Proofs

We first consider the main result, and then turn to the remaining proofs.

Proof of the Main Theorem

The proof of the main result is provided is several steps. First observe that any efficient social choice function $F : \mathcal{S}^n \to X$ must be onto. By Theorem A above, we can thus restrict our analysis to voting by issues in all what follows.

Step 1: Efficiency requires weak neutrality

By contradiction, suppose that $H \perp \tilde{H}$ but $\mathcal{W}_H \neq \mathcal{W}_{\tilde{H}}$. We will show that the corresponding voting by issues is not efficient. Without loss of generality, assume that $W \in \mathcal{W}_H$ but $W \notin \mathcal{W}_{\tilde{H}}$ for some coalition $W \subseteq N$. Note that by (2), this implies $W^c \notin \mathcal{W}_{H^c}$ and $W^c \in \mathcal{W}_{\tilde{H}^c}$.

Choose $x \in H \cap \tilde{H}$, $y \in H^c \cap \tilde{H}^c$ and $w \in H^c \cap \tilde{H}$, and consider any two single-peaked preference orderings $\succ'$ with peak x and $\succ''$ with peak y. Now assign to all voters in W the preference ordering $\succ'$ and to all voters in W^c the preference ordering $\succ''$. Denote by $(\succ'; W, \succ''; W^c)$ the resulting preference profile, and let $z := F(\succ'; W, \succ''; W^c)$ be the chosen outcome. Since W is winning for H, we have $z \in H$. Since $W^c \in \mathcal{W}_{\tilde{H}^c}$, we have $z \in \tilde{H}^c$, i.e. $z \in H \cap \tilde{H}^c$. By construction, z is neither between x and w, nor between y and w. Hence, by the richness condition R2, $\succ'$ and $\succ''$ can be chosen such that $w \succ' z$ and $w \succ'' z$, i.e. such that all voters strictly prefer w to the chosen outcome. Thus, F is not efficient.

Step 2: The structure of weakly neutral voting by issues

Suppose that $\dim(X, \mathcal{H}) \geq 2$, i.e. suppose that the set

$$\mathcal{H}^0 := \{H \in \mathcal{H} : \text{ there exists } H' \text{ such that } H \perp H'\}$$

is non-empty. Furthermore, define

$$\begin{aligned}\mathcal{H}^* &:= \{H \in \mathcal{H}: \text{ there exists } H_1, H_2 \in \mathcal{H}^0 \text{ such that } H_1 \subseteq H \subseteq H_2\},\\ \mathcal{G}^+ &:= \{G \in \mathcal{H} \setminus \mathcal{H}^* : G \subseteq H \text{ for some } H \in \mathcal{H}^*\}, \text{ and}\\ \mathcal{G}^- &:= \{G \in \mathcal{H} \setminus \mathcal{H}^* : G^c \in \mathcal{G}^+\}.\end{aligned}$$

Note that $\mathcal{H}^0$ and $\mathcal{H}^*$ are both closed under taking complements. The following result will be useful. For graphic illustration, see Figure 8 below in which $\mathcal{H}^0 = \mathcal{H}^* = \{H_1, H_1^c, H_2, H_2^c\}$ and $\mathcal{G}^+ = \{G_1, ..., G_5\}$.

Claim a) $G \in \mathcal{G}^+$ if and only if, for all $H \in \mathcal{H}^*$, $H \cap G^c \neq \emptyset$.
b) $\{\mathcal{H}^*, \mathcal{G}^+, \mathcal{G}^-\}$ is a partition of $\mathcal{H}$.
c) For all $G, \tilde{G} \in \mathcal{G}^+$ with $G \cap \tilde{G} \neq \emptyset$, $G \subseteq \tilde{G}$ or $\tilde{G} \subseteq G$.

For verification, let $G \in \mathcal{G}^+$, i.e. $G \subseteq H$ for some $H \in \mathcal{H}^*$, and consider any $H' \in \mathcal{H}^*$. If $H' \cap G^c = \emptyset$, one would obtain $H' \subseteq G \subseteq H$, and hence $G \in \mathcal{H}^*$. Since by definition $\mathcal{G}^+ \cap \mathcal{H}^* = \emptyset$, we must therefore have $H \cap G^c \neq \emptyset$ for all $H \in \mathcal{H}^*$. Conversely, suppose that $H \cap G^c \neq \emptyset$ for all $H \in \mathcal{H}^*$. Then, $G \notin \mathcal{H}^*$; in particular, for any $H \in \mathcal{H}^*$, $G \not\perp H$ which implies either $H \cap G = \emptyset$, or $H^c \cap G = \emptyset$ (note that $H^c \cap G^c \neq \emptyset$ since $H^c \in \mathcal{H}^*$). In the first case, $G \subseteq H^c \in \mathcal{H}^*$, and in the second case $G \subseteq H \in \mathcal{H}^*$; hence, $G \in \mathcal{G}^+$ in either case. This shows part a) of the assertion. From this, part b) is immediate since, if $H \cap G^c = \emptyset$ for some $H \in \mathcal{H}^*$, either $H = G$ or $G^c \in \mathcal{G}^+$. Finally, to verify c), consider $G, \tilde{G} \in \mathcal{G}^+$ with $G \cap \tilde{G} \neq \emptyset$. One must have $G^c \cap \tilde{G}^c \neq \emptyset$, since otherwise $G^c \subseteq \tilde{G}$ and therefore $G^c \in \mathcal{G}^+$, i.e. $G \in \mathcal{G}^-$. Thus, since $G \not\perp \tilde{G}$, either $G \cap \tilde{G}^c = \emptyset$, or $G^c \cap \tilde{G} = \emptyset$. In the first case, $G \subseteq \tilde{G}$ and in the second case, $\tilde{G} \subseteq G$.

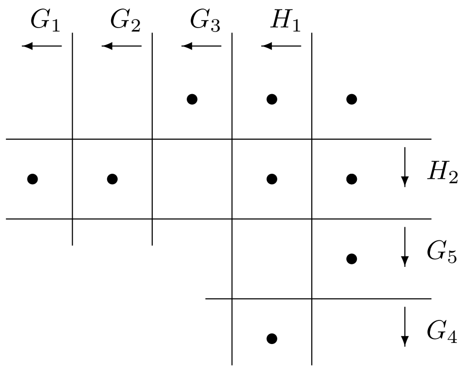


Figure 8: Decomposition into independent and dependent basic properties

Let $F_{\mathcal{W}}$ be voting by issues satisfying weak neutrality. We show that $\mathcal{W}_H = \mathcal{W}^*$ for all $H \in \mathcal{H}^*$ and some $\mathcal{W}^*$, i.e. all families of winning coalitions corresponding to the basic properties in $\mathcal{H}^*$ are identical. The following straightforward implication of the Intersection Property will be useful. For all H, H',

$$H \subseteq H' \Rightarrow \mathcal{W}_H \subseteq \mathcal{W}_{H'}. \tag{4}$$

First observe that, by weak neutrality, $\mathcal{W}_H = \mathcal{W}_{H^c}$ for all $H \in \mathcal{H}^0$. Indeed, $H \perp H'$ implies $H^c \perp H'$, hence $\mathcal{W}_H = \mathcal{W}_{H'} = \mathcal{W}_{H^c}$. Next, consider $H_1, H_2 \in \mathcal{H}^0$. If $H_1 \perp H_2$, weak neutrality immediately implies $\mathcal{W}_{H_1} = \mathcal{W}_{H_2}$. Otherwise, one has $H_1 \not\perp H_2$ along with $H_1 \perp H_1'$ and $H_2 \perp H_2'$ for some H_1', H_2'. One thus obtains $\mathcal{W}_{H_1} = \mathcal{W}_{H_1^c}$, $\mathcal{W}_{H_2} = \mathcal{W}_{H_2^c}$ and exactly one of the following four cases: $[H_1 \subseteq H_2^c$ and $H_2 \subseteq H_1^c]$, $[H_1 \subseteq H_2$ and $H_2^c \subseteq H_1^c]$, $[H_1^c \subseteq H_2^c$ and $H_2 \subseteq H_1]$, or $[H_1^c \subseteq H_2$ and $H_2^c \subseteq H_1^c]$. In each case, $\mathcal{W}_{H_1} = \mathcal{W}_{H_2}$ by twofold application of (4). Thus, $\mathcal{W}_H = \mathcal{W}^*$ for some $\mathcal{W}^*$ and all $H \in \mathcal{H}^0$. But by (4) again, one clearly has $\mathcal{W}_H = \mathcal{W}^*$ even for all $H \in \mathcal{H}^*$.

Finally, by (4) and the definition of $\mathcal{G}^+$, one obtains for all $H \in \mathcal{H}^*$ and all $G \in \mathcal{G}^+$,

$$\mathcal{W}_G \subseteq \mathcal{W}_H = \mathcal{W}^* \subseteq \mathcal{W}_{G^c}. \tag{5}$$

Step 3: Under weak neutrality a local dictator is an almost dictator

Denote by $\hat{\mathcal{H}} := \{H \in \mathcal{H} : \{i\} \in \mathcal{W}_H \cap \mathcal{W}_{H^c}$ for some $i\}$. If F is locally dictatorial, $\hat{\mathcal{H}} \neq \emptyset$; in that case, it is immediate from (5) that $\{i\} \in \mathcal{W}_H \cup \mathcal{W}_{H^c}$ for all $H \in \mathcal{H}$, i.e. that voter i is an almost-dictator. Note in particular that $\hat{\mathcal{H}} \supseteq \mathcal{H}^*$.

Step 4: Weak neutrality and no local dictatorship requires a median space

By contradiction, suppose that $(X, \mathcal{H})$ is not a median space. By NP (2004, Proposition 4.1), there exists a critical family $\{H_1, ..., H_l\}$ with $l \geq 3$. If $H_j^c \cap H_k^c = \emptyset$, for all $j, k \in \{1, ..., l\}$ with $j \neq k$, then there exists a local dictator by Nehring and Puppe (2002, Proposition 5.2). Hence, $H_j^c \cap H_k^c \neq \emptyset$ for some j, k, say $H_1^c \cap H_2^c \neq \emptyset$. Together with the criticality of $\{H_1, ..., H_l\}$ this implies $H_1 \perp H_2$, hence by weak neutrality, $\mathcal{W}_{H_1} = \mathcal{W}_{H_1^c} = \mathcal{W}_{H_2} = \mathcal{W}_{H_2^c} = \mathcal{W}^*$. Next, consider H_3. If $H_3 \in \mathcal{H}^*$, then $\mathcal{W}_{H_3} = \mathcal{W}^*$; if not, we must have $H_2^c \cap H_3^c = \emptyset$, i.e. $H_3^c \subseteq H_2$, and hence $H_3 \in \mathcal{G}^-$ as in Step 2. Thus, $\mathcal{W}^* \subseteq \mathcal{W}_{H_3}$ in any case.

By criticality of $\{H_1, ..., H_l\}$, one can choose $x_j \in H_j^c \cap \bigcap_{k \neq j} H_k$ for $j = 1, 2, 3$. Furthermore let $W \in \mathcal{W}^*$ be a minimal winning coalition with $i \in W$. Consider a preference profile $(\succ_1, ..., \succ_n)$ where all voters in $W \setminus \{i\}$ have peak x_1, all voters in W^c have peak x_2, and voter i has the peak x_3. Since $W \setminus \{i\} \notin \mathcal{W}^*$, one has $(W \setminus \{i\})^c = W^c \cup \{i\} \in \mathcal{W}^*$ by (2). But all voters in $W^c \cup \{i\}$ have their peak in H_1, hence $F(\succ_1, ..., \succ_n) \in H_1$. Since $W \in \mathcal{W}^*$ and all voters in W have their peak in H_2, one also has $F(\succ_1, ..., \succ_n) \in H_2$. Since F is not locally dictatorial, $\{i\} \notin \mathcal{W}^*$, hence $N \setminus \{i\} \in \mathcal{W}^*$, by (2), and therefore $N \setminus \{i\} \in \mathcal{W}_{H_3}$ by the above observation. This implies $F(\succ_1, ..., \succ_n) \in H_3$. Since $F(\succ_1, ..., \succ_n) \in H_j$ for all $j \in \{4, ..., l\}$, we thus obtain $F(\succ_1, ..., \succ_n) \in \bigcap_{j=1}^{l} H_j$, contradicting the criticality of $\{H_1, ..., H_l\}$.

Step 5: Impossibility in spaces of dimension three or higher

Let F be efficient and not locally dictatorial on $(X, \mathcal{H})$; we show that $\dim(X, \mathcal{H}) \leq 2$ by contraposition. Thus, assume that H_1, H_2, H_3 are pairwise independent. By Step 4, $(X, \mathcal{H})$ is a median space, thus all critical families have two elements. As is easily verified this implies $G_1 \cap G_2 \cap G_3 \neq \emptyset$ whenever $G_j \in \{H_j, H_j^c\}$ for $j = 1, 2, 3$. Consider any three single-peaked preferences $\succ^1$, $\succ^2$ and $\succ^3$ such that $\succ^1$ has peak $x \in H_1^c \cap H_2 \cap H_3$, $\succ^2$ has peak $y \in H_1 \cap H_2^c \cap H_3$ and $\succ^3$ has peak $z \in H_1 \cap H_2 \cap H_3^c$. Also, choose an element $w \in H_1^c \cap H_2^c \cap H_3^c$. By weak neutrality, $\mathcal{W}_{H_j} = \mathcal{W}_{H_j^c} = \mathcal{W}^*$. Let W be minimal in $\mathcal{W}^*$, and let $i \in W$. Consider a preference profile where all voters in $W \setminus \{i\}$ have preference $\succ^1$, all voters in W^c have preference $\succ^2$, and i has preference $\succ^3$. Since F is not locally dictatorial, the outcome v under F lies in $H_1 \cap H_2 \cap H_3$ as in Step 4 above. By construction, v is not between any of the peaks x, y, z and w. By the richness condition R2, the preferences can be chosen such that $w \succ^j v$ for $j = 1, 2, 3$, hence F is not efficient.

Combining Steps 1 to 5, we have thus far proved part a) of our main theorem, i.e. the necessary conditions for efficiency. We now turn to part b), i.e. the sufficient conditions.

Step 6: Almost dictatorship and weak neutrality implies efficiency

Let F be weakly neutral voting by issues with almost-dictator i. Partition $\mathcal{H}$ into

$$\begin{aligned} \hat{\mathcal{H}} &:= \{H \in \mathcal{H} : \{i\} \in \mathcal{W}_H \cap \mathcal{W}_{H^c}\}, \\ \hat{\mathcal{G}}^+ &:= \{G \in \mathcal{H} : \{i\} \notin \mathcal{W}_G\}, \text{ and} \\ \hat{\mathcal{G}}^- &:= \{G \in \mathcal{H} : G^c \in \hat{\mathcal{G}}^+\}. \end{aligned}$$

Observe that, since i is an almost-dictator, $\{i\} \in \mathcal{W}_G$ for all $G \in \hat{\mathcal{G}}^-$. For a given preference profile $(\succ_1, ..., \succ_n)$, let $x = F(\succ_1, ..., \succ_n)$ and denote by x_i^* the peak of $\succ_i$. Clearly, x is efficient if $x = x_i^*$. Thus, assume that $x \neq x_i^*$. Suppose that $G \in \mathcal{H}$ separates x_i^* and x such that $x_i^* \in G$ and $x \in G^c$. Then, $\{i\} \notin \mathcal{W}_G$, hence $G \in \hat{\mathcal{G}}^+$. Now consider any other separating $\tilde{G}$ such that $x_i^* \in \tilde{G}$ and $x \in \tilde{G}^c$. Clearly, one has $\tilde{G} \in \hat{\mathcal{G}}^+$; moreover, $G^c \cap \tilde{G}^c \neq \emptyset$, since otherwise $G^c \subseteq \tilde{G}$ and hence, using (4), $\{i\} \in \mathcal{W}_{G^c} \subseteq \mathcal{W}_{\tilde{G}}$. By weak neutrality, we must have either $G^c \cap \tilde{G} = \emptyset$, or $G \cap \tilde{G}^c = \emptyset$; indeed, otherwise $G \perp \tilde{G}$, which would imply $\mathcal{W}_G = \mathcal{W}_{G^c}$ as in Step 2. Therefore, one has either $G \subseteq \tilde{G}$ or $\tilde{G} \subseteq G$. The family $\{G_1, ..., G_k\}$ of all separating basic properties with $x_i^* \in G_j$ and $x \in G_j^c$ can thus be ordered such that $G_1 \subseteq G_2 \subseteq ... \subseteq G_k$.

We will now show that no $y \neq x$ can be unanimously preferred to x. First, suppose $y \in G_k^c$; then, $y \in G_j^c$ for all $j \in \{1, ..., k\}$, hence $x \in [x_i^*, y]$, i.e. x is between i's peak and y. By single-peakedness, $x \succ_i y$.

Next suppose $y \in G_k$. Since $x = F(\succ_1, ..., \succ_n) \in G_k^c$, there must exist some voter h with peak x_h^* in G_k^c. We claim that $x \in [x_h^*, y]$, and therefore by single-peakedness, $x \succ_h y$. Suppose, by way of contradiction, $x \notin [x_h^*, y]$. Then, there exists $H \in \mathcal{H}$ such that $\{x_h^*, y\} \subseteq H$ and $x \in H^c$. We have $x_h^* \in G_k^c \cap H$, $x \in G_k^c \cap H^c$ and $y \in G_k \cap H$. Since $G_k \in \hat{\mathcal{G}}^+$, $G_k \not\perp H$ by weak neutrality, hence $G_k \cap H^c = \emptyset$, i.e. $G_k \subseteq H$. But then $x_i^* \in H$ and $x \in H^c$ which implies $H = G_k$ by definition of G_k. This contradicts the fact that $x_h^* \in G_k^c \cap H$.

Step 7: Weak neutrality implies efficiency in two-dimensional spaces

Let $x = F(\succ_1, ..., \succ_n)$, and consider any $y \neq x$. We will show that y cannot be unanimously preferred to x. Denote by $\mathcal{H}_{x\neg y} = \{H \in \mathcal{H} : x \in H, y \notin H\}$. We consider two cases. First, suppose that for all $G, \tilde{G} \in \mathcal{H}_{x\neg y}$, $G \not\perp \tilde{G}$. This implies $G \subseteq \tilde{G}$, or $\tilde{G} \subseteq G$, hence the family $\mathcal{H}_{x\neg y} = \{G_1, ..., G_k\}$ can be ordered such that $G_1 \subseteq G_2 \subseteq ... \subseteq G_k$. Since $x \in G_1$, there must be at least one voter i with peak x_i^* in G_1. By construction, $x \in [x_i^*, y]$, hence by single-peakedness, $x \succ_i y$.

Thus, suppose now that there exist $G, H \in \mathcal{H}_{x\neg y}$ such that $G \perp H$. Since $\dim(X, \mathcal{H}) \leq 2$, one has $\tilde{G} \not\perp G$ or $\tilde{G} \not\perp H$ for any $\tilde{G} \in \mathcal{H}$. In particular, for any $\tilde{G} \in \mathcal{H}_{x\neg y}$, either $[\tilde{G} \subseteq G$ or $G \subseteq \tilde{G}]$ or $[\tilde{G} \subseteq H$ or $H \subseteq \tilde{G}]$. Thus, $\mathcal{H}_{x\neg y}$ can be partitioned into $\mathcal{H}^1_{x\neg y} = \{G_1, ..., G_k\}$ and $\mathcal{H}^2_{x\neg y} = \{H_1, ..., H_l\}$ such that $G \in \mathcal{H}^1_{x\neg y}$, $H \in \mathcal{H}^2_{x\neg y}$, $G_1 \subseteq G_2 \subseteq ... \subseteq G_k$, and $H_1 \subseteq H_2 \subseteq ... \subseteq H_l$. Since $x \in G_1$, $W := \{i : x_i^* \in G_1\} \in \mathcal{W}_{G_1}$. Moreover, since $G_1 \subseteq G$ and $G \in \mathcal{H}^*$, $W \in \mathcal{W}^*$ by weak neutrality using (5). Similarly, $W' := \{i : x_i^* \in H_1\} \in \mathcal{W}_{H_1} \subseteq \mathcal{W}^*$. By (3), $W \cap W' \neq \emptyset$, i.e. there exists $x_i^* \in G_1 \cap H_1$. By construction, $x \in [x_i^*, y]$, hence by single-peakedness, $x \succ_i y$.

Remaining Proofs

Proof of Fact 1 The result follows from van de Vel (1993, Theorem II.4.16 and Exercise II.4.30.5), in view of the fact that in median spaces the set of all halfspaces is the unique subbase of halfspaces.

Proof of Fact 2 Evidently, a voter i can enforce the outcome x in the sense that $F(\succ_1, ..., \succ_n) = x$ whenever $x_i^* = x$ if and only if $\{i\} \in \mathcal{W}_H$ for all $H \in \mathcal{H}$ with $H \ni x$. This implies part a) since a dictator can enforce any outcome.

To verify part b), suppose that i is a local dictator over the subdomain $\mathcal{D}$ such that x and y are two distinct possible peaks of the preferences in $\mathcal{D}$. Then, $\{i\} \in \mathcal{W}_H \cap \mathcal{W}_{H^c}$ for any H that separates x and y. Conversely, suppose that $\{i\} \in \mathcal{W}_H \cap \mathcal{W}_{H^c}$ for some H; choose $x \in H$ and $y \in H^c$ such that the segment $[x, y]$ is inclusion minimal. Using the transitivity of the betweenness relation (cf. NP (2004)), minimality implies $[x, y] = \{x, y\}$. Consider a profile $(\succ_1, ..., \succ_n)$ of single-peaked preferences such that i has peak at x and all other voters have their peak at y. Since i is winning for H, one has $F(\succ_1, ..., \succ_n) \in H$; moreover, $F(\succ_1, ..., \succ_n) \in [x, y]$ by Proposition 2, hence in fact $F(\succ_1, ..., \succ_n) = x$ by the choice of x and y. Analogously, if i has peak at y and all others at x, one can conclude $F(\succ_1, ..., \succ_n) = y$. Thus, i is a dictator on an appropriate subdomain of preferences with peaks at x or at y.

Finally, consider part c). Given $H \in \mathcal{H}$, choose $x \in H$ and $y \in H^c$ such that the segment $[x, y]$ is inclusion minimal. As above, this implies $[x, y] = \{x, y\}$. Without loss of generality suppose that i can veto y when all peaks are in $\{x, y\}$. If all voters but i have their peak at y but at the same time $F(\succ_1, ..., \succ_n) \neq y$, we must have that i has peak at x, and thus $F(\succ_1, ..., \succ_n) = x$ by Proposition 2. This implies $\{i\} \in \mathcal{W}_H$. To show the converse implication, consider any pair $x \neq y$, and choose H such that $x \in H$ and $y \in H^c$. Obviously, if $\{i\} \in \mathcal{W}_H$ then voter i can veto y, and if $\{i\} \in \mathcal{W}_{H^c}$ then voter i can veto x.

Proof of Proposition 1 Suppose that $f(\xi) \in Co\{\xi_1, ..., \xi_n\}$. Consider, for each i, the segment $[\xi_i, f(\xi)]$. It suffices to show that $\cap_{i=1}^n [\xi_i, f(\xi)] = \{f(\xi)\}$; from this, the weak efficiency is immediate. Suppose, by way of contradiction that each segment $[\xi_i, f(\xi)]$ contains an element $x \neq f(\xi)$. Choose a basic property $H \in \mathcal{H}$ such that $f(\xi) \in H$ and $x \in H^c$. Since H^c is convex, it is not possible that $\{\xi_1, ..., \xi_n\} \subseteq H^c$ (otherwise, the convex hull of $\{\xi_1, ..., \xi_n\}$ would be contained in H^c). Hence, there exists j such that $\xi_j \in H$. Since H is convex, $[\xi_j, f(\xi)] \subseteq H$, hence $x \notin [\xi_j, f(\xi)]$.

Conversely, let $(X, \mathcal{H})$ be a median space. By Nehring and Puppe (2002, Proposition 4.1), there exist a unique "projection" of $f(\xi)$ on the convex hull of the voters' peaks, i.e. an element $y \in Co\{\xi_1, ..., \xi_n\}$ such that $y \in [\xi_i, f(\xi)]$ for all i. Obviously, if $y \neq f(\xi)$, f is not weakly efficient.

Proof of Proposition 2 It is clear that F is onto if the social choice is always contained in the convex hull of the voters' peaks. Conversely, suppose that F is strategy-proof and onto. By Theorem A, F is voting by issues. Since the set of all voters is always winning, one has $F(\succ_1, ..., \succ_n) \in H$ for all $H \supseteq \{x_1^*, ..., x_n^*\}$; by definition, this means that $F(\succ_1, ..., \succ_n) \in Co\{x_1^*, ..., x_n^*\}$.

Proof of Proposition 3 a) Clearly, 1-efficiency implies voter sovereignty. Conversely, if F is onto it is voting by issues by Theorem A. Let $x = F(\succ_1, ..., \succ_n)$, and consider any $y \neq x$. If $d(x, y) = 1$, the family $\mathcal{H}_{x\neg y}$ is ordered by set inclusion; let $G \in \mathcal{H}_{x\neg y}$ be

inclusion minimal. Since $x \in G$, there exists a voter i with $x_i^* \in G$. By construction, $x \in [x_i^*, y]$, thus by single-peakedness $x \succ_i y$, and therefore F is 1-efficient.
b) The proof of necessity of weak neutrality parallels Step 1 in the proof of the main theorem. In addition, one needs to verify that, given the basic properties H and $\tilde{H}$ as in Step 1, one can choose the elements $x \in H \cap \tilde{H}$, $y \in H^c \cap \tilde{H}^c$ and $w \in H^c \cap \tilde{H}$ such that $d(w, z) = 2$, where $z = F(\succ'; W, \succ''; W^c)$ as in Step 1 above. This follows from the fact that any median space is a graphic space with an underlying graph that is bipartite (cf. van de Vel (1993)). The proof of sufficiency is the same as in Step 7 in the main theorem.
c) Let F be 3-efficient. By parts a) and b), F is weakly neutral voting by issues. If F is locally dictatorial, it is in fact almost dictatorial by Step 3 of the proof of the main theorem, hence efficient by Step 6 of that proof. Thus, suppose that F is not locally dictatorial. Then one must have $\dim(X, \mathcal{H}) \leq 2$, in which case 3-efficiency trivially coincides with efficiency. Indeed, assume by way of contradiction, that $\dim(X, \mathcal{H}) \geq 3$. By van de Vel (1993, Theorem II.4.19, p.235), $(X, \mathcal{H})$ contains a graphic 3-hypercube. In this hypercube, a counterexample to the 3-efficiency can be constructed using the arguments of Step 5 in the proof of the main theorem.

References

[1] Barberá, S., F. Gul and E. Stacchetti (1993), Generalized Median Voter Schemes and Committees, *Journal of Economic Theory* **61**, 262-289.

[2] Barberá, S., J. Massò and A. Neme (1997), Voting under Constraints, *Journal of Economic Theory* **76**, 298-321.

[3] Barberá, S., H. Sonnenschein and L. Zhou (1991), Voting by Committees, *Econometrica* **59**, 595-609.

[4] Border, K. and J.S. Jordan (1983), Straightforward Elections, Unanimity and Phantom Voters, *Review of Economic Studies* **50**, 153-170.

[5] Brams, S., D.M. Kilgour and W. Zwicker (1998), The Paradox of Multiple Elections, *Social Choice and Welfare* **15**, 211-236.

[6] Demange, G. (1982), Single Peaked Orders on a Tree, *Mathematical Social Sciences* **3**, 389-396.

[7] Dietrich, F. and C. List (2005), Strategy-Proof Judgment Aggregation, *mimeo.*

[8] Green, J.R. and J.-J. Laffont (1979), *Incentives in Public Decision Making* North-Holland, Amsterdam.

[9] Kim, K.H. and F.W. Roush (1984), Non-Manipulability in Two Dimensions, *Mathematical Social Sciences* **8**, 29-43.

[10] Kornhauser, L.A. and L.G. Sager (1986), Unpacking the Court, *Yale Law Journal* **96**, 82-117.

[11] List, C. and P. Pettit (2002), Aggregating Sets of Judgements: An Impossibility Theorem, *Economics and Philosophy* **18**, 89-110.

[12] Moulin, H. (1980), On Strategy-Proofness and Single-Peakedness, *Public Choice* **35**, 437-455.

[13] Nehring, K. (1999), Diversity and the Geometry of Similarity, *mimeo.*

[14] Nehring, K. (2004), Social Aggregation without Veto, *mimeo.*

[15] Nehring, K. and C. Puppe (2002), Strategy-Proof Social Choice on Single-Peaked Domains. Possibility, Impossibility and the Space Between, *mimeo.*

[16] Nehring, K. and C. Puppe (2004), The Structure of Strategy-Proof Social Choice: General Characterization and Possibility Results on Median Spaces, *Journal of Economic Theory*, forthcoming.

[17] Nehring, K. and C. Puppe (2005a), On the Possibility of Strategy-Proof Social Choice: Non-Dictatorship, Anonymity and Neutrality, *mimeo.*

[18] Nehring, K. and C. Puppe (2005b), Consistent Judgement Aggregation: A Characterization, *mimeo.*

[19] Peters, H., H. van der Stel and T. Storcken (1992), Pareto Optimality, Anonymity, and Strategy-Proofness in Location Problems, *International Journal of Game Theory* **21**, 221-235.

[20] Peters, H., H. van der Stel and T. Storcken (1993), Generalized Median Solutions, Strategy-Proofness and Strictly Convex Norms, *ZOR - Methods and Models of Operations Research* **38**, 19-53.

[21] Schummer, J. and R. Vohra (2002), Strategy-Proof Location on a Network, *Journal of Economic Theory* **104**, 405-428.

[22] Shimomura, K.-I. (1996), Partially Efficient Voting by Committees, *Social Choice and Welfare* **13**, 327-342.

[23] van de Vel, M. (1993), *Theory of Convex Structures*, North Holland, Amsterdam.