# How Often Are You Decisive: an Enquiry About the Pivotality of Voting Rules

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**Summary.** The probability that an individual is decisive in an election is an important criterion for the evaluation of voting rules. It depends on factors such as the number and the behaviour of the other voters, the available alternatives and, of course, also on the voting rule itself. Classical power indices like the Banzhaf- or the Shapley-Shubik-Index are only applicable in special cases. In this paper, an approach is proposed that can be viewed as a natural extension of the Banzhaf-Index for more than two alternatives and for different stochastic assumptions.

The approach is applied to plurality voting with two and more alternatives and computed for variations of the number of voters and alternatives. For three alternatives the pivotality is also computed for the Borda-Rule. The comparison of the computations for three alternatives shows that the probability of being decisive under the Borda-Rule is uniformly larger than under plurality voting.

## 1 Introduction

The influence of a participant in a decision or voting procedure is an important criterion for many aspects regarding the evaluation of voting rules: On the one hand, a voter can use his influence to manipulate the result of an election, i.e. she can give a vote that does not correspond to her true preferences over the alternatives. The well-known Gibbard-Satthernwaite Theorem states that every social choice function over three or more alternatives can be manipulated by misrepresentation of preferences. An important question is which voting rules have a high chance and which a low chance of being susceptible to strategic manipulation.

On the other hand in democratic decision procedures like parliament elections the participation of many voters is desired, and therefore the chance that one's vote makes a difference should be high. There is the well known problem of lack of incentive for voters to participate in (large) elections when the probability that they are decisive is low. Also, in decision procedures like comittees the probability that a participant has influence on the outcome should be high, in order to induce comittee members to carefully consider their decisions.

One way to measure influence is via so-called power indices like the Banzhaf- and the Shapley-Shubik-Index, originally used in cooperative game theory. In fact, these indices are only applicable in special situations with two alternatives and under specific probabilistic assumptions about the behaviour of voters. For example, the probabilistic assumption of the Banzhaf-Index is that voters act independently and vote with 0.5 for the first and 0.5 for the second alternative (see [5]).

In this paper we propose a general approach to measure the influence of an individual in a voting situation called pivotality. By pivotality we mean the probability that one's vote can change the winner of an election, i.e. that one is a pivotal figure in the voting situation. Manipulability is a special case: a voting rule is said to be manipulable, if by *misrepresenting* her preferences a voter can change the result of the election *to her own benefit*.

Pivotality depends on four factors: the number $n$ of voters and the number $k$ of alternatives, the behaviour of the voters (modelled by a discrete probability distrubition $P$ over the possible votes) and finally the voting rule itself (denoted by $f$). Furthermore we assume in what follows that voters act indepedently.

## 2 The Model

Let $X$ be a finite set of alternatives (e. g. candidates, policies) and $|X| = k$. Formally, a voting rule is a social choice function $f : V \to X$ which maps a vector $v = (\succ_1, \ldots, \succ_n)$ of the votes of $n$ individuals to a chosen alternative ("the winner") $r \in X$. Each vote is a preference ordering (i.e. a strict linear ordering) of the $k$ alternatives. Note that with $k$ alternatives there are $m = k!$ different votes.

We will only consider voting rules that satisfy anonymity, i.e. we assume $f(v) = f(\sigma(v))$ for all permutations $\sigma$ of the votes. Thus, the relevant information for determining the result of an election is the anonymous profile, i.e. the frequency distribution of the different votes, which will be denoted by $a = (a_1, \ldots, a_m)$, where $a_i$ is the frequency of votes of the i-th type, where votes ("types") are ordered in some way. The behaviour of a voter is modelled by a discrete probability distribution $P = (p_1, \ldots, p_m)$ over the possible votes. The probability to observe a particular anonymous profile follows a multinomial distribution with parameters $n$ and $P$:

$$p(a = (a_1, a_2, \ldots, a_m)) = \binom{n}{a_1, a_2, \ldots, a_m} p_1^{a_1}\ p_2^{a_2}\ \ldots\ p_m^{a_m} \qquad (1)$$

To calculate the pivotality of a voting rule in dependence of $n$, $k$ and $P$ we have to sum the probability of all anonymous profiles at which an additional voter can change the outcome of the voting rule. We denote the set of all anonymous profiles by $A$ and the subset of the pivotal profiles by $A^p$. This set, in turn, consists of two subsets $A_1^p$ and $A_2^p$ corresponding to two different ways to be pivotal: first, one can influence the election in a way that one's own vote unambiguously decides the winner, for example if there is a tie among the other votes (set $A_1^p$); alternatively, it is possible that one can only produce a tie and the final outcome is decided by a random device (set $A_2^p$). The pivotality of every voting rule that satisfies anonymity and where voters act independently (and are identically distributed) is described by:

$$\sum_{a \in A_1^p} \binom{n}{a_1, a_2, \ldots, a_m} p_1^{a_1}\ p_2^{a_2}\ \ldots\ p_m^{a_m} \qquad (2)$$

$$+\frac{1}{2} \sum_{a \in A_2^p} \binom{n}{a_1, a_2, \ldots, a_m} p_1^{a_1}\ p_2^{a_2}\ \ldots\ p_m^{a_m} \qquad (3)$$

The pivotality of a voter consists of two sums: the first is the sum of the probability of those situations, where one can unambiguously decide the winner, and the second is the sum of those situations, where one can produce a tie. The latter is weighted by $\frac{1}{2}$, the probability that the following tie-breaker leads to an alternative outcome[1].

The elements of $A^p$ and thus the value of the expression above were computed using Matlab. First, the elements of $A_1^p$ and $A_2^p$ are determined. Subsequently, the probabilities of their occurence are computed

[1] From the viewpoint of pivotality, ties with more than two alternatives are subsumed under the set $A_1^p$. This is in contrast to the case of manipulability, i.e. the probability of being able to influence the outcome to one's own benefit.

using the multinomial distribution. Although the runtime of this procedure can be long, the advantage of the approach is that the pivotality is computed exactly instead of estimating it by Monte-Carlo-Simulation (for more details see [3]).

For most voting rules, pivotality is equivalent to the (non-normalized) Banzhaf-Index $\beta'$, provided that there are only two alternatives with $P = (0.5; 0.5)$. To see this, let $Z$ be a random variable that counts the votes for the first alternative and let $s$ be the number of so-called swing coalitions (i.e. situations with a pivotal voter). If $n$ is even, one is pivotal if and only if $\frac{n}{2}$ individuals are voting for one alternative (there is a tie and one breaks it with one's vote). But these are exactly the situations in which one is the "swing voter." For $p = 0.5$ one obtains the Banzhaf-Index:[2]

$$P\left(Z = \frac{n}{2}\right) = \binom{n}{\frac{n}{2}} \cdot p^{\frac{n}{2}} \cdot (1-p)^{n-\frac{n}{2}} \tag{4}$$

$$= \binom{n}{\frac{n}{2}} \cdot \left(\frac{1}{2}\right)^{\frac{n}{2}} \cdot \left(\frac{1}{2}\right)^{n-\frac{n}{2}} \tag{5}$$

$$= \frac{s}{2^n} = \beta' \tag{6}$$

The expression in (4) has been first described by [4] and [1] and can be approximated by Stirling's Formula for large $n$.

## 3 Applications

We will use now our general approach to measure pivotality in three examples; in each pivotality is calculated under a variation of one of the four influencing factors.

### 3.1 Comparison of stochastic assumptions

As already noted in the introduction, the Banzhaf-Index is only applicable for special cases. We examine the pivotality of a simple election with two alternatives under majority rule by computing the values of (4) for $p \in (0; 1)$ and $n = 2, 3, \ldots, 100$ which is displayed in Figure 1. The result is "knife-edged": For $p = 0.5$ the pivotality of this election

[2] If $n$ is odd, then there two situations where one's vote can produce a tie - in each there is a chance of 0.5 to win.

is just the Banzhaf-Index; even for small deviations from $p = 0.5$ the pivotality is declining rapidly (see also [2]).

One important application is to determine for a given level of pivotality the appropiate groupsize of voters, if $p$ is known or can be estimated reliably. For this, we have to calculate “iso-pivotality-lines”, i.e. combinations of $p$ and $n$ with the same pivotality (see Figure 1 right).

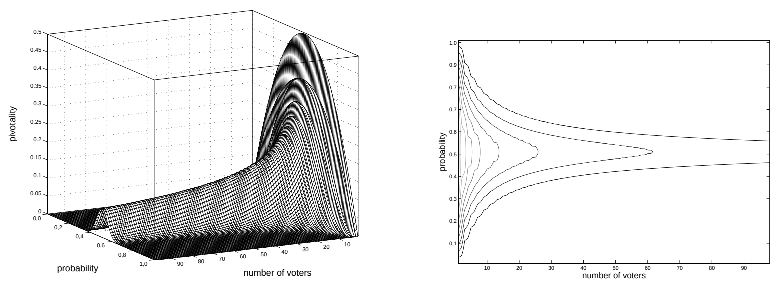


**Fig. 1.** Pivotality of majority rule with two alternatives

### 3.2 Comparison of the quantity of alternatives

While more voters usually yield less pivotality we can also examine the influence of more alternatives in an election. We calculate the pivotality of the plurality rule for $k = 2, \ldots, 10$ under the impartial culture assumption (i.e. under the assumption that each preference ordering has equal probability for each voter). Therefore, we can compare the pivotality for variations of the number of alternatives. The values are shown in Figure 2: more alternatives give rise to more pivotality under our assumptions.

### 3.3 Comparison of voting rules

Finally, we compare different voting rules in a given situation. We examine the pivotality of the plurality rule and the Borda Count for $n = 2, \ldots, 30$ voters and three alternatives under the impartial culture assumption. The comparison in Figure 3 shows that the pivotality of the Borda Count is uniformly larger than the one of the plurality rule in this situation.

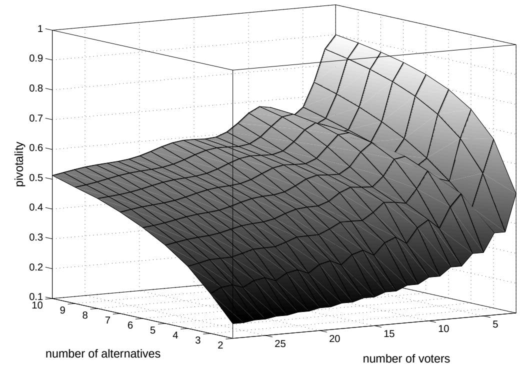


**Fig. 2.** Pivotality of plurality rule for different quantities of alternatives

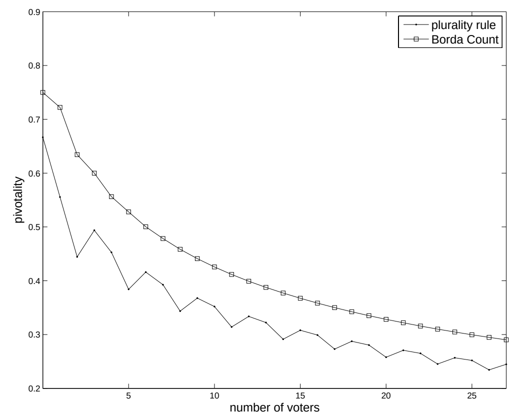


**Fig. 3.** Comparision of plurality rule and Borda Count

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